DISCRETE-TO-CONTINUUM LIMITS OF PARTICLES WITH AN ANNIHILATION RULE*

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Abstract. In the recent trend of extending discrete-to-continuum limit passages for gradient flows of single-species particle systems with singular and nonlocal interactions to particles of opposite sign, any annihilation effect of particles with opposite sign has been side-stepped. We present the first rigorous discrete-to-continuum limit passage which includes annihilation. This result paves the way to applications such as vortices, charged particles, and dislocations. In more detail, the discrete setting of our discrete-to-continuum limit passage is given by particles on the real line. Particles of the same type interact by a singular interaction kernel; those of opposite sign interact by a regular one. If two particles of opposite sign collide, they annihilate, *i.e.*, they are taken out of the system. The challenge for proving a discrete-to-continuum limit is that annihilation is an intrinsically discrete effect where particles vanish instantaneously in time, while on the continuum scale the mass of the particle density decays continuously in time. The proof contains two novelties: (i) the observation that empirical measures of the discrete dynamics (with annihilation rule) satisfy the continuum evolution equation that only implicitly encodes annihilation, and (ii) the fact that, by imposing a relatively mild separation assumption on the initial data, we can identify the limiting particle density as a solution to the same continuum evolution equation.

Key words. Particle system, discrete-to-continuum asymptotics, annihilation, gradient flows

AMS subject classifications. 82C22, (82C21, 35A15, 74G10).

1. Introduction.

A recent trend in discrete-to-continuum limit passages in overdamped particle systems with singular and nonlocal interactions (with applications to, *e.g.*, vortices [9, 19, 38], charged particles [36], dislocations [18, 27, 30], and dislocation walls [13, 47, 48]) is to extend such results to two-species particle systems. The singularity in the interaction potential imposes the immediate problem that the evolution of the particle system is only defined up to the first collision time between particles of opposite sign. This problem is dealt with by either *regularising* the singular interaction potential (see [11, 12]) or by limiting the geometry such that particles of opposite sign cannot collide (see [7, 46]). However, more realistic models of vortices, charged particles, and dislocations include the *annihilation* of particles of opposite sign. While annihilation has been analysed on the discrete scale [40, 41] and continuum scale [3, 6] separately, there is no rigorous discrete-to-continuum limit passage known between these two scales.

The main result in this paper establishes the first result on a discrete-to-continuum limit passage in two-species particle systems in one dimension with annihilation.

Below, we first describe the physical context of our main result. Then, we introduce the discrete and continuum problems. Our main result is the connection between them in terms of the limit passage as the number of particles n tends to ∞. Then, we put our discrete and continuum problems in the perspective of the literature, and com-

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ment how our proof combines known techniques with novel ideas. We conclude with an exposition of possible extensions to work towards singular interspecies interactions and higher dimensions.

1.1. Application to plasticity and dislocations. The main application we have in mind is to increase the understanding of the plastic behaviour of metals. Plasticity in metals is the emergent behaviour of large groups of dislocations moving and interacting on microscopic time- and length-scales. Dislocations are stacking faults in the atomic lattice. We keep the description of dislocations concise, and refer to the classical textbooks [21, 24] for a detailed description. In two-dimensional elastic bodies, dislocations are often represented as points in the elastic body at which the stress has a prescribed singularity. This singularity depends on the orientation of the dislocation, which is described by the so-called Burgers vector. While dislocations themselves exert a stress field, they can also move in response to the stress induced by other dislocations in the elastic body. The simplest model to capture such effects is an interacting particle system which fits to the setting in this paper.

One of the main unsolved problems in plasticity is how to describe the group behaviour of many dislocations in terms of a dislocation density. While there are many different models available in the engineering literature for the dislocation density [5, 15, 16, 22, 25, 26, 42], it is not clear which of these models describes the group behaviour of a given collection of dislocations for a given set of parameters. This problem arises from a lack of rigour in the derivation of these continuum dislocation models from the dynamics of a large group of interacting dislocations (called discrete dislocation models).

To resolve this lack of rigour, over the course of two decades a large mathematical community has established rigorous connections between discrete and continuum dislocation models; see [1, 10, 11, 13, 18, 29, 30] for a few examples of different discrete dislocation models and different techniques. The final aim is to lift all the currently required simplifications on the discrete dislocation models without losing the rigorous connection(s) with the related continuum model(s).

In recent years, the simplification that all dislocations have the same Burgers vector is being lifted. This generalisation corresponds to particle systems with multiple species. It has the difficulty that dislocations with different Burgers vector may collide in finite time (due to the singular stress they exert). In particular, two (screw) dislocations with opposite Burgers vector are known to collide in finite time [23], and disappear upon collision. Such a collision is called annihilation. In the current literature, the difficulty of including annihilation or other collision rules is side-stepped by either enforcing geometrical restrictions [7, 46], or by introducing an artificial regularisation of the singularity in the stress field (see [12] and [44, Chap. 9]). A common observation in these papers is that, depending on the geometrical restrictions or the regularisation, rigid micro-structures can appear over time which are not recovered by the expected continuum dislocation model. In fact, the simulations in [44, Chap. 9] show that the group behaviour of dislocations can depend on the choice of regularisation, which would imply that the continuum model has to depend on the choice of regularisation.

Therefore, to avoid the dependence of the continuum model on the choice of regularisation or geometrical restrictions, we aim to make the first step for including dislocation annihilation in connecting discrete to continuum dislocation models. Our novel result includes an annihilation rule, but sidesteps the additional difficulty that prior to collision, the speed of the colliding dislocations becomes unbounded. To avoid unbounded velocities prior to collision, we replace the singular interaction between dislocations of opposite Burgers vector by a regular one. This choice induces the further restriction of a one-dimensional spatial setting, which is needed to enforce collisions. Indeed, for regular interactions in higher dimensions, dislocations of opposite Burgers vector need not collide in finite time.

In Section 1.7 we demonstrate how our main result can be used as a stepping stone for considering annihilation with singular interactions between dislocations of opposite Burgers vector.

1.2. The discrete problem (particle system with annihilation). We return our attention from dislocations to a more general particle system with two species and an annihilation rule. We introduce the related evolution problem by first specifying the state of the system, then the related interaction energy, and finally the evolution law. The state of the system is described by $x \coloneqq (x_1, \dots, x_n) \in \mathbb{R}^n$ and $b \coloneqq (b_1, \dots, b_n) \in \{-1, 0, 1\}^n$, with $n \geq 2$ the number of particles. The point x_i is the location of the i-th particle, and b_i is its charge (or Burgers vector, in the setting of dislocations).

To any state (x, b) we assign the interaction energy $E_n \colon \mathbb{R}^n \times \{-1, 0, 1\}^n \to \mathbb{R} \cup \{+\infty\}$ by

$$E_n(x; b) \coloneqq \frac{1}{2n^2} \sum_{i=1}^n \Bigg(\sum_{\substack{j=1 \\ j \neq i \\ b_i b_j = 1}}^n V(x_i - x_j) + \sum_{\substack{j=1 \\ b_i b_j = -1}}^n W(x_i - x_j) \Bigg), \tag{1.1}$$

where V and W are the interaction potentials between particles of equal and opposite charge, respectively. For V and W, we have three choices in mind, all of which are of separate interest:

(i) $V(r) = -\log|r|$ and $W \equiv 0$. This corresponds to the easiest case in which the two species only interact with their own kind. It is distinct from the single-particle case solely by the annihilation rule which we specify below. We consider this setting as a convenient benchmark problem, but we have no direct application in mind.

(ii) $V(r) = -\log|r|$ and W a regularisation of $-V$ (as illustrated in Figure 1). This is a first step to considering the case of positive and negative charges (or positive and negative dislocations) in which $W = -V$ is chosen in a two-dimensional setup [40, 41, 43]. After stating our main result for regular W, we comment in Subsection 1.7 on how this result helps in passing to the limit in the particle dynamics corresponding to regular potentials W_δ which converge to the singular $-V$ as the regularisation parameter δ tends to 0.

(iii) $V(r) = r \coth r - \log|2 \sinh r|$ and W a regularisation of $-V$. This setting corresponds to that of dislocation walls, *i.e.*, infinite arrays of equi-spaced dislocations. The explicit expression for V is found by summing over all dislocations in such a wall; see [21, (19-75)] or [46, Sec. 2]. This potential V has several pleasant properties: it has a logarithmic singularity at 0, it is decreasing on $(0, \infty)$, and it is positive with integrable tails. Discrete-to-continuum limits of particle systems consisting of interacting dislocation walls are established in [13, 17, 21, 46, 47, 48] for either single-sign scenarios or without annihilation.

For our analysis, we propose a unified setting which includes the three cases above: we consider a class of potentials V and W which satisfy a certain set of assumptions

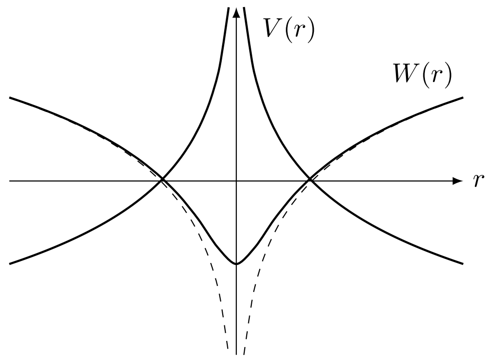

FIGURE 1. *Plots of $V(r) = -\log|r|$ and a typical regularisation W of $-V$.*

specified in Assumption 2.1. The crucial assumptions are that the singularity of V at 0 is at most logarithmic, that $V(r) \to +\infty$ as $r \to 0$, that W is regular, and that V and W have at most logarithmic growth at infinity. In view of other typical assumptions in the literature, we *do not* rely on convexity or monotonicity. In Subsection 1.6 we elaborate on the necessity of these assumptions to our main discrete-to-continuum result.

Finally, we make three observations on the structure of (1.1). First, if the i-th particle has 0 charge (*i.e.*, $b_i = 0$), then it does not contribute to E_n. Second, the factor $1/2$ in front of the energy is common; it corrects the fact that all interactions are counted twice in the summation. Third, the condition $j \neq i$ prevents self-interaction.

Equation (1.2) formally describes the dynamics; for a rigorous definition see Problem 4.1 and Definition 4.2.

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} x_i = -\dfrac{1}{n} \displaystyle\sum_{j \,:\, b_i b_j = 1} V'(x_i - x_j) - \dfrac{1}{n} \displaystyle\sum_{j \,:\, b_i b_j = -1} W'(x_i - x_j) \quad \text{on } (0,T) \setminus T_{\mathrm{col}}, \\ \text{annihilation rule at } T_{\mathrm{col}}. \end{cases} \tag{1.2}$$

Here, $T_{\mathrm{col}} = \{t_1, \ldots, t_K\}$ is a finite set of collision times, outside of which $x(t)$ is the gradient flow of E_n. The version of (1.2) in two dimensions and in which $W(r) = -V(r) = \log|r|$ is discussed in great detail in [41].

Next we explain the "annihilation rule at T_{col}". Given that at $t = 0$ all particles are at different positions, (1.2) follows for at least a small time interval simply the gradient flow of $E_n(\cdot\,; b)$ in which b is constant in time. Since V is a singular, repelling interaction potential and W is regular, particles of the same sign will not cross each other, but particles of opposite sign may. We call the first time instance at which such a crossing happens a *collision time*, and denote it by t_1. At t_1, the annihilation rule states that those particles of opposite sign which are at the same position are 'removed' from the system, and that the system is restarted at time t_1 with the remaining particles at their current positions. It again follows the gradient flow of E_n (but now with fewer particles) until the next collision time t_2 at which two particles of opposite sign cross. At t_2, an analogous annihilation rule is applied. In this manner, T_{col} is constructed. We allow for more than one pair of particles to annihilate at the same time instance t_k. Because of the singularity of V, annihilations that happen at the same time always occur at different points in space.

For technical reasons, we encode the removal of particles by putting their charge $b_i(t)$ from ± 1 to 0 as opposed to making n dependent on t. We note that, if particle i has zero charge, then

- $x_i(t)$ remains stationary,
- the velocity of all other particles does not depend on $x_i(t)$, and
- particle i cannot annihilate any more with any other particle.

We note that each $b_i(t)$ is a shifted Heaviside functions that jumps at some collision time t_k.

Next we motivate the applicability of (1.2) by two related examples. The first example is that of dislocations, whose dynamical law naturally includes annihilation effects. The linear relation in (1.2) between the velocity and the gradient of the energy is purely phenomenological, and is, due to its simplicity and lack of consensus for a better alternative, the most commonly used relation in dislocation dynamics models. We refer to [43] for simulations of a generalized version of (1.2) in the context of dislocations.

The second example of a system related to (1.2) is that in [40] and [41, Theorems 1.3 and 1.4], where the limit of the Ginzburg-Landau equation on the dynamics of vortices is studied as the phase-field parameter ε tends to 0. In the limiting equation, the vortices are characterised as points with a charge whose dynamics are given by the version of (1.2) in which $W(r) = -V(r) = \log|r|$ and the particles are two-dimensional. While detailed properties of the particles trajectories are proven, a precise solution concept to this version of (1.2) remains elusive. In our one-dimensional setting, we establish a solution concept to (1.2) in Definition 4.2 and Proposition 4.5.

1.3. The continuum problem (PDE for the particle density). On the continuum level, the state of the system is described by the nonnegative measures $\rho^\pm$, which represent the density of the positive/negative particles (including those that are annihilated). We further set

$$\rho := \rho^+ + \rho^- \quad \text{and} \quad \kappa := \rho^+ - \rho^-,$$

and require the total mass of ρ to be 1. We note that ρ^+ and ρ^- need not be mutually singular, and thus $\rho^\pm \geq [\kappa]_\pm$, where $[\kappa]_\pm$ denotes the positive/negative part of the signed measure κ. We interpret $[\kappa]_\pm$ as the density of positive/negative particles that have not been annihilated yet.

For $\rho^\pm(t)$ we consider the following set of evolution equations

$$\begin{cases} \partial_t \rho^+ = \big([\kappa]_+ \left(V' * [\kappa]_+ + W' * [\kappa]_- \right) \big)' & \text{in } \mathcal{D}'((0,T) \times \mathbb{R}), \\ \partial_t \rho^- = \big([\kappa]_- \left(V' * [\kappa]_- + W' * [\kappa]_+ \right) \big)' & \text{in } \mathcal{D}'((0,T) \times \mathbb{R}), \end{cases} \tag{1.3}$$

where we denote by the prime symbol $'$ the derivative with respect to the spatial variable. We remark that no annihilation rule is specified; the annihilation is encoded in taking the positive/negative part of κ. Indeed, it is easy to imagine that while the integral of $\rho = \rho^+ + \rho^-$ is conserved in time, the integral of $[\kappa]_+ + [\kappa]_- = |\rho^+ - \rho^-|$ may not be conserved.

1.4. Main result: discrete-to-continuum limit. Our main theorem (Theorem 5.1) states that the solutions to (1.2) converge to a solution of (1.3) as $n \to \infty$. It specifies the concept of solution to both problems, the required conditions on the sequence of initial data of (1.2), and guarantees that the so-constructed solution to (1.3) at time 0 corresponds to the limit of the initial conditions as $n \to \infty$. The convergence is uniform in time on $[0, T]$ for any $T > 0$. The convergence in space is with respect to the weak convergence. As a by-product of Theorem 5.1, we obtain global-in-time existence of a solution (ρ^+, ρ^-) to (1.3) for which the masses of $\rho^\pm$ are conserved in time.

In order to give effectively an outline of the proof and the motivation for the main assumptions under which Theorem 5.1 holds (Subsection 1.6), we first describe the related literature.

1.5. Related literature. We start by relating (1.3) formally to its singular counterpart. Replacing W by $-V$, we obtain from a formal calculation that the difference of the two equations in (1.3) is given by

$$\partial_t \kappa = \big(|\kappa| (V' * \kappa) \big)'. \tag{1.4}$$

For $V(r) = -\log |r|$, equation (1.4) was introduced by [20] and later proven in [6] to attain unique solutions when posed on $\mathbb{R}$ with proper initial data.

In the remainder of this subsection, we put our main result Theorem 5.1 in the perspective of the literature. We start by describing those specifications of [10, 28, 29] which are closest to our main result. A specification of [10, Theorems 2.1–2.3] proves a 'discrete'-to-continuum result from (1.2) to (1.4), in the case where $V(r) = -W(r)$ is a regularisation of $-\log|r|$ on the length-scale $1/n$. We put 'discrete' in apostrophes, because their equivalent of (1.2), given by [10, equation (5)], is a Hamilton-Jacobi equation, which includes the solution to (1.2) only if all particles have the same sign. It is not clear if this Hamilton-Jacobi equation relates to (1.2) if the particles have opposite sign.

As opposed to [10], [29] starts from a different Hamilton-Jacobi equation, which corresponds to the Peierls-Nabarro model [32, 33]. This model is a phase-field model for the dynamics of dislocations which naturally includes annihilation. In this model, opposite to encoding dislocations as points on the line, the dislocations are identified by the pulses of the derivative of a multi-layer phase field on the real line. In [29], the width of these pulses is taken to be on the same length-scale as the typical distance between neighbouring dislocations. Then, in the joint limit when the regularisation length-scale (and thus simultaneously $1/n$) tend to 0, an *implicit* Hamilton-Jacobi equation is recovered [29]. In [28, Theorem 1.2] it is shown that this implicit Hamilton-Jacobi equation converges to (1.4) in the dilute dislocation density limit. While this framework seems promising for a direct 'discrete'-to-continuum result ('discrete' being the Peierls-Nabarro model) to (1.3), it only applies to co-dimension 1 objects, *i.e.*, particles in 1D and curves in 2D.

Regarding the continuum problem (1.3), we have not found this set of equations in the literature. Nonetheless, we believe the case $W = 0$ to be of independent interest, since then (1.3) serves as the easiest benchmark problem for future studies on annihilating particles. Also, since our discrete-to-continuum result holds for taking W as a regularisation of $-V$, we expect that (1.4) can be obtained from (1.3) as the regularisation length-scale tends to 0 (see Subsection 1.7). Therefore, we review the literature on (1.4).

Equation (1.4) as posed on $\mathbb{R}$ with $V(r) = -\log|r|$, or even $V(r) = |r|^{-a}$ with $0 < a < 1$, attains a self-similar solution [6, Theorem 2.4] in which κ has a sign. The self-similar solution is expanding in time (due to the repelling interaction force $V'(r)$), and describes the long-time behaviour of the unique viscosity solutions to (1.4) [6, Theorem 2.5] for appropriate initial data. Moreover, for $V(r) = -\log|r|$ and initial condition $\kappa^\circ \in L^1(\mathbb{R})$, the viscosity solution κ to (1.4) satisfies $\kappa(t) \in L^p(\mathbb{R})$ for all $1 \leq p \leq \infty$ [6, Theorem 2.7]. In conclusion, despite (1.4) being the singular counterpart of (1.3), it has a well-defined global-in-time solution concept.

Lastly, we compare our result to that of [3]. There, the authors are interested in deriving a gradient flow structure of (1.4) on $\mathbb{R}^2$ with V having a logarithmic singularity at 0 by defining a discrete in time minimising movement scheme and passing to the limit as the time step size tends to 0. The related convergence result is [3, Theorem 1.4]. However, the limit equation is not fully characterised as (1.4), since in that equation $|\kappa|$ is replaced by an unknown measure $\mu \geq |\kappa|$ which is obtained from compactness. The connection to our main result is that we faced a similar problem. Due to our 1D setup and by a technical assumption on the initial data, we were able to characterise the corresponding μ as $|\kappa|$.

1.6. Discussion on the proof, assumptions, and possible extensions. We divide this section into several topics regarding the proof, assumptions, and possible extensions of Theorem 5.1 (outlined in Subsection 1.4).

Summary of the proof. A crucial step is the observation that the solution to (1.2), seen as a pair of empirical measures $\mu_n^\pm$, is a solution to (1.3), *i.e.*,

$$\begin{cases} \partial_t \mu_n^+ = \big([\kappa_n]_+ \, (V' * [\kappa_n]_+ + W' * [\kappa_n]_-)\big)' & \text{in } \mathcal{D}'((0,T)\times\mathbb{R}), \\ \partial_t \mu_n^- = \big([\kappa_n]_- \, (V' * [\kappa_n]_- + W' * [\kappa_n]_+)\big)' & \text{in } \mathcal{D}'((0,T)\times\mathbb{R}), \end{cases} \tag{1.5}$$

where $\kappa_n := \mu_n^+ - \mu_n^-$. The annihilation is completely covered by taking the positive and negative part of κ_n. This property is the reason for encoding annihilation in the charges $b_i(t)$ rather than removing particles from the dynamics. Then, relying on the gradient flow structure underlying (1.2) and the boundedness of W, we find, by the usual compactness arguments *à la* Arzelà-Ascoli, limiting curves $\rho^\pm(t)$. It then remains to pass to the limit $n \to \infty$ in (1.5). The difficulty is in characterising the limit of $[\kappa_n]_\pm$, which only accounts for the particles that have not collided yet. Indeed, the convergence of measures is not invariant with respect to taking the positive and negative part. It is here that we heavily rely on the one-dimensional setting and on a technical assumption on the initial data (Assumption 2.2), which provides an n-independent bound on the number of neighbouring pairs of particles with opposite sign. This bound allows us to characterise the limit of $[\kappa_n]_\pm$ as $[\kappa]_\pm$.

Motivation for Assumption 2.2. Assumption 2.2 prevents small-scale oscillations between ± 1 phases. A similar assumption is made in [29], where the initial data for the particles is constructed from the continuum initial datum. While one might expect that small-scale oscillations cancel out on small time scales, the simulations in [45, Chapter 9] suggest otherwise. The problem with such small-scale oscillations is that they cause the limit of $[\kappa_n]_\pm$ to be larger than $[\kappa]_\pm$, which makes it difficult to characterise the limit as $n \to \infty$ of (1.5) as (1.3).

Singularity of V. Assuming the singularity of V to be at most logarithmic is needed to apply the discrete-to-continuum limit passage technique in [38].

In fact, we also *require* that $V(r) \to \infty$ as $r \to 0$, *i.e.*, we do not allow for a regular V. While regular V and W (in particular $W = -V$) would simplify the equations and many steps in the proof of our main theorem, it may result in two technical difficulties: collision between three or more particles, and the limiting signed measure κ having atoms. These difficulties complicate the convergence proof of $[\kappa_n]_\pm$ to $[\kappa]_\pm$ as $n \to \infty$. Since all our intended applications correspond to singular potentials V, we choose to side-step these technical difficulties by simply requiring V to have a singularity at 0.

Regularity of W. W being bounded around 0 results in a lower bound on the energy along the evolution, which we need for equicontinuity and thus for compactness of $\mu_n^\pm$. Also, while passing to the limit $n \to \infty$ in (1.5), we need W' regular enough (the technique in [38] does not apply for logarithmic W).

Logarithmic tails of V, W. While it would be easier to assume that V is bounded from below and W is globally bounded, we *also* allow for logarithmic tails to include all three scenarios in Subsection 1.2. The logarithmic tails of V and W result in the energy E_n to be unbounded from below. However, following the idea in [38] to prove a priori bounds on the *moments* of $\mu_n^\pm(t)$, we easily obtain that $E(\mu_n^\pm(t))$ is bounded from below by $-C(1+t)$ for some $C > 0$ independent of n and t.

Uniqueness of solutions to (1.3). While Theorem 5.1 provides a solution of (1.3) that exists globally in time, we have not investigated uniqueness. We rather interpret (1.3) as a stepping stone for a future convergence result to (1.4), for which a uniqueness result is established in [6].

1.7. Conclusion and outlook. We intend our main result to open a new thread of research on including annihilation in discrete-to-continuum limits. Here we discuss several open ends.

$W = -V$ *singular.* This setting corresponds to charges (or dislocations) on the real line. On the continuum level, see (1.4), this equation is well-understood [6], but on the discrete level we have not found a closed set of equations to describe the discrete counterpart of (1.2) (other than [40, 41], whose results are discussed in Subsection 1.5). Since our main result does allow for $-W$ to be a regularisation V_δ of V (δ denotes the arbitrarily small, but fixed, length-scale of the regularisation), this calls for three interesting limit passages:

(a) $\delta \to 0$ *with n fixed.* This limit seems the easiest out of the three. Similar to [40, 41], the idea is to pass to the limit, and *describe* the limit rather than posing a closed set of equations for it. One challenge is that in the limiting curves prior to collision at t_*, the particles' speed blows up as $\sim 1/\sqrt{t_* - t}$ (this is easily seen by considering only two particles; one positive and one negative). While the resulting curves are not Lipschitz in time, they are $\mathrm{C}^{1/2}$ in time. However, such collisions correspond to $-\infty$ wells in the energy, which require the development of a proper renormalisation of E_n.
Another challenge is that particles need not collide if they come close, regardless how small $\delta > 0$ is. To see this, consider two particles with opposite sign and with mutual distance smaller than δ. Since V_δ is regular, the particles will come exponentially close, but they will not collide in finite time. In the case of many particles, such a close pair will only collide if the external force (induced by the other particles) acts in the right direction. If it does not collide, then the pair remains in the system (as opposed to the case of singular W), and may even interact with or annihilate other particles that come close.

(b) *Connecting* (1.3) *to* (1.4) *by $\delta \to 0$.* Taking $W = -V_\delta$ and setting $\rho_\delta^\pm$ as a corresponding solution to (1.3), it is impossible to pass directly to the limit in (1.3) due to the term $[\kappa_\delta]_\pm (V_\delta' * [\kappa_\delta]_\mp)$. Instead, the structure of (1.4) in terms of viscosity solutions (see [6]) seems promising. We leave it to future research to find out whether (1.3) enjoys a similar structure, and if not, whether there is a different continuum model for annihilating particles that does.

(c) *Connecting* (1.2) *to* (1.4) *by a joint limit $n \to \infty$ and $\delta_n \to 0$.* This approach fits to the convergence result obtained in [29], where roughly speaking $\delta_n \sim$

$1/n$ is considered, but where a different equation than (1.4) is obtained in the limit. It would be interesting to see whether those results can be extended to the case $\delta_n \ll 1/n$, in which case the expected limit is (1.4) (see [28]).

Different regularisations of collisions. In the spirit of proving any of the above limit passages, we discuss alternative regularisations other than taking W regular. One idea is 'premature annihilation', where particles are removed from the system when they come δ-close, with $\delta > 0$ a regularisation parameter. This approach is commonly adapted in numerical simulations of discrete systems with an annihilation rule. However, it is not obvious what the limiting equation as $n \to \infty$ (counterpart of (1.4)) is for $\delta > 0$ fixed, because we expect the supports of $[\kappa]_+$ and $[\kappa]_-$ to be separated by at least δ. A third option is to mollify the jump of the charge $b_i(t)$ from ± 1 to 0, possibly by an additional ODE for $b_i(t)$. We have not found a proper rule for this that would still allow for a discrete-to-continuum convergence result.

Higher dimensions. In this paragraph we consider the extension to two dimensions; the discussion easily extends to higher dimensions. The one ingredient in our proof which intrinsically relies on our 1D setting, is the *separation* condition on the initial data. This condition limits the collisions to happen only at a finite number of points. In 2D, collisions are bound to happen along curves (or more complicated subsets of $\mathbb{R}^2$), which makes it challenging to characterise the limit of $[\kappa_n]_\pm$. A similar problem occurred in [3] as discussed in Subsection 1.5. In future research we plan to relax our 'separation' assumption, possibly by considering a different regularisation of collisions.

The remainder of the paper is organised as follows. In Section 2 we fix our notation and list the assumptions on V, W and the initial data. In Section 3 we recall known results and provide the preliminaries. In Section 4 we give a rigorous definition of (1.2), show that it attains a unique solution, and establish several properties of it. In Section 5 we state and prove our main result, Theorem 5.1.

2. Notation and standing assumptions. Here we list the symbols and notation which we use in the remainder of this paper:

$\mathcal{B}(\mathbb{R})$	space of Borel sets on $\mathbb{R}$	Section 3
$f(a-)$	$\lim_{y\uparrow a} f(y)$	
$[f]_\pm$	positive or negative part of f	
$\mu \otimes \nu$	product measure; $(\mu \otimes \nu)(A \times B) = \mu(A)\nu(B)$	Section 3
$C > 0$	constant whose value can possibly change from line to line	
$\boldsymbol{\mu}$	$\boldsymbol{\mu} := (\mu^+, \mu^-) \in \mathcal{P}(\mathbb{R} \times \{\pm 1\})$	(3.2)
$\mathcal{M}(\mathbb{R})$	space of finite, signed Borel measures on $\mathbb{R}$	Section 3
$\mathcal{M}_+(\mathbb{R})$	space of the non-negative measures in $\mathcal{M}(\mathbb{R})$	Section 3
$\mathbb{N}$	$\{1, 2, 3, \dots\}$	
$\mathcal{P}(\mathbb{R})$	space of probability measures; $\mathcal{P}(\mathbb{R}) = \{\mu \in \mathcal{M}_+(\mathbb{R}) : \mu(\mathbb{R}) = 1\}$	Section 3
$\mathcal{P}_2(\mathbb{R})$	probability measures with finite second moment; $\mathcal{P}_2(\mathbb{R}) = \{\mu \in \mathcal{P}_2(\mathbb{R}) : \int_{-\infty}^{\infty} x^2 \,\mathrm{d}\mu(x) < \infty\}$	Section 3
V	interaction potential for equally signed particles	Assumption 2.1
W	interaction potential for oppositely signed particles	Assumption 2.1
$W(\mu,\nu)$	2-Wasserstein distance between $\mu, \nu \in \mathcal{P}(\mathbb{R})$	[2]
$\mathbf{W}(\boldsymbol{\mu},\boldsymbol{\nu})$	2-Wasserstein distance between $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{P}_2(\mathbb{R})$	(3.3)

Assumption 2.1 lists the standing properties which we impose on V and W.

ASSUMPTION 2.1. *We require that the interaction potentials* $V : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *and* $W : \mathbb{R} \to \mathbb{R}$ *satisfy the following conditions:*

(2.1a) $V \in C^1(\mathbb{R} \setminus \{0\})$, $W \in C^1(\mathbb{R})$, $V' \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R} \setminus \{0\})$, *and* $W' \in \mathrm{Lip}(\mathbb{R})$,

(2.1b) V *and* W *are even;*

(2.1c) $V(r) \to +\infty$ *as* $r \to 0$*;*

(2.1d) $r \mapsto rV'(r)$ *and* $r \mapsto rW'(r)$ *are in* $L^\infty(\mathbb{R})$.

For convenience, we set $V'(0) := 0$. Below we list two remarks on Assumption 2.1:

- we assume no monotonicity on V or W;
- Condition (2.1d) implies that V has at most a logarithmic singularity (as mentioned in Subsection 1.2), and that V and W have at most logarithmically diverging tails, namely

$$|V(r)| + |W(r)| \le C\big(\big|\log|r|\big| + 1\big), \quad \text{for all } r \neq 0. \tag{2.2}$$

  Due to condition (2.1c), and keeping (2.1a) into account, we can sharpen this inequality around 0 by

$$(V + W)(r) \ge -Cr^2, \quad \text{for all } r \neq 0. \tag{2.3}$$

The following assumption on the initial data states that no pair of particles of opposite sign should start at the same position.

ASSUMPTION 2.2 (Separation assumption on the initial data $(x^\circ; b^\circ)$). *There exist* $-\infty < a_0 \le a_1 \le \ldots \le a_{2L} < +\infty$ *such that*

$$\{x_i^\circ : b_i^\circ = 1\} \subset \bigcup_{\ell=1}^{L} (a_{2\ell-2}, a_{2\ell-1}), \qquad \{x_i^\circ : b_i^\circ = -1\} \subset \bigcup_{\ell=1}^{L} (a_{2\ell-1}, a_{2\ell}).$$

The importance of this assumption is clarified later when the limit $n \to \infty$ is considered, in which the number L is assumed to be n-independent (see also Subsection 1.6). Moreover, we will show in Proposition 4.5 that this assumption is conserved in time.

3. Preliminary results. We collect here some basic definitions and known results that will be useful in the sequel.

3.1. Probability spaces and the Wasserstein distance. On $\mathcal{P}_2(\mathbb{R})$ (space of probability measures with finite second moment; see Section 2), the square of the 2-Wasserstein distance $W(\mu, \nu)$ with $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$ is defined as

$$W^2(\mu, \nu) := \inf_{\gamma \in \Gamma(\mu,\nu)} \iint_{\mathbb{R}^2} |x - y|^2 \, \mathrm{d}\gamma(x, y), \tag{3.1}$$

where $\Gamma(\mu, \nu)$ is the set of couplings of μ and ν, namely,

$$\Gamma(\mu, \nu) := \big\{\gamma \in \mathcal{P}(\mathbb{R}^2) : \gamma(A \times \mathbb{R}) = \mu(A),\ \gamma(\mathbb{R} \times A) = \nu(A) \text{ for all } A \in \mathcal{B}(\mathbb{R})\big\}.$$

We refer to [4] for the basic properties of W. As usual, we set $\Gamma_\circ(\mu, \nu) \subset \Gamma(\mu, \nu)$ as the set of transport plans γ which minimise (3.1).

Since we are working with positive and negative particles, we follow [12] by defining a space of probability measures on $\mathbb{R} \times \{\pm 1\}$, where $\mathbb{R} \times \{\pm 1\}$ is endowed with the distance

$$\mathsf{d}^2(\bar{x}, \bar{y}) := |x-y|^2 + |p-q|, \qquad \bar{x} = (x,p) \in \mathbb{R} \times \{\pm 1\},\ \bar{y} = (y,q) \in \mathbb{R} \times \{\pm 1\}.$$

We denote this probability space by $\mathcal{P}(\mathbb{R} \times \{\pm 1\})$, and its elements by $\boldsymbol{\mu}$ or (μ^+, μ^-), with the understanding that

$$\boldsymbol{\mu}(A^+, A^-) = \mu^+(A^+) + \mu^-(A^-), \qquad \text{for all } A^+, A^- \in \mathcal{B}(\mathbb{R}). \tag{3.2}$$

On

$$\mathcal{P}_2(\mathbb{R} \times \{\pm 1\}) := \left\{ \boldsymbol{\mu} \in \mathcal{P}(\mathbb{R} \times \{\pm 1\}) : \int_{\mathbb{R}} |x|^2 \, \mathrm{d}\mu^{\pm}(x) < +\infty \right\}$$

we define the (square of the) 2-Wasserstein distance between $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ as

$$\mathbf{W}^2(\boldsymbol{\mu}, \boldsymbol{\nu}) := \inf_{\boldsymbol{\gamma} \in \boldsymbol{\Gamma}(\boldsymbol{\mu}, \boldsymbol{\nu})} \iint_{(\mathbb{R} \times \{\pm 1\})^2} \mathsf{d}^2(\bar{x}, \bar{y}) \, \mathrm{d}\boldsymbol{\gamma}(\bar{x}, \bar{y}), \tag{3.3}$$

where $\boldsymbol{\Gamma}(\boldsymbol{\mu}, \boldsymbol{\nu})$ is the set of couplings of $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$, namely,

$$\begin{aligned} \boldsymbol{\Gamma}(\boldsymbol{\mu}, \boldsymbol{\nu}) := \big\{ \boldsymbol{\gamma} \in \mathcal{P}\big((\mathbb{R} \times \{\pm 1\})^2 \big) : \boldsymbol{\gamma}(A \times (\mathbb{R} \times \{\pm 1\})) = \boldsymbol{\mu}(A), \\ \boldsymbol{\gamma}((\mathbb{R} \times \{\pm 1\}) \times A) = \boldsymbol{\nu}(A) \text{ for all } A \in \mathcal{B}(\mathbb{R} \times \{\pm 1\}) \big\}. \end{aligned}$$

Since it turns out that (1.3) has a mass-preserving solution $\boldsymbol{\rho}(t) := (\rho^+(t), \rho^-(t))$ belonging to $\mathcal{P}_2(\mathbb{R} \times \{\pm 1\})$, for which also the mass of $\rho^+(t)$ and $\rho^-(t)$ is conserved in time, we define the corresponding subspace

$$\mathcal{P}_2^m(\mathbb{R} \times \{\pm 1\}) := \{ \boldsymbol{\mu} \in \mathcal{P}_2(\mathbb{R} \times \{\pm 1\}) : \mu^+(\mathbb{R}) = m \};$$

where $m \in [0,1]$ is the total mass of the positive particle density. Clearly, if $\boldsymbol{\mu} \in \mathcal{P}_2^m(\mathbb{R} \times \{\pm 1\})$, then $\mu^-(\mathbb{R}) = 1 - m$. For any $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{P}_2^m(\mathbb{R} \times \{\pm 1\})$ we have that

$$\mathbf{W}^2(\boldsymbol{\mu}, \boldsymbol{\nu}) \leq W^2(\mu^+, \nu^+) + W^2(\mu^-, \nu^-), \tag{3.4}$$

which simply follows by shrinking the set of couplings $\boldsymbol{\Gamma}(\boldsymbol{\mu}, \boldsymbol{\nu})$ in (3.3).

3.2. Weak form of the continuum problem (1.3). We use the following notation convention. For any $\boldsymbol{\rho} \in \mathcal{P}(\mathbb{R} \times \{\pm 1\})$, we set

$$\rho := \rho^+ + \rho^- \in \mathcal{P}(\mathbb{R}), \qquad \kappa := \rho^+ - \rho^- \in \mathcal{M}(\mathbb{R}), \qquad \tilde{\rho}^{\pm} := [\kappa]_{\pm} \in \mathcal{M}_+(\mathbb{R}). \tag{3.5}$$

We consider the following weak form of (1.3): given an initial condition $\boldsymbol{\rho}^\circ \in \mathcal{P}_2(\mathbb{R} \times \{\pm 1\})$, find ρ satisfying

$$\begin{aligned} 0 = &\int_0^T \int_{\mathbb{R}} \partial_t \varphi^{\pm}(x) \, \mathrm{d}\rho^{\pm}(x) \mathrm{d}t \\ &- \frac{1}{2} \int_0^T \iint_{\mathbb{R} \times \mathbb{R}} \big((\varphi^{\pm})'(x) - (\varphi^{\pm})'(y) \big) \, V'(x-y) \, \mathrm{d}([\kappa]_{\pm} \otimes [\kappa]_{\pm})(x,y) \mathrm{d}t \\ &- \int_0^T \int_{\mathbb{R}} (\varphi^{\pm})'(x) \, (W' * [\kappa]_{\mp})(x) \, \mathrm{d}[\kappa]_{\pm}(x) \mathrm{d}t, \end{aligned} \tag{3.6}$$

for all $\varphi^{\pm} \in \mathrm{C}_c^\infty((0,T) \times \mathbb{R})$, where we have exploited that V' is odd. We seek a solution of (3.6) in $\mathrm{AC}(0,T; \mathcal{P}_2^m(\mathbb{R} \times \{\pm 1\}))$ with $m = \rho^{\circ,+}(\mathbb{R}) \in [0,1]$.

3.3. Several topologies and their connections. Next we define the space of absolutely continuous curves and their metric derivatives. While the following definitions work on any complete metric space, we limit our exposition to $(\mathcal{P}_2(\mathbb{R} \times \{\pm 1\}), \mathbf{W})$. For any $1 \leq p < \infty$, $\mathrm{AC}^p(0,T;\mathcal{P}_2(\mathbb{R} \times \{\pm 1\}))$ denotes the space of all curves $\boldsymbol{\mu} : (0,T) \to \mathcal{P}_2(\mathbb{R} \times \{\pm 1\})$ for which there exists a function $f \in \mathrm{L}^p(0,T)$ such that

$$\mathbf{W}\big(\boldsymbol{\mu}(s), \boldsymbol{\mu}(t)\big) \leq \int_s^t |f(r)|^p \,\mathrm{d}r, \qquad \text{for all } 0 < s \leq t < T. \tag{3.7}$$

We set $\mathrm{AC}(0,T;\mathcal{P}_2(\mathbb{R} \times \{\pm 1\})) := \mathrm{AC}^1(0,T;\mathcal{P}_2(\mathbb{R} \times \{\pm 1\}))$. By [2, Theorem 1.1.2], the metric derivative

$$|\boldsymbol{\mu}'|_{\mathbf{W}}(t) := \lim_{s \to t} \frac{\mathbf{W}\big(\boldsymbol{\mu}(s), \boldsymbol{\mu}(t)\big)}{|s-t|} \tag{3.8}$$

is defined for any $\boldsymbol{\mu} \in \mathrm{AC}(0,T;\mathcal{P}_2(\mathbb{R}\times\{\pm 1\}))$ and for a.e. $t \in (0,T)$. Moreover, $|\boldsymbol{\mu}'|_{\mathbf{W}}$ is a possible choice for f in (3.7).

The following theorem is a simplified version of [31, Theorem 47.1] applied to the metric space $(\mathcal{P}_2(\mathbb{R} \times \{\pm 1\}), \mathbf{W})$.

Lemma 3.1 (Ascoli-Arzelà). *$\mathcal{F} \subset \mathrm{C}([0,T];\mathcal{P}_2(\mathbb{R} \times \{\pm 1\}))$ is pre-compact if and only if*

(i) $\{\boldsymbol{\mu}(t) : \boldsymbol{\mu} \in \mathcal{F}\}$ is pre-compact in $\mathcal{P}_2(\mathbb{R} \times \{\pm 1\})$ for all $t \in [0,T]$,
(ii) $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ such that $\forall\, \boldsymbol{\mu} \in \mathcal{F}$, $\forall\, t,s \in [0,T] : |t-s| < \delta \implies \mathbf{W}\big(\boldsymbol{\mu}(t), \boldsymbol{\mu}(s)\big) < \varepsilon$.

The following theorem provides a lower semi-continuity result on the $L^2(0,T)$-norm of the metric derivative. We expect it to be well-known, but we only found it proven in the PhD thesis [45, Lemma 8.2.8].

Theorem 3.2 (Lower semi-continuity of metric derivatives). *Let $\boldsymbol{\mu}_n, \boldsymbol{\mu} : [0,T] \to \mathcal{P}_2(\mathbb{R} \times \{\pm 1\})$. If $\mathbf{W}(\boldsymbol{\mu}_n(t), \boldsymbol{\mu}(t)) \to 0$ as $n \to \infty$ pointwise for a.e. $t \in (0,T)$, then*

$$\liminf_{n \to \infty} \int_0^T |\boldsymbol{\mu}_n'|_{\mathbf{W}}^2(t) \,\mathrm{d}t \geq \int_0^T |\boldsymbol{\mu}'|_{\mathbf{W}}^2(t) \,\mathrm{d}t. \tag{3.9}$$

Proof. We start with several preparations. First, we take a dense subset $(t_\ell)_\ell$ of $[0,T]$ for which $\mathbf{W}(\boldsymbol{\mu}_n(t_\ell), \boldsymbol{\mu}(t_\ell)) \to 0$ as $n \to \infty$ for any $\ell \in \mathbb{N}$. Second, without loss of generality, we assume that there exists $C > 0$ such that for all n

$$\int_0^T |\boldsymbol{\mu}_n'|_{\mathbf{W}}^2(t) \,\mathrm{d}t \leq C. \tag{3.10}$$

In particular, this means that $\boldsymbol{\mu}_n$ has a representative in $\mathrm{AC}^2(0,T;\mathcal{P}_2(\mathbb{R} \times \{\pm 1\}))$ which is defined for all $t \in (0,T)$. Taking this representative, we set $D_n^\ell(t) := \mathbf{W}(\boldsymbol{\mu}_n(t_\ell), \boldsymbol{\mu}_n(t))$, and obtain from [2, Theorem 1.1.2] that

$$|\boldsymbol{\mu}_n'|_{\mathbf{W}}(t) = \sup_{\ell \in \mathbb{N}} \big|(D_n^\ell)'(t)\big| \qquad \text{for a.e. } t \in (0,T). \tag{3.11}$$

Next we prove (3.9). Firstly, since $\mathbf{W}(\boldsymbol{\mu}_n(t), \boldsymbol{\mu}(t)) \to 0$ as $n \to \infty$ for a.e. $t \in (0,T)$, we have for fixed $\ell \in \mathbb{N}$ and for a.e. $t \in (0,T)$ that

$$\big|D_n^\ell(t) - D^\ell(t)\big| \xrightarrow{n \to \infty} 0, \qquad \text{where } D^\ell(t) := \mathbf{W}\big(\boldsymbol{\mu}(t_\ell), \boldsymbol{\mu}(t)\big). \tag{3.12}$$

Secondly, $\|D_n^\ell\|_{H^1(0,T)}$ and $\|D^\ell\|_{H^1(0,T)}$ are bounded uniformly in n and ℓ. To see this, we have by the definition of the metric derivative and (3.10) that

$$D_n^\ell(t) \leq \left| \int_{t_\ell}^t |\boldsymbol{\mu}_n'|_{\mathbf{W}}(s)\,\mathrm{d}s \right| \leq C\sqrt{T}.$$

Hence, $\|D_n^\ell\|_{L^2(0,T)}$ is uniformly bounded. With the characterisation of $|\boldsymbol{\mu}_n'|_{\mathbf{W}}$ in (3.11), we estimate

$$C \geq \int_0^T |\boldsymbol{\mu}_n'|_{\mathbf{W}}^2(t)\,\mathrm{d}t \geq \int_0^T \big((D_n^\ell)'(t)\big)^2\,\mathrm{d}t \qquad \text{for all } \ell \in \mathbb{N}, \tag{3.13}$$

and thus $\|D_n^\ell\|_{H^1(0,T)}$ is uniformly bounded. Therefore, in view of (3.12), we have

$$D_n^\ell \rightharpoonup D^\ell \qquad \text{in } H^1(0,T) \text{ as } n \to \infty. \tag{3.14}$$

In particular, we observe from (3.14) that $D^\ell \in H^1(0,T)$ and that

$$C \geq \liminf_{n\to\infty} \|D_n^\ell\|_{H^1(0,T)} \geq \|D^\ell\|_{H^1(0,T)} \qquad \text{for all } \ell \in \mathbb{N}.$$

To establish (3.9), we carefully perform a joint limit passage as $n \to \infty$ and a maximisation over ℓ in (3.13). With this aim, we take a large fixed $L \in \mathbb{N}$, and choose a partition $\{A_\ell\}_{\ell=1}^L$ of Borel sets of $(0,T)$ such that for all $\ell = 1, \ldots, L$,

$$\big|(D^\ell)'(t)\big| = \sup_{1\leq\tilde{\ell}\leq L} \big|(D^{\tilde{\ell}})'(t)\big| \qquad \text{for a.e. } t \in A_\ell.$$

We estimate

$$\int_0^T |\boldsymbol{\mu}_n'|_{\mathbf{W}}^2(t)\,\mathrm{d}t \geq \int_0^T \sup_{1\leq\ell\leq L} \big((D_n^\ell)'(t)\big)^2\,\mathrm{d}t \geq \sum_{\ell=1}^L \int_{A_\ell} \big((D_n^\ell)'(t)\big)^2\,\mathrm{d}t.$$

Using (3.14), we pass to the limit $n \to \infty$ to obtain

$$\liminf_{n\to\infty} \int_0^T |\boldsymbol{\mu}_n'|_{\mathbf{W}}^2(t)\,\mathrm{d}t \geq \sum_{\ell=1}^L \int_{A_\ell} \big((D^\ell)'(t)\big)^2\,\mathrm{d}t = \int_0^T \sup_{1\leq\ell\leq L} \big((D^\ell)'(t)\big)^2\,\mathrm{d}t.$$

By using the Monotone Convergence Theorem, we take the supremum over $L \in \mathbb{N}$ to deduce that

$$\liminf_{n\to\infty} \int_0^T |\boldsymbol{\mu}_n'|_{\mathbf{W}}^2(t)\,\mathrm{d}t \geq \int_0^T \sup_{\ell\in\mathbb{N}} \big((D^\ell)'(t)\big)^2\,\mathrm{d}t.$$

We conclude by using [2, Theorem 1.1.2] to identify $\sup_{\ell\in\mathbb{N}} |(D^\ell)'|$ in $L^2(0,T)$ by $|\boldsymbol{\mu}'|_{\mathbf{W}}$. ∎

Next we introduce the *narrow convergence* of measures. For $\nu_n, \nu \in \mathcal{M}(\mathbb{R})$, we say that ν_n converges in the narrow topology to ν (and write $\nu_n \rightharpoonup \nu$) as $n \to \infty$ if

$$\int \varphi\,\mathrm{d}\nu_n \xrightarrow{n\to\infty} \int \varphi\,\mathrm{d}\nu.$$

for any bounded test function $\varphi \in \mathrm{C}(\mathbb{R})$. The following lemma extends this notion for non-negative measures by allowing for discontinuous test functions.

Lemma 3.3 ([34, Lemma 2.1]). *Let $\nu_n \rightharpoonup \nu$ in $\mathcal{M}_+(\mathbb{R}^d)$. Let $A \in \mathcal{B}(\mathbb{R}^d)$ such that $\nu(A) = 0$. Then for every bounded $\varphi \in \mathrm{C}(\mathbb{R}^d \setminus A)$ it holds that*

$$\int \varphi \, \mathrm{d}\nu_n \xrightarrow{n\to\infty} \int \varphi \, \mathrm{d}\nu.$$

Proofs can be found in [39, Theorems 62-63, chapter IV, paragraph 6] and in [8, 14], or [37] in the case where A is closed.

Finally, we state and prove a lemma which allows us to show that Assumption 2.2 is conserved in the limit as $n \to \infty$.

Lemma 3.4 (Narrow topology preserves separation of supports). *Let $(\nu_\varepsilon)_{\varepsilon>0}$, $(\rho_\varepsilon)_{\varepsilon>0} \subset \mathcal{M}_+(\mathbb{R})$ converge in the narrow topology as $\varepsilon \to 0$ to ν and ρ, respectively. If*

$$\forall\, \varepsilon > 0 : \sup(\operatorname{supp} \nu_\varepsilon) \le \inf(\operatorname{supp} \rho_\varepsilon),$$

then also $\sup(\operatorname{supp} \nu) \le \inf(\operatorname{supp} \rho)$.

Proof. We reason by contradiction. Suppose $M := \sup(\operatorname{supp} \nu) > \inf(\operatorname{supp} \rho) =: m$. Take a non-decreasing test function $\varphi \in \mathrm{C}_b(\mathbb{R})$ which satisfies

$$\varphi \equiv 0 \text{ on } \Big(-\infty, \frac{m+2M}{3} \Big], \quad \text{and} \quad \varphi \equiv 1 \text{ on } [M, \infty).$$

Since $M = \sup(\operatorname{supp} \nu)$, it holds that $\int \varphi \, \mathrm{d}\nu > 0$. Hence, from $\nu_\varepsilon \xrightarrow{\varepsilon\to 0} \nu$ we infer that for all ε small enough, it also holds that $\int \varphi \, \mathrm{d}\nu_\varepsilon > 0$, and thus

$$\sup(\operatorname{supp} \nu_\varepsilon) \ge \frac{m+2M}{3}.$$

With a similar argument, we can deduce that $\inf(\operatorname{supp} \rho_\varepsilon) \le \frac{2m+M}{3}$, which contradicts with $m < M$. ∎

4. Definition and properties of the discrete problem (1.2).

In this section we give a rigorous definition to the discrete dynamics formally given by (1.2). We start by formulating it as Problem 4.1, which may have several solutions. Then, we define a precise solution concept to Problem 4.1 (see Definition 4.2) which encodes the annihilation rule and selects a unique solution to Problem 4.1. After establishing some properties of the energy E_n introduced in (1.1), we prove an existence and uniqueness result (see Proposition 4.5). Finally, we state the discrete problem in the language of measures (see Lemma 4.6).

Problem 4.1. *Given $(x^\circ, b^\circ) \in \mathbb{R}^n \times \{\pm 1\}^n$ such that $x_1^\circ < x_2^\circ < \ldots < x_n^\circ$, find $(x, b) \colon [0,T] \to \mathbb{R}^n \times \{-1, 0, 1\}^n$ such that*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} x_i = -\dfrac{1}{n} \displaystyle\sum_{j \,:\, b_i b_j = 1} V'(x_i - x_j) - \dfrac{1}{n} \sum_{j \,:\, b_i b_j = -1} W'(x_i - x_j) & \text{on } (0,T) \setminus T_{\mathrm{col}} \\ (x_i(0), b_i(0)) = (x_i^\circ, b_i^\circ) \end{cases} \tag{4.1}$$

for all $i = 1, \ldots, n$, where T_{col} is the jump set of b.

We encode the annihilation rule in the solution concept below. With this aim, we set $H \colon \mathbb{R} \cup \{+\infty\} \to [0,1]$ as the usual Heaviside function, with $H(0) := 0$ and $H(+\infty) := 1$.

DEFINITION 4.2 (Solution to Problem 4.1). *We say that* $(x, b)\colon [0,T] \to \mathbb{R}^n \times \{-1,0,1\}^n$ *is a solution to* Problem 4.1 *if*

(a) *there exists a vector of collision times* $\tau = (\tau_1, \ldots, \tau_n)$ *with* $\tau_i \in (0,T) \cup \{+\infty\}$ *such that, setting*

$$T_{\mathrm{col}} \coloneqq \{\tau_i : 1 \le i \le n\} \setminus \{+\infty\} = \{t_1, t_2, \ldots, t_K\} \subset (0,T) \tag{4.2}$$

with $0 < t_1 < \ldots < t_K < T$, *there holds*

$$b_i(t) \coloneqq b_i^\circ H(\tau_i - t) \qquad \text{for all } i = 1, \ldots, n; \tag{4.3}$$

(b) $x \in \mathrm{Lip}([0,T];\mathbb{R}^n) \cap C^1((0,T) \setminus T_{\mathrm{col}};\mathbb{R}^n)$;

(c) (4.1) *is satisfied in the classical sense;*

(d) *setting* $t_0 \coloneqq 0$, *for all* $k = 1, \ldots, K$,

$$\begin{aligned} t_k = \inf \big\{ t \in (0,T) : \exists\, (i,j) \text{ such that} \\ b_i(t_{k-1}) b_j(t_{k-1}) = -1 \text{ and } x_i(t) = x_j(t) \big\} > t_{k-1}; \end{aligned}$$

(e) *at each time* $t \in [0,T]$, *there is a bijection*

$$\alpha\colon \{i : b_i^\circ = 1,\, \tau_i \le t\} \to \{j : b_j^\circ = -1,\, \tau_j \le t\}$$

such that $x_i(t) = x_{\alpha(i)}(t)$.

Remark 4.3 (Comments on Definition 4.2). We collect here some remarks on the notion of solution presented above.

- τ_i is the time at which particle x_i gets annihilated: equation (4.3) describes this by putting to zero the charge b_i at time τ_i. If $\tau_i = +\infty$, then it means that the particle x_i does not collide in the time interval $(0,T)$.
- (t_k) is the ordered list of collision times at which at least one collision occurs.
- In equation (4.1), both x_i and b_i depend on time. However, on each open component of $(0,T) \setminus T_{\mathrm{col}}$, the charges b_i remain constant.
- Since V is singular and W is regular, straight-forward a priori energy estimates show that particles of the same type can never come closer than some positive distance. Hence, the only type of collision that can occur is that of two particles with opposite sign. We prove precise energy estimates in Proposition 4.5.
- Property (d) ensures that for each pair of two colliding particles, at least one gets annihilated. Property (e) ensures that both particles are getting annihilated, and that annihilation can only occur for colliding particles with non-zero charge. These two properties are the mathematical formulation of the annihilation process described in Subsection 1.2.
- Recalling (4.1), by (4.3), it follows that colliding particles are stationary after collision.

With reference to the collision times $t_1 < \ldots < t_K$ in (4.2), we define the set of indices of the particles colliding at t_k and its cardinality by

$$\Gamma_k \coloneqq \{i : \tau_i = t_k\}, \qquad \gamma_k \coloneqq \#\Gamma_k. \tag{4.4}$$

We observe that γ_k is even for every k and that

$$\sum_{k=1}^K \gamma_k \le \frac{n}{2}. \tag{4.5}$$

We first establish some properties of E_n defined in (1.1). For convenience, we display

$$\frac{\partial}{\partial x_i} E_n(x;b) = \frac{1}{n^2} \sum_{j\,:\,b_i b_j = 1} V'(x_i - x_j) + \frac{1}{n^2} \sum_{j\,:\,b_i b_j = -1} W'(x_i - x_j), \tag{4.6}$$

where we rely on the choice $V'(0) = 0$. We also introduce

$$M_k : \mathbb{R}^n \to [0,\infty), \qquad M_k(x) := \frac{1}{n} \sum_{i=1}^n |x_i|^k, \qquad k = 1, 2, \ldots$$

which is the k-th moment of the empirical measure related to the particles $x_1, \ldots, x_n$.

LEMMA 4.4 (Properties of E_n). *Let $n \geq 2$. For any $x \in \mathbb{R}^n$ and $b \in \{-1, 0, 1\}^n$, the following properties hold:*

(i) $E_n(x;b) < +\infty$ if and only if $\forall\, i \neq j : x_i = x_j \;\Rightarrow\; b_i b_j \neq 1$;
(ii) $E_n + M_2$ is bounded from below;
(iii) ∇E_n is Lipschitz continuous on the sublevelsets of $y \mapsto E_n(y;b) + 2M_2(y)$;
(iv) if $E_n(x;b) < +\infty$ and if there exists an index pair (I,J) which satisfies $b_I b_J = -1$ and $x_I = x_J$, then, there exists $C > 0$ independent of n such that

$$E_n(x;\bar{b}) \leq E_n(x;b) + \frac{C}{n}(M_2(x) + x_I^2 + 1),$$

where $\bar{b}$ is the modification of b in which b_I and b_J are put to 0.

Proof. Property (i) is a direct consequences of the properties of V, W (see Assumption 2.1). Property (ii) is a matter of a simple estimate. Using Assumption 2.1) (in particular (2.2)), some manipulations inspired by [37], and $r \mapsto r^2 - C \log r$ being bounded from below, we obtain

$$\begin{aligned} E_n(x;b) + M_2(x) &= \frac{1}{2n^2} \Big(\sum_{\substack{i \neq j \\ b_i b_j = 1}} V(x_i - x_j) + \sum_{\substack{i,j \\ b_i b_j = -1}} W(x_i - x_j) + \sum_{i,j=1}^n (x_i^2 + x_j^2) \Big) \\ &\geq \frac{1}{2n^2} \sum_{i,j=1}^n \Big(-C\big([\log|x_i - x_j|]_+ + 1\big) + \frac{1}{2}(x_i - x_j)^2 \Big) \geq C. \end{aligned}$$

Property (iii) follows easily from property (ii) by (2.1a) and (2.1c). To prove (iv), we set $y := x_I = x_J$ and assume for convenience that $b_I = 1$ and $b_J = -1$. Then, we compute

$$\begin{aligned} E_n(x;b) - E_n(x;\bar{b}) &= \frac{1}{2n^2} \Big(\sum_{\substack{j \neq I \\ b_j = 1}} V(x_I - x_j) + \sum_{\substack{i \neq J \\ b_i = -1}} V(x_i - x_J) \Big) \\ &\quad + \frac{1}{2n^2} \Big(\sum_{j\,:\,b_j = -1} W(x_I - x_j) + \sum_{i\,:\,b_i = 1} W(x_i - x_J) \Big) - \frac{W(0)}{2n^2} \\ &= \frac{1}{2n^2} \Big(\sum_{\substack{i=1 \\ i \neq I,J}}^n |b_i| V(x_i - y) + \sum_{i=1}^n |b_i| W(x_i - y) \Big) - \frac{W(0)}{2n^2} \end{aligned}$$

$$= \frac{1}{2n^2} \sum_{\substack{i=1 \\ i \neq I,J}}^{n} |b_i|(V+W)(x_i - y) + \frac{W(0)}{2n^2}$$

$$\geq -\frac{C}{n^2} \sum_{i=1}^{n} (x_i - y)^2 + \frac{W(0)}{2n^2} \geq -\frac{C}{n}(M_2(x) + y^2 + 1),$$

where we have used (2.3). □

We now prove that Problem 4.1 has a unique solution. In addition, we establish several properties of it.

Proposition 4.5. *Let $n \geq 2$, $T > 0$, and $(x^\circ, b^\circ) \in \mathbb{R}^n \times \{\pm 1\}^n$ be such that $x_1^\circ < x_2^\circ < \ldots < x_n^\circ$. Then there exists a unique solution (x, b) to Problem 4.1 in the sense of Definition 4.2. Moreover, the following properties are satisfied:*

(i) there exists $C > 0$ independent of n such that

$$M_2(x(t)) \leq Ct + M_2(x^\circ), \quad M_4(x(t)) \leq Ct(M_2(x^\circ) + t) + M_4(x^\circ)$$

for all $t \in [0, T]$;

(ii) $\inf_{0<t<T} \min\{|x_i(t) - x_j(t)| : b_i(t) b_j(t) = 1\} > 0$*;*

(iii) the energy function $e \colon [0, T) \to \mathbb{R}$ defined by $e(t) := E_n(x(t); b(t))$ is left-continuous on $[0, T)$, differentiable on $(0, T) \setminus T_{\mathrm{col}}$, and $e'(t) \leq 0$ for all $t \in (0, T) \setminus T_{\mathrm{col}}$. Moreover, denoting by $[\![e(t_k)]\!] := e(t_k) - e(t_k-)$ the jump of e at t_k, we have that

$$[\![e(t_k)]\!] \leq \frac{C}{n} \Big(\gamma_k M_2(x(t_k)) + \gamma_k + \sum_{i \in \Gamma_k} x_i^2(t_k) \Big) \tag{4.7}$$

for every $k = 1, \ldots, K$, and

$$\sum_{k=1}^{K} [\![e(t_k)]\!] \leq C(T + M_2(x^\circ) + 1), \tag{4.8}$$

where γ_k and Γ_k are defined in (4.4), and $C > 0$ is a constant independent of n;

(iv) $E_n(x(t); b(t)) - E_n(x^\circ; b^\circ) \leq C(t + M_2(x^\circ) + 1) - \frac{1}{n} \int_0^t |\dot{x}(s)|^2 \, \mathrm{d}s$ for all $t \in (0, T]$;

(v) there exists an $L \in \mathbb{N}$ (independent of n) such that for all $t \in [0, T)$, $(x(t), b(t))$ satisfies Assumption 2.2, i.e., *there exist $-\infty < a_0(t) \leq a_1(t) \leq \ldots \leq a_{2L}(t) < +\infty$ such that*

$$\{x_i(t) : b_i(t) = 1\} \subset \bigcup_{\ell=1}^{L} \big(a_{2\ell-2}(t), a_{2\ell-1}(t) \big),$$

$$\{x_i(t) : b_i(t) = -1\} \subset \bigcup_{\ell=1}^{L} \big(a_{2\ell-1}(t), a_{2\ell}(t) \big).$$

Proof. Step 1: Construction of (x,b)*, properties (i) and (ii), and* (4.7). We define the counterpart of (4.1) in which no collision occurs, *i.e.*, we seek n trajectories $y_i : [0,T] \to \mathbb{R}$ such that $y_i(0) = x_i^\circ$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} y_i = -\frac{1}{n} \sum_{j \,:\, b_i^\circ b_j^\circ = 1} V'(y_i - y_j) - \frac{1}{n} \sum_{j \,:\, b_i^\circ b_j^\circ = -1} W'(y_i - y_j) \quad \text{on } (0, +\infty). \tag{4.9}$$

for all $i = 1, \ldots, n$. From (4.6) we observe that (4.9) is the gradient flow of $E_n(\cdot; b^\circ)$ given by

$$\begin{cases} \dot{y}(t) = -n \nabla E_n(y(t); b^\circ), \\ y(0) = x^\circ. \end{cases} \tag{4.10}$$

From Lemma 4.4 we observe that (4.10) has a unique, classical solution $y(t)$ locally in time. In particular, $t \mapsto E_n(y(t); b^\circ)$ is non-increasing.

Next we show that the solution y can be extended to the complete time interval $[0,T]$. With this aim, we prove that the second moment $M_2(y(t))$ (and for later use the fourth moment $M_4(y(t))$) are finite as long as $t \mapsto y(t)$ exists. We follow the argument in [38]. From (4.9), using (2.1b) and (2.1d), we estimate

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} M_2(y(t)) &= \frac{2}{n} \sum_{i=1}^n y_i(t) \dot{y}_i(t) \\
&= -\frac{2}{n^2} \sum_{i=1}^n \bigg(\sum_{j \,:\, b_i b_j = 1} y_i V'(y_i - y_j) + \sum_{j \,:\, b_i b_j = -1} y_i W'(y_i - y_j) \bigg) \\
&= -\frac{1}{n^2} \sum_{i,j :\, b_i b_j = 1} (y_i - y_j) V'(y_i - y_j) - \frac{1}{n^2} \sum_{i,j :\, b_i b_j = -1} (y_i - y_j) W'(y_i - y_j) \leq C.
\end{aligned}$$

Hence,

$$M_2(y(t)) \leq M_2(y(0)) + Ct \leq M_2(x^\circ) + CT, \qquad \text{for all } t \in [0,T]. \tag{4.11}$$

Similarly, using the identity $a^3 - b^3 = (a^2 + ab + b^2)(a - b)$, we compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} M_4(y(t)) &= \frac{4}{n} \sum_{i=1}^n y_i^3(t) \dot{y}_i(t) \\
&= -\frac{4}{n^2} \sum_{i=1}^n \bigg(\sum_{j \,:\, b_i b_j = 1} y_i^3 V'(y_i - y_j) + \sum_{j \,:\, b_i b_j = -1} y_i^3 W'(y_i - y_j) \bigg) \\
&= -\frac{2}{n^2} \sum_{i,j \,:\, b_i b_j = 1} (y_i^3 - y_j^3) V'(y_i - y_j) - \frac{2}{n^2} \sum_{i,j \,:\, b_i b_j = -1} (y_i^3 - y_j^3) W'(y_i - y_j) \\
&\leq \frac{C}{n^2} \sum_{i,j \,:\, b_i b_j = 1} (y_i^2 + y_i y_j + y_j^2) + \frac{C}{n^2} \sum_{i,j \,:\, b_i b_j = -1} (y_i^2 + y_i y_j + y_j^2) \\
&\leq \frac{C}{n^2} \sum_{i=1}^n \sum_{j=1}^n (y_i^2(t) + y_j^2(t)) = C M_2(y(t)) \leq C(t + M_2(x^\circ)),
\end{aligned}$$

where we have used (4.11). Hence,

$$M_4(y(t)) \leq M_4(x^\circ) + CT\big(M_2(x^\circ) + T\big), \qquad \text{for all } t \in [0,T]. \tag{4.12}$$

In conclusion, (4.11) and (4.12) provide a priori bounds for $M_2(y(t))$ and $M_4(y(t))$ that are uniform in n and t. Finally, from (4.11) and Lemma 4.4(i)–(iii) we obtain that the solution y to (4.10) is defined and unique at least up to time T.

Next we identify t_1 and choose those b_i that jump at $t = t_1$ (see (4.3)). For this choice, it is enough to specify the collision times τ_i (see (4.2)). We note that

$$t^* := \inf \left\{ t \in (0, T] : \exists\, (i, j) : b_i^\circ b_j^\circ = -1 \text{ and } y_i(t) = y_j(t) \right\}$$

is either attained or $t^* = +\infty$. If $t^* \geq T$, we set $x = y$ and $\tau_i = +\infty$ for all i, and observe that properties (d) and (e) of Definition 4.2 are satisfied. If $t^* < T$, we observe that t_1 in Definition 4.2(d) has to be equal to t^*. We set $x|_{[0,t_1]} := y|_{[0,t^*]}$ and observe from (4.11) and (4.12) that property (i) is satisfied up to $t = t_1$. For the choice of τ_i, we follow the algorithm explained in Subsection 1.2, *i.e.*, for each pair of particles that collide at t_1, we set the corresponding τ_i equal to t_1. We choose the remaining values for $\tau_j > t_1$ later on in the construction. With this choice for τ_i, it follows from the continuity of x_i that properties (d) and (e) of Definition 4.2 are satisfied by construction. Since $E_n(x(t)) \leq E_n(x^\circ)$ for all $t \in [0, t_1)$, it follows that (ii) holds on $[0, t_1]$.

Next we show that we can continue the construction above for $t > t_1$. First, applying Lemma 4.4(iv) $\frac{1}{2}\gamma_1$ times (recall from (4.4) that γ_1 is even), we find that

$$E_n(x(t_1); b(t_1)) \leq E_n(x(t_1); b(t_1-)) + \frac{C}{2n}\bigg(\gamma_1 M_2(x(t_1)) + \gamma_1 + \sum_{i \in \Gamma_1} x_i^2(t_1) \bigg).$$

Hence, (4.7) is satisfied for $k = 1$. Furthermore, we obtain that $E_n(x(t_1); b(t_1)) < \infty$, and thus we can continue the construction above for $t > t_1$ by putting $x(t_1), b(t_1)$ as the initial condition at $t = t_1$.

Iterating over k, this construction identifies all $\tau_i < T$ (for $i \notin \cup_{k=1}^K \Gamma_k$, we set $\tau_i := +\infty$) and t_k, and guarantees that x is piecewise C^1 on $[t_k, t_{k+1}]$ and globally Lipschitz. In addition, (4.7) holds for all $k = 1, \ldots, K$.

Step 2: Uniqueness of (x, b). Let x and τ be as constructed in Step 1, and set b accordingly. Since (4.10) has a unique solution, Definition 4.2(d) defines uniquely the time t_1 until which $x(t)$ is uniquely defined. By Definition 4.2(e), b has to be constant on $[0, t_1)$. Since x satisfies Property (ii) at $t = t_1$, all collisions at t_1 are collisions of two particles with opposite type. Then, from the explanation in Remark 4.3, it is obvious that properties (d) and (e) of Definition 4.2 define uniquely the set of indices i for which $\tau_i = t_1$. Hence, $b(t_1)$ is uniquely determined. We conclude by iterating over k.

Step 3: The remaining Properties (iii)–(v). Estimate (4.7) is already proved; summing over k reads

$$\sum_{k=1}^K [\![e(t_k)]\!] \leq \frac{C}{n} \Big(\sum_{k=1}^K \gamma_k M_2(x(t_k)) + \sum_{k=1}^K \gamma_k + \sum_{k=1}^K \sum_{i \in \Gamma_k} x_i^2(t_k) \Big). \tag{4.13}$$

The first and second sums in the right-hand side above can be easily estimated using (i) and (4.5). We estimate the third sum by using that the sets Γ_k for $k = 1, \ldots, K$ are disjoint, and that for every $k = 1, \ldots, K$ and for every $i \in \Gamma_k$ we have that $x_i(t) = x_i(t_k)$ for all $t \geq t_k$. Hence, the third sum is bounded by $M_2(x(T))$. Collecting our estimates, we obtain (4.8) from (4.13).

With (iii) proven, we prove (iv) for $t = T$ by the following computation (the case $t < T$ follows by a similar estimate). Setting $t_{K+1} \coloneqq T$, we compute

$$\begin{aligned} E_n(x(T); b(T)) - E_n(x^\circ; b^\circ) &= E_n(x(T); b(T)) - E_n(x(t_K); b(t_K)) \\ &\quad + \sum_{k=1}^{K} \big[[\![e(t_k)]\!] + \big(E_n(x(t_k-); b(t_k-)) - E_n(x(t_{k-1}); b(t_{k-1})) \big) \big] \\ &\le \sum_{k=1}^{K+1} \int_{t_{k-1}}^{t_k} \frac{\mathrm{d}}{\mathrm{d}t} E_n(x(t); b(t)) \, \mathrm{d}t + C(T + M_2(x^\circ) + 1) \\ &= - \sum_{k=1}^{K+1} \frac{1}{n} \int_{t_{k-1}}^{t_k} |\dot{x}(t)|^2 \, \mathrm{d}t + C(T + M_2(x^\circ) + 1) \\ &= - \frac{1}{n} \int_0^T |\dot{x}(t)|^2 \, \mathrm{d}t + C(T + M_2(x^\circ) + 1), \end{aligned}$$

where we have used in the second-to-last equality that $x(t)$ satisfies (4.1).

Finally, we prove (v). First, we claim that the strict ordering of the particles $\{x_i(t) : |b_i(t)| = 1\}$ is conserved in time. Clearly, this ordering holds at $t = 0$. From (ii) it follows that any two particles, say with corresponding indices $i \neq j$ such that $b_i(t)b_j(t) = 1$, can never swap position. Similarly, any pair $(x_i(t), x_j(t))$ with $b_i(t)b_j(t) = -1$ cannot swap either, because Definition 4.2(d) ensures that $b_i(t)$ and $b_j(t)$ jump to 0 at the first t at which $x_i(t) = x_j(t)$. In fact, as soon as this happens, the particles cease to move (see the last bullet in Remark 4.3 and also the first bullet in Subsection 1.2 regarding the properties of particles with zero charge).

Next we construct $a_\ell(t)$. We start with $t = 0$, and set $a_0(0), a_1(0), \ldots$ sequentially. We set $a_0(0) \coloneqq x_1^\circ - 1$, and, if $b_1^\circ = -1$, we also put $a_1(0) \coloneqq x_1^\circ - 1$. For each pair of consecutive particles x_i°, x_{i+1}° of opposite sign, we define a new point

$$a_\ell(0) \coloneqq \frac{1}{2}(x_i^\circ + x_{i+1}^\circ).$$

If the current value of ℓ is odd, we define $L := (\ell + 1)/2$ and set $a_{2L}(0) \coloneqq x_n^0 + 1$. If ℓ is even, we define $L := (\ell + 2)/2$ and set $a_{2L-1}(0) \coloneqq a_{2L}(0) \coloneqq x_n^\circ + 1$.

Since the strict ordering of the particles $\{x_i(t) : |b_i(t)| = 1\}$ is conserved in time, we can construct $a_\ell(t)$ analogously, but for a time-dependent L_t. Next we show how to modify this construction such that L_t can be chosen independently of t. Because of the ordering of $\{x_i(t) : |b_i(t)| = 1\}$ and that its cardinality is non-increasing in time, the numbers of pairs of consecutive particles $x_i(t), x_{i+1}(t)$ of opposite non-zero charge is also non-increasing in time. Hence, $t \mapsto L_t$ is non-increasing in time. In case $L_t < L$, we modify the construction of $a_\ell(t)$ above simply by adding a surplus of points $a_\ell(t)$ which all equal $a_{2L_t}(t)$. ∎

Next we establish several properties of the empirical measures associated to the solution $(x; b)$ of Problem 4.1 with initial condition (x°, b°) as in Proposition 4.5. With this aim, we set

$$n^\pm := \#\{i : b_i^\circ = \pm 1\} \tag{4.14}$$

as the number of positive/negative particles at time 0, and note that $n^+ + n^- = n$. The empirical measures associated to $(x(t); b(t))$ are

$$\mu_n^{\circ,\pm} \coloneqq \frac{1}{n} \sum_{i \,:\, b_i^\circ = \pm 1} \delta_{x_i^\circ}, \qquad \mu_n^\pm(t) \coloneqq \frac{1}{n} \sum_{i \,:\, b_i^\circ = \pm 1} \delta_{x_i(t)}, \tag{4.15}$$

which both have total mass equal to $n^\pm/n$ for all $t \in [0,T)$. As in (3.5), we also set

$$\kappa_n(t) \coloneqq \frac{1}{n}\sum_{i=1}^n b_i^\circ \delta_{x_i(t)}, \qquad \mu_n(t) \coloneqq \frac{1}{n}\sum_{i=1}^n \delta_{x_i(t)}, \qquad \tilde\mu_n^\pm(t) \coloneqq [\kappa_n(t)]_\pm. \tag{4.16}$$

LEMMA 4.6 (Proposition 4.5 in terms of measures). *Given the setting as in Proposition 4.5 with (x,b) the solution to (4.1), let $\boldsymbol{\mu}_n \coloneqq (\mu_n^+,\mu_n^-)$, $\tilde{\boldsymbol{\mu}}_n \coloneqq (\tilde\mu_n^+,\tilde\mu_n^-)$, and κ_n as constructed from (x,b) through (4.15) and (4.16). Then,*

(i) $\tilde\mu_n^\pm(t) = \frac{1}{n}\sum_{i=1}^n [b_i(t)]_\pm \delta_{x_i(t)}$;

(ii) $\boldsymbol{\mu}_n \in \mathrm{AC}^2(0,T;\mathcal{P}_2^m(\mathbb{R}^2))$ *with* $m = n^+/n$ *(see (4.14)), and*

$$|\boldsymbol{\mu}_n'|^2_{\mathbf{W}}(t) \le \frac{1}{n}\sum_{i=1}^n \Big(\frac{\mathrm{d}}{\mathrm{d}t}x_i(t)\Big)^2 \quad \textit{for all } 0<t<T; \tag{4.17}$$

(iii) $\boldsymbol{\mu}_n$ *is a solution to (1.3) with initial condition* $\boldsymbol{\mu}_n^\circ = (\mu_n^{\circ,+},\mu_n^{\circ,-})$.

Proof. Property (i) is a corollary of Proposition 4.5. Indeed, Proposition 4.5(v) implies that $[\kappa_n(t)]_\pm \ge \frac{1}{n}\sum_{i=1}^n [b_i(t)]_\pm \delta_{x_i(t)}$, while Definition 4.2(e) implies that $|\kappa_n(t)|(\mathbb{R}) \le \frac{1}{n}\sum_{i=1}^n |b_i(t)|$. We conclude (i).

Next we prove (ii). From the definition of $\boldsymbol{\mu}_n$ in (4.15) we observe that $\boldsymbol{\mu}_n(t) \in \mathcal{P}_2^m(\mathbb{R}^2)$ for all $0<t<T$. Hence, (3.4) applies, and we obtain

$$\mathbf{W}^2\big(\boldsymbol{\mu}_n(s),\boldsymbol{\mu}_n(t)\big) \le W^2\big(\mu_n^+(s),\mu_n^+(t)\big) + W^2\big(\mu_n^-(s),\mu_n^-(t)\big) \tag{4.18}$$

for all $0<s\le t<T$. To estimate the right-hand side, we let $0<s\le t<T$ be given, and introduce the coupling

$$\gamma_n^\pm \coloneqq \frac{1}{n}\sum_{i\,:\,b_i^\circ=\pm1} \delta_{(x_i(s),x_i(t))} \in \Gamma\big(\mu_n^\pm(s),\mu_n^\pm(t)\big).$$

By definition of the Wasserstein distance (3.1), we obtain

$$W^2\big(\mu_n^\pm(s),\mu_n^\pm(t)\big) \le \iint_{\mathbb{R}^2\times\mathbb{R}^2} |x-y|^2 \,\mathrm{d}\gamma_n^\pm(x,y) = \frac{1}{n}\sum_{i\,:\,b_i^\circ=\pm1}\big(x_i(s)-x_i(t)\big)^2. \tag{4.19}$$

Finally, using in sequence the estimates (3.8), (4.18), and (4.19), we conclude (4.17). Since $x \in \mathrm{Lip}([0,T];\mathbb{R}^n)$, we obtain that $\boldsymbol{\mu}_n \in \mathrm{AC}^2(0,T;\mathcal{P}_2^m(\mathbb{R}^2\times\{\pm1\}))$.

Next we prove (iii). We rewrite (4.1) as

$$\begin{aligned}
\dot x_i(t) &= -b_i(t)\big(V' * \tilde\mu_n^+(t) + W' * \tilde\mu_n^-(t)\big)(x_i(t)), &&\text{for } i \text{ such that } b_i^\circ = 1,\\
\dot x_i(t) &= -b_i(t)\big(W' * \tilde\mu_n^+(t) + V' * \tilde\mu_n^-(t)\big)(x_i(t)), &&\text{for } i \text{ such that } b_i^\circ = -1.
\end{aligned}$$

Let $\varphi \in \mathrm{C}_c^\infty((0,T)\times\mathbb{R})$ be any test function. Since x_i is Lipschitz, the Fundamental

Theorem of Calculus applies, and thus we obtain, using (i),

$$\begin{aligned}
0 &= \frac{1}{n} \sum_{i\,:\,b_i^\circ=1} \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\varphi(t, x_i(t))\,\mathrm{d}t \\
&= \frac{1}{n} \sum_{i\,:\,b_i^\circ=1} \bigg[\int_0^T \partial_t\varphi(t, x_i(t))\,\mathrm{d}t + \int_0^T \varphi'(t, x_i(t))\,\dot{x}_i(t)\,\mathrm{d}t \bigg] \\
&= \int_0^T \int_{\mathbb{R}} \partial_t\varphi\,\mathrm{d}\mu_n^+\mathrm{d}t - \int_0^T \frac{1}{n} \sum_{i\,:\,b_i=1} \varphi'(x_i)\big(V' * \tilde{\mu}_n^+ + W' * \tilde{\mu}_n^-\big)(x_i)\,\mathrm{d}t \\
&= \int_0^T \int_{\mathbb{R}} \partial_t\varphi\,\mathrm{d}\mu_n^+\mathrm{d}t - \int_0^T \int_{\mathbb{R}} \varphi'\big(V' * [\kappa_n]_+ + W' * [\kappa_n]_-\big)\,\mathrm{d}[\kappa_n]_+\mathrm{d}t,
\end{aligned}$$

where φ' denotes the partial derivative with respect to the spatial variable. Since φ is arbitrary and V' is odd, we conclude that μ_n^+ satisfies (3.6). From a similar argument, it follows that also μ_n^- satisfies (3.6). ∎

5. Statement and proof of the main convergence theorem. In this section, we state and prove our main convergence theorem.

THEOREM 5.1 (Discrete-to-continuum limit). *Let the potentials V and W satisfy Assumption 2.1. Let $(x^{n,\circ}, b^{n,\circ})_n$ be a sequence of initial conditions such that*

(i) $E_n(x^{n,\circ}; b^{n,\circ})$ is bounded uniformly in n,
(ii) $(\boldsymbol{\mu}_n^\circ)_n$ (see (4.15)) has bounded fourth moment uniformly in n,
(iii) there exists an $L \in \mathbb{N}$ independent of n such that Assumption 2.2 is satisfied for all n.

Then for every $T > 0$ the curves $\boldsymbol{\mu}_n \in \mathrm{AC}^2(0, T; \mathcal{P}_2(\mathbb{R} \times \{\pm 1\}))$ determined by the solution (x^n, b^n) to Problem 4.1 through (4.15) for each n, converge in measure uniformly in time along a subsequence to a solution $\boldsymbol{\rho}$ of (3.6), whose initial condition $\boldsymbol{\rho}^\circ$ is the limit of $(\boldsymbol{\mu}_n^\circ)_n$ along the same subsequence.

The proof is divided in three steps. In the first step we use compactness of $\boldsymbol{\mu}_n(t)$ to extract a subsequence n_k along which $\boldsymbol{\mu}_n(t)$ converges to some $\boldsymbol{\rho}(t)$. In the remaining two steps we pass to the limit in (3.6) as $k \to \infty$ to show that the limiting curve $\boldsymbol{\rho}(t)$ also satisfies (3.6). Step 2 contains the main novelty; relying on Assumption 2.2 with an n_k-independent number L, we prove that $[\kappa_{n_k}(t)]_\pm \rightharpoonup [\kappa(t)]_\pm$ as $k \to \infty$ pointwise in t.

Proof. Step 1: $\boldsymbol{\mu}_n$ converges along a subsequence $n_k \to \infty$ in $\mathrm{C}([0, T]; \mathcal{P}_2(\mathbb{R} \times \{\pm 1\}))$ to $\boldsymbol{\rho} \in \mathrm{AC}^2(0, T; \mathcal{P}_2^m(\mathbb{R} \times \{\pm 1\}))$ with $m := \rho^{\circ,+}(\mathbb{R})$. We prove this statement by means of the Ascoli-Arzelà Theorem (see Lemma 3.1) applied to the metric space $(\mathcal{P}_2(\mathbb{R} \times \{\pm 1\}), \mathbf{W})$.

First, we show that, for fixed $t \in [0, T]$, the sequence $(\boldsymbol{\mu}_n(t))_n$ is pre-compact in $\mathcal{P}_2(\mathbb{R} \times \{\pm 1\})$. From the assumption on the initial data and Proposition 4.5(i) we observe that the second and fourth moments of the measures $\mu_n(t)$ defined in (4.16), given by

$$M_2(x^n(t)) = \int_{\mathbb{R}} y^2\,\mathrm{d}\mu_n(t)(y), \qquad M_4(x^n(t)) = \int_{\mathbb{R}} y^4\,\mathrm{d}\mu_n(t)(y),$$

are bounded uniformly in n and $t \in [0, T]$. Then, from [47, Lemma B.3] and [2, Proposition 7.1.5] we find that $(\boldsymbol{\mu}_n(t))_n$ is pre-compact in the Wasserstein distance $\mathbf{W}$.

Second, we show that the sequence $(\boldsymbol{\mu}_n)_n \subset \mathrm{C}([0,T];\mathcal{P}_2(\mathbb{R}\times\{\pm1\}))$ is equicontinuous (*i.e.*, $(\boldsymbol{\mu}_n)_n$ satisfies Lemma 3.1(ii)). For any $0 \le s < t \le T$, we estimate

$$\mathbf{W}^2\big(\boldsymbol{\mu}_n(t),\boldsymbol{\mu}_n(s)\big) \le \bigg(\int_s^t |\boldsymbol{\mu}_n'|_{\mathbf{W}}(r)\,\mathrm{d}r\bigg)^2 \le (t-s)\int_0^T |\boldsymbol{\mu}_n'|_{\mathbf{W}}^2(r)\,\mathrm{d}r. \tag{5.1}$$

To estimate the last integral above, we use the estimates in Lemma 4.6(ii) and Proposition 4.5(iv) to obtain

$$\begin{aligned}\int_0^T |\boldsymbol{\mu}_n'|_{\mathbf{W}}^2(r)\,\mathrm{d}r \le& \frac{1}{n}\int_0^T \sum_{i=1}^n \Big(\frac{\mathrm{d}}{\mathrm{d}t}x_i^n(r)\Big)^2 \mathrm{d}r = \frac{1}{n}\int_0^T |\dot{x}^n(r)|^2\,\mathrm{d}r \\ \le& C(T + M_2(x^{n,\circ}) + 1) + E_n(x^{n,\circ};b^{n,\circ}) - E_n(x^n(T);b^n(T)).\end{aligned} \tag{5.2}$$

Since, by Lemma 4.4(ii) and Proposition 4.5(i), we have

$$\begin{aligned}E_n(x^n(T);b^n(T)) =& [E_n(x^n(T);b^n(T)) + M_2(x^n(T))] - M_2(x^n(T)) \\ \ge& -C - [\tilde{C}T + M_2(x^{n,\circ})],\end{aligned}$$

we obtain from (5.2) that

$$\int_0^T |\boldsymbol{\mu}_n'|_{\mathbf{W}}^2(r)\,\mathrm{d}r \le C(T + M_2(x^{n,\circ}) + 1) + E_n(x^{n,\circ};b^{n,\circ}). \tag{5.3}$$

By the assumptions on the initial data, the right-hand side is bounded uniformly in n. Hence, the right-hand side in (5.1) is bounded by $C(t-s)$, and thus $(\boldsymbol{\mu}_n)_n$ is equicontinuous.

From the pre-compactness of $(\boldsymbol{\mu}_n(t))_n$ and the equicontinuity of $(\boldsymbol{\mu}_n)_n$, we obtain from Lemma 3.1 the existence of a subsequence n_k along which $(\boldsymbol{\mu}_n)_n$ converges in $\mathrm{C}([0,T];\mathcal{P}_2(\mathbb{R}\times\{\pm1\}))$ to some limiting curve $\boldsymbol{\rho} \in \mathrm{C}([0,T];\mathcal{P}_2(\mathbb{R}\times\{\pm1\}))$. In fact, combining the lower semi-continuity obtained in Theorem 3.2 with (5.3), we obtain that $\boldsymbol{\rho} \in \mathrm{AC}^2(0,T;\mathcal{P}_2(\mathbb{R}\times\{\pm1\}))$. Moreover, since the total mass of $\mu_n^+(t)$ is conserved in time, and since the narrow topology conserves mass, we conclude that $\boldsymbol{\rho}(t) \in \mathcal{P}_2^m(\mathbb{R}\times\{\pm1\})$ for all $t\in[0,T]$. This completes the proof of Step 1. For later use, we set as in (3.5)

$$\rho := \rho^+ + \rho^-, \qquad \kappa := \rho^+ - \rho^-, \qquad \tilde{\rho}^\pm := [\kappa]_\pm.$$

Step 2: $\tilde{\boldsymbol{\mu}}_{n_k}(t) \rightharpoonup \tilde{\boldsymbol{\rho}}(t)$ as $k\to\infty$ pointwise for all $t\in[0,T]$. We set $\tilde{\mu}_{n_k}^\pm = [\kappa_{n_k}]_\pm$ as in (4.16) and $\tilde{\boldsymbol{\mu}}_{n_k}$ as in Lemma 4.6. We keep $t\in[0,T]$ fixed, and remove it from the notation in the remainder of this step. The structure of the proof of Step 2 is to show by compactness that $(\tilde{\boldsymbol{\mu}}_{n_k})_k$ has a converging subsequence, and to characterise the limit as $\tilde{\boldsymbol{\rho}}$. Since $\tilde{\boldsymbol{\rho}}$ is independent of the choice of subsequence, we then conclude that the full sequence $(\tilde{\boldsymbol{\mu}}_{n_k})_k$ converges to $\tilde{\boldsymbol{\rho}}$. Keeping this in mind, in the following we omit all labels of subsequences of n.

Since the second moments of $\tilde{\boldsymbol{\mu}}_n$ are obviously bounded by $M_2(x^n)$, the sequence $(\tilde{\boldsymbol{\mu}}_n)$ is tight, and thus, by Prokhorov's Theorem, $(\tilde{\boldsymbol{\mu}}_n)$ converges narrowly along a subsequence to some $\tilde{\boldsymbol{\mu}} \in \mathcal{M}_+(\mathbb{R}\times\{\pm1\})$.

We claim that $\tilde{\boldsymbol{\mu}}$ does not have atoms. We reason by contradiction. Suppose that $\tilde{\mu}^+$ has an atom at y of mass $\alpha > 0$ (the case of $\tilde{\mu}^-$ can be treated analogously). Then, setting $B_\eta(y)$ as the ball around y with radius η, we infer from $\tilde{\mu}_n^+ \rightharpoonup \tilde{\mu}^+$ that

$\liminf_{n\to\infty} \tilde\mu_n^+(B_\eta(y)) \geq \alpha > 0$ for any $\eta > 0$. By choosing $\eta > 0$ small enough, the contribution of the particles in $B_\eta(y)$ to the energy $E_n(x^n; b^n)$ can be made arbitrarily large, which contradicts with the uniform bound on $E_n(x^n; b^n)$ given by Proposition 4.5(iv).

In the remainder of this step we show that $\tilde\mu^\pm = [\kappa]_\pm$, regardless of the choice of the subsequence. It is enough to show that

$$[\kappa]_\pm \leq \tilde\mu^\pm \tag{5.4}$$

$$[\kappa]_\pm(\mathbb{R}) \geq \tilde\mu^\pm(\mathbb{R}) \tag{5.5}$$

Regarding (5.4), we obtain from Step 1 that

$$\tilde\mu_n^+ - \tilde\mu_n^- = \kappa_n \rightharpoonup \kappa \quad \text{as } n \to \infty.$$

Hence, $\tilde\mu^+ - \tilde\mu^- = \kappa$, which implies (5.4). To prove (5.5), we let $\{a_\ell^n\}_{\ell=0}^{2L}$ be as in Proposition 4.5(v), and set

$$\tilde\mu_n^\ell := \begin{cases} \tilde\mu_n^+|_{(a_{\ell-1}^n, a_\ell^n)} & \ell \text{ odd} \\ \tilde\mu_n^-|_{(a_{\ell-1}^n, a_\ell^n)} & \ell \text{ even} \end{cases}$$

for all $\ell \in \{1, \ldots, 2L\}$. By construction,

$$\sum_{\ell=1}^L \tilde\mu_n^{2\ell-1} = \tilde\mu_n^+ \quad \text{and} \quad \sum_{\ell=1}^L \tilde\mu_n^{2\ell} = \tilde\mu_n^-.$$

Together with $\tilde{\boldsymbol\mu}_n \rightharpoonup \tilde{\boldsymbol\mu}$, we conclude that $(\tilde\mu_n^\ell)_n$ are tight for any ℓ, and thus, applying Prokhorov's Theorem once more, each sequence $(\tilde\mu_n^\ell)_n$ converges along a subsequence in the narrow topology to some $\tilde\mu^\ell \in \mathcal{M}_+(\mathbb{R})$. In particular, from $\tilde{\boldsymbol\mu}_n \rightharpoonup \tilde{\boldsymbol\mu}$ and

$$\tilde\mu_n^- = \sum_{\ell=1}^L \tilde\mu_n^{2\ell} \rightharpoonup \sum_{\ell=1}^L \tilde\mu^{2\ell},$$

we infer that $\tilde\mu^- = \sum_{\ell=1}^L \tilde\mu^{2\ell}$. By a similar argument, it follows that $\tilde\mu^+ = \sum_{\ell=1}^L \tilde\mu^{2\ell-1}$. Finally, since $\sup(\operatorname{supp}\tilde\mu_n^\ell) < \inf(\operatorname{supp}\tilde\mu_n^{\ell+1})$ for all $1 \leq \ell \leq 2L-1$, we obtain from Lemma 3.4 that $\sup(\operatorname{supp}\tilde\mu^\ell) < \inf(\operatorname{supp}\tilde\mu^{\ell+1})$ for all $1 \leq \ell \leq 2L-1$. Hence, there exists $A := \{a_\ell\}_{\ell=1}^{2L-1}$ such that

$$\begin{aligned} \operatorname{supp}\tilde\mu^+ \cap \operatorname{supp}\tilde\mu^- &= \left(\bigcup_{\ell=1}^L \operatorname{supp}\tilde\mu^{2\ell-1} \right) \cap \left(\bigcup_{k=1}^L \operatorname{supp}\tilde\mu^{2k} \right) \\ &= \bigcup_{\ell=1}^L \bigcup_{k=1}^L \left(\operatorname{supp}\tilde\mu^{2\ell-1} \cap \operatorname{supp}\tilde\mu^{2k} \right) = \bigcup_{\ell=1}^{2L-1} \left(\operatorname{supp}\tilde\mu^\ell \cap \operatorname{supp}\tilde\mu^{\ell+1} \right) \subset A. \end{aligned}$$

Since $\tilde\mu^\pm$ does not have atoms, $\tilde\mu^\pm(A) = 0$. Together with $\tilde\mu^+ - \tilde\mu^- = \kappa$, it is easy to construct a Hahn decomposition of κ (see, *e.g.*, [35, Theorem 6.14]). We conclude (5.5).

Step 3: $\boldsymbol\rho$ is a solution to (1.3). To ease notation, we replace n_k by n. We show that $\boldsymbol\rho$ satisfies (3.6). With this aim, let $\varphi^\pm \in C_c^\infty((0,T) \times \mathbb{R})$ be arbitrary. We recall from Lemma 4.6(iii) that $\boldsymbol{\mu}_n$ satisfies

$$
\begin{aligned}
0 = \int_0^T \int_{\mathbb{R}} \partial_t \varphi^\pm(x)\, \mathrm{d}\mu_n^\pm(x)\mathrm{d}t - \int_0^T \int_{\mathbb{R}} (\varphi^\pm)'(x)\, (W' * [\kappa_n]_\mp)(x)\, \mathrm{d}[\kappa_n]_\pm(x)\mathrm{d}t \\
- \frac{1}{2}\int_0^T \iint_{\mathbb{R}\times\mathbb{R}} \big((\varphi^\pm)'(x) - (\varphi^\pm)'(y)\big)\, V'(x-y)\, \mathrm{d}([\kappa_n]_\pm \otimes [\kappa_n]_\pm)(x,y)\mathrm{d}t.
\end{aligned}
\tag{5.6}
$$

We show that we can pass to the limit in all three terms separately. From Step 1 it follows that $\boldsymbol{\mu}_n \rightharpoonup \boldsymbol{\rho}$, and thus the limit of the first integral equals

$$
\int_0^T \int_{\mathbb{R}} \partial_t \varphi^\pm(x)\, \mathrm{d}\rho^\pm(x)\mathrm{d}t.
$$

Regarding the other two integrals in (5.6), we recall from Step 2 that $[\kappa_n(t)]_\pm \rightharpoonup [\kappa(t)]_\pm$ as $n \to \infty$ pointwise for all $t \in [0,T]$. Then, for the second term, since $(x,y) \mapsto (\varphi^\pm)'(x)\, W'(x-y)$ is bounded and continuous on $\mathbb{R}^2$, we obtain that

$$
\int_{\mathbb{R}} (\varphi^\pm)'(x)\, (W' * [\kappa_n]_\mp)(x)\, \mathrm{d}[\kappa_n]_\pm(x) = \iint_{\mathbb{R}^2} (\varphi^\pm)'(x)\, W'(x-y)\, \mathrm{d}([\kappa_n]_\pm \otimes [\kappa_n]_\mp)(x,y)
$$

converges, as $n \to \infty$, to

$$
\iint_{\mathbb{R}^2} (\varphi^\pm)'(x)\, W'(x-y)\, \mathrm{d}([\kappa]_\pm \otimes [\kappa]_\mp)(x,y) = \int_{\mathbb{R}} (\varphi^\pm)'(x)\, (W' * [\kappa]_\mp)(x)\, \mathrm{d}[\kappa]_\pm(x).
$$

Finally, we pass to the limit in the third integral in (5.6). We employ Lemma 3.3 with $d = 2$ and $\Delta = \{(y,y) : y \in \mathbb{R}\}$ the diagonal in $\mathbb{R}^2$. To show that the conditions of Lemma 3.3 are satisfied, we observe from the fact that $r \mapsto rV'(r)$ is bounded and belongs to $\mathrm{C}(\mathbb{R}\setminus\{0\})$, it holds that $(x,y) \mapsto [(\varphi^\pm)'(x) - (\varphi^\pm)'(y)]\, V'(x-y)$ is bounded and belongs to $\mathrm{C}(\mathbb{R}^2 \setminus \Delta)$. Moreover, by Step 2, $([\kappa]_\pm \otimes [\kappa]_\pm)(\Delta) = (\tilde{\mu}^\pm \otimes \tilde{\mu}^\pm)(\Delta) = 0$. Hence, by Lemma 3.3 we can pass to the limit in the third term in (5.6), whose limit reads

$$
-\frac{1}{2}\int_0^T \iint_{\mathbb{R}\times\mathbb{R}} \big((\varphi^\pm)'(x) - (\varphi^\pm)'(y)\big)\, V'(x-y)\, \mathrm{d}([\kappa]_\pm \otimes [\kappa]_\pm)(x,y)\mathrm{d}t.
$$

Combining the three limits above, and recalling the time regularity of $\boldsymbol{\rho}$ from Step 1, we conclude that $\boldsymbol{\rho}$ is a solution to (1.3). ◻

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