

Discrete-to-continuum limits of particles with an annihilation rule

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1     **DISCRETE-TO-CONTINUUM LIMITS OF PARTICLES WITH AN**  
2                                   **ANNIHILATION RULE\***

3                   PATRICK VAN MEURS<sup>†</sup> AND MARCO MORANDOTTI<sup>‡</sup>

4     **Abstract.** In the recent trend of extending discrete-to-continuum limit passages for gradient  
5 flows of single-species particle systems with singular and nonlocal interactions to particles of opposite  
6 sign, any annihilation effect of particles with opposite sign has been side-stepped. We present the  
7 first rigorous discrete-to-continuum limit passage which includes annihilation. This result paves the  
8 way to applications such as vortices, charged particles, and dislocations. In more detail, the discrete  
9 setting of our discrete-to-continuum limit passage is given by particles on the real line. Particles of  
10 the same type interact by a singular interaction kernel; those of opposite sign interact by a regular  
11 one. If two particles of opposite sign collide, they annihilate, *i.e.*, they are taken out of the system.  
12 The challenge for proving a discrete-to-continuum limit is that annihilation is an intrinsically discrete  
13 effect where particles vanish instantaneously in time, while on the continuum scale the mass of the  
14 particle density decays continuously in time. The proof contains two novelties: (i) the observation  
15 that empirical measures of the discrete dynamics (with annihilation rule) satisfy the continuum  
16 evolution equation that only implicitly encodes annihilation, and (ii) the fact that, by imposing a  
17 relatively mild separation assumption on the initial data, we can identify the limiting particle density  
18 as a solution to the same continuum evolution equation.

19     **Key words.** Particle system, discrete-to-continuum asymptotics, annihilation, gradient flows

20     **AMS subject classifications.** 82C22, (82C21, 35A15, 74G10).

21     **1. Introduction.** A recent trend in discrete-to-continuum limit passages in over-  
22 damped particle systems with singular and nonlocal interactions (with applications  
23 to, *e.g.*, vortices [9, 19, 38], charged particles [36], dislocations [18, 27, 30], and dis-  
24 location walls [13, 47, 48]) is to extend such results to two-species particle systems.  
25 The singularity in the interaction potential imposes the immediate problem that the  
26 evolution of the particle system is only defined up to the first collision time between  
27 particles of opposite sign. This problem is dealt with by either *regularising* the singu-  
28 lar interaction potential (see [11, 12]) or by limiting the geometry such that particles of  
29 opposite sign cannot collide (see [7, 46]). However, more realistic models of vortices,  
30 charged particles, and dislocations include the *annihilation* of particles of opposite  
31 sign. While annihilation has been analysed on the discrete scale [40, 41] and contin-  
32 uum scale [3, 6] separately, there is no rigorous discrete-to-continuum limit passage  
33 known between these two scales.

34     The main result in this paper establishes the first result on a discrete-to-continuum  
35 limit passage in two-species particle systems in one dimension with annihilation.

36     Below, we first describe the physical context of our main result. Then, we intro-  
37 duce the discrete and continuum problems. Our main result is the connection between  
38 them in terms of the limit passage as the number of particles  $n$  tends to  $\infty$ . Then, we  
39 put our discrete and continuum problems in the perspective of the literature, and com-

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40 ment how our proof combines known techniques with novel ideas. We conclude with  
 41 an exposition of possible extensions to work towards singular interspecies interactions  
 42 and higher dimensions.

43 **1.1. Application to plasticity and dislocations.** The main application we  
 44 have in mind is to increase the understanding of the plastic behaviour of metals.  
 45 Plasticity in metals is the emergent behaviour of large groups of dislocations moving  
 46 and interacting on microscopic time- and length-scales. Dislocations are stacking  
 47 faults in the atomic lattice. We keep the description of dislocations concise, and refer  
 48 to the classical textbooks [21, 24] for a detailed description. In two-dimensional elastic  
 49 bodies, dislocations are often represented as points in the elastic body at which the  
 50 stress has a prescribed singularity. This singularity depends on the orientation of the  
 51 dislocation, which is described by the so-called Burgers vector. While dislocations  
 52 themselves exert a stress field, they can also move in response to the stress induced  
 53 by other dislocations in the elastic body. The simplest model to capture such effects  
 54 is an interacting particle system which fits to the setting in this paper.

55 One of the main unsolved problems in plasticity is how to describe the group  
 56 behaviour of many dislocations in terms of a dislocation density. While there are  
 57 many different models available in the engineering literature for the dislocation density  
 58 [5, 15, 16, 22, 25, 26, 42], it is not clear which of these models describes the group  
 59 behaviour of a given collection of dislocations for a given set of parameters. This  
 60 problem arises from a lack of rigour in the derivation of these continuum dislocation  
 61 models from the dynamics of a large group of interacting dislocations (called discrete  
 62 dislocation models).

63 To resolve this lack of rigour, over the course of two decades a large mathematical  
 64 community has established rigorous connections between discrete and continuum dis-  
 65 location models; see [1, 10, 11, 13, 18, 29, 30] for a few examples of different discrete  
 66 dislocation models and different techniques. The final aim is to lift all the currently  
 67 required simplifications on the discrete dislocation models without losing the rigorous  
 68 connection(s) with the related continuum model(s).

69 In recent years, the simplification that all dislocations have the same Burgers  
 70 vector is being lifted. This generalisation corresponds to particle systems with mul-  
 71 tiple species. It has the difficulty that dislocations with different Burgers vector may  
 72 collide in finite time (due to the singular stress they exert). In particular, two (screw)  
 73 dislocations with opposite Burgers vector are known to collide in finite time [23], and  
 74 disappear upon collision. Such a collision is called annihilation. In the current litera-  
 75 ture, the difficulty of including annihilation or other collision rules is side-stepped by  
 76 either enforcing geometrical restrictions [7, 46], or by introducing an artificial regu-  
 77 larisation of the singularity in the stress field (see [12] and [44, Chap. 9]). A common  
 78 observation in these papers is that, depending on the geometrical restrictions or the  
 79 regularisation, rigid micro-structures can appear over time which are not recovered by  
 80 the expected continuum dislocation model. In fact, the simulations in [44, Chap. 9]  
 81 show that the group behaviour of dislocations can depend on the choice of regulari-  
 82 sation, which would imply that the continuum model has to depend on the choice of  
 83 regularisation.

84 Therefore, to avoid the dependence of the continuum model on the choice of regu-  
 85 larisation or geometrical restrictions, we aim to make the first step for including  
 86 dislocation annihilation in connecting discrete to continuum dislocation models. Our  
 87 novel result includes an annihilation rule, but sidesteps the additional difficulty that  
 88 prior to collision, the speed of the colliding dislocations becomes unbounded. To avoid

89 unbounded velocities prior to collision, we replace the singular interaction between  
 90 dislocations of opposite Burgers vector by a regular one. This choice induces the  
 91 further restriction of a one-dimensional spatial setting, which is needed to enforce col-  
 92 lisions. Indeed, for regular interactions in higher dimensions, dislocations of opposite  
 93 Burgers vector need not collide in finite time.

94 In Section 1.7 we demonstrate how our main result can be used as a stepping  
 95 stone for considering annihilation with singular interactions between dislocations of  
 96 opposite Burgers vector.

97 **1.2. The discrete problem (particle system with annihilation).** We re-  
 98 turn our attention from dislocations to a more general particle system with two species  
 99 and an annihilation rule. We introduce the related evolution problem by first spec-  
 100 ifying the state of the system, then the related interaction energy, and finally the  
 101 evolution law. The state of the system is described by  $x := (x_1, \dots, x_n) \in \mathbb{R}^n$  and  
 102  $b := (b_1, \dots, b_n) \in \{-1, 0, 1\}^n$ , with  $n \geq 2$  the number of particles. The point  $x_i$  is  
 103 the location of the  $i$ -th particle, and  $b_i$  is its charge (or Burgers vector, in the setting  
 104 of dislocations).

105 To any state  $(x, b)$  we assign the interaction energy  $E_n: \mathbb{R}^n \times \{-1, 0, 1\}^n \rightarrow$   
 106  $\mathbb{R} \cup \{+\infty\}$  by

$$107 \quad (1.1) \quad E_n(x; b) := \frac{1}{2n^2} \sum_{i=1}^n \left( \sum_{\substack{j=1 \\ j \neq i \\ b_i b_j = 1}}^n V(x_i - x_j) + \sum_{\substack{j=1 \\ b_i b_j = -1}}^n W(x_i - x_j) \right),$$

108 where  $V$  and  $W$  are the interaction potentials between particles of equal and opposite  
 109 charge, respectively. For  $V$  and  $W$ , we have three choices in mind, all of which are of  
 110 separate interest:

- 111 (i)  $V(r) = -\log|r|$  and  $W \equiv 0$ . This corresponds to the easiest case in which  
 112 the two species only interact with their own kind. It is distinct from the  
 113 single-particle case solely by the annihilation rule which we specify below.  
 114 We consider this setting as a convenient benchmark problem, but we have no  
 115 direct application in mind.
- 116 (ii)  $V(r) = -\log|r|$  and  $W$  a regularisation of  $-V$  (as illustrated in Figure 1).  
 117 This is a first step to considering the case of positive and negative charges  
 118 (or positive and negative dislocations) in which  $W = -V$  is chosen in a two-  
 119 dimensional setup [40, 41, 43]. After stating our main result for regular  $W$ , we  
 120 comment in Subsection 1.7 on how this result helps in passing to the limit in  
 121 the particle dynamics corresponding to regular potentials  $W_\delta$  which converge  
 122 to the singular  $-V$  as the regularisation parameter  $\delta$  tends to 0.
- 123 (iii)  $V(r) = r \coth r - \log|2 \sinh r|$  and  $W$  a regularisation of  $-V$ . This setting  
 124 corresponds to that of dislocation walls, *i.e.*, infinite arrays of equi-spaced  
 125 dislocations. The explicit expression for  $V$  is found by summing over all  
 126 dislocations in such a wall; see [21, (19-75)] or [46, Sec. 2]. This potential  
 127  $V$  has several pleasant properties: it has a logarithmic singularity at 0, it  
 128 is decreasing on  $(0, \infty)$ , and it is positive with integrable tails. Discrete-  
 129 to-continuum limits of particle systems consisting of interacting dislocation  
 130 walls are established in [13, 17, 21, 46, 47, 48] for either single-sign scenarios  
 131 or without annihilation.

132 For our analysis, we propose a unified setting which includes the three cases above:  
 133 we consider a class of potentials  $V$  and  $W$  which satisfy a certain set of assumptions

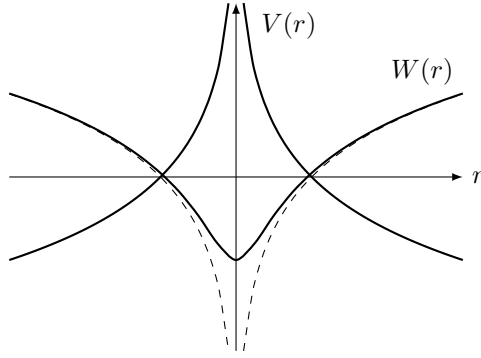


FIGURE 1. Plots of  $V(r) = -\log|r|$  and a typical regularisation  $W$  of  $-V$ .

134 specified in [Assumption 2.1](#). The crucial assumptions are that the singularity of  $V$  at 0  
 135 is at most logarithmic, that  $V(r) \rightarrow +\infty$  as  $r \rightarrow 0$ , that  $W$  is regular, and that  $V$  and  
 136  $W$  have at most logarithmic growth at infinity. In view of other typical assumptions  
 137 in the literature, we *do not* rely on convexity or monotonicity. In [Subsection 1.6](#) we  
 138 elaborate on the necessity of these assumptions to our main discrete-to-continuum  
 139 result.

140 Finally, we make three observations on the structure of [\(1.1\)](#). First, if the  $i$ -th  
 141 particle has 0 charge (*i.e.*,  $b_i = 0$ ), then it does not contribute to  $E_n$ . Second, the  
 142 factor  $1/2$  in front of the energy is common; it corrects the fact that all interactions are  
 143 counted twice in the summation. Third, the condition  $j \neq i$  prevents self-interaction.  
 144

145 Equation [\(1.2\)](#) formally describes the dynamics; for a rigorous definition see [Prob-](#)  
 146 [lem 4.1](#) and [Definition 4.2](#).

$$147 \quad (1.2) \quad \begin{cases} \frac{d}{dt}x_i = -\frac{1}{n} \sum_{j: b_i b_j = 1} V'(x_i - x_j) - \frac{1}{n} \sum_{j: b_i b_j = -1} W'(x_i - x_j) & \text{on } (0, T) \setminus T_{\text{col}}, \\ \text{annihilation rule at } T_{\text{col}}. \end{cases}$$

148 Here,  $T_{\text{col}} = \{t_1, \dots, t_K\}$  is a finite set of collision times, outside of which  $x(t)$  is the  
 149 gradient flow of  $E_n$ . The version of [\(1.2\)](#) in two dimensions and in which  $W(r) =$   
 150  $-V(r) = \log|r|$  is discussed in great detail in [\[41\]](#).

151 Next we explain the “annihilation rule at  $T_{\text{col}}$ ”. Given that at  $t = 0$  all particles  
 152 are at different positions, [\(1.2\)](#) follows for at least a small time interval simply the  
 153 gradient flow of  $E_n(\cdot; b)$  in which  $b$  is constant in time. Since  $V$  is a singular, repelling  
 154 interaction potential and  $W$  is regular, particles of the same sign will not cross each  
 155 other, but particles of opposite sign may. We call the first time instance at which  
 156 such a crossing happens a *collision time*, and denote it by  $t_1$ . At  $t_1$ , the annihilation  
 157 rule states that those particles of opposite sign which are at the same position are  
 158 ‘removed’ from the system, and that the system is restarted at time  $t_1$  with the  
 159 remaining particles at their current positions. It again follows the gradient flow of  $E_n$   
 160 (but now with fewer particles) until the next collision time  $t_2$  at which two particles of  
 161 opposite sign cross. At  $t_2$ , an analogous annihilation rule is applied. In this manner,  
 162  $T_{\text{col}}$  is constructed. We allow for more than one pair of particles to annihilate at the  
 163 same time instance  $t_k$ . Because of the singularity of  $V$ , annihilations that happen at  
 164 the same time always occur at different points in space.

165 For technical reasons, we encode the removal of particles by putting their charge  
 166  $b_i(t)$  from  $\pm 1$  to 0 as opposed to making  $n$  dependent on  $t$ . We note that, if particle  
 167  $i$  has zero charge, then

- 168 •  $x_i(t)$  remains stationary,
- 169 • the velocity of all other particles does not depend on  $x_i(t)$ , and
- 170 • particle  $i$  cannot annihilate any more with any other particle.

171 We note that each  $b_i(t)$  is a shifted Heaviside functions that jumps at some collision  
 172 time  $t_k$ .

173 Next we motivate the applicability of (1.2) by two related examples. The first  
 174 example is that of dislocations, whose dynamical law naturally includes annihilation  
 175 effects. The linear relation in (1.2) between the velocity and the gradient of the energy  
 176 is purely phenomenological, and is, due to its simplicity and lack of consensus for a  
 177 better alternative, the most commonly used relation in dislocation dynamics models.  
 178 We refer to [43] for simulations of a generalized version of (1.2) in the context of  
 179 dislocations.

180 The second example of a system related to (1.2) is that in [40] and [41, Theo-  
 181 rems 1.3 and 1.4], where the limit of the Ginzburg-Landau equation on the dynamics  
 182 of vortices is studied as the phase-field parameter  $\varepsilon$  tends to 0. In the limiting equa-  
 183 tion, the vortices are characterised as points with a charge whose dynamics are given  
 184 by the version of (1.2) in which  $W(r) = -V(r) = \log|r|$  and the particles are two-  
 185 dimensional. While detailed properties of the particles trajectories are proven, a pre-  
 186 cise solution concept to this version of (1.2) remains elusive. In our one-dimensional  
 187 setting, we establish a solution concept to (1.2) in Definition 4.2 and Proposition 4.5.

188 **1.3. The continuum problem (PDE for the particle density).** On the  
 189 continuum level, the state of the system is described by the nonnegative measures  $\rho^\pm$ ,  
 190 which represent the density of the positive/negative particles (including those that  
 191 are annihilated). We further set

$$192 \quad \rho := \rho^+ + \rho^- \quad \text{and} \quad \kappa := \rho^+ - \rho^-,$$

193 and require the total mass of  $\rho$  to be 1. We note that  $\rho^+$  and  $\rho^-$  need not be mutually  
 194 singular, and thus  $\rho^\pm \geq [\kappa]_\pm$ , where  $[\kappa]_\pm$  denotes the positive/negative part of the  
 195 signed measure  $\kappa$ . We interpret  $[\kappa]_\pm$  as the density of positive/negative particles that  
 196 have not been annihilated yet.

197 For  $\rho^\pm(t)$  we consider the following set of evolution equations

$$198 \quad (1.3) \quad \begin{cases} \partial_t \rho^+ = ([\kappa]_+ (V' * [\kappa]_+ + W' * [\kappa]_-))' & \text{in } \mathcal{D}'((0, T) \times \mathbb{R}), \\ \partial_t \rho^- = ([\kappa]_- (V' * [\kappa]_- + W' * [\kappa]_+))' & \text{in } \mathcal{D}'((0, T) \times \mathbb{R}), \end{cases}$$

199 where we denote by the prime symbol  $'$  the derivative with respect to the spatial  
 200 variable. We remark that no annihilation rule is specified; the annihilation is encoded  
 201 in taking the positive/negative part of  $\kappa$ . Indeed, it is easy to imagine that while the  
 202 integral of  $\rho = \rho^+ + \rho^-$  is conserved in time, the integral of  $[\kappa]_+ + [\kappa]_- = |\rho^+ - \rho^-|$   
 203 may not be conserved.

204 **1.4. Main result: discrete-to-continuum limit.** Our main theorem (Theo-  
 205 rem 5.1) states that the solutions to (1.2) converge to a solution of (1.3) as  $n \rightarrow \infty$ .  
 206 It specifies the concept of solution to both problems, the required conditions on the  
 207 sequence of initial data of (1.2), and guarantees that the so-constructed solution to  
 208 (1.3) at time 0 corresponds to the limit of the initial conditions as  $n \rightarrow \infty$ . The

209 convergence is uniform in time on  $[0, T]$  for any  $T > 0$ . The convergence in space is  
 210 with respect to the weak convergence. As a by-product of [Theorem 5.1](#), we obtain  
 211 global-in-time existence of a solution  $(\rho^+, \rho^-)$  to [\(1.3\)](#) for which the masses of  $\rho^\pm$  are  
 212 conserved in time.

213 In order to give effectively an outline of the proof and the motivation for the main  
 214 assumptions under which [Theorem 5.1](#) holds ([Subsection 1.6](#)), we first describe the  
 215 related literature.

216 **1.5. Related literature.** We start by relating [\(1.3\)](#) formally to its singular  
 217 counterpart. Replacing  $W$  by  $-V$ , we obtain from a formal calculation that the  
 218 difference of the two equations in [\(1.3\)](#) is given by

$$219 \quad (1.4) \quad \partial_t \kappa = (|\kappa|(V' * \kappa))'.$$

220 For  $V(r) = -\log|r|$ , equation [\(1.4\)](#) was introduced by [\[20\]](#) and later proven in [\[6\]](#) to  
 221 attain unique solutions when posed on  $\mathbb{R}$  with proper initial data.

222 In the remainder of this subsection, we put our main result [Theorem 5.1](#) in the  
 223 perspective of the literature. We start by describing those specifications of [\[10, 28, 29\]](#)  
 224 which are closest to our main result. A specification of [\[10, Theorems 2.1–2.3\]](#) proves a  
 225 ‘discrete’-to-continuum result from [\(1.2\)](#) to [\(1.4\)](#), in the case where  $V(r) = -W(r)$  is  
 226 a regularisation of  $-\log|r|$  on the length-scale  $1/n$ . We put ‘discrete’ in apostrophes,  
 227 because their equivalent of [\(1.2\)](#), given by [\[10, equation \(5\)\]](#), is a Hamilton-Jacobi  
 228 equation, which includes the solution to [\(1.2\)](#) only if all particles have the same sign.  
 229 It is not clear if this Hamilton-Jacobi equation relates to [\(1.2\)](#) if the particles have  
 230 opposite sign.

231 As opposed to [\[10\]](#), [\[29\]](#) starts from a different Hamilton-Jacobi equation, which  
 232 corresponds to the Peierls-Nabarro model [\[32, 33\]](#). This model is a phase-field model  
 233 for the dynamics of dislocations which naturally includes annihilation. In this model,  
 234 opposite to encoding dislocations as points on the line, the dislocations are identified  
 235 by the pulses of the derivative of a multi-layer phase field on the real line. In [\[29\]](#), the  
 236 width of these pulses is taken to be on the same length-scale as the typical distance  
 237 between neighbouring dislocations. Then, in the joint limit when the regularisation  
 238 length-scale (and thus simultaneously  $1/n$ ) tend to 0, an *implicit* Hamilton-Jacobi  
 239 equation is recovered [\[29\]](#). In [\[28, Theorem 1.2\]](#) it is shown that this implicit Hamilton-  
 240 Jacobi equation converges to [\(1.4\)](#) in the dilute dislocation density limit. While this  
 241 framework seems promising for a direct ‘discrete’-to-continuum result (‘discrete’ being  
 242 the Peierls-Nabarro model) to [\(1.3\)](#), it only applies to co-dimension 1 objects, *i.e.*,  
 243 particles in 1D and curves in 2D.

244 Regarding the continuum problem [\(1.3\)](#), we have not found this set of equations  
 245 in the literature. Nonetheless, we believe the case  $W = 0$  to be of independent  
 246 interest, since then [\(1.3\)](#) serves as the easiest benchmark problem for future studies  
 247 on annihilating particles. Also, since our discrete-to-continuum result holds for taking  
 248  $W$  as a regularisation of  $-V$ , we expect that [\(1.4\)](#) can be obtained from [\(1.3\)](#) as the  
 249 regularisation length-scale tends to 0 (see [Subsection 1.7](#)). Therefore, we review the  
 250 literature on [\(1.4\)](#).

251 Equation [\(1.4\)](#) as posed on  $\mathbb{R}$  with  $V(r) = -\log|r|$ , or even  $V(r) = |r|^{-a}$  with  
 252  $0 < a < 1$ , attains a self-similar solution [\[6, Theorem 2.4\]](#) in which  $\kappa$  has a sign.  
 253 The self-similar solution is expanding in time (due to the repelling interaction force  
 254  $V'(r)$ ), and describes the long-time behaviour of the unique viscosity solutions to  
 255 [\(1.4\)](#) [\[6, Theorem 2.5\]](#) for appropriate initial data. Moreover, for  $V(r) = -\log|r|$  and

256 initial condition  $\kappa^\circ \in L^1(\mathbb{R})$ , the viscosity solution  $\kappa$  to (1.4) satisfies  $\kappa(t) \in L^p(\mathbb{R})$   
 257 for all  $1 \leq p \leq \infty$  [6, Theorem 2.7]. In conclusion, despite (1.4) being the singular  
 258 counterpart of (1.3), it has a well-defined global-in-time solution concept.

259 Lastly, we compare our result to that of [3]. There, the authors are interested  
 260 in deriving a gradient flow structure of (1.4) on  $\mathbb{R}^2$  with  $V$  having a logarithmic  
 261 singularity at 0 by defining a discrete in time minimising movement scheme and  
 262 passing to the limit as the time step size tends to 0. The related convergence result is  
 263 [3, Theorem 1.4]. However, the limit equation is not fully characterised as (1.4), since  
 264 in that equation  $|\kappa|$  is replaced by an unknown measure  $\mu \geq |\kappa|$  which is obtained from  
 265 compactness. The connection to our main result is that we faced a similar problem.  
 266 Due to our 1D setup and by a technical assumption on the initial data, we were able  
 267 to characterise the corresponding  $\mu$  as  $|\kappa|$ .

268 **1.6. Discussion on the proof, assumptions, and possible extensions.** We  
 269 divide this section into several topics regarding the proof, assumptions, and possible  
 270 extensions of Theorem 5.1 (outlined in Subsection 1.4).

271 *Summary of the proof.* A crucial step is the observation that the solution to (1.2),  
 272 seen as a pair of empirical measures  $\mu_n^\pm$ , is a solution to (1.3), *i.e.*,

$$273 \quad (1.5) \quad \begin{cases} \partial_t \mu_n^+ = ([\kappa_n]_+ (V' * [\kappa_n]_+ + W' * [\kappa_n]_-))' & \text{in } \mathcal{D}'((0, T) \times \mathbb{R}), \\ \partial_t \mu_n^- = ([\kappa_n]_- (V' * [\kappa_n]_- + W' * [\kappa_n]_+))' & \text{in } \mathcal{D}'((0, T) \times \mathbb{R}), \end{cases}$$

274 where  $\kappa_n := \mu_n^+ - \mu_n^-$ . The annihilation is completely covered by taking the positive  
 275 and negative part of  $\kappa_n$ . This property is the reason for encoding annihilation in the  
 276 charges  $b_i(t)$  rather than removing particles from the dynamics. Then, relying on  
 277 the gradient flow structure underlying (1.2) and the boundedness of  $W$ , we find, by  
 278 the usual compactness arguments *à la* Arzelà-Ascoli, limiting curves  $\rho^\pm(t)$ . It then  
 279 remains to pass to the limit  $n \rightarrow \infty$  in (1.5). The difficulty is in characterising the  
 280 limit of  $[\kappa_n]_\pm$ , which only accounts for the particles that have not collided yet. Indeed,  
 281 the convergence of measures is not invariant with respect to taking the positive and  
 282 negative part. It is here that we heavily rely on the one-dimensional setting and  
 283 on a technical assumption on the initial data (Assumption 2.2), which provides an  
 284  $n$ -independent bound on the number of neighbouring pairs of particles with opposite  
 285 sign. This bound allows us to characterise the limit of  $[\kappa_n]_\pm$  as  $[\kappa]_\pm$ .

286 *Motivation for Assumption 2.2.* Assumption 2.2 prevents small-scale oscillations  
 287 between  $\pm 1$  phases. A similar assumption is made in [29], where the initial data  
 288 for the particles is constructed from the continuum initial datum. While one might  
 289 expect that small-scale oscillations cancel out on small time scales, the simulations in  
 290 [45, Chapter 9] suggest otherwise. The problem with such small-scale oscillations is  
 291 that they cause the limit of  $[\kappa_n]_\pm$  to be larger than  $[\kappa]_\pm$ , which makes it difficult to  
 292 characterise the limit as  $n \rightarrow \infty$  of (1.5) as (1.3).

293 *Singularity of  $V$ .* Assuming the singularity of  $V$  to be at most logarithmic is  
 294 needed to apply the discrete-to-continuum limit passage technique in [38].

295 In fact, we also *require* that  $V(r) \rightarrow \infty$  as  $r \rightarrow 0$ , *i.e.*, we do not allow for a regular  
 296  $V$ . While regular  $V$  and  $W$  (in particular  $W = -V$ ) would simplify the equations and  
 297 many steps in the proof of our main theorem, it may result in two technical difficulties:  
 298 collision between three or more particles, and the limiting signed measure  $\kappa$  having  
 299 atoms. These difficulties complicate the convergence proof of  $[\kappa_n]_\pm$  to  $[\kappa]_\pm$  as  $n \rightarrow \infty$ .  
 300 Since all our intended applications correspond to singular potentials  $V$ , we choose to  
 301 side-step these technical difficulties by simply requiring  $V$  to have a singularity at 0.



302 *Regularity of  $W$ .*  $W$  being bounded around 0 results in a lower bound on the  
 303 energy along the evolution, which we need for equicontinuity and thus for compactness  
 304 of  $\mu_n^\pm$ . Also, while passing to the limit  $n \rightarrow \infty$  in (1.5), we need  $W'$  regular enough  
 305 (the technique in [38] does not apply for logarithmic  $W$ ).

306 *Logarithmic tails of  $V, W$ .* While it would be easier to assume that  $V$  is bounded  
 307 from below and  $W$  is globally bounded, we *also* allow for logarithmic tails to include  
 308 all three scenarios in Subsection 1.2. The logarithmic tails of  $V$  and  $W$  result in the  
 309 energy  $E_n$  to be unbounded from below. However, following the idea in [38] to prove  
 310 a priori bounds on the *moments* of  $\mu_n^\pm(t)$ , we easily obtain that  $E(\mu_n^\pm(t))$  is bounded  
 311 from below by  $-C(1+t)$  for some  $C > 0$  independent of  $n$  and  $t$ .

312 *Uniqueness of solutions to (1.3).* While Theorem 5.1 provides a solution of (1.3)  
 313 that exists globally in time, we have not investigated uniqueness. We rather interpret  
 314 (1.3) as a stepping stone for a future convergence result to (1.4), for which a uniqueness  
 315 result is established in [6].

316 **1.7. Conclusion and outlook.** We intend our main result to open a new thread  
 317 of research on including annihilation in discrete-to-continuum limits. Here we discuss  
 318 several open ends.

319  $W = -V$  *singular.* This setting corresponds to charges (or dislocations) on the  
 320 real line. On the continuum level, see (1.4), this equation is well-understood [6],  
 321 but on the discrete level we have not found a closed set of equations to describe  
 322 the discrete counterpart of (1.2) (other than [40, 41], whose results are discussed in  
 323 Subsection 1.5). Since our main result does allow for  $-W$  to be a regularisation  $V_\delta$  of  
 324  $V$  ( $\delta$  denotes the arbitrarily small, but fixed, length-scale of the regularisation), this  
 325 calls for three interesting limit passages:

326 (a)  $\delta \rightarrow 0$  *with  $n$  fixed.* This limit seems the easiest out of the three. Similar to  
 327 [40, 41], the idea is to pass to the limit, and *describe* the limit rather than  
 328 posing a closed set of equations for it. One challenge is that in the limiting  
 329 curves prior to collision at  $t_*$ , the particles' speed blows up as  $\sim 1/\sqrt{t_* - t}$   
 330 (this is easily seen by considering only two particles; one positive and one  
 331 negative). While the resulting curves are not Lipschitz in time, they are  $C^{1/2}$   
 332 in time. However, such collisions correspond to  $-\infty$  wells in the energy, which  
 333 require the development of a proper renormalisation of  $E_n$ .

334 Another challenge is that particles need not collide if they come close, regard-  
 335 less how small  $\delta > 0$  is. To see this, consider two particles with opposite sign  
 336 and with mutual distance smaller than  $\delta$ . Since  $V_\delta$  is regular, the particles  
 337 will come exponentially close, but they will not collide in finite time. In the  
 338 case of many particles, such a close pair will only collide if the external force  
 339 (induced by the other particles) acts in the right direction. If it does not col-  
 340 lide, then the pair remains in the system (as opposed to the case of singular  
 341  $W$ ), and may even interact with or annihilate other particles that come close.

342 (b) *Connecting (1.3) to (1.4) by  $\delta \rightarrow 0$ .* Taking  $W = -V_\delta$  and setting  $\rho_\delta^\pm$  as a  
 343 corresponding solution to (1.3), it is impossible to pass directly to the limit in  
 344 (1.3) due to the term  $[\kappa_\delta]_\pm (V'_\delta * [\kappa_\delta]_\mp)$ . Instead, the structure of (1.4) in terms  
 345 of viscosity solutions (see [6]) seems promising. We leave it to future research  
 346 to find out whether (1.3) enjoys a similar structure, and if not, whether there  
 347 is a different continuum model for annihilating particles that does.

348 (c) *Connecting (1.2) to (1.4) by a joint limit  $n \rightarrow \infty$  and  $\delta_n \rightarrow 0$ .* This approach  
 349 fits to the convergence result obtained in [29], where roughly speaking  $\delta_n \sim$

350  $1/n$  is considered, but where a different equation than (1.4) is obtained in the  
 351 limit. It would be interesting to see whether those results can be extended to  
 352 the case  $\delta_n \ll 1/n$ , in which case the expected limit is (1.4) (see [28]).

353 *Different regularisations of collisions.* In the spirit of proving any of the above  
 354 limit passages, we discuss alternative regularisations other than taking  $W$  regular.  
 355 One idea is ‘premature annihilation’, where particles are removed from the system  
 356 when they come  $\delta$ -close, with  $\delta > 0$  a regularisation parameter. This approach is  
 357 commonly adapted in numerical simulations of discrete systems with an annihilation  
 358 rule. However, it is not obvious what the limiting equation as  $n \rightarrow \infty$  (counterpart  
 359 of (1.4)) is for  $\delta > 0$  fixed, because we expect the supports of  $[\kappa]_+$  and  $[\kappa]_-$  to be  
 360 separated by at least  $\delta$ . A third option is to mollify the jump of the charge  $b_i(t)$  from  
 361  $\pm 1$  to 0, possibly by an additional ODE for  $b_i(t)$ . We have not found a proper rule  
 362 for this that would still allow for a discrete-to-continuum convergence result.

363 *Higher dimensions.* In this paragraph we consider the extension to two dimen-  
 364 sions; the discussion easily extends to higher dimensions. The one ingredient in our  
 365 proof which intrinsically relies on our 1D setting, is the *separation* condition on the  
 366 initial data. This condition limits the collisions to happen only at a finite number  
 367 of points. In 2D, collisions are bound to happen along curves (or more complicated  
 368 subsets of  $\mathbb{R}^2$ ), which makes it challenging to characterise the limit of  $[\kappa_n]_{\pm}$ . A similar  
 369 problem occurred in [3] as discussed in Subsection 1.5. In future research we plan to  
 370 relax our ‘separation’ assumption, possibly by considering a different regularisation  
 371 of collisions.

372 The remainder of the paper is organised as follows. In Section 2 we fix our notation  
 373 and list the assumptions on  $V$ ,  $W$  and the initial data. In Section 3 we recall known  
 374 results and provide the preliminaries. In Section 4 we give a rigorous definition of  
 375 (1.2), show that it attains a unique solution, and establish several properties of it. In  
 376 Section 5 we state and prove our main result, Theorem 5.1.

377 **2. Notation and standing assumptions.** Here we list the symbols and nota-  
 378 tion which we use in the remainder of this paper:

$\mathcal{B}(\mathbb{R})$	space of Borel sets on $\mathbb{R}$	Section 3
$f(a-)$	$\lim_{y \uparrow a} f(y)$	
$[f]_{\pm}$	positive or negative part of $f$	
$\mu \otimes \nu$	product measure; $(\mu \otimes \nu)(A \times B) = \mu(A)\nu(B)$	Section 3
$C > 0$	constant whose value can possibly change from line to line	
$\boldsymbol{\mu}$	$\boldsymbol{\mu} := (\mu^+, \mu^-) \in \mathcal{P}(\mathbb{R} \times \{\pm 1\})$	(3.2)
$\mathcal{M}(\mathbb{R})$	space of finite, signed Borel measures on $\mathbb{R}$	Section 3
$\mathcal{M}_+(\mathbb{R})$	space of the non-negative measures in $\mathcal{M}(\mathbb{R})$	Section 3
$\mathbb{N}$	$\{1, 2, 3, \dots\}$	
$\mathcal{P}(\mathbb{R})$	space of probability measures; $\mathcal{P}(\mathbb{R}) = \{\mu \in \mathcal{M}_+(\mathbb{R}) : \mu(\mathbb{R}) = 1\}$	Section 3
$\mathcal{P}_2(\mathbb{R})$	probability measures with finite second moment; $\mathcal{P}_2(\mathbb{R}) = \{\mu \in \mathcal{P}(\mathbb{R}) : \int_{-\infty}^{\infty} x^2 d\mu(x) < \infty\}$	Section 3
$V$	interaction potential for equally signed particles	Assumption 2.1
$W$	interaction potential for oppositely signed particles	Assumption 2.1
$W(\mu, \nu)$	2-Wasserstein distance between $\mu, \nu \in \mathcal{P}(\mathbb{R})$	[2]
$\mathbf{W}(\boldsymbol{\mu}, \boldsymbol{\nu})$	2-Wasserstein distance between $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{P}_2(\mathbb{R})$	(3.3)

379 **Assumption 2.1** lists the standing properties which we impose on  $V$  and  $W$ .

380 **ASSUMPTION 2.1.** *We require that the interaction potentials  $V: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$  and*  
 381  *$W: \mathbb{R} \rightarrow \mathbb{R}$  satisfy the following conditions:*

382 (2.1a)  $V \in C^1(\mathbb{R} \setminus \{0\})$ ,  $W \in C^1(\mathbb{R})$ ,  $V' \in \text{Lip}_{\text{loc}}(\mathbb{R} \setminus \{0\})$ , and  $W' \in \text{Lip}(\mathbb{R})$ ,

383 (2.1b)  $V$  and  $W$  are even;

384 (2.1c)  $V(r) \rightarrow +\infty$  as  $r \rightarrow 0$ ;

385 (2.1d)  $r \mapsto rV'(r)$  and  $r \mapsto rW'(r)$  are in  $L^\infty(\mathbb{R})$ .

386 For convenience, we set  $V'(0) := 0$ . Below we list two remarks on **Assumption 2.1**:

- 387 • we assume no monotonicity on  $V$  or  $W$ ;
- 388 • Condition (2.1d) implies that  $V$  has at most a logarithmic singularity (as
- 389 mentioned in **Subsection 1.2**), and that  $V$  and  $W$  have at most logarithmically
- 390 diverging tails, namely

391 (2.2)  $|V(r)| + |W(r)| \leq C(|\log |r|| + 1)$ , for all  $r \neq 0$ .

392 Due to condition (2.1c), and keeping (2.1a) into account, we can sharpen this  
 393 inequality around 0 by

394 (2.3)  $(V + W)(r) \geq -Cr^2$ , for all  $r \neq 0$ .

395 The following assumption on the initial data states that no pair of particles of  
 396 opposite sign should start at the same position.

397 **ASSUMPTION 2.2** (Separation assumption on the initial data  $(x^\circ; b^\circ)$ ). *There*  
 398 *exist  $-\infty < a_0 \leq a_1 \leq \dots \leq a_{2L} < +\infty$  such that*

399 
$$\{x_i^\circ : b_i^\circ = 1\} \subset \bigcup_{\ell=1}^L (a_{2\ell-2}, a_{2\ell-1}), \quad \{x_i^\circ : b_i^\circ = -1\} \subset \bigcup_{\ell=1}^L (a_{2\ell-1}, a_{2\ell}).$$

400 The importance of this assumption is clarified later when the limit  $n \rightarrow \infty$  is consid-  
 401 ered, in which the number  $L$  is assumed to be  $n$ -independent (see also **Subsection 1.6**).  
 402 Moreover, we will show in **Proposition 4.5** that this assumption is conserved in time.

403 **3. Preliminary results.** We collect here some basic definitions and known re-  
 404 sults that will be useful in the sequel.

405 **3.1. Probability spaces and the Wasserstein distance.** On  $\mathcal{P}_2(\mathbb{R})$  (space  
 406 of probability measures with finite second moment; see **Section 2**), the square of the  
 407 2-Wasserstein distance  $W(\mu, \nu)$  with  $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$  is defined as

408 (3.1) 
$$W^2(\mu, \nu) := \inf_{\gamma \in \Gamma(\mu, \nu)} \iint_{\mathbb{R}^2} |x - y|^2 d\gamma(x, y),$$

410 where  $\Gamma(\mu, \nu)$  is the set of couplings of  $\mu$  and  $\nu$ , namely,

411 
$$\Gamma(\mu, \nu) := \{\gamma \in \mathcal{P}(\mathbb{R}^2) : \gamma(A \times \mathbb{R}) = \mu(A), \gamma(\mathbb{R} \times A) = \nu(A) \text{ for all } A \in \mathcal{B}(\mathbb{R})\}.$$

412 We refer to [4] for the basic properties of  $W$ . As usual, we set  $\Gamma_\circ(\mu, \nu) \subset \Gamma(\mu, \nu)$  as  
 413 the set of transport plans  $\gamma$  which minimise (3.1).

414 Since we are working with positive and negative particles, we follow [12] by defin-  
 415 ing a space of probability measures on  $\mathbb{R} \times \{\pm 1\}$ , where  $\mathbb{R} \times \{\pm 1\}$  is endowed with  
 416 the distance

$$417 \quad d^2(\bar{x}, \bar{y}) := |x - y|^2 + |p - q|, \quad \bar{x} = (x, p) \in \mathbb{R} \times \{\pm 1\}, \quad \bar{y} = (y, q) \in \mathbb{R} \times \{\pm 1\}.$$

418 We denote this probability space by  $\mathcal{P}(\mathbb{R} \times \{\pm 1\})$ , and its elements by  $\boldsymbol{\mu}$  or  $(\mu^+, \mu^-)$ ,  
 419 with the understanding that

$$420 \quad (3.2) \quad \boldsymbol{\mu}(A^+, A^-) = \mu^+(A^+) + \mu^-(A^-), \quad \text{for all } A^+, A^- \in \mathcal{B}(\mathbb{R}).$$

421 On

$$422 \quad \mathcal{P}_2(\mathbb{R} \times \{\pm 1\}) := \left\{ \boldsymbol{\mu} \in \mathcal{P}(\mathbb{R} \times \{\pm 1\}) : \int_{\mathbb{R}} |x|^2 d\mu^\pm(x) < +\infty \right\}$$

423 we define the (square of the) 2-Wasserstein distance between  $\boldsymbol{\mu}$  and  $\boldsymbol{\nu}$  as

$$424 \quad (3.3) \quad \mathbf{W}^2(\boldsymbol{\mu}, \boldsymbol{\nu}) := \inf_{\boldsymbol{\gamma} \in \boldsymbol{\Gamma}(\boldsymbol{\mu}, \boldsymbol{\nu})} \iint_{(\mathbb{R} \times \{\pm 1\})^2} d^2(\bar{x}, \bar{y}) d\boldsymbol{\gamma}(\bar{x}, \bar{y}),$$

425 where  $\boldsymbol{\Gamma}(\boldsymbol{\mu}, \boldsymbol{\nu})$  is the set of couplings of  $\boldsymbol{\mu}$  and  $\boldsymbol{\nu}$ , namely,

$$426 \quad \boldsymbol{\Gamma}(\boldsymbol{\mu}, \boldsymbol{\nu}) := \left\{ \boldsymbol{\gamma} \in \mathcal{P}((\mathbb{R} \times \{\pm 1\})^2) : \boldsymbol{\gamma}(A \times (\mathbb{R} \times \{\pm 1\})) = \boldsymbol{\mu}(A), \right. \\ 427 \quad \left. \boldsymbol{\gamma}((\mathbb{R} \times \{\pm 1\}) \times A) = \boldsymbol{\nu}(A) \text{ for all } A \in \mathcal{B}(\mathbb{R} \times \{\pm 1\}) \right\}.$$

428 Since it turns out that (1.3) has a mass-preserving solution  $\boldsymbol{\rho}(t) := (\rho^+(t), \rho^-(t))$   
 429 belonging to  $\mathcal{P}_2(\mathbb{R} \times \{\pm 1\})$ , for which also the mass of  $\rho^+(t)$  and  $\rho^-(t)$  is conserved  
 430 in time, we define the corresponding subspace

$$431 \quad \mathcal{P}_2^m(\mathbb{R} \times \{\pm 1\}) := \{ \boldsymbol{\mu} \in \mathcal{P}_2(\mathbb{R} \times \{\pm 1\}) : \mu^+(\mathbb{R}) = m \};$$

432 where  $m \in [0, 1]$  is the total mass of the positive particle density. Clearly, if  $\boldsymbol{\mu} \in$   
 433  $\mathcal{P}_2^m(\mathbb{R} \times \{\pm 1\})$ , then  $\mu^-(\mathbb{R}) = 1 - m$ . For any  $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{P}_2^m(\mathbb{R} \times \{\pm 1\})$  we have that

$$434 \quad (3.4) \quad \mathbf{W}^2(\boldsymbol{\mu}, \boldsymbol{\nu}) \leq W^2(\mu^+, \nu^+) + W^2(\mu^-, \nu^-),$$

435 which simply follows by shrinking the set of couplings  $\boldsymbol{\Gamma}(\boldsymbol{\mu}, \boldsymbol{\nu})$  in (3.3).

436 **3.2. Weak form of the continuum problem (1.3).** We use the following  
 437 notation convention. For any  $\boldsymbol{\rho} \in \mathcal{P}(\mathbb{R} \times \{\pm 1\})$ , we set

$$438 \quad (3.5) \quad \rho := \rho^+ + \rho^- \in \mathcal{P}(\mathbb{R}), \quad \kappa := \rho^+ - \rho^- \in \mathcal{M}(\mathbb{R}), \quad \tilde{\rho}^\pm := [\kappa]_\pm \in \mathcal{M}_+(\mathbb{R}).$$

439 We consider the following weak form of (1.3): given an initial condition  $\boldsymbol{\rho}^\circ \in \mathcal{P}_2(\mathbb{R} \times$   
 440  $\{\pm 1\})$ , find  $\boldsymbol{\rho}$  satisfying

$$441 \quad (3.6) \quad 0 = \int_0^T \int_{\mathbb{R}} \partial_t \varphi^\pm(x) d\rho^\pm(x) dt \\ - \frac{1}{2} \int_0^T \iint_{\mathbb{R} \times \mathbb{R}} ((\varphi^\pm)'(x) - (\varphi^\pm)'(y)) V'(x - y) d([\kappa]_\pm \otimes [\kappa]_\pm)(x, y) dt \\ - \int_0^T \int_{\mathbb{R}} (\varphi^\pm)'(x) (W' * [\kappa]_{\mp})(x) d[\kappa]_\pm(x) dt,$$

442 for all  $\varphi^\pm \in C_c^\infty((0, T) \times \mathbb{R})$ , where we have exploited that  $V'$  is odd. We seek a  
 443 solution of (3.6) in  $\text{AC}(0, T; \mathcal{P}_2^m(\mathbb{R} \times \{\pm 1\}))$  with  $m = \rho^{\circ,+}(\mathbb{R}) \in [0, 1]$ .

444 **3.3. Several topologies and their connections.** Next we define the space  
 445 of absolutely continuous curves and their metric derivatives. While the following  
 446 definitions work on any complete metric space, we limit our exposition to  $(\mathcal{P}_2(\mathbb{R} \times$   
 447  $\{\pm 1\}), \mathbf{W})$ . For any  $1 \leq p < \infty$ ,  $\text{AC}^p(0, T; \mathcal{P}_2(\mathbb{R} \times \{\pm 1\}))$  denotes the space of all  
 448 curves  $\boldsymbol{\mu} : (0, T) \rightarrow \mathcal{P}_2(\mathbb{R} \times \{\pm 1\})$  for which there exists a function  $f \in L^p(0, T)$  such  
 449 that

$$450 \quad (3.7) \quad \mathbf{W}(\boldsymbol{\mu}(s), \boldsymbol{\mu}(t)) \leq \int_s^t |f(r)|^p dr, \quad \text{for all } 0 < s \leq t < T.$$

451 We set  $\text{AC}(0, T; \mathcal{P}_2(\mathbb{R} \times \{\pm 1\})) := \text{AC}^1(0, T; \mathcal{P}_2(\mathbb{R} \times \{\pm 1\}))$ . By [2, Theorem 1.1.2],  
 452 the metric derivative

$$453 \quad (3.8) \quad |\boldsymbol{\mu}'|_{\mathbf{W}}(t) := \lim_{s \rightarrow t} \frac{\mathbf{W}(\boldsymbol{\mu}(s), \boldsymbol{\mu}(t))}{|s - t|}$$

454 is defined for any  $\boldsymbol{\mu} \in \text{AC}(0, T; \mathcal{P}_2(\mathbb{R} \times \{\pm 1\}))$  and for a.e.  $t \in (0, T)$ . Moreover,  $|\boldsymbol{\mu}'|_{\mathbf{W}}$   
 455 is a possible choice for  $f$  in (3.7).

456 The following theorem is a simplified version of [31, Theorem 47.1] applied to the  
 457 metric space  $(\mathcal{P}_2(\mathbb{R} \times \{\pm 1\}), \mathbf{W})$ .

458 **LEMMA 3.1** (Ascoli-Arzelà).  $\mathcal{F} \subset C([0, T]; \mathcal{P}_2(\mathbb{R} \times \{\pm 1\}))$  is pre-compact if and  
 459 only if

- 460 (i)  $\{\boldsymbol{\mu}(t) : \boldsymbol{\mu} \in \mathcal{F}\}$  is pre-compact in  $\mathcal{P}_2(\mathbb{R} \times \{\pm 1\})$  for all  $t \in [0, T]$ ,  
 461 (ii)  $\forall \varepsilon > 0 \exists \delta > 0$  such that  $\forall \boldsymbol{\mu} \in \mathcal{F}, \forall t, s \in [0, T] : |t - s| < \delta \implies$   
 462  $\mathbf{W}(\boldsymbol{\mu}(t), \boldsymbol{\mu}(s)) < \varepsilon$ .

463 The following theorem provides a lower semi-continuity result on the  $L^2(0, T)$ -  
 464 norm of the metric derivative. We expect it to be well-known, but we only found it  
 465 proven in the PhD thesis [45, Lemma 8.2.8].

466 **THEOREM 3.2** (Lower semi-continuity of metric derivatives). Let  $\boldsymbol{\mu}_n, \boldsymbol{\mu} : [0, T] \rightarrow$   
 467  $\mathcal{P}_2(\mathbb{R} \times \{\pm 1\})$ . If  $\mathbf{W}(\boldsymbol{\mu}_n(t), \boldsymbol{\mu}(t)) \rightarrow 0$  as  $n \rightarrow \infty$  pointwise for a.e.  $t \in (0, T)$ , then

$$468 \quad (3.9) \quad \liminf_{n \rightarrow \infty} \int_0^T |\boldsymbol{\mu}'_n|_{\mathbf{W}}^2(t) dt \geq \int_0^T |\boldsymbol{\mu}'|_{\mathbf{W}}^2(t) dt.$$

469 *Proof.* We start with several preparations. First, we take a dense subset  $(t_\ell)_\ell$  of  
 470  $[0, T]$  for which  $\mathbf{W}(\boldsymbol{\mu}_n(t_\ell), \boldsymbol{\mu}(t_\ell)) \rightarrow 0$  as  $n \rightarrow \infty$  for any  $\ell \in \mathbb{N}$ . Second, without loss  
 471 of generality, we assume that there exists  $C > 0$  such that for all  $n$

$$472 \quad (3.10) \quad \int_0^T |\boldsymbol{\mu}'_n|_{\mathbf{W}}^2(t) dt \leq C.$$

473 In particular, this means that  $\boldsymbol{\mu}_n$  has a representative in  $\text{AC}^2(0, T; \mathcal{P}_2(\mathbb{R} \times \{\pm 1\}))$   
 474 which is defined for all  $t \in (0, T)$ . Taking this representative, we set  $D_n^\ell(t) :=$   
 475  $\mathbf{W}(\boldsymbol{\mu}_n(t_\ell), \boldsymbol{\mu}_n(t))$ , and obtain from [2, Theorem 1.1.2] that

$$476 \quad (3.11) \quad |\boldsymbol{\mu}'_n|_{\mathbf{W}}(t) = \sup_{\ell \in \mathbb{N}} |(D_n^\ell)'(t)| \quad \text{for a.e. } t \in (0, T).$$

477 Next we prove (3.9). Firstly, since  $\mathbf{W}(\boldsymbol{\mu}_n(t), \boldsymbol{\mu}(t)) \rightarrow 0$  as  $n \rightarrow \infty$  for a.e.  $t \in$   
 478  $(0, T)$ , we have for fixed  $\ell \in \mathbb{N}$  and for a.e.  $t \in (0, T)$  that

$$479 \quad (3.12) \quad |D_n^\ell(t) - D^\ell(t)| \xrightarrow{n \rightarrow \infty} 0, \quad \text{where } D^\ell(t) := \mathbf{W}(\boldsymbol{\mu}(t_\ell), \boldsymbol{\mu}(t)).$$

480 Secondly,  $\|D_n^\ell\|_{H^1(0,T)}$  and  $\|D^\ell\|_{H^1(0,T)}$  are bounded uniformly in  $n$  and  $\ell$ . To see  
 481 this, we have by the definition of the metric derivative and (3.10) that

$$482 \quad D_n^\ell(t) \leq \left| \int_{t_\ell}^t |\mu'_n|_{\mathbf{W}}(s) \, ds \right| \leq C\sqrt{T}.$$

483 Hence,  $\|D_n^\ell\|_{L^2(0,T)}$  is uniformly bounded. With the characterisation of  $|\mu'_n|_{\mathbf{W}}$  in  
 484 (3.11), we estimate

$$485 \quad (3.13) \quad C \geq \int_0^T |\mu'_n|_{\mathbf{W}}^2(t) \, dt \geq \int_0^T ((D_n^\ell)'(t))^2 \, dt \quad \text{for all } \ell \in \mathbb{N},$$

486 and thus  $\|D_n^\ell\|_{H^1(0,T)}$  is uniformly bounded. Therefore, in view of (3.12), we have

$$487 \quad (3.14) \quad D_n^\ell \rightharpoonup D^\ell \quad \text{in } H^1(0,T) \text{ as } n \rightarrow \infty.$$

488 In particular, we observe from (3.14) that  $D^\ell \in H^1(0,T)$  and that

$$489 \quad C \geq \liminf_{n \rightarrow \infty} \|D_n^\ell\|_{H^1(0,T)} \geq \|D^\ell\|_{H^1(0,T)} \quad \text{for all } \ell \in \mathbb{N}.$$

490 To establish (3.9), we carefully perform a joint limit passage as  $n \rightarrow \infty$  and a  
 491 maximisation over  $\ell$  in (3.13). With this aim, we take a large fixed  $L \in \mathbb{N}$ , and choose  
 492 a partition  $\{A_\ell\}_{\ell=1}^L$  of Borel sets of  $(0,T)$  such that for all  $\ell = 1, \dots, L$ ,

$$493 \quad |(D^\ell)'(t)| = \sup_{1 \leq \tilde{\ell} \leq L} |(D^{\tilde{\ell}})'(t)| \quad \text{for a.e. } t \in A_\ell.$$

494 We estimate

$$495 \quad \int_0^T |\mu'_n|_{\mathbf{W}}^2(t) \, dt \geq \int_0^T \sup_{1 \leq \ell \leq L} ((D_n^\ell)'(t))^2 \, dt \geq \sum_{\ell=1}^L \int_{A_\ell} ((D_n^\ell)'(t))^2 \, dt.$$

496 Using (3.14), we pass to the limit  $n \rightarrow \infty$  to obtain

$$497 \quad \liminf_{n \rightarrow \infty} \int_0^T |\mu'_n|_{\mathbf{W}}^2(t) \, dt \geq \sum_{\ell=1}^L \int_{A_\ell} ((D^\ell)'(t))^2 \, dt = \int_0^T \sup_{1 \leq \ell \leq L} ((D^\ell)'(t))^2 \, dt.$$

498 By using the Monotone Convergence Theorem, we take the supremum over  $L \in \mathbb{N}$  to  
 499 deduce that

$$500 \quad \liminf_{n \rightarrow \infty} \int_0^T |\mu'_n|_{\mathbf{W}}^2(t) \, dt \geq \int_0^T \sup_{\ell \in \mathbb{N}} ((D^\ell)'(t))^2 \, dt.$$

501 We conclude by using [2, Theorem 1.1.2] to identify  $\sup_{\ell \in \mathbb{N}} |(D^\ell)'|$  in  $L^2(0,T)$  by  
 502  $|\mu'|_{\mathbf{W}}$ .  $\square$

503 Next we introduce the *narrow convergence* of measures. For  $\nu_n, \nu \in \mathcal{M}(\mathbb{R})$ , we  
 504 say that  $\nu_n$  converges in the narrow topology to  $\nu$  (and write  $\nu_n \rightharpoonup \nu$ ) as  $n \rightarrow \infty$  if

$$505 \quad \int \varphi \, d\nu_n \xrightarrow{n \rightarrow \infty} \int \varphi \, d\nu.$$

506 for any bounded test function  $\varphi \in C(\mathbb{R})$ . The following lemma extends this notion  
 507 for non-negative measures by allowing for discontinuous test functions.

508 LEMMA 3.3 ([34, Lemma 2.1]). *Let  $\nu_n \rightharpoonup \nu$  in  $\mathcal{M}_+(\mathbb{R}^d)$ . Let  $A \in \mathcal{B}(\mathbb{R}^d)$  such*  
 509 *that  $\nu(A) = 0$ . Then for every bounded  $\varphi \in C(\mathbb{R}^d \setminus A)$  it holds that*

$$510 \quad \int \varphi \, d\nu_n \xrightarrow{n \rightarrow \infty} \int \varphi \, d\nu.$$

511 Proofs can be found in [39, Theorems 62-63, chapter IV, paragraph 6] and in [8, 14],  
 512 or [37] in the case where  $A$  is closed.

513 Finally, we state and prove a lemma which allows us to show that [Assumption 2.2](#)  
 514 is conserved in the limit as  $n \rightarrow \infty$ .

515 LEMMA 3.4 (Narrow topology preserves separation of supports). *Let  $(\nu_\varepsilon)_{\varepsilon>0}$ ,*  
 516  *$(\rho_\varepsilon)_{\varepsilon>0} \subset \mathcal{M}_+(\mathbb{R})$  converge in the narrow topology as  $\varepsilon \rightarrow 0$  to  $\nu$  and  $\rho$ , respectively.*  
 517 *If*

$$518 \quad \forall \varepsilon > 0 : \sup(\text{supp } \nu_\varepsilon) \leq \inf(\text{supp } \rho_\varepsilon),$$

519 *then also  $\sup(\text{supp } \nu) \leq \inf(\text{supp } \rho)$ .*

520 *Proof.* We reason by contradiction. Suppose  $M := \sup(\text{supp } \nu) > \inf(\text{supp } \rho) =:$   
 521  $m$ . Take a non-decreasing test function  $\varphi \in C_b(\mathbb{R})$  which satisfies

$$522 \quad \varphi \equiv 0 \text{ on } \left(-\infty, \frac{m+2M}{3}\right], \quad \text{and} \quad \varphi \equiv 1 \text{ on } [M, \infty).$$

523 Since  $M = \sup(\text{supp } \nu)$ , it holds that  $\int \varphi \, d\nu > 0$ . Hence, from  $\nu_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \nu$  we infer  
 524 that for all  $\varepsilon$  small enough, it also holds that  $\int \varphi \, d\nu_\varepsilon > 0$ , and thus

$$525 \quad \sup(\text{supp } \nu_\varepsilon) \geq \frac{m+2M}{3}.$$

526 With a similar argument, we can deduce that  $\inf(\text{supp } \rho_\varepsilon) \leq \frac{2m+M}{3}$ , which contradicts  
 527 with  $m < M$ .  $\square$

528 **4. Definition and properties of the discrete problem (1.2).** In this section  
 529 we give a rigorous definition to the discrete dynamics formally given by (1.2). We  
 530 start by formulating it as [Problem 4.1](#), which may have several solutions. Then, we  
 531 define a precise solution concept to [Problem 4.1](#) (see [Definition 4.2](#)) which encodes the  
 532 annihilation rule and selects a unique solution to [Problem 4.1](#). After establishing some  
 533 properties of the energy  $E_n$  introduced in (1.1), we prove an existence and uniqueness  
 534 result (see [Proposition 4.5](#)). Finally, we state the discrete problem in the language of  
 535 measures (see [Lemma 4.6](#)).

536 **PROBLEM 4.1.** *Given  $(x^\circ, b^\circ) \in \mathbb{R}^n \times \{\pm 1\}^n$  such that  $x_1^\circ < x_2^\circ < \dots < x_n^\circ$ , find*  
 537  *$(x, b) : [0, T] \rightarrow \mathbb{R}^n \times \{-1, 0, 1\}^n$  such that*

$$538 \quad (4.1) \quad \begin{cases} \frac{d}{dt} x_i = -\frac{1}{n} \sum_{j: b_i b_j = 1} V'(x_i - x_j) - \frac{1}{n} \sum_{j: b_i b_j = -1} W'(x_i - x_j) & \text{on } (0, T) \setminus T_{\text{col}} \\ (x_i(0), b_i(0)) = (x_i^\circ, b_i^\circ) \end{cases}$$

539 *for all  $i = 1, \dots, n$ , where  $T_{\text{col}}$  is the jump set of  $b$ .*

540 We encode the annihilation rule in the solution concept below. With this aim,  
 541 we set  $H : \mathbb{R} \cup \{+\infty\} \rightarrow [0, 1]$  as the usual Heaviside function, with  $H(0) := 0$  and  
 542  $H(+\infty) := 1$ .

543 DEFINITION 4.2 (Solution to [Problem 4.1](#)). We say that  $(x, b): [0, T] \rightarrow \mathbb{R}^n \times$   
 544  $\{-1, 0, 1\}^n$  is a solution to [Problem 4.1](#) if

545 (a) there exists a vector of collision times  $\tau = (\tau_1, \dots, \tau_n)$  with  $\tau_i \in (0, T) \cup \{+\infty\}$   
 546 such that, setting

$$547 \quad (4.2) \quad T_{\text{col}} := \{\tau_i : 1 \leq i \leq n\} \setminus \{+\infty\} = \{t_1, t_2, \dots, t_K\} \subset (0, T)$$

548 with  $0 < t_1 < \dots < t_K < T$ , there holds

$$549 \quad (4.3) \quad b_i(t) := b_i^\circ H(\tau_i - t) \quad \text{for all } i = 1, \dots, n;$$

550 (b)  $x \in \text{Lip}([0, T]; \mathbb{R}^n) \cap C^1((0, T) \setminus T_{\text{col}}; \mathbb{R}^n)$ ;

551 (c) [\(4.1\)](#) is satisfied in the classical sense;

552 (d) setting  $t_0 := 0$ , for all  $k = 1, \dots, K$ ,

$$553 \quad t_k = \inf \{t \in (0, T) : \exists (i, j) \text{ such that}$$

$$554 \quad b_i(t_{k-1})b_j(t_{k-1}) = -1 \text{ and } x_i(t) = x_j(t)\} > t_{k-1};$$

555 (e) at each time  $t \in [0, T]$ , there is a bijection

$$556 \quad \alpha : \{i : b_i^\circ = 1, \tau_i \leq t\} \rightarrow \{j : b_j^\circ = -1, \tau_j \leq t\}$$

557 such that  $x_i(t) = x_{\alpha(i)}(t)$ .

560 *Remark 4.3* (Comments on [Definition 4.2](#)). We collect here some remarks on the  
 561 notion of solution presented above.

- 562 •  $\tau_i$  is the time at which particle  $x_i$  gets annihilated: equation [\(4.3\)](#) describes  
 563 this by putting to zero the charge  $b_i$  at time  $\tau_i$ . If  $\tau_i = +\infty$ , then it means  
 564 that the particle  $x_i$  does not collide in the time interval  $(0, T)$ .
- 565 •  $(t_k)$  is the ordered list of collision times at which at least one collision occurs.
- 566 • In equation [\(4.1\)](#), both  $x_i$  and  $b_i$  depend on time. However, on each open  
 567 component of  $(0, T) \setminus T_{\text{col}}$ , the charges  $b_i$  remain constant.
- 568 • Since  $V$  is singular and  $W$  is regular, straight-forward a priori energy esti-  
 569 mates show that particles of the same type can never come closer than some  
 570 positive distance. Hence, the only type of collision that can occur is that  
 571 of two particles with opposite sign. We prove precise energy estimates in  
 572 [Proposition 4.5](#).
- 573 • Property (d) ensures that for each pair of two colliding particles, at least  
 574 one gets annihilated. Property (e) ensures that both particles are getting  
 575 annihilated, and that annihilation can only occur for colliding particles with  
 576 non-zero charge. These two properties are the mathematical formulation of  
 577 the annihilation process described in [Subsection 1.2](#).
- 578 • Recalling [\(4.1\)](#), by [\(4.3\)](#), it follows that colliding particles are stationary after  
 579 collision.

580 With reference to the collision times  $t_1 < \dots < t_K$  in [\(4.2\)](#), we define the set of  
 581 indices of the particles colliding at  $t_k$  and its cardinality by

$$582 \quad (4.4) \quad \Gamma_k := \{i : \tau_i = t_k\}, \quad \gamma_k := \#\Gamma_k.$$

583 We observe that  $\gamma_k$  is even for every  $k$  and that

$$584 \quad (4.5) \quad \sum_{k=1}^K \gamma_k \leq \frac{n}{2}.$$



585 We first establish some properties of  $E_n$  defined in (1.1). For convenience, we  
586 display

$$587 \quad (4.6) \quad \frac{\partial}{\partial x_i} E_n(x; b) = \frac{1}{n^2} \sum_{j: b_i b_j = 1} V'(x_i - x_j) + \frac{1}{n^2} \sum_{j: b_i b_j = -1} W'(x_i - x_j),$$

588 where we rely on the choice  $V'(0) = 0$ . We also introduce

$$589 \quad M_k : \mathbb{R}^n \rightarrow [0, \infty), \quad M_k(x) := \frac{1}{n} \sum_{i=1}^n |x_i|^k, \quad k = 1, 2, \dots$$

590 which is the  $k$ -th moment of the empirical measure related to the particles  $x_1, \dots, x_n$ .

591 LEMMA 4.4 (Properties of  $E_n$ ). *Let  $n \geq 2$ . For any  $x \in \mathbb{R}^n$  and  $b \in \{-1, 0, 1\}^n$ ,  
592 the following properties hold:*

- 593 (i)  $E_n(x; b) < +\infty$  if and only if  $\forall i \neq j : x_i = x_j \Rightarrow b_i b_j \neq 1$ ;
- 594 (ii)  $E_n + M_2$  is bounded from below;
- 595 (iii)  $\nabla E_n$  is Lipschitz continuous on the sublevelsets of  $y \mapsto E_n(y; b) + 2M_2(y)$ ;
- 596 (iv) if  $E_n(x; b) < +\infty$  and if there exists an index pair  $(I, J)$  which satisfies  
597  $b_I b_J = -1$  and  $x_I = x_J$ , then, there exists  $C > 0$  independent of  $n$  such that

$$598 \quad E_n(x; \bar{b}) \leq E_n(x; b) + \frac{C}{n} (M_2(x) + x_I^2 + 1),$$

599 where  $\bar{b}$  is the modification of  $b$  in which  $b_I$  and  $b_J$  are put to 0.

600 *Proof.* Property (i) is a direct consequences of the properties of  $V, W$  (see [As-](#)  
601 [sumption 2.1](#)). Property (ii) is a matter of a simple estimate. Using [Assumption 2.1](#)  
602 (in particular (2.2)), some manipulations inspired by [37], and  $r \mapsto r^2 - C \log r$  being  
603 bounded from below, we obtain

$$604 \quad \begin{aligned} E_n(x; b) + M_2(x) &= \frac{1}{2n^2} \left( \sum_{\substack{i \neq j \\ b_i b_j = 1}} V(x_i - x_j) + \sum_{\substack{i, j \\ b_i b_j = -1}} W(x_i - x_j) + \sum_{i, j=1}^n (x_i^2 + x_j^2) \right) \\ &\geq \frac{1}{2n^2} \sum_{i, j=1}^n \left( -C([\log |x_i - x_j|]_+ + 1) + \frac{1}{2}(x_i - x_j)^2 \right) \geq C. \end{aligned}$$

605 Property (iii) follows easily from property (ii) by (2.1a) and (2.1c). To prove (iv),  
606 we set  $y := x_I = x_J$  and assume for convenience that  $b_I = 1$  and  $b_J = -1$ . Then, we  
607 compute

$$\begin{aligned} 608 \quad E_n(x; b) - E_n(x; \bar{b}) &= \frac{1}{2n^2} \left( \sum_{\substack{j \neq I \\ b_j = 1}} V(x_I - x_j) + \sum_{\substack{i \neq J \\ b_i = -1}} V(x_i - x_J) \right) \\ &\quad + \frac{1}{2n^2} \left( \sum_{j: b_j = -1} W(x_I - x_j) + \sum_{i: b_i = 1} W(x_i - x_J) \right) - \frac{W(0)}{2n^2} \\ &= \frac{1}{2n^2} \left( \sum_{\substack{i=1 \\ i \neq I, J}}^n |b_i| V(x_i - y) + \sum_{i=1}^n |b_i| W(x_i - y) \right) - \frac{W(0)}{2n^2} \end{aligned}$$

609

$$\begin{aligned}
&= \frac{1}{2n^2} \sum_{\substack{i=1 \\ i \neq I, J}}^n |b_i|(V+W)(x_i - y) + \frac{W(0)}{2n^2} \\
&\geq -\frac{C}{n^2} \sum_{i=1}^n (x_i - y)^2 + \frac{W(0)}{2n^2} \geq -\frac{C}{n}(M_2(x) + y^2 + 1),
\end{aligned}$$

610

611 where we have used (2.3).  $\square$ 

612 We now prove that [Problem 4.1](#) has a unique solution. In addition, we establish  
613 several properties of it.

614 **PROPOSITION 4.5.** *Let  $n \geq 2$ ,  $T > 0$ , and  $(x^\circ, b^\circ) \in \mathbb{R}^n \times \{\pm 1\}^n$  be such that  
615  $x_1^\circ < x_2^\circ < \dots < x_n^\circ$ . Then there exists a unique solution  $(x, b)$  to [Problem 4.1](#) in the  
616 sense of [Definition 4.2](#). Moreover, the following properties are satisfied:*

617 (i) *there exists  $C > 0$  independent of  $n$  such that*

$$618 \quad M_2(x(t)) \leq Ct + M_2(x^\circ), \quad M_4(x(t)) \leq Ct(M_2(x^\circ) + t) + M_4(x^\circ)$$

619 for all  $t \in [0, T]$ ;

620 (ii)  $\inf_{0 < t < T} \min\{|x_i(t) - x_j(t)| : b_i(t)b_j(t) = 1\} > 0$ ;

621 (iii) *the energy function  $e: [0, T) \rightarrow \mathbb{R}$  defined by  $e(t) := E_n(x(t); b(t))$  is left-  
622 continuous on  $[0, T)$ , differentiable on  $(0, T) \setminus T_{\text{col}}$ , and  $e'(t) \leq 0$  for all  
623  $t \in (0, T) \setminus T_{\text{col}}$ . Moreover, denoting by  $\llbracket e(t_k) \rrbracket := e(t_k) - e(t_k^-)$  the jump of  
624  $e$  at  $t_k$ , we have that*

$$625 \quad (4.7) \quad \llbracket e(t_k) \rrbracket \leq \frac{C}{n} \left( \gamma_k M_2(x(t_k)) + \gamma_k + \sum_{i \in \Gamma_k} x_i^2(t_k) \right)$$

626 for every  $k = 1, \dots, K$ , and

$$627 \quad (4.8) \quad \sum_{k=1}^K \llbracket e(t_k) \rrbracket \leq C(T + M_2(x^\circ) + 1),$$

628 where  $\gamma_k$  and  $\Gamma_k$  are defined in (4.4), and  $C > 0$  is a constant independent  
629 of  $n$ ;

630 (iv)  $E_n(x(t); b(t)) - E_n(x^\circ; b^\circ) \leq C(t + M_2(x^\circ) + 1) - \frac{1}{n} \int_0^t |\dot{x}(s)|^2 ds$  for all  $t \in$   
631  $(0, T]$ ;

632 (v) *there exists an  $L \in \mathbb{N}$  (independent of  $n$ ) such that for all  $t \in [0, T)$ ,  $(x(t), b(t))$   
633 satisfies [Assumption 2.2](#), i.e., there exist  $-\infty < a_0(t) \leq a_1(t) \leq \dots \leq$   
634  $a_{2L}(t) < +\infty$  such that*

$$\begin{aligned}
\{x_i(t) : b_i(t) = 1\} &\subset \bigcup_{\ell=1}^L (a_{2\ell-2}(t), a_{2\ell-1}(t)), \\
\{x_i(t) : b_i(t) = -1\} &\subset \bigcup_{\ell=1}^L (a_{2\ell-1}(t), a_{2\ell}(t)).
\end{aligned}$$

635

636 *Proof. Step 1: Construction of  $(x, b)$ , properties (i) and (ii), and (4.7).* We  
 637 define the counterpart of (4.1) in which no collision occurs, *i.e.*, we seek  $n$  trajectories  
 638  $y_i : [0, T] \rightarrow \mathbb{R}$  such that  $y_i(0) = x_i^\circ$  and

$$639 \quad (4.9) \quad \frac{d}{dt} y_i = -\frac{1}{n} \sum_{j: b_i^\circ b_j^\circ = 1} V'(y_i - y_j) - \frac{1}{n} \sum_{j: b_i^\circ b_j^\circ = -1} W'(y_i - y_j) \quad \text{on } (0, +\infty).$$

640 for all  $i = 1, \dots, n$ . From (4.6) we observe that (4.9) is the gradient flow of  $E_n(\cdot; b^\circ)$   
 641 given by

$$642 \quad (4.10) \quad \begin{cases} \dot{y}(t) = -n \nabla E_n(y(t); b^\circ), \\ y(0) = x^\circ. \end{cases}$$

643 From Lemma 4.4 we observe that (4.10) has a unique, classical solution  $y(t)$  locally  
 644 in time. In particular,  $t \mapsto E_n(y(t); b^\circ)$  is non-increasing.

645 Next we show that the solution  $y$  can be extended to the complete time interval  
 646  $[0, T]$ . With this aim, we prove that the second moment  $M_2(y(t))$  (and for later use  
 647 the fourth moment  $M_4(y(t))$ ) are finite as long as  $t \mapsto y(t)$  exists. We follow the  
 648 argument in [38]. From (4.9), using (2.1b) and (2.1d), we estimate

$$649 \quad \frac{d}{dt} M_2(y(t)) = \frac{2}{n} \sum_{i=1}^n y_i(t) \dot{y}_i(t)$$

$$650 \quad = -\frac{2}{n^2} \sum_{i=1}^n \left( \sum_{j: b_i b_j = 1} y_i V'(y_i - y_j) + \sum_{j: b_i b_j = -1} y_i W'(y_i - y_j) \right)$$

$$651 \quad = -\frac{1}{n^2} \sum_{i,j: b_i b_j = 1} (y_i - y_j) V'(y_i - y_j) - \frac{1}{n^2} \sum_{i,j: b_i b_j = -1} (y_i - y_j) W'(y_i - y_j) \leq C.$$

653 Hence,

$$654 \quad (4.11) \quad M_2(y(t)) \leq M_2(y(0)) + Ct \leq M_2(x^\circ) + CT, \quad \text{for all } t \in [0, T].$$

655 Similarly, using the identity  $a^3 - b^3 = (a^2 + ab + b^2)(a - b)$ , we compute

$$656 \quad \frac{d}{dt} M_4(y(t)) = \frac{4}{n} \sum_{i=1}^n y_i^3(t) \dot{y}_i(t)$$

$$657 \quad = -\frac{4}{n^2} \sum_{i=1}^n \left( \sum_{j: b_i b_j = 1} y_i^3 V'(y_i - y_j) + \sum_{j: b_i b_j = -1} y_i^3 W'(y_i - y_j) \right)$$

$$658 \quad = -\frac{2}{n^2} \sum_{i,j: b_i b_j = 1} (y_i^3 - y_j^3) V'(y_i - y_j) - \frac{2}{n^2} \sum_{i,j: b_i b_j = -1} (y_i^3 - y_j^3) W'(y_i - y_j)$$

$$659 \quad \leq \frac{C}{n^2} \sum_{i,j: b_i b_j = 1} (y_i^2 + y_i y_j + y_j^2) + \frac{C}{n^2} \sum_{i,j: b_i b_j = -1} (y_i^2 + y_i y_j + y_j^2)$$

$$660 \quad \leq \frac{C}{n^2} \sum_{i=1}^n \sum_{j=1}^n (y_i^2(t) + y_j^2(t)) = CM_2(y(t)) \leq C(t + M_2(x^\circ)),$$

661  
 662 where we have used (4.11). Hence,

$$663 \quad (4.12) \quad M_4(y(t)) \leq M_4(x^\circ) + CT(M_2(x^\circ) + T), \quad \text{for all } t \in [0, T].$$

664 In conclusion, (4.11) and (4.12) provide a priori bounds for  $M_2(y(t))$  and  $M_4(y(t))$   
 665 that are uniform in  $n$  and  $t$ . Finally, from (4.11) and Lemma 4.4(i)–(iii) we obtain  
 666 that the solution  $y$  to (4.10) is defined and unique at least up to time  $T$ .

667 Next we identify  $t_1$  and choose those  $b_i$  that jump at  $t = t_1$  (see (4.3)). For this  
 668 choice, it is enough to specify the collision times  $\tau_i$  (see (4.2)). We note that

$$669 \quad t^* := \inf \{t \in (0, T] : \exists (i, j) : b_i^\circ b_j^\circ = -1 \text{ and } y_i(t) = y_j(t)\}$$

670 is either attained or  $t^* = +\infty$ . If  $t^* \geq T$ , we set  $x = y$  and  $\tau_i = +\infty$  for all  $i$ ,  
 671 and observe that properties (d) and (e) of Definition 4.2 are satisfied. If  $t^* < T$ , we  
 672 observe that  $t_1$  in Definition 4.2(d) has to be equal to  $t^*$ . We set  $x|_{[0, t_1]} := y|_{[0, t^*]}$   
 673 and observe from (4.11) and (4.12) that property (i) is satisfied up to  $t = t_1$ . For the  
 674 choice of  $\tau_i$ , we follow the algorithm explained in Subsection 1.2, *i.e.*, for each pair  
 675 of particles that collide at  $t_1$ , we set the corresponding  $\tau_i$  equal to  $t_1$ . We choose the  
 676 remaining values for  $\tau_j > t_1$  later on in the construction. With this choice for  $\tau_i$ ,  
 677 it follows from the continuity of  $x_i$  that properties (d) and (e) of Definition 4.2 are  
 678 satisfied by construction. Since  $E_n(x(t)) \leq E_n(x^\circ)$  for all  $t \in [0, t_1]$ , it follows that  
 679 (ii) holds on  $[0, t_1]$ .

680 Next we show that we can continue the construction above for  $t > t_1$ . First,  
 681 applying Lemma 4.4(iv)  $\frac{1}{2}\gamma_1$  times (recall from (4.4) that  $\gamma_1$  is even), we find that

$$682 \quad E_n(x(t_1); b(t_1)) \leq E_n(x(t_1); b(t_1^-)) + \frac{C}{2n} \left( \gamma_1 M_2(x(t_1)) + \gamma_1 + \sum_{i \in \Gamma_1} x_i^2(t_1) \right).$$

683 Hence, (4.7) is satisfied for  $k = 1$ . Furthermore, we obtain that  $E_n(x(t_1); b(t_1)) < \infty$ ,  
 684 and thus we can continue the construction above for  $t > t_1$  by putting  $x(t_1), b(t_1)$  as  
 685 the initial condition at  $t = t_1$ .

686 Iterating over  $k$ , this construction identifies all  $\tau_i < T$  (for  $i \notin \cup_{k=1}^K \Gamma_k$ , we set  
 687  $\tau_i := +\infty$ ) and  $t_k$ , and guarantees that  $x$  is piecewise  $C^1$  on  $[t_k, t_{k+1}]$  and globally  
 688 Lipschitz. In addition, (4.7) holds for all  $k = 1, \dots, K$ .

689 *Step 2: Uniqueness of  $(x, b)$ .* Let  $x$  and  $\tau$  be as constructed in Step 1, and set  $b$   
 690 accordingly. Since (4.10) has a unique solution, Definition 4.2(d) defines uniquely the  
 691 time  $t_1$  until which  $x(t)$  is uniquely defined. By Definition 4.2(e),  $b$  has to be constant  
 692 on  $[0, t_1]$ . Since  $x$  satisfies Property (ii) at  $t = t_1$ , all collisions at  $t_1$  are collisions  
 693 of two particles with opposite type. Then, from the explanation in Remark 4.3, it is  
 694 obvious that properties (d) and (e) of Definition 4.2 define uniquely the set of indices  
 695  $i$  for which  $\tau_i = t_1$ . Hence,  $b(t_1)$  is uniquely determined. We conclude by iterating  
 696 over  $k$ .

697 *Step 3: The remaining Properties (iii)–(v).* Estimate (4.7) is already proved;  
 698 summing over  $k$  reads

$$699 \quad (4.13) \quad \sum_{k=1}^K \llbracket e(t_k) \rrbracket \leq \frac{C}{n} \left( \sum_{k=1}^K \gamma_k M_2(x(t_k)) + \sum_{k=1}^K \gamma_k + \sum_{k=1}^K \sum_{i \in \Gamma_k} x_i^2(t_k) \right).$$

700 The first and second sums in the right-hand side above can be easily estimated using  
 701 (i) and (4.5). We estimate the third sum by using that the sets  $\Gamma_k$  for  $k = 1, \dots, K$   
 702 are disjoint, and that for every  $k = 1, \dots, K$  and for every  $i \in \Gamma_k$  we have that  
 703  $x_i(t) = x_i(t_k)$  for all  $t \geq t_k$ . Hence, the third sum is bounded by  $M_2(x(T))$ . Collecting  
 704 our estimates, we obtain (4.8) from (4.13).

705 With (iii) proven, we prove (iv) for  $t = T$  by the following computation (the case  
706  $t < T$  follows by a similar estimate). Setting  $t_{K+1} := T$ , we compute

$$\begin{aligned}
E_n(x(T); b(T)) - E_n(x^\circ; b^\circ) &= E_n(x(T); b(T)) - E_n(x(t_K); b(t_K)) \\
&\quad + \sum_{k=1}^K \left[ [e(t_k)] + (E_n(x(t_k-); b(t_k-)) - E_n(x(t_{k-1}); b(t_{k-1}))) \right] \\
707 &\leq \sum_{k=1}^{K+1} \int_{t_{k-1}}^{t_k} \frac{d}{dt} E_n(x(t); b(t)) dt + C(T + M_2(x^\circ) + 1) \\
&= - \sum_{k=1}^{K+1} \frac{1}{n} \int_{t_{k-1}}^{t_k} |\dot{x}(t)|^2 dt + C(T + M_2(x^\circ) + 1) \\
&= - \frac{1}{n} \int_0^T |\dot{x}(t)|^2 dt + C(T + M_2(x^\circ) + 1),
\end{aligned}$$

708 where we have used in the second-to-last equality that  $x(t)$  satisfies (4.1).

709 Finally, we prove (v). First, we claim that the strict ordering of the particles  
710  $\{x_i(t) : |b_i(t)| = 1\}$  is conserved in time. Clearly, this ordering holds at  $t = 0$ .  
711 From (ii) it follows that any two particles, say with corresponding indices  $i \neq j$  such  
712 that  $b_i(t)b_j(t) = 1$ , can never swap position. Similarly, any pair  $(x_i(t), x_j(t))$  with  
713  $b_i(t)b_j(t) = -1$  cannot swap either, because Definition 4.2(d) ensures that  $b_i(t)$  and  
714  $b_j(t)$  jump to 0 at the first  $t$  at which  $x_i(t) = x_j(t)$ . In fact, as soon as this happens,  
715 the particles cease to move (see the last bullet in Remark 4.3 and also the first bullet  
716 in Subsection 1.2 regarding the properties of particles with zero charge).

717 Next we construct  $a_\ell(t)$ . We start with  $t = 0$ , and set  $a_0(0), a_1(0), \dots$  sequentially.  
718 We set  $a_0(0) := x_1^\circ - 1$ , and, if  $b_1^\circ = -1$ , we also put  $a_1(0) := x_1^\circ - 1$ . For each pair of  
719 consecutive particles  $x_i^\circ, x_{i+1}^\circ$  of opposite sign, we define a new point

$$720 \quad a_\ell(0) := \frac{1}{2}(x_i^\circ + x_{i+1}^\circ).$$

721 If the current value of  $\ell$  is odd, we define  $L := (\ell + 1)/2$  and set  $a_{2L}(0) := x_n^0 + 1$ . If  
722  $\ell$  is even, we define  $L := (\ell + 2)/2$  and set  $a_{2L-1}(0) := a_{2L}(0) := x_n^0 + 1$ .

723 Since the strict ordering of the particles  $\{x_i(t) : |b_i(t)| = 1\}$  is conserved in time,  
724 we can construct  $a_\ell(t)$  analogously, but for a time-dependent  $L_t$ . Next we show how  
725 to modify this construction such that  $L_t$  can be chosen independently of  $t$ . Because  
726 of the ordering of  $\{x_i(t) : |b_i(t)| = 1\}$  and that its cardinality is non-increasing in  
727 time, the numbers of pairs of consecutive particles  $x_i(t), x_{i+1}(t)$  of opposite non-zero  
728 charge is also non-increasing in time. Hence,  $t \mapsto L_t$  is non-increasing in time. In  
729 case  $L_t < L$ , we modify the construction of  $a_\ell(t)$  above simply by adding a surplus of  
730 points  $a_\ell(t)$  which all equal  $a_{2L_t}(t)$ .  $\square$

731 Next we establish several properties of the empirical measures associated to the  
732 solution  $(x; b)$  of Problem 4.1 with initial condition  $(x^\circ, b^\circ)$  as in Proposition 4.5.  
733 With this aim, we set

$$734 \quad (4.14) \quad n^\pm := \#\{i : b_i^\circ = \pm 1\}$$

735 as the number of positive/negative particles at time 0, and note that  $n^+ + n^- = n$ .  
736 The empirical measures associated to  $(x(t); b(t))$  are

$$737 \quad (4.15) \quad \mu_n^{\circ, \pm} := \frac{1}{n} \sum_{i: b_i^\circ = \pm 1} \delta_{x_i^\circ}, \quad \mu_n^\pm(t) := \frac{1}{n} \sum_{i: b_i^\circ = \pm 1} \delta_{x_i(t)},$$

738 which both have total mass equal to  $n^\pm/n$  for all  $t \in [0, T]$ . As in (3.5), we also set

$$739 \quad (4.16) \quad \kappa_n(t) := \frac{1}{n} \sum_{i=1}^n b_i^\circ \delta_{x_i(t)}, \quad \mu_n(t) := \frac{1}{n} \sum_{i=1}^n \delta_{x_i(t)}, \quad \tilde{\mu}_n^\pm(t) := [\kappa_n(t)]_\pm.$$

740 LEMMA 4.6 (Proposition 4.5 in terms of measures). *Given the setting as in*  
 741 *Proposition 4.5 with  $(x, b)$  the solution to (4.1), let  $\mu_n := (\mu_n^+, \mu_n^-)$ ,  $\tilde{\mu}_n := (\tilde{\mu}_n^+, \tilde{\mu}_n^-)$ ,*  
 742 *and  $\kappa_n$  as constructed from  $(x, b)$  through (4.15) and (4.16). Then,*

$$743 \quad (i) \quad \tilde{\mu}_n^\pm(t) = \frac{1}{n} \sum_{i=1}^n [b_i(t)]_\pm \delta_{x_i(t)};$$

744 (ii)  $\mu_n \in \text{AC}^2(0, T; \mathcal{P}_2^m(\mathbb{R}^2))$  with  $m = n^+/n$  (see (4.14)), and

$$745 \quad (4.17) \quad |\mu_n'|_{\mathbf{W}}^2(t) \leq \frac{1}{n} \sum_{i=1}^n \left( \frac{d}{dt} x_i(t) \right)^2 \quad \text{for all } 0 < t < T;$$

746 (iii)  $\mu_n$  is a solution to (1.3) with initial condition  $\mu_n^\circ = (\mu_n^{\circ,+}, \mu_n^{\circ,-})$ .

747 *Proof.* Property (i) is a corollary of Proposition 4.5. Indeed, Proposition 4.5(v)  
 748 implies that  $[\kappa_n(t)]_\pm \geq \frac{1}{n} \sum_{i=1}^n [b_i(t)]_\pm \delta_{x_i(t)}$ , while Definition 4.2(e) implies that  
 749  $|\kappa_n(t)|(\mathbb{R}) \leq \frac{1}{n} \sum_{i=1}^n |b_i(t)|$ . We conclude (i).

750 Next we prove (ii). From the definition of  $\mu_n$  in (4.15) we observe that  $\mu_n(t) \in$   
 751  $\mathcal{P}_2^m(\mathbb{R}^2)$  for all  $0 < t < T$ . Hence, (3.4) applies, and we obtain

$$752 \quad (4.18) \quad \mathbf{W}^2(\mu_n(s), \mu_n(t)) \leq W^2(\mu_n^+(s), \mu_n^+(t)) + W^2(\mu_n^-(s), \mu_n^-(t))$$

753 for all  $0 < s \leq t < T$ . To estimate the right-hand side, we let  $0 < s \leq t < T$  be given,  
 754 and introduce the coupling

$$755 \quad \gamma_n^\pm := \frac{1}{n} \sum_{i: b_i^\circ = \pm 1} \delta_{(x_i(s), x_i(t))} \in \Gamma(\mu_n^\pm(s), \mu_n^\pm(t)).$$

756 By definition of the Wasserstein distance (3.1), we obtain

$$757 \quad (4.19) \quad W^2(\mu_n^\pm(s), \mu_n^\pm(t)) \leq \iint_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y|^2 d\gamma_n^\pm(x, y) = \frac{1}{n} \sum_{i: b_i^\circ = \pm 1} (x_i(s) - x_i(t))^2.$$

758 Finally, using in sequence the estimates (3.8), (4.18), and (4.19), we conclude (4.17).  
 759 Since  $x \in \text{Lip}([0, T]; \mathbb{R}^n)$ , we obtain that  $\mu_n \in \text{AC}^2(0, T; \mathcal{P}_2^m(\mathbb{R}^2 \times \{\pm 1\}))$ .

760 Next we prove (iii). We rewrite (4.1) as

$$761 \quad \dot{x}_i(t) = -b_i(t)(V' * \tilde{\mu}_n^+(t) + W' * \tilde{\mu}_n^-(t))(x_i(t)), \quad \text{for } i \text{ such that } b_i^\circ = 1,$$

$$762 \quad \dot{x}_i(t) = -b_i(t)(W' * \tilde{\mu}_n^+(t) + V' * \tilde{\mu}_n^-(t))(x_i(t)), \quad \text{for } i \text{ such that } b_i^\circ = -1.$$

764 Let  $\varphi \in C_c^\infty((0, T) \times \mathbb{R})$  be any test function. Since  $x_i$  is Lipschitz, the Fundamental

765 Theorem of Calculus applies, and thus we obtain, using (i),

$$\begin{aligned}
766 \quad 0 &= \frac{1}{n} \sum_{i: b_i^\circ=1} \int_0^T \frac{d}{dt} \varphi(t, x_i(t)) dt \\
767 \quad &= \frac{1}{n} \sum_{i: b_i^\circ=1} \left[ \int_0^T \partial_t \varphi(t, x_i(t)) dt + \int_0^T \varphi'(t, x_i(t)) \dot{x}_i(t) dt \right] \\
768 \quad &= \int_0^T \int_{\mathbb{R}} \partial_t \varphi d\mu_n^+ dt - \int_0^T \frac{1}{n} \sum_{i: b_i^\circ=1} \varphi'(x_i) (V' * \tilde{\mu}_n^+ + W' * \tilde{\mu}_n^-)(x_i) dt \\
769 \quad &= \int_0^T \int_{\mathbb{R}} \partial_t \varphi d\mu_n^+ dt - \int_0^T \int_{\mathbb{R}} \varphi' (V' * [\kappa_n]_+ + W' * [\kappa_n]_-) d[\kappa_n]_+ dt, \\
770
\end{aligned}$$

771 where  $\varphi'$  denotes the partial derivative with respect to the spatial variable. Since  $\varphi$  is  
772 arbitrary and  $V'$  is odd, we conclude that  $\mu_n^+$  satisfies (3.6). From a similar argument,  
773 it follows that also  $\mu_n^-$  satisfies (3.6).  $\square$

774 **5. Statement and proof of the main convergence theorem.** In this section,  
775 we state and prove our main convergence theorem.

776 **THEOREM 5.1** (Discrete-to-continuum limit). *Let the potentials  $V$  and  $W$  satisfy*  
777 *Assumption 2.1. Let  $(x^{n,\circ}, b^{n,\circ})_n$  be a sequence of initial conditions such that*

- 778 (i)  $E_n(x^{n,\circ}; b^{n,\circ})$  is bounded uniformly in  $n$ ,
- 779 (ii)  $(\mu_n^\circ)_n$  (see (4.15)) has bounded fourth moment uniformly in  $n$ ,
- 780 (iii) there exists an  $L \in \mathbb{N}$  independent of  $n$  such that Assumption 2.2 is satisfied  
781 for all  $n$ .

782 Then for every  $T > 0$  the curves  $\mu_n \in \text{AC}^2(0, T; \mathcal{P}_2(\mathbb{R} \times \{\pm 1\}))$  determined by the  
783 solution  $(x^n, b^n)$  to Problem 4.1 through (4.15) for each  $n$ , converge in measure uni-  
784 formly in time along a subsequence to a solution  $\rho$  of (3.6), whose initial condition  
785  $\rho^\circ$  is the limit of  $(\mu_n^\circ)_n$  along the same subsequence.

786 The proof is divided in three steps. In the first step we use compactness of  $\mu_n(t)$  to  
787 extract a subsequence  $n_k$  along which  $\mu_{n_k}(t)$  converges to some  $\rho(t)$ . In the remaining  
788 two steps we pass to the limit in (3.6) as  $k \rightarrow \infty$  to show that the limiting curve  $\rho(t)$   
789 also satisfies (3.6). Step 2 contains the main novelty; relying on Assumption 2.2 with  
790 an  $n_k$ -independent number  $L$ , we prove that  $[\kappa_{n_k}(t)]_\pm \rightarrow [\kappa(t)]_\pm$  as  $k \rightarrow \infty$  pointwise  
791 in  $t$ .

792 *Proof. Step 1:  $\mu_n$  converges along a subsequence  $n_k \rightarrow \infty$  in  $C([0, T]; \mathcal{P}_2(\mathbb{R} \times$   
793  $\{\pm 1\}))$  to  $\rho \in \text{AC}^2(0, T; \mathcal{P}_2^m(\mathbb{R} \times \{\pm 1\}))$  with  $m := \rho^{\circ,+}(\mathbb{R})$ .* We prove this statement  
794 by means of the Ascoli-Arzelà Theorem (see Lemma 3.1) applied to the metric space  
795  $(\mathcal{P}_2(\mathbb{R} \times \{\pm 1\}), \mathbf{W})$ .

796 First, we show that, for fixed  $t \in [0, T]$ , the sequence  $(\mu_n(t))_n$  is pre-compact in  
797  $\mathcal{P}_2(\mathbb{R} \times \{\pm 1\})$ . From the assumption on the initial data and Proposition 4.5(i) we  
798 observe that the second and fourth moments of the measures  $\mu_n(t)$  defined in (4.16),  
799 given by

$$800 \quad M_2(x^n(t)) = \int_{\mathbb{R}} y^2 d\mu_n(t)(y), \quad M_4(x^n(t)) = \int_{\mathbb{R}} y^4 d\mu_n(t)(y),$$

801 are bounded uniformly in  $n$  and  $t \in [0, T]$ . Then, from [47, Lemma B.3] and [2,  
802 Proposition 7.1.5] we find that  $(\mu_n(t))_n$  is pre-compact in the Wasserstein distance  
803  $\mathbf{W}$ .

804 Second, we show that the sequence  $(\boldsymbol{\mu}_n)_n \subset C([0, T]; \mathcal{P}_2(\mathbb{R} \times \{\pm 1\}))$  is equicon-  
 805 tinuous (i.e.,  $(\boldsymbol{\mu}_n)_n$  satisfies Lemma 3.1(ii)). For any  $0 \leq s < t \leq T$ , we estimate

$$806 \quad (5.1) \quad \mathbf{W}^2(\boldsymbol{\mu}_n(t), \boldsymbol{\mu}_n(s)) \leq \left( \int_s^t |\boldsymbol{\mu}'_n|_{\mathbf{W}}(r) \, dr \right)^2 \leq (t-s) \int_0^T |\boldsymbol{\mu}'_n|_{\mathbf{W}}^2(r) \, dr.$$

807 To estimate the last integral above, we use the estimates in Lemma 4.6(ii) and Propo-  
 808 sition 4.5(iv) to obtain

$$809 \quad (5.2) \quad \int_0^T |\boldsymbol{\mu}'_n|_{\mathbf{W}}^2(r) \, dr \leq \frac{1}{n} \int_0^T \sum_{i=1}^n \left( \frac{d}{dt} x_i^n(r) \right)^2 \, dr = \frac{1}{n} \int_0^T |\dot{x}^n(r)|^2 \, dr \\ \leq C(T + M_2(x^{n,\circ}) + 1) + E_n(x^{n,\circ}; b^{n,\circ}) - E_n(x^n(T); b^n(T)).$$

810 Since, by Lemma 4.4(ii) and Proposition 4.5(i), we have

$$811 \quad \begin{aligned} E_n(x^n(T); b^n(T)) &= [E_n(x^n(T); b^n(T)) + M_2(x^n(T))] - M_2(x^n(T)) \\ &\geq -C - [\tilde{C}T + M_2(x^{n,\circ})], \end{aligned}$$

812 we obtain from (5.2) that

$$813 \quad (5.3) \quad \int_0^T |\boldsymbol{\mu}'_n|_{\mathbf{W}}^2(r) \, dr \leq C(T + M_2(x^{n,\circ}) + 1) + E_n(x^{n,\circ}; b^{n,\circ}).$$

814 By the assumptions on the initial data, the right-hand side is bounded uniformly in  
 815  $n$ . Hence, the right-hand side in (5.1) is bounded by  $C(t-s)$ , and thus  $(\boldsymbol{\mu}_n)_n$  is  
 816 equicontinuous.

817 From the pre-compactness of  $(\boldsymbol{\mu}_n(t))_n$  and the equicontinuity of  $(\boldsymbol{\mu}_n)_n$ , we obtain  
 818 from Lemma 3.1 the existence of a subsequence  $n_k$  along which  $(\boldsymbol{\mu}_n)_n$  converges  
 819 in  $C([0, T]; \mathcal{P}_2(\mathbb{R} \times \{\pm 1\}))$  to some limiting curve  $\boldsymbol{\rho} \in C([0, T]; \mathcal{P}_2(\mathbb{R} \times \{\pm 1\}))$ . In  
 820 fact, combining the lower semi-continuity obtained in Theorem 3.2 with (5.3), we  
 821 obtain that  $\boldsymbol{\rho} \in AC^2(0, T; \mathcal{P}_2(\mathbb{R} \times \{\pm 1\}))$ . Moreover, since the total mass of  $\mu_n^+(t)$  is  
 822 conserved in time, and since the narrow topology conserves mass, we conclude that  
 823  $\boldsymbol{\rho}(t) \in \mathcal{P}_2^m(\mathbb{R} \times \{\pm 1\})$  for all  $t \in [0, T]$ . This completes the proof of Step 1. For later  
 824 use, we set as in (3.5)

$$825 \quad \rho := \rho^+ + \rho^-, \quad \kappa := \rho^+ - \rho^-, \quad \tilde{\rho}^\pm := [\kappa]_\pm.$$

826 *Step 2:*  $\tilde{\boldsymbol{\mu}}_{n_k}(t) \rightarrow \tilde{\boldsymbol{\rho}}(t)$  as  $k \rightarrow \infty$  pointwise for all  $t \in [0, T]$ . We set  $\tilde{\mu}_{n_k}^\pm = [\kappa_{n_k}]_\pm$   
 827 as in (4.16) and  $\tilde{\boldsymbol{\mu}}_{n_k}$  as in Lemma 4.6. We keep  $t \in [0, T]$  fixed, and remove it from  
 828 the notation in the remainder of this step. The structure of the proof of Step 2 is to  
 829 show by compactness that  $(\tilde{\boldsymbol{\mu}}_{n_k})_k$  has a converging subsequence, and to characterise  
 830 the limit as  $\tilde{\boldsymbol{\rho}}$ . Since  $\tilde{\boldsymbol{\rho}}$  is independent of the choice of subsequence, we then conclude  
 831 that the full sequence  $(\tilde{\boldsymbol{\mu}}_{n_k})_k$  converges to  $\tilde{\boldsymbol{\rho}}$ . Keeping this in mind, in the following  
 832 we omit all labels of subsequences of  $n$ .

833 Since the second moments of  $\tilde{\boldsymbol{\mu}}_n$  are obviously bounded by  $M_2(x^n)$ , the sequence  
 834  $(\tilde{\boldsymbol{\mu}}_n)$  is tight, and thus, by Prokhorov's Theorem,  $(\tilde{\boldsymbol{\mu}}_n)$  converges narrowly along a  
 835 subsequence to some  $\tilde{\boldsymbol{\mu}} \in \mathcal{M}_+(\mathbb{R} \times \{\pm 1\})$ .

836 We claim that  $\tilde{\boldsymbol{\mu}}$  does not have atoms. We reason by contradiction. Suppose that  
 837  $\tilde{\mu}^+$  has an atom at  $y$  of mass  $\alpha > 0$  (the case of  $\tilde{\mu}^-$  can be treated analogously).  
 838 Then, setting  $B_\eta(y)$  as the ball around  $y$  with radius  $\eta$ , we infer from  $\tilde{\mu}_n^+ \rightarrow \tilde{\mu}^+$  that



839  $\liminf_{n \rightarrow \infty} \tilde{\mu}_n^+(B_\eta(y)) \geq \alpha > 0$  for any  $\eta > 0$ . By choosing  $\eta > 0$  small enough,  
 840 the contribution of the particles in  $B_\eta(y)$  to the energy  $E_n(x^n; b^n)$  can be made  
 841 arbitrarily large, which contradicts with the uniform bound on  $E_n(x^n; b^n)$  given by  
 842 [Proposition 4.5\(iv\)](#).

843 In the remainder of this step we show that  $\tilde{\mu}^\pm = [\kappa]_\pm$ , regardless of the choice of  
 844 the subsequence. It is enough to show that

$$845 \quad (5.4) \quad [\kappa]_\pm \leq \tilde{\mu}^\pm$$

$$846 \quad (5.5) \quad [\kappa]_\pm(\mathbb{R}) \geq \tilde{\mu}^\pm(\mathbb{R})$$

848 Regarding (5.4), we obtain from Step 1 that

$$849 \quad \tilde{\mu}_n^+ - \tilde{\mu}_n^- = \kappa_n \rightarrow \kappa \quad \text{as } n \rightarrow \infty.$$

850 Hence,  $\tilde{\mu}^+ - \tilde{\mu}^- = \kappa$ , which implies (5.4). To prove (5.5), we let  $\{a_\ell^n\}_{\ell=0}^{2L}$  be as in  
 851 [Proposition 4.5\(v\)](#), and set

$$852 \quad \tilde{\mu}_n^\ell := \begin{cases} \tilde{\mu}_n^+|_{(a_{\ell-1}^n, a_\ell^n)} & \ell \text{ odd} \\ \tilde{\mu}_n^-|_{(a_{\ell-1}^n, a_\ell^n)} & \ell \text{ even} \end{cases}$$

853 for all  $\ell \in \{1, \dots, 2L\}$ . By construction,

$$854 \quad \sum_{\ell=1}^L \tilde{\mu}_n^{2\ell-1} = \tilde{\mu}_n^+ \quad \text{and} \quad \sum_{\ell=1}^L \tilde{\mu}_n^{2\ell} = \tilde{\mu}_n^-.$$

855 Together with  $\tilde{\mu}_n \rightarrow \tilde{\mu}$ , we conclude that  $(\tilde{\mu}_n^\ell)_n$  are tight for any  $\ell$ , and thus, applying  
 856 Prokhorov's Theorem once more, each sequence  $(\tilde{\mu}_n^\ell)_n$  converges along a subsequence  
 857 in the narrow topology to some  $\tilde{\mu}^\ell \in \mathcal{M}_+(\mathbb{R})$ . In particular, from  $\tilde{\mu}_n \rightarrow \tilde{\mu}$  and

$$858 \quad \tilde{\mu}_n^- = \sum_{\ell=1}^L \tilde{\mu}_n^{2\ell} \rightarrow \sum_{\ell=1}^L \tilde{\mu}^{2\ell},$$

859 we infer that  $\tilde{\mu}^- = \sum_{\ell=1}^L \tilde{\mu}^{2\ell}$ . By a similar argument, it follows that  $\tilde{\mu}^+ = \sum_{\ell=1}^L \tilde{\mu}^{2\ell-1}$ .  
 860 Finally, since  $\sup(\text{supp } \tilde{\mu}_n^\ell) < \inf(\text{supp } \tilde{\mu}_n^{\ell+1})$  for all  $1 \leq \ell \leq 2L-1$ , we obtain from  
 861 [Lemma 3.4](#) that  $\sup(\text{supp } \tilde{\mu}^\ell) < \inf(\text{supp } \tilde{\mu}^{\ell+1})$  for all  $1 \leq \ell \leq 2L-1$ . Hence, there  
 862 exists  $A := \{a_\ell\}_{\ell=1}^{2L-1}$  such that

$$863 \quad \text{supp } \tilde{\mu}^+ \cap \text{supp } \tilde{\mu}^- = \left( \bigcup_{\ell=1}^L \text{supp } \tilde{\mu}^{2\ell-1} \right) \cap \left( \bigcup_{k=1}^L \text{supp } \tilde{\mu}^{2k} \right)$$

$$864 \quad = \bigcup_{\ell=1}^L \bigcup_{k=1}^L (\text{supp } \tilde{\mu}^{2\ell-1} \cap \text{supp } \tilde{\mu}^{2k}) = \bigcup_{\ell=1}^{2L-1} (\text{supp } \tilde{\mu}^\ell \cap \text{supp } \tilde{\mu}^{\ell+1}) \subset A.$$

867 Since  $\tilde{\mu}^\pm$  does not have atoms,  $\tilde{\mu}^\pm(A) = 0$ . Together with  $\tilde{\mu}^+ - \tilde{\mu}^- = \kappa$ , it is easy  
 868 to construct a Hahn decomposition of  $\kappa$  (see, e.g., [\[35, Theorem 6.14\]](#)). We conclude  
 869 [\(5.5\)](#).

870 *Step 3:  $\rho$  is a solution to (1.3).* To ease notation, we replace  $n_k$  by  $n$ . We show  
 871 that  $\rho$  satisfies [\(3.6\)](#). With this aim, let  $\varphi^\pm \in C_c^\infty((0, T) \times \mathbb{R})$  be arbitrary. We recall

872 from [Lemma 4.6\(iii\)](#) that  $\mu_n$  satisfies

$$873 \quad (5.6) \quad 0 = \int_0^T \int_{\mathbb{R}} \partial_t \varphi^\pm(x) d\mu_n^\pm(x) dt - \int_0^T \int_{\mathbb{R}} (\varphi^\pm)'(x) (W' * [\kappa_n]_\mp)(x) d[\kappa_n]_\pm(x) dt \\ - \frac{1}{2} \int_0^T \iint_{\mathbb{R} \times \mathbb{R}} ((\varphi^\pm)'(x) - (\varphi^\pm)'(y)) V'(x-y) d([\kappa_n]_\pm \otimes [\kappa_n]_\pm)(x, y) dt.$$

874 We show that we can pass to the limit in all three terms separately. From Step 1  
875 it follows that  $\mu_n \rightharpoonup \rho$ , and thus the limit of the first integral equals

$$876 \quad \int_0^T \int_{\mathbb{R}} \partial_t \varphi^\pm(x) d\rho^\pm(x) dt.$$

877 Regarding the other two integrals in (5.6), we recall from Step 2 that  $[\kappa_n(t)]_\pm \rightharpoonup$   
878  $[\kappa(t)]_\pm$  as  $n \rightarrow \infty$  pointwise for all  $t \in [0, T]$ . Then, for the second term, since  
879  $(x, y) \mapsto (\varphi^\pm)'(x) W'(x-y)$  is bounded and continuous on  $\mathbb{R}^2$ , we obtain that

$$880 \quad \int_{\mathbb{R}} (\varphi^\pm)'(x) (W' * [\kappa_n]_\mp)(x) d[\kappa_n]_\pm(x) = \iint_{\mathbb{R}^2} (\varphi^\pm)'(x) W'(x-y) d([\kappa_n]_\pm \otimes [\kappa_n]_\mp)(x, y)$$

881 converges, as  $n \rightarrow \infty$ , to

$$882 \quad \iint_{\mathbb{R}^2} (\varphi^\pm)'(x) W'(x-y) d([\kappa]_\pm \otimes [\kappa]_\mp)(x, y) = \int_{\mathbb{R}} (\varphi^\pm)'(x) (W' * [\kappa]_\mp)(x) d[\kappa]_\pm(x).$$

883 Finally, we pass to the limit in the third integral in (5.6). We employ [Lemma 3.3](#)  
884 with  $d = 2$  and  $\Delta = \{(y, y) : y \in \mathbb{R}\}$  the diagonal in  $\mathbb{R}^2$ . To show that the conditions  
885 of [Lemma 3.3](#) are satisfied, we observe from the fact that  $r \mapsto rV'(r)$  is bounded and  
886 belongs to  $C(\mathbb{R} \setminus \{0\})$ , it holds that  $(x, y) \mapsto [(\varphi^\pm)'(x) - (\varphi^\pm)'(y)] V'(x-y)$  is bounded  
887 and belongs to  $C(\mathbb{R}^2 \setminus \Delta)$ . Moreover, by Step 2,  $([\kappa]_\pm \otimes [\kappa]_\pm)(\Delta) = (\tilde{\mu}^\pm \otimes \tilde{\mu}^\pm)(\Delta) = 0$ .  
888 Hence, by [Lemma 3.3](#) we can pass to the limit in the third term in (5.6), whose limit  
889 reads

$$890 \quad -\frac{1}{2} \int_0^T \iint_{\mathbb{R} \times \mathbb{R}} ((\varphi^\pm)'(x) - (\varphi^\pm)'(y)) V'(x-y) d([\kappa]_\pm \otimes [\kappa]_\pm)(x, y) dt.$$

891 Combining the three limits above, and recalling the time regularity of  $\rho$  from Step 1,  
892 we conclude that  $\rho$  is a solution to (1.3).  $\square$

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