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In memory of those participants of the Voronezh Winter Mathematical School who have already passed into another world where all problems are solved

Abstract. Using the natural notion of *Hasse–Schmidt derivations on an exterior algebra*, we relate two classical and seemingly unrelated subjects. The first is the famous Cayley–Hamilton theorem of linear algebra, "*each endomorphism of a finite-dimensional vector space is a root of its own characteristic polynomial*", and the second concerns the expression of the bosonic vertex operators occurring in the representation theory of the (infinite-dimensional) Heinsenberg algebra.

1. Introduction

In 1937, Hasse and Schmidt introduced the notion of higher derivations [**8**], nowadays called *Hasse–Schmidt (HS) derivations*. Let $(A, *)$ be an algebra over a ring B, not necessarily commutative or associative. A *HS–derivation on A* is a B-algebra homomorphism, $D(t) : A \to A[[t]]$, that is, a B-linear mapping satisfying

$$D(t)(a_1 * a_2) = D(t)a_1 * D(t)a_2, \quad \forall a_1, a_2 \in A.$$

A fundamental example of a HS–derivation is given by the map sending any function $f = f(z)$, holomorphic in some domain of the complex plane, to its *formal Taylor series*,

$$f(z) \mapsto T(t)[f(z)] = \left(\exp\left(t\frac{d}{dz}\right)\right) f(z).$$

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The property $T(t)[f(z)g(z)] = T(t)[f(z)]T(t)[g(z)]$ encodes the full set of the Leibniz's rules,

$$\frac{d^i(fg)}{dz^i} = \sum_{j=0}^{i} \binom{i}{j} \frac{d^j f}{dz^i} \cdot \frac{d^{i-j} f}{dz^{i-j}}\,.$$

In general, if $(A, *)$ is any commutative $\mathbb{Q}$-algebra and $\delta(t) : A \to A[[t]]$ is a derivation in the Leibniz rule sense, i.e., $\delta(t)(a * b) = \delta(t)a * b + a * \delta(t)b$, then $\exp(\delta(t))$ is a HS–derivation.

The aim of Hasse and Schmidt was to find a counterpart to the Taylor series that would work in positive characteristic. Their definition does not require division by integers and is therefore particularly suitable for this purpose. Schmidt later applied the theory to investigate Weierstrass points and Wronskians on curves in positive characteristic [**15**].

In a number of papers motivated by Schubert Calculus [**3, 6**] (see also the book [**5**]), one of us proposed to study HS–derivations for exterior algebras.

If A is a commutative ring with unit, M a module over A, and $\bigwedge M$ its exterior algebra, then a *HS–derivation on* $\bigwedge M$ is a $\wedge$-homomorphism

$$\mathcal{D}(t) : \bigwedge M \to \left(\bigwedge M\right)[[t]],$$

that is, a linear mapping satisfying

$$\mathcal{D}(t)(u \wedge v) = \mathcal{D}(t)u \wedge \mathcal{D}(t)v, \quad \forall u, v \in \bigwedge M. \tag{1.1}$$

In this paper we consider HS–derivations $\mathcal{D}(t) = \sum_{i\geq 0} \mathcal{D}_i \cdot t^i$ with $\mathcal{D}_0 = \mathbf{1}$, the identity on $\bigwedge M$. Such $\mathcal{D}(t)$ is invertible as an element of $\mathrm{End}(\bigwedge M)[[t]]$, that is, there exists $\overline{\mathcal{D}}(t) \in \mathrm{End}(\bigwedge M)[[t]]$ satisfying

$$\overline{\mathcal{D}}(t)D(t) = \mathcal{D}(t)\overline{\mathcal{D}}(t) = \mathbf{1}. \tag{1.2}$$

A straightforward calculation shows that $\overline{\mathcal{D}}(t)$ is also a HS–derivation on $\bigwedge M$. Hence,

$$\mathcal{D}(t)(\overline{\mathcal{D}}(t)u \wedge v) = u \wedge \mathcal{D}(t)v,\ \overline{\mathcal{D}}(t)(\mathcal{D}(t)u \wedge v) = u \wedge \overline{\mathcal{D}}(t)v, \quad \forall u, v \in \bigwedge M. \tag{1.3}$$

The coefficients of t in these equations give $u \wedge \mathcal{D}_1 v = \mathcal{D}_1(u \wedge v) - \mathcal{D}_1 u \wedge v$. That is why in [**3**] we call (1.3) the *integration by parts* formulas.

In the present article we show how these simple formulas link two classical and seemingly unrelated subjects (one finite-dimensional and the other infinite-dimensional), apparently leading to a unified interpretation.

One topic is the classical Cayley–Hamilton Theorem of Linear Algebra saying that *each endomorphism f of an r-dimensional vector space M is a root of its own characteristic polynomial* $\det(t\mathbf{1} - f)$. Let us reformulate this theorem as a linear recurrence relation on the sequence of endomorphisms $(f^j)_{j\geq 0}$,

$$f^{r+k} - e_1 f^{r+k-1} + \cdots + (-1)^k e_r f^k = \mathbf{0}, \quad \forall k \geq 0, \tag{1.4}$$

where $\det(t\mathbf{1} - f) = t^r - e_1 t^{r-1} + \cdots + (-1)^r e_r$, and $\mathbf{0}$ denotes the zero endomorphism. Consider $D(t) = \sum_{i\geq 0} D_i \cdot t^i$, the unique HS–derivation on the exterior algebra of M such that $D_{i|M} = f^i$, $i \geq 0$. It turns out that the sequence $(D_i)_{i\geq 0}$ of endomorphisms of $\bigwedge M$ satisfies relations similar to (1.4), see Theorem 2.3 in Section 2.1 for the exact formulation, and Section 3 for the proof.

The other topic concerns bosonic vertex operators arising in the representation theory of the (infinite-dimensional) Heisenberg algebra (see, for example, [**9**]). As we observe in Section 2.2, any countably generated vector space over the rationals can be equipped with the structure of a free module of finite rank r over a ring of polynomials in r variables with rational coefficients, for any integer $r > 0$. We present the construction in Section 4, and in Section 5 we apply it to obtain the "finite-dimensional approximation" to the well-known expressions of the vertex operators $\Gamma(t)$ and $\Gamma^*(t)$ generating the bosonic Heisenberg vertex algebra (see [**9**, p. 56]). We interpret $\Gamma(t)$ and $\Gamma^*(t)$ as the limit, when $r \to \infty$, of the ratio of two characteristic polynomials associated to the shift endomorphisms of steps $+1$ and -1, respectively. The precise formulation can be found in Section 2.3.

Our work is based on interpreting (1.3) as a sort of abstract Cayley–Hamilton theorem, holding for general invertible HS–derivations on exterior algebras of arbitrary modules (not necessarily free). If the module is countably generated, then (1.3) produces a sequence of Cayley-Hamilton relations (3.5) which specialize to the classical Cayley–Hamilton formulas (2.7) when considering Hasse–Schmidt derivations associated to endomorphisms of finitely generated free modules.

Plan of the paper. In Section 2 we formulate the main statements. Section 2.1 is devoted to our extention of the Cayley–Hamilton theorem on an exterior algebra of a finite rank free module, which is Theorem 2.3. The proof can be found in Section 3, which also includes the necessary information on the Hasse–Schmidt derivations, and the discussion concerning the case when the ring A contains the rationals.

In Section 2.2, we equip a countably infinite-dimensional $\mathbb{Q}$-vector space with a natural structure of a free module of rank r over the ring of polynomials of r variables with rational coefficients, for any integer $r > 0$. The construction is based on a Giambelli's type formula. The details are explained in Section 4.

In Section 2.3 we apply our construction to the bosonic Heisenberg vertex algebra and interpret the truncation of bosonic vertex operators as the ratio of two characteristic polynomials, respectively associated to the shift endomorphisms of step ± 1; see Section 5 for detailed explanation.

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2. Formulation of the results

2.1. Cayley–Hamilton theorem for exterior algebras. Let M be a free A-module of at most countable (i.e., either finite or countable) rank. If $b_1, b_2, \ldots$ is a basis of M, then $\left\{b_{i_1} \wedge \ldots \wedge b_{i_j}\right\}_{1 \le i_1 < \ldots < i_j}$ is a basis of $\bigwedge^j M$, $j \ge 1$. Following a referee's suggestion, we observe that any A-endomorphism $f : M \to M$ naturally extends to the endomorphism $\bigwedge f$ of $\bigwedge M$ as follows.

Define $\bigwedge f = \bigoplus_{j \ge 0} \bigwedge^j f$ by setting $\bigwedge^0 f$ to be the identity on $\bigwedge^0 M = A$, and by defining the action of $\bigwedge^j f$ on the basis of $\bigwedge^j M$ as follows,

$$b_{i_1} \wedge \ldots \wedge b_{i_j} \;\mapsto\; f b_{i_1} \wedge \ldots \wedge f b_{i_j}, \quad j \ge 1. \tag{2.1}$$

As a consequence, any formal power series $P(t) = \sum_{i\geq 0} p_i t^i$, where $p_i \in \mathrm{End}_A M$, clearly defines a HS–derivation $\bigwedge P(t)$ on $\bigwedge M$. If, in addition, $p_0 = \mathbf{1}$, the identity on M, then the formal inverse $P^{-1}(t) \in (\mathrm{End}_A M)[[t]]$ defines the inverse HS–derivation,

$$\overline{\bigwedge P}(t) = \bigwedge P^{-1}(t).$$

Proposition 2.1. Let M be an A-module of at most countable rank, and $f \in \mathrm{End}_A M$. Set $f^0 = \mathbf{1}$. Then

$$D(t) = \bigwedge \left(\sum_{i\geq 0} f^i t^i \right) \tag{2.2}$$

is an invertible HS–derivation, and its inverse is

$$\overline{D}(t) = \bigwedge (\mathbf{1} - ft). \tag{2.3}$$ ■

Our extension of the Cayley–Hamilton theorem concerns modules of finite rank. If M has rank r over A, then $\bigwedge^r M$ has rank 1, and so $D(t)$ and $\overline{D}(t)$ act on $\bigwedge^r M$ by multiplication by some formal power series. Indeed, $\bigwedge^r M = \mathrm{Span}_A \{b_1 \wedge \ldots \wedge b_r\}$, and, by definition (2.3),

$$\overline{D}(t)\,(b_1 \wedge \ldots \wedge b_r) = (\mathbf{1} - ft)b_1 \wedge \ldots \wedge (\mathbf{1} - ft)b_r = \det(\mathbf{1} - ft)\,(b_1 \wedge \ldots \wedge b_r)\,.$$

Thus, the “eigenvalue” of $\overline{D}(t)$ on $\bigwedge^r M$ is $\det(\mathbf{1} - ft)$, considered as an element of $A[[t]]$. Let us write $\det(\mathbf{1} - ft) = E_r(t)$, where

$$E_r(t) = 1 - e_1 t + \cdots + (-1)^r t^r. \tag{2.4}$$

Remark 2.2. (1) Clearly “the eigenvalue” $H_r(t)$ of $D(t)$ on $\bigwedge^r M$ is the formal inverse of $E_r(t)$, i.e.,

$$H_r(t) E_r(t) = 1. \tag{2.5}$$

If we write $H_r(t) = \sum_{j\geq 0} h_j t^j$, then (2.4) and (2.5) determine each h_j as a polynomial of $e_1, \ldots, e_r$. For example, $h_0 = 1$, $h_1 = e_1$, $h_2 = e_1^2 - e_2$ etc.

(2) It is worth emphasizing that $\{e_i\}$ and $\{h_j\}$ are related in exactly the same way as *the elementary* and *the complete symmetric functions* of r variables, since $E_r(t)$ and $H_r(t)$ are their generating functions, respectively (see, for example, [**13**, I 2]).

For $f \in \mathrm{End}_A M$, one can write HS–derivations $D(t)$ and $\overline{D}(t)$, defined by (2.2) and (2.3), respectively, in the form

$$D(t) = \sum_{i\geq 0} D_i \cdot t^i, \quad \overline{D}(t) = \sum_{i\geq 0} (-1)^i \overline{D}_i \cdot t^i.$$

We have $D_0 = \mathbf{1} = \overline{D}_0$, $D_1 = \overline{D}_1$, and $D_i \,|_M = f^i$, $i \geq 0$.

Denote $\bigwedge^{>i} M = \bigoplus_{j=i+1}^r \bigwedge^j M$.

Theorem 2.3. *For $1 \leq k < r$, the endomorphism*

$$D_k - e_1 D_{k-1} + \cdots + (-1)^k e_k \mathbf{1} \tag{2.6}$$

vanishes on $\bigwedge^{>(r-k)} M$, and for $i \geq r$ the endomorphism

$$D_i - e_1 D_{i-1} + \cdots + (-1)^r e_r D_{i-r} \tag{2.7}$$

vanishes on the whole of $\bigwedge M$.

According to (2.4), the characteristic polynomial of f is

$$\det(t\mathbf{1}-f)=t^rE_r(1/t)=t^r-e_1t^{r-1}+\cdots+(-1)^re_r. \tag{2.8}$$

Hence, for $i=r+k$, the restriction of (2.7) to M gives the classical Cayley–Hamilton theorem (1.4).

2.2. A look at the infinite-dimensional case. Let M_0 be a $\mathbb{Q}$-vector space with a countable basis, and $\bigwedge M_0=\bigoplus_{j\geq 0}\bigwedge^j M_0$ its exterior algebra.

We equip M_0 with a structure of a free module of rank r over the ring of polynomials of r variables with rational coefficients, for any integer $r>0$. See Section 4 for details.

We fix a basis $(b_j)_{j\geq 1}$ of M_0, and define the shift operators σ_{+1}, σ_{-1} on M_0 by their action on the basis,

$$\sigma_{+1}(b_j)=b_{j+1},\ j\geq 1,\ \text{and}\ \sigma_{-1}(b_1)=0,\ \sigma_{-1}(b_j)=b_{j-1},\ j>1.$$

One can attach to each of the endomorphisms σ_{+1}, σ_{-1} a unique HS–derivation and its inverse, as in (2.2), (2.3). In this subsection we need only the HS–derivations

$$\sigma_+(t),\overline{\sigma}_+(t):\bigwedge M_0\to\left(\bigwedge M_0\right)[[t]]$$

generated by σ_{+1}. We shall denote by $\sigma_{+i},\overline{\sigma}_{+i}:\bigwedge M_0\to\bigwedge M_0$ the coefficients of t^i in $\sigma_+(t)$ and $\overline{\sigma}_+(t)$ respectively. In the next subsection, the HS–derivations corresponding to σ_{-1} will also appear.

Let us fix $r>0$. It is convenient to enumerate the basis of $\bigwedge^r M_0$, which corresponds to $(b_j)_{j\geq 1}$, by partitions $\boldsymbol{\lambda}=(\lambda_1\geq\cdots\geq\lambda_r\geq 0)$ of lenght at most r. We write $\mathcal{P}_r$ for the set of all such partitions, and denote the basis vectors as follows,

$$[\mathbf{b}]^r_{\boldsymbol{\lambda}}=b_{1+\lambda_r}\wedge b_{2+\lambda_{r-1}}\wedge\cdots\wedge b_{r+\lambda_1}\,,\quad \boldsymbol{\lambda}\in\mathcal{P}_r\,. \tag{2.9}$$

In particular, the zero partition $\mathbf{0}=(\lambda_1=0)$ gives $[\mathbf{b}]^r_{\mathbf{0}}=b_1\wedge\cdots\wedge b_r$.

Now consider $(e_i)_{1\leq i\leq r}$ as indeterminates, and the polynomial ring

$$B_r=\mathbb{Q}[e_1,e_2,\ldots,e_r]. \tag{2.10}$$

Let us equip $\bigwedge^r M_0$ with a B_r-module structure via

$$E_r(t)[\mathbf{b}]^r_{\boldsymbol{\lambda}}=\overline{\sigma}_+(t)[\mathbf{b}]^r_{\boldsymbol{\lambda}}\,, \tag{2.11}$$

where $E_r(t)$ is given by (2.4). In terms of the inverses, see (2.5), the same structure is given by

$$H_r(t)[\mathbf{b}]^r_{\boldsymbol{\lambda}}=\sigma_+(t)[\mathbf{b}]^r_{\boldsymbol{\lambda}}\,. \tag{2.12}$$

The interpretation of e_j's and h_j's as the elementary and the complete symmetric functions of r variables, see Remark 2.2 (2), suggests to consider B_r as a $\mathbb{Q}$-vector spaces generated by the Schur polynomials (see, for example, [**13**, I 3]),

$$\Delta_{\boldsymbol{\lambda}}(H_r)=\det(h_{\lambda_j-j+i})_{1\leq i,j\leq r}\,,\quad \boldsymbol{\lambda}\in\mathcal{P}_r. \tag{2.13}$$

Here h_j's are defined as in Remark 2.2 (1) for $j\geq 0$, and $h_j=0$ for $j<0$.

According to *Giambelli's formula* as in [**3**, p. 321]),

$$[\mathbf{b}]^r_{\boldsymbol{\lambda}}=\Delta_{\boldsymbol{\lambda}}(H_r)[\mathbf{b}]^r_{\mathbf{0}}, \tag{2.14}$$

that is, $\bigwedge^r M_0$ is a free B_r-module of rank 1 generated by $[\mathbf{b}]^r_{\mathbf{0}}$. This allows us to equip M_0 with a multiplicative structure over B_r, see Proposition 4.3. Denote M_0, endowed with this multiplicative structure, by M_r. In Section 4, we check that

- M_r is a B_r-module of rank r freely generated by $b_1, \ldots, b_r$.
- $\bigwedge^r M_r$ is $\bigwedge^r M_0$ with the B_r-module structure defined by (2.11) or (2.12).
- e_i is the eigenvalue of $\overline{\sigma}_{+i}$ restricted to $\bigwedge^r M_r$, $1 \le i \le r$.
- h_j is the eigenvalue of the restriction of σ_{+j} to $\bigwedge^r M_r$, $j \ge 0$.

Remark 2.4. The notion of HS–derivation on an exterior algebra enables one to extend some finite-dimensional linear algebra concepts (like eigenvalues and characteristic polynomials) to an infinite-dimensional situation. Indeed, an endomorphism of an infinite-dimensional vector space does not have a characteristic polynomial, whereas the corresponding HS–derivation is still defined.

2.3. Finite-dimensional approximations of bosonic vertex operators. We apply the construction of the previous subsection in order to get a "finite-dimensional approximation" of the well-known expression of the vertex operators occurring in the boson-fermion correspondence. We interpret this approximation as the ratio of certain characteristic polynomials.

Details are in Section 5, see also [**5**].

Take the polynomial ring of countably many indeterminates, $B = \mathbb{Q}[x_1, x_2, \ldots]$, and define the *bosonic vertex operators*, following [**9**, p. 56],

$$\Gamma(t) = \exp\left(\sum_{i\ge1} x_i t^i\right) \cdot \exp\left(-\sum_{i\ge1} \frac{1}{it^i}\frac{\partial}{\partial x_i}\right) : B \to B[t^{-1}, t]],$$

$$\Gamma^*(t) = \exp\left(-\sum_{i\ge1} x_i t^i\right) \cdot \exp\left(\sum_{i\ge1} \frac{1}{it^i}\frac{\partial}{\partial x_i}\right) : B \to B[t^{-1}, t]].$$

We find finite-dimensional counterparts of these operators using the symmetric functions interpretation. Namely, similarly to the finite-dimensional case, define $E_\infty(t)$ and $H_\infty(t)$,

$$E_\infty(t) = 1 - e_1 t + e_2 t^2 + \cdots + (-1)^k e_k t^k + \ldots\ , \quad H_\infty(t) = 1/E_\infty(t),$$

as the generating functions of the elementary and the complete symmetric functions of a countable set of variables, say, $(\xi_k)_{k\ge1}$. Consider also $x_j = j\sum_{k\ge1}\xi_k^j$, *the power sum symmetric functions*, see [**13**, I 3]. We have $\mathbb{Q}[x_1, x_2, \ldots] = \mathbb{Q}[e_1, e_2, \ldots]$. Moreover, $X_\infty(t) = \sum_{i\ge1} x_i t^i$, the generating function of $(x_i)_{i\ge1}$, satisfies

$$\exp\left(\sum_{i\ge1} x_i t^i\right) = \sum_{i\ge0} h_i t^i.$$

Clearly, $E_r(t)$, $H_r(t)$, $X_r(t)$ are obtained from $E_\infty(t)$, $H_\infty(t)$, $X_\infty(t)$ by setting $\tau_k = 0$ for $k > r$.

In order to define $\Gamma_r(t)$ and $\Gamma^*_r(t)$ for $r > 0$, use the notation of Section 2.2. In particular, the ring (2.10) is freely generated by the Schur polynomials (2.13), and $\bigwedge^r M_r$ is spanned over B_r by $[\mathbf{b}]^r_{\mathbf{0}}$, according to (2.14). Juxtaposing (2.11) or (2.12) and (2.14) we get, respectively,

$$\overline{\sigma}_+(t)[\mathbf{b}]^r_{\boldsymbol{\lambda}} = \overline{\sigma}_+(t)\left(\Delta_{\boldsymbol{\lambda}}(H_r)[\mathbf{b}]^r_{\mathbf{0}}\right) = E_r(t)\Delta_{\boldsymbol{\lambda}}(H_r)[\mathbf{b}]^r_{\mathbf{0}},, \tag{2.15}$$

$$\sigma_+(t)[\mathbf{b}]^r_{\boldsymbol\lambda} = \sigma_+(t)\left(\Delta_{\boldsymbol\lambda}(H_r)[\mathbf{b}]^r_{\mathbf{0}}\right) = H_r(t)\Delta_{\boldsymbol\lambda}(H_r)[\mathbf{b}]^r_{\mathbf{0}}\,. \tag{2.16}$$

Thus each of $\overline{\sigma}_+(t)$, $\sigma_+(t)$ defines certain homomorphism $B_r \to B_r[[t]]$, which we denote in the same way.

For the HS–derivations generated by the shift operator σ_{-1} of Section 2.2, we use the indeterminate t^{-1} instead of t, and denote them by $\sigma_-(t^{-1})$, $\overline{\sigma}_-(t^{-1})$. The corresponding homomorphisms are defined via

$$(\sigma_-(t^{-1})\Delta_{\boldsymbol\lambda}(H_r))[\mathbf{b}]^r_0 = \sigma_-(t^{-1})[\mathbf{b}]^r_{\boldsymbol\lambda}, \quad (\overline{\sigma}_-(t^{-1})\Delta_{\boldsymbol\lambda}(H_r))[\mathbf{b}]^r_0 = \overline{\sigma}_-(t^{-1})[\mathbf{b}]^r_{\boldsymbol\lambda}.$$

Definition of σ_{-1} implies that $\sigma_-(t^{-1})b_i = b_i + b_{i-1}t^{-1} + b_{i-2}t^{-2} + \cdots + b_1 t^{1-i}$ is a polynomial of t^{-1} for each $i > 0$. It follows that $\sigma_-(t^{-1})$, $\overline{\sigma}_-(t^{-1})$ also send all $\Delta_{\boldsymbol\lambda}(H_r)$'s to B_r-polynomials of t^{-1}.

Now we are ready to define the homomorphisms $\Gamma_r(t), \Gamma^*_r(t) : B_r \to B_r[t^{-1}, t]]$ by their values on $\Delta_{\boldsymbol\lambda}(H_r)$'s, as follows,

$$\begin{aligned}\Gamma_r(t)\left(\Delta_{\boldsymbol\lambda}(H_r)\right) &= \frac{1}{E_r(t)}\left(\overline{\sigma}_-(t^{-1})\Delta_{\boldsymbol\lambda}(H_r)\right),\\ \Gamma^*_r(t)\left(\Delta_{\boldsymbol\lambda}(H_r)\right) &= E_r(t)\cdot\left(\sigma_-(t^{-1})\Delta_{\boldsymbol\lambda}(H_r)\right).\end{aligned}$$

For $r_1 < r_2$, the natural projection $B_{r_2} \to B_{r_1}$ sending each of $e_{r_1+1}, \ldots, e_{r_2}$ to zero, sends $E_{r_2}(t)$ to $E_{r_1}(t)$, $H_{r_2}(t)$ to $H_{r_1}(t)$, and $X_{r_2}(t)$ to $X_{r_1}(t)$. In this sense, $E_r(t) \to E_\infty(t)$, $H_r(t) \to H_\infty(t)$, $X_r(t) \to X_\infty(t)$ as $r \to \infty$.

Thus, $\Gamma_r(t)$ and $\Gamma^*_r(t)$ tend to $\Gamma(t)$ and $\Gamma^*(t)$ when $r \to \infty$.

3. Cayley–Hamilton Theorem revisited

3.1. Hasse-Schmidt derivations on exterior algebras [3, 6]. Let A be a commutative ring with unit, M a free A-module of rank r, and $b_1, \ldots, b_r$ some A-basis of M.

Set $\bigwedge^0 M = A$. For $1 \le j \le r$, denote by $\bigwedge^j M$ the A-module generated by all $b_{i_1} \wedge \ldots \wedge b_{i_j}$ modulo permutations,

$$b_{i_{\tau(1)}} \wedge \ldots \wedge b_{i_{\tau(j)}} = \mathrm{sgn}(\tau) b_{i_1} \wedge \ldots \wedge b_{i_j},$$

where $\mathrm{sgn}(\tau)$ is the sign of permutation τ. In particular, $\bigwedge^1 M = M$.

The exterior algebra $\bigwedge M = \bigoplus_{j=0}^r \bigwedge^j M$ possesses the natural graduation given by juxtaposition $\wedge : \bigwedge^i M \times \bigwedge^j M \to \bigwedge^{i+j} M$.

We denote by $(\bigwedge M)[[t]]$ the ring of formal power series of t with coefficients in $\bigwedge M$, and by $(\mathrm{End}_A(\bigwedge M))[[t]]$ the ring of formal power series of t with coefficients in $\mathrm{End}_A(\bigwedge M)$.

For $\mathcal{D}(t) = \sum_{i\ge0} \mathcal{D}_i t^i$, $\mathcal{D}(t) = \sum_{j\ge0} \widetilde{\mathcal{D}}_j t^j \in (\mathrm{End}_A(\bigwedge M))[[t]]$, their product is defined as follows,

$$\mathcal{D}(t)\widetilde{\mathcal{D}}(t)u = \mathcal{D}(t)\sum_{j\ge0}\widetilde{\mathcal{D}}_j u\cdot t^j = \sum_{j\ge0}(\mathcal{D}(t)\widetilde{\mathcal{D}}_j u)\cdot t^j, \quad \forall u \in \bigwedge M.$$

Given series $\mathcal{D}(t)$, we use the same notation for the induced A-homomorphism,

$$\mathcal{D}(t) : \bigwedge M \to \bigwedge M[[t]], \quad u \mapsto \mathcal{D}(t)u = \sum_{i\ge0}\mathcal{D}_i u\cdot t^i, \quad \forall u \in \bigwedge M.$$

The series $\mathcal{D}(t)=\sum_{i\geq 0}\mathcal{D}_it^i$ is *invertible* in $(\mathrm{End}_A(\bigwedge M))[[t]]$, if there exists $\overline{\mathcal{D}}(t)\in(\mathrm{End}_A(\bigwedge M))[[t]]$ such that

$$\mathcal{D}(t)\overline{\mathcal{D}}(t)=\overline{\mathcal{D}}(t)\mathcal{D}(t)=\mathbf{1}_{\bigwedge M}. \tag{3.1}$$

We call $\overline{\mathcal{D}}(t)$ the *inverse* series and write it in the form $\overline{\mathcal{D}}(t)=\sum_{i\geq 0}(-1)^i\overline{\mathcal{D}}_it^i$. Then (3.1) is equivalent to

$$\mathcal{D}_j-\overline{\mathcal{D}}_1\mathcal{D}_{j-1}+\ldots+(-1)^j\overline{\mathcal{D}}_j=0,\ \forall j\geq 1. \tag{3.2}$$

One can check that $\mathcal{D}(t)$ invertible if and only if $\mathcal{D}_0$ is an automorphism of $\bigwedge M$.

Proposition 3.1. *The following two statements are equivalent:*

i) $\mathcal{D}(t)(u\wedge v)=\mathcal{D}(t)u\wedge\mathcal{D}(t)v,\ \ \forall u,v\in\bigwedge M;$
ii) $\mathcal{D}_i(u\wedge v)=\sum_{j=0}^{i}\mathcal{D}_ju\wedge\mathcal{D}_{i-j}v,\ \ \forall u,v\in\bigwedge M,\ \ \forall i\geq 0.$

Proof. *i)* $\Rightarrow$ *ii)* By definition of $\mathcal{D}(t)$, one can write *i)* as

$$\sum_{i\geq 0}\mathcal{D}_i(u\wedge v)t^i=\sum_{j_1\geq 0}\mathcal{D}_{j_1}u\cdot t^{j_1}\wedge\sum_{j_2\geq 0}\mathcal{D}_{j_2}v\cdot t^{j_2}. \tag{3.3}$$

Hence $\mathcal{D}_i(u\wedge v)$ is the coefficient of t^i on the right hand side of (3.3), which is $\sum_{j_1+j_2=i}\mathcal{D}_{j_1}u\wedge\mathcal{D}_{j_2}v=\sum_{j=0}^{i}\mathcal{D}_ju\wedge\mathcal{D}_{i-j}v.$

ii) $\Rightarrow$ *i)* We have

$$\begin{aligned}\mathcal{D}(t)(u\wedge v) &= \sum_{i\geq 0}\mathcal{D}_i(u\wedge v)t^i=\sum_{i\geq 0}\left(\sum_{i_1+i_2=j}\mathcal{D}_{i_1}u\wedge\mathcal{D}_{i_2}v\right)t^i\\ &= \sum_{i\geq 0}\left(\sum_{i_1}\mathcal{D}_{i_1}u\cdot t^{i_1}\wedge\sum_{i_2}\mathcal{D}_{i_2}v\cdot t^{i_2}\right)=\mathcal{D}(t)u\wedge\mathcal{D}(t)v.\end{aligned}$$

∎

Definition 3.2. (Cf. [**3**]) Let $\mathcal{D}(t)\in(\mathrm{End}_A(\bigwedge M))[[t]]$. The induced map $\mathcal{D}(t):\bigwedge M\to(\bigwedge M)[[t]]$ is called a *Hasse–Schmidt derivation* (or, for brevity, a *HS–derivation*) on $\bigwedge M$, if it satisfies the (equivalent) conditions of Proposition 3.1.

Proposition 3.3. (Cf. [**3, 6**]) *The product of two HS–derivations is a HS–derivation. The inverse of a HS–derivation is a HS–derivation.*

Proof. For the product of HS–derivations $\mathcal{D}(t)$ and $\tilde{\mathcal{D}}(t)$, the statement *i)* of Proposition 3.1 holds. Indeed, $\forall u,v\in\bigwedge M$,

$$\begin{aligned}\mathcal{D}(t)\tilde{\mathcal{D}}(t)(u\wedge v) &= \mathcal{D}(t)\left(\sum_{j\geq 0}\sum_{j_1+j_2=j}\tilde{\mathcal{D}}_{j_1}u\wedge\tilde{\mathcal{D}}_{j_2}v\right)t^j\\ &=\sum_{j\geq 0}\sum_{j_1+j_2=j}\mathcal{D}(t)D_{j_1}u\cdot t^{j_1}\wedge\mathcal{D}(t)\mathcal{D}_{j_2}v\cdot t^{j_2}\\ &=\mathcal{D}(t)\tilde{\mathcal{D}}(t)u\wedge\mathcal{D}(t)\tilde{\mathcal{D}}(t)v.\end{aligned}$$

Similarly, if $\overline{\mathcal{D}}(t)$ is the inverse of the HS–derivation $\mathcal{D}(t)$, then $\forall u,v\in\bigwedge M$,

$$\begin{aligned}\overline{\mathcal{D}}(t)(u\wedge v) &= \overline{\mathcal{D}}(t)(\mathcal{D}(t)\overline{\mathcal{D}}(t)u\wedge\mathcal{D}(t)\overline{\mathcal{D}}(t)v)\\ &= (\overline{\mathcal{D}}(t)\mathcal{D}(t))(\overline{\mathcal{D}}(t)u\wedge\overline{\mathcal{D}}(t)v)\\ &= \overline{\mathcal{D}}(t)u\wedge\overline{\mathcal{D}}(t)v.\end{aligned}$$

∎

Corollary 3.4. [**6**] If $\overline{\mathcal{D}}(t)$ is the inverse of a HS–derivation $\mathcal{D}(t)$, then

$$\begin{aligned} \mathcal{D}(t)u\wedge v &= \mathcal{D}(t)u\wedge\mathcal{D}(t)\overline{\mathcal{D}}(t)v=\mathcal{D}(t)(u\wedge\overline{\mathcal{D}}(t)v),\\ u\wedge\overline{\mathcal{D}}(t)v &= \overline{\mathcal{D}}(t)\mathcal{D}(t)u\wedge\overline{\mathcal{D}}(t)v=\overline{\mathcal{D}}(t)(\mathcal{D}(t)u\wedge v)\end{aligned}\tag{3.4}$$

for all $u,v\in\bigwedge M$. Equivalenly, for any $k\geq 1$,

$$\begin{aligned} \mathcal{D}_k u\wedge v &= \mathcal{D}_k(u\wedge v)-\mathcal{D}_{k-1}(u\wedge\overline{\mathcal{D}}_1 v)+\ldots+(-1)^k u\wedge\overline{\mathcal{D}}_k v,\\ u\wedge\overline{D}_k v &= \overline{D}_k(u\wedge v)-\overline{D}_{k-1}(D_1u\wedge v)+\ldots+(-1)^k D_k u\wedge v\end{aligned}\tag{3.5}$$

3.2. Proof of the Theorem 2.3. As we have seen in Proposition 2.1, any endomorphism $f\in\mathrm{End}_A(M)$ defines two graded mutually inverse HS–derivations,

$$\overline{D}(t)=\bigwedge(\mathbf{1}-ft)\quad\text{and}\quad D(t)=\bigwedge\left(\sum_{i\geq 0}f^it^i\right),$$

where $\mathbf{1}$ denotes the identity endomorphism. Write $\overline{D}(t)$ and $D(t)$ in the form

$$\overline{D}(t)=\sum_{i\geq 0}(-1)^i\overline{D}_it^i\quad\text{and}\quad D(t)=\sum_{i\geq 0}D_it^i,$$

then these HS–derivations satisfy the following properties.

Lemma 3.5. *We have*
(i) $\overline{D}_0=D_0=\mathbf{1}_{\bigwedge M}$ *and* $D_1=\overline{D}_1$,
(ii) $D_i\mid_M=f^i$, $i\geq 0$,
(iii) $\overline{D}_k u=0$, *for all* $u\in\bigwedge^i M$ *with* $i<k$.

Indeed, $\overline{D}(t)\mid_{\bigwedge^k M}$ is a polynomial of t of degree k, $1\leq k\leq r$. ∎

As before, we assume that our A-module M is freely generated by $(b_j)_{1\leq j\leq r}$. Thus $\bigwedge^r M$ has rank 1 and is spanned by $[\mathbf{b}]^r_{\mathbf{0}}=b_1\wedge\cdots\wedge b_r$. The restriction of each $\overline{D}_i$ to $\bigwedge^r M$ is a multiplication by some scalar $e_i\in A$,

$$\overline{D}_i\left([\mathbf{b}]^r_{\mathbf{0}}\right)=e_i[\mathbf{b}]^r_{\mathbf{0}},\quad 1\leq i\leq r.\tag{3.6}$$

Take now $u\in\bigwedge^i M$ and $v\in\bigwedge^{r-i}M$. Then $D_ju\wedge v\in\bigwedge^r M$ for $1\leq j\leq k$. Applying (3.5) to our situation, we can write

$$D_ku\wedge v-e_1(D_{k-1}u\wedge v)+\cdots+(-1)^ke_k(u\wedge v)=(-1)^ku\wedge\overline{D}_kv$$

for $1\leq k\leq r$, and

$$D_ku\wedge v-e_1(D_{k-1}u\wedge v)+\cdots+(-1)^ke_r(D_{k-r}u\wedge v)=(-1)^ku\wedge\overline{D}_kv$$

for $k>r$.

Equivalently, we have

$$\big(D_ku-e_1D_{k-1}u+\cdots+(-1)^ke_ku\big)\wedge v=(-1)^ku\wedge\overline{D}_kv,\ 1\leq k\leq r,\tag{3.7}$$

and

$$\big(D_ku-e_1D_{k-1}u+\cdots+(-1)^ke_rD_{k-r}u\big)\wedge v=(-1)^ku\wedge\overline{D}_kv,\ k>r.\tag{3.8}$$

Of course, one can set $e_k=0$ for $k>r$, in order do not distinguish between the two cases. However, we prefer a division into cases.

Assume now $i>r-k>0$. Then, according to Remark 3.5, *(iii)*, the right hand side of (3.7) vanishes $\forall v\in\bigwedge^{r-i}M$, as $r-i<k$. This means that

$$D_ku-e_1D_{k-1}u+\cdots+(-1)e_ku=0$$

for any $u \in \bigwedge^i M$ with $i > r - k > 0$. This proves the first part of Theorem 2.3.

If $k > r$, then the left hand side of (3.8) vanishes for each $i \geq 0$, and this proves the second part. ∎

Remark 3.6. Thus we understand (1.3) as an abstract Cayley–Hamilton theorem valid for general invertible HS–derivations on exterior algebras of arbitrary (not necessarily free) modules. If the module is free and at most countably generated, then (1.3) produces a sequence of Cayley–Hamilton relations (3.5). This sequence turns into the classical Cayley–Hamilton formulas (2.7) when the HS–derivation corresponds to an endomorphism of a finitely generated free module.

3.3. Example. Take $M = \mathbb{R}^3$. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ have an eigenbasis,

$$f\mathbf{u} = a\mathbf{u}, \quad f\mathbf{v} = b\mathbf{v}, \quad f\mathbf{w} = c\mathbf{w}.$$

Then $\det(\mathbf{1}t - f) = t^3 - e_1 t^2 + e_2 t - e_3$, where

$$e_1 = a + b + c, \quad e_2 = ab + ac + bc, \quad e_3 = abc.$$

In notation of Theorem 2.3, we have $r = 3$.

1) Let us take $k = 2$ and check that $D_2 - e_1 D_1 + e_2 \mathbf{1}$ vanishes on $\mathbb{R}^3 \wedge \mathbb{R}^3$, as it should be, according to (2.6). We show the calculation of $(D_2 - e_1 D_1 + e_2 \mathbf{1})(\mathbf{u} \wedge \mathbf{v})$; for two other basis vectors $\mathbf{u} \wedge \mathbf{w}$ and $\mathbf{v} \wedge \mathbf{w}$ it is completely similar.

First, we find the action of D_1, D_2 on $\mathbf{u} \wedge \mathbf{v}$. We have

$$(\mathbf{1} + ft + f^2t^2 + \circ(t^2))\mathbf{u} \wedge (\mathbf{1} + ft + f^2t^2 + \circ(t^2))\mathbf{v} =$$

$$= \mathbf{u} \wedge \mathbf{v} + (f\mathbf{u} \wedge \mathbf{v} + \mathbf{u} \wedge f\mathbf{v})t + (f^2\mathbf{u} \wedge \mathbf{v} + f\mathbf{u} \wedge f\mathbf{v} + \mathbf{u} \wedge f^2\mathbf{v})t^2 + \circ(t^2).$$

Thus $D_1(\mathbf{u} \wedge \mathbf{v}) = (a + b)(\mathbf{u} \wedge \mathbf{v})$, $D_2(\mathbf{u} \wedge \mathbf{v}) = (a^2 + ab + b^2)(\mathbf{u} \wedge \mathbf{v})$, and

$$(D_2 - e_1 D_1 + e_2 \mathbf{1})(\mathbf{u} \wedge \mathbf{v}) = (a^2 + ab + b^2 - e_1 a - e_1 b + e_2)(\mathbf{u} \wedge \mathbf{v}).$$

Now, we substitude the expressions for e_1, e_2, and get

$$a^2+ab+b^2-e_1a-e_1b+e_2 = a^2+ab+b^2-a^2-ab-ac-ab-b^2-bc+ab+ac+bc = 0.$$

2) Take $k = 4$ and check that $D_4 - e_1 D_3 + e_2 D_2 - e_3 D_1$ vanishes on $\mathbb{R}^3 \wedge \mathbb{R}^3$. According to (2.7), this endomorphism vanishes on the whole of $\bigwedge \mathbb{R}^3$, and in fact the verification for the rest of the direct summands is simpler. Again we calculate the image of $\mathbf{u} \wedge \mathbf{v}$.

First, we obtain $D_3(\mathbf{u} \wedge \mathbf{v})$ and $D_4(\mathbf{u} \wedge \mathbf{v})$ in the standard way, writing

$$(\mathbf{1} + ft + f^2t^2 + f^3t^3 + f^4t^4 + \circ(t^5))\mathbf{u} \wedge (\mathbf{1} + ft + f^2t^2 + f^3t^3 + f^4t^4 + \circ(t^5))\mathbf{v},$$

and collecting the coefficients of t^3, t^4, respectively. We get

$$D_3(\mathbf{u}\wedge\mathbf{v}) = (a^3+a^2b+ab^2+b^3)(\mathbf{u}\wedge\mathbf{v}), D_4(\mathbf{u}\wedge\mathbf{v}) = (a^4+a^3b+a^2b^2+ab^3+b^4)(\mathbf{u}\wedge\mathbf{v}),$$

substitute all the expressions for D_4, D_3, D_2, D_1, e_1, e_2, e_3 in terms of a, b, c into $D_4 - e_1 D_3 + e_2 D_2 - e_3 D_1$, and safely get 0.

Remark 3.7. In general, if $f \in \mathrm{End}(M)$ is diagonalizable, and if $(\mathbf{v}_i)_{1 \leq i \leq r}$ is an eigenbasis, $f\mathbf{v}_i = x_i \mathbf{v}_i$, $1 \leq i \leq r$, then the vector $\mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_l \in \bigwedge^l M$ is an eigenvector of D_k with the eigenvalue which is the complete symmetric polynomial of $x_1, \ldots, x_l$ of degree k. Therefore, for a diagonizable endomorphism our Theorem 2.3 is reduced to the following identity. Denote by $h_i(\mathbf{x}_j)$ the complete symmetric polynomial of degree i in $x_1, \ldots, x_j$, and by $e_k(\mathbf{x}_n)$ the elementary symmetric polynomial of degree k in $x_1, \ldots, x_n$. Then, for $n \geq 1$ and all $1 \leq j \leq n$, we have

$$h_n(\mathbf{x}_j) - e_1(\mathbf{x}_n)h_{n-1}(\mathbf{x}_j) + \ldots + (-1)^n e_n(\mathbf{x}_n) = 0.$$

One can deduce the identity, for example, from the formula (*) of [**13**, I 3 28].

This remark can be turned into a rigorous general proof, using a standard (though rather long) reasoning. Another possible way, which was suggested by our referee, is based on the Frobenius proof of the classical Cayley–Hamilton theorem for the complex matrices, [**2**]. We were not aware of that 1896 paper by Frobenius. Probably one could translate our arguments into the language of matrix minors. However, our approach, through the relationship to symmetric functions, is short, easy, and, in addition, allows us to concern with the infinite-dimensional case.

3.4. The case of a $\mathbb{Q}$-algebra. If A is a $\mathbb{Q}$-algebra, then for $f \in \mathrm{End}_A M$ the exponential

$$\exp(ft) := \sum_{k\geq 0} \frac{f^k t^k}{k!} \in (\mathrm{End}_A M)[[t]]$$

is well-defined. In the ring $(\mathrm{End}_A M)[[t]]$, there is the formal derivative by t,

$$y(t) = \sum_{k\geq 0} g_k t^k \ \Rightarrow \ y'(t) = \sum_{k\geq 0} k g_k t^{k-1}, \quad g_k \in \mathrm{End}_A M.$$

Recall the notation $E_r(t)$ given by (2.4), and write the characterictic polynomial of f as in (2.8). In [**4**] for any commutative ring R containing the rational numbers, *the formal Laplace transform* $L : R[[t]] \to R[[t]]$ and its inverse L^{-1} are defined as follows,

$$L \sum_{n\geq 0} a_n t^n = \sum_{n\geq 0} n! a_n t^n, \quad L^{-1} \sum_{n\geq 0} c_n t^n = \sum_{n\geq 0} c_n \frac{t^n}{n!}, \quad a_n, c_n \in R.$$

Take the *inverse formal Laplace transform of the HS–derivation* $D(t) = \sum_{i\geq 0} D_i t^i$ corresponding to $f \in \mathrm{End}_A M$,

$$D^*(t) = L^{-1} \sum_{k\geq 0} D_k t^k = \sum_{k\geq 0} \frac{D_k t^k}{k!} \in \Big(\mathrm{End}_A(\bigwedge M)\Big)[[t]]. \tag{3.9}$$

Define $\mathtt{p}_k(D)$ as the coefficient of t^k in $E_r(t)D(t)$. We have

$$\mathtt{p}_0(D) = \mathbf{1}, \ \mathtt{p}_1 = D_1 - e_1\mathbf{1}, \ \mathtt{p}_j = D_j - e_1 D_{j-1} + \ldots + (-1)^j e_j \mathbf{1}, \ 1 < j < r,$$

and

$$\mathtt{p}_{r+j}(D) = D_{r+j} - e_1 D_{r+j-1} + \ldots + (-1)^r e_r D_j = \mathbf{0}, \quad j \geq 0,$$

according to Theorem (2.3). Therefore,

Proposition 3.8. We have

$$D(t) = \frac{\mathbf{1} + \mathtt{p}_1(D)t + \ldots + \mathtt{p}_{r-1}(D)t^{r-1}}{E_r(t)}. \tag{3.10}$$

■

Corollary 3.9. Let $\mathbb{Q} \subseteq A$ and the characteristic polynomial of $f \in \mathrm{End}_A M$ be given by (2.8). Then the series $D^*(t)$ defined in (3.9) solves the ordinary differential equation

$$y^{(r)}(t) - e_1 y^{(r-1)}(t) + \ldots + (-1)^r e_r y(t) = 0 \tag{3.11}$$

in $(\mathrm{End}_A(\bigwedge M))[[t]]$.

Proof. Take the inverse formal Laplace transform of (3.10). We obtain

$$D^*(t) = u_0 + \mathtt{p}_1(D)u_{-1} + \ldots + \mathtt{p}_{r-1}(D)u_{-r+1},$$

where

$$u_{-j} = u_{-j}(t) = L^{-1}\left(\frac{t^j}{E_r(t)}\right), \quad 0 \le j \le r-1.$$

Let us re-write the series $u_0, u_{-1}, \ldots, u_{-r+1}$ in terms of $H_r(t) = 1/E_r(t)$, see Remark 2.2(1),

$$u_{-j} = L^{-1}(t^j H_r(t)) = \sum_{n \ge j} h_{n-j}\frac{t^n}{n!}, \quad 0 \le j \le r-1.$$

In [**7**], we proved that these series form an A-basis of solutions to the ODE (3.11) in $R[[t]]$. For $R = \mathrm{End}_A(\bigwedge M)$ we get the claim. ■

3.5. Elementary remarks. We finish this section with a few remarks relevant to the case when A is a $\mathbb{Q}$-algebra.

(1) The characteristic polynomial of $f \in \mathrm{End}_A M$ is given by (2.8) if and only if $y(t) = \exp(ft)$ satisfies the linear ordinary differential equation (3.11). This is our Corollary 3.9 restricted to M.

In particular,

$$\exp(ft) = v_0(t)\mathbf{1}_M + v_1(t)f + \cdots + v_{r-1}(t)f^{r-1},$$

where $(v_j(t))_{0 \le j \le r-1}$ is the *standard* A-basis of solutions to (3.11) in $A[[t]]$, that is, $v_j^{(i)}(t) = \delta_{ij}$, $0 \le i, j \le r-1$. Indeed, $\mathbf{1}_M, f, \ldots, f^{r-1}$ are the initial conditions of the solution $\exp(ft)$.

In the context of endomorphisms of complex vector spaces, the formula for $\exp(ft)$ was obtained in 1966 by Putzer [**14**], and then re-obtained in 1998 by Leonard and Liz, [**11, 12**], in a different way.

(2) The relation between the standard fundamental system $v_j(t))_{0 \le j \le r-1}$ and the fundamental system $u_{-j}(t)_{0 \le j \le r-1}$ appeared in the proof of Corollary 3.9 is as follows. Consider the linear system of first order differential equations equivalent to our ODE (3.11),

$$y_1' = y_2, \quad y_2' = y_3, \ \ldots, \ y_{r-2}' = y_{r-1}, \quad y_{r-1}' = e_1 y_{(r-1)} - \ldots + (-1)^{r-1} e_r y_1.$$

Denote the matrix of this system by P_r. Then $Q = \exp(P_r t)$ is the Wronski matrix of $v_1(t), \ldots, \ v_{r-1}(t)$,

$$(Q)_{ij} = v_j^{(i)}(t), \quad 0 \le i, j \le r-1,$$

and $u_0(t), u_{-1}(t), \ldots, \ u_{1-r}(t)$ is the last column of Q,

$$v_{r-1}(t) = u_{1-r}(t), \ v_{r-1}'(t) = u_{2-r}(t), \ldots, \ v_{r-1}^{(r-1)}(t) = u_0(t).$$

(3) As another elementary corollary of our considerations, we get formulas for the coefficients e_k of the characterictic polynomial of $f \in \mathrm{End}_A M$ in terms of its

matrix elements. If $C = (c_{ij})$ is the $r \times r$ matrix of f in some A-basis of M, denote by $D(i_1, \dots, i_k)$ the determinant of the $(r-k) \times (r-k)$-matrix obtained from the matrix C by deleting the i_1-th,…,i_k-th rows and columns. Then

$$(-1)^k e_k = \sum_{1 \le i_1 < \dots < i_k \le r} D(i_1, \dots, i_k), \ 1 \le k \le r.$$

This formula was obtained differently by Brooks in [**1**].

4. Countably generated $\mathbb{Q}$-vector spaces

Let M_0 be a $\mathbb{Q}$-vector space generated by $(b_j)_{j \ge 1}$ and $\bigwedge M_0 = \bigoplus_{r \ge 0} \bigwedge^r M_0$ be its exterior algebra.

As in Section 2.2, we take shift operators $\sigma_{\pm 1} \in \mathrm{End}_{\mathbb{Q}} M_0$, and denote by $\sigma_\pm(t^{\pm 1}), \overline{\sigma}_\pm(t^{\pm 1})$ the corresponding HS-derivations. Below we will write

$$\sigma_+(t) = \sum_{j \ge 0} \sigma_j t^j, \quad \overline{\sigma}_+(t) = \sum_{j \ge 0} \overline{\sigma}_j t^j, \tag{4.1}$$

skipping sign + in the subscript.

In this section, we treat $e_1, \dots, e_r$ as indeterminates. As we already pointed out in Section 2.2, the ring B_r, given by (2.10), has a basis formed by Schur polynomials $\Delta_{\boldsymbol{\lambda}}(H_r)$, see (2.13). The structure of a principal B_r-module on $\bigwedge^r M_0$ is defined via any of the two equivalent equalities (2.11) and (2.12).

Let $(\beta_i)_{i \ge 1}$ be linear forms on M_0 defined by $\beta_i(b_j) = \delta_{ij}$. Their linear span is, by definition, the restricted dual M_0^*. Each β_j induces a $\mathbb{Q}$-linear contraction map $\beta_j : \bigwedge^r M_0 \to \bigwedge^{r-1} M_0$ defined by $\beta_j \lrcorner m = \beta_j(m)$ for all $m \in M_0$ and

$$\beta_j \lrcorner (m \wedge \eta) = \beta_j(m)\eta - m \wedge \beta_j \lrcorner \eta. \tag{4.2}$$

As each $\zeta \in \bigwedge M_0$ is a sum of homogeneous elements of the form $m \wedge \eta$, equation (4.2) defines the contraction operator over the entire exterior algebra $\bigwedge M_0$.

Lemma 4.1. *Let $m, m' \in M_0$ satisfy*

$$m \wedge \eta = m' \wedge \eta \tag{4.3}$$

for all $\eta \in \bigwedge^{r-1} M_0$. Then $m = m'$.

Proof. Under the hypothesis (4.3), suppose first that $m' = am$ for some $a \neq 1$. If $m \neq 0$, then there is $\mu \in M_0^*$ such that $\mu(m) \neq 0$. Because of the isomorphism $\bigwedge^{r-1} M_0 \cong M_0^*$, there is then $\eta \in \bigwedge^{r-1} M_0$ such that $m \wedge \eta \neq 0$. We get $m' \wedge \eta = a(m \wedge \eta)$, hence $m \wedge \eta \neq m' \wedge \eta$.

If m and m' are not proportional, take their duals $\mu, \mu' \in M_0^*$ and choose $\zeta_{m,m'} = \mu' \lrcorner (\mu \lrcorner [\mathbf{b}]_0^r) \in \bigwedge^{r-1} M_0$. Then $\mu(m) = \mu'(m') = 1$ and so $m \wedge m \wedge \zeta_{m,m'} = 0$ while $m' \wedge m \wedge \zeta_{m,m'} \neq 0$. ∎

Corollary 4.2. For each e_i, $1 \le i \le r$, there exists the unique mapping $M_0 \to M_0$, called the *multiplication by* e_i, $m \mapsto e_i m$, such that

$$(e_i m) \wedge \eta = e_i (m \wedge \eta), \tag{4.4}$$

for all $\eta \in \bigwedge^{r-1} M_0$.

The vector space M_0 endowed with the multiplications by e_i, $1 \le i \le r$, becomes a B_r-module, denoted by M_r.

Proposition 4.3. The B_r-module M_r is freely generated by $(b_i)_{1\leq i\leq r}$, and $\sigma_+(t)\in End_{B_r}(M_r)[[t]]$.

Proof. Let us check first that $b_1, b_2, \ldots, b_r$ are B_r-linearly independent. Denote by $(\beta_j)_{j\geq 0}$ the generators of M_0^* dual to $(b_i)_{i\geq 1}$. Notice that $a_1b_1+\cdots+a_rb_r=0$ implies $0=a_ib_i\wedge\eta_i$, where $\eta_i=\beta_i\lrcorner[\mathbf{b}]^r_{\mathbf{0}}$, that is, $a_i=0$ for all $1\leq i\leq r$.

Now, let us show that $b_{i+r}-e_1b_{i+r-1}+\cdots+e_rb_i=0$ for all $i\geq 0$. This will prove, by induction, that M_r is generated over B_r by $b_1, b_2, \ldots, b_r$. It is enough to observe that

$$\sum_{i=1}^{\infty}(b_i-e_1b_{i-1}+(-1)^re_rb_{r-1+i})t^i=E_r(t)\sigma_+(t)b_1$$

is a polynomial of degree r (here we set $b_j=0$ for $j<1$). By definition of the module structure, for each $\eta\in\bigwedge^{r-1}M_0$ we have

$$\begin{aligned}
E_r(t)(\sigma_+(t)b_0)\wedge\eta &= E_r(t)(\sigma_+(t)b_0\wedge\eta) && \text{(definition of the } B_r\text{-module structure)}\\
&= E_r(t)\sigma_+(t)(b_0\wedge\overline{\sigma}_+(t)\eta) && \text{(Proposition 3.4)}\\
&= E_r(t)\frac{1}{E_r(t)}(b_0\wedge\overline{\sigma}_+(t)\eta) && \text{(definition of the } B_r\text{-module structure)}\\
&= b_0\wedge\overline{\sigma}_+(t)\eta.
\end{aligned}$$

We use now the agreement (4.1). As in Remark 3.5, we see that $\overline{\sigma}_r$ vanishes on $\bigwedge^{r-1}M_0$, hence the expression obtained above is a polynomial in t of degree $r-1$. We have so proven that M_r is a B_r-module of rank r. Moreover σ_1 is B_r-linear. In fact,

$$\sigma_1(e_im)\wedge\eta=\sigma_1(e_im\wedge\eta)-e_im\wedge\sigma_1\eta=e_i\sigma_1(m\wedge\eta)-e_im\wedge\sigma_1\eta=e_1\sigma_1m\wedge\eta. \qquad \blacksquare$$

Corollary 4.4. The elements e_i, $1\leq i\leq r$, and h_j, $j\geq 0$, of B_r are the eigenvalues of $\overline{\sigma}_i$ and σ_j, respectively, thought of as endomorphisms of $\bigwedge^r M_r$. $\blacksquare$

Similarly to (2.15) and (2.16), the mappings $\sigma_-(t^{-1})$ and $\overline{\sigma}_-(t^{-1})$ define two homomorphisms $B_r\to B_r[t^{-1}]$, via the equalities

$$(\sigma_-(t^{-1})\Delta_{\boldsymbol{\lambda}}(H_r))[\mathbf{b}]^r_0:=\sigma_-(t^{-1})[\mathbf{b}]^r_{\boldsymbol{\lambda}}, \tag{4.5}$$

and

$$(\overline{\sigma}_-(t^{-1})\Delta_{\boldsymbol{\lambda}}(H_r))[\mathbf{b}]^r_0:=\overline{\sigma}_-(t^{-1})[\mathbf{b}]^r_{\boldsymbol{\lambda}}. \tag{4.6}$$

By abuse of notation, we denote the homomorphisms in the same way.

5. Bosonic Vertex Operators

5.1. Let $B:=\mathbb{Q}[x_1,x_2,\ldots]$ be the polynomial ring in infinitely many indeterminates and $M_0:=\bigoplus_{i\geq 0}\mathbb{Q}b_i$. The aim of this section is to show that the *bosonic vertex operators*

$$\Gamma(t):=\exp(\sum_{i\geq 1}x_it^i)\cdot\exp\left(-\sum_{i\geq 1}\frac{1}{it^i}\frac{\partial}{\partial x_i}\right):B\to B[t^{-1},z]]$$

and

$$\Gamma^*(t) := \exp(-\sum_{i\geq 1} x_i t^i)\cdot \exp\left(\sum_{i\geq 1}\frac{1}{it^i}\frac{\partial}{\partial x_i}\right) : B \to B[t^{-1}, z]]$$

may be identified with ratios of characteristic series operators associated to the shift endomorphisms of step ± 1 of M_0.

Let $\Gamma_r(t), \Gamma_r^*(t) : B_r \to B_r[t^{-1}, t]]$ be defined by

$$\Gamma_r(t)\Delta_{\boldsymbol{\lambda}}(H_r)[\mathbf{b}]_0^r := \sigma_+(t)\overline{\sigma}_-(t^{-1})\Delta_{\boldsymbol{\lambda}}(H_r))[\mathbf{b}]_0^r$$

and

$$\Gamma_r^*(t)\Delta_{\boldsymbol{\lambda}}(H_r)[\mathbf{b}]_0^r := \overline{\sigma}_+(t)\sigma_-(t^{-1})\Delta_{\boldsymbol{\lambda}}(H_r)[\mathbf{b}]_0^r.$$

Then, due to (2.15) and (2.16) one can write:

$$\Gamma_r(t) = \frac{1}{E_r(t)}\cdot\overline{\sigma}_-(t^{-1}) \qquad\text{and}\qquad \Gamma_r^*(t) = E_r(t)\cdot\frac{1}{\overline{\sigma}_-(t^{-1})}\,.$$

Remark 5.1. Notice that $E_r(t)$ is indeed the characteristic polynomial of σ_1, thought of as endomorphism of M_r, and $\overline{\sigma}_-(t^{-1})$ is the characteristic series operator associated to σ_{-1}.

Proposition 5.2. [**5**] The operators $\Gamma_r(t), \Gamma_r^*(t)$ tend to $\Gamma(t), \Gamma^*(t)$ as r goes to infinity.

Proof. We sketch the arguments of [**5**]. First of all, notice that for all $r \geq 1$

$$\overline{\sigma}_-(t^{-1})h_n = h_n - \frac{h_{n-1}}{t} \quad\text{and}\quad \sigma_-(t^{-1})h_n = \sum_{i\geq 0}\frac{h_{n-i}}{t^i}\,. \tag{5.1}$$

Let $\overline{\sigma}_-(t^{-1})H_r = (\sigma_-(t^{-1})h_n)_{n\in\mathbb{Z}}$ and $\sigma_-(t^{-1})H_r = (\sigma_-(t^{-1})h_n)_{n\in\mathbb{Z}}$. Then

$$\overline{\sigma}_-(t^{-1})\Delta_{\boldsymbol{\lambda}}(H_r) = \Delta_{\boldsymbol{\lambda}}(\overline{\sigma}_-(t^{-1})H_r),\quad \sigma_-(t^{-1})\Delta_{\boldsymbol{\lambda}}(H_r) = \Delta_{\boldsymbol{\lambda}}(\sigma_-(t^{-1})H_r), \tag{5.2}$$

by [**5**, Propositions 6.2.10 and 6.2.13]. Now, according to [**5**, Corollaries 6.2.11 and 6.2.14],

$$\sigma_-(t^{-1})(h_{i_1}\cdot\ldots\cdot h_{i_r}) = \sigma_-(t^{-1})h_{i_1}\cdot\ldots\cdot\sigma_-(t^{-1})h_{i_r} \tag{5.3}$$

and

$$\overline{\sigma}_-(t^{-1})(h_{i_1}\cdot\ldots\cdot h_{i_r}) = \overline{\sigma}_-(t^{-1})h_{i_1}\cdot\ldots\cdot\overline{\sigma}_-(t^{-1})h_{i_r}. \tag{5.4}$$

Clearly formulas (5.1) and (5.2) do not depend on r when r is big enough (that is, at least the length of the partition $\boldsymbol{\lambda}$). Thus these formulas hold for $r = \infty$ as well. We set

$$E_\infty(t) = 1 - e_1 z + e_2 t^2 + \cdots \quad\text{and}\quad H_\infty(t) = 1/E_\infty(t).$$

Define now

$$\exp\left(\sum_{i\geq 1} x_i t^i\right) := \sum_{n\geq 0} h_n t^n.$$

Then we have

$$B = \mathbb{Q}[e_1, e_2, \ldots] = \mathbb{Q}[h_1, h_2, \ldots] = \mathbb{Q}[x_1, x_2, \ldots],$$

see, for example, [**13**, I 3]. Moreover,

$$\frac{\partial h_n}{\partial x_i} = h_{n-i} \quad\text{and}\quad \frac{\partial^j h_n}{\partial x_1^j} = \frac{\partial h_n}{\partial x_j}. \tag{5.5}$$

As it follows from (5.3) and (5.4), $\overline{\sigma}_-(t^{-1})$ and $\sigma_-(t^{-1})$, for $r=\infty$, become ring homomorphisms $B\to B[t^{-1}]$. Thus

$$\begin{aligned}
\overline{\sigma}_-(t^{-1})h_n &= h_n-\frac{h_{n-1}}{t} && \text{(first formula in (5.1))}\\
&= \left(1-\frac{1}{t}\frac{\partial}{\partial x_1}\right)h_n\\
&= \exp\left(-\sum_{i\geq 1}\frac{1}{it}\frac{\partial^i}{\partial x_1^i}\right)h_n && \text{(definition of the logarithm}\\
& && \text{of a formal power series)}\\
&= \exp\left(-\sum_{i\geq 1}\frac{1}{it}\frac{\partial}{\partial x_i}\right)h_n && \text{(second equality in (5.5)).}
\end{aligned}$$

Notice that $\exp\left(-\sum_{i\geq 1}\frac{1}{it}\frac{\partial}{\partial x_i}\right)$, being the exponent of a first order differential operator, is a ring homomorphism whose value at h_n coincides with $\overline{\sigma}_-(t^{-1})h_n$. This means that

$$\overline{\sigma}_-(t^{-1})=\exp\left(-\sum_{i\geq 1}\frac{1}{it}\frac{\partial}{\partial x_i}\right).$$

Similarly one shows that

$$\sigma_-(t^{-1})=\exp\left(\sum_{i\geq 1}\frac{1}{it}\frac{\partial}{\partial x_i}\right)$$

Thus

$$\Gamma_\infty(t)=\frac{1}{E_\infty(t)}\overline{\sigma}_-(t)=\exp\left(\sum_{i\geq 1}x_it^i\right)\exp\left(-\sum_{i\geq 1}\frac{1}{it}\frac{\partial}{\partial x_i}\right)=\Gamma(t)$$

and

$$\Gamma^*_\infty(t)=E_\infty(t)\sigma_-(t^{-1})=\exp\left(-\sum_{i\geq 1}x_it^i\right)\exp\left(\sum_{i\geq 1}\frac{1}{it}\frac{\partial}{\partial x_i}\right)=\Gamma^*(t)$$

as claimed. ∎

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