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# ON NUMBERS DIVISIBLE BY THE PRODUCT OF THEIR NONZERO BASE $b$ DIGITS

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ABSTRACT. For each integer  $b \geq 3$  and every  $x \geq 1$ , let  $\mathcal{N}_{b,0}(x)$  be the set of positive integers  $n \leq x$  which are divisible by the product of their nonzero base  $b$  digits. We prove bounds of the form  $x^{\rho_{b,0} + o(1)} < \#\mathcal{N}_{b,0}(x) < x^{\eta_{b,0} + o(1)}$ , as  $x \rightarrow +\infty$ , where  $\rho_{b,0}$  and  $\eta_{b,0}$  are constants in  $]0, 1[$  depending only on  $b$ . In particular, we show that  $x^{0.526} < \#\mathcal{N}_{10,0}(x) < x^{0.787}$ , for all sufficiently large  $x$ . This improves the bounds  $x^{0.495} < \#\mathcal{N}_{10,0}(x) < x^{0.901}$ , which were proved by De Koninck and Luca.

## 1. INTRODUCTION

Let  $b \geq 2$  be an integer. Then, every positive integer  $n$  has a unique representation as

$$n = \sum_{j=0}^{\ell} d_j b^j, \quad d_0, \dots, d_{\ell} \in \{0, \dots, b-1\}, \quad d_{\ell} \neq 0,$$

where  $d_0, \dots, d_{\ell}$  are the *base  $b$  digits* of  $n$ . Positive integers whose base  $b$  digits obey certain restrictions have been investigated by several authors. For instance, an asymptotic formula for the counting function of  *$b$ -Niven numbers*, that is, positive integers divisible by the sum of their base  $b$  digits, has been proved by De Koninck, Doyon, and Kátai [4], and (independently) by Mauduit, Pomerance, and Sárközy [9]. Also, arithmetic properties of integers with a fixed sum of their base  $b$  digits have been studied by Luca [8], Mauduit and Sárközy [10]. Moreover, prime numbers with specific restrictions on their base  $b$  digits have been investigated by Bourgain [1, 2] and Maynard [11, 12] (see [3, 7] for similar works on almost primes and squarefree numbers).

Let  $p_b(n)$  be the product of the base  $b$  digits of  $n$ , and let  $p_{b,0}(n)$  be the product of the nonzero base  $b$  digits of  $n$ . For all  $x \geq 1$ , define the sets

$$\mathcal{N}_b(x) := \{n \leq x : p_b(n) \mid n\} \quad \text{and} \quad \mathcal{N}_{b,0}(x) := \{n \leq x : p_{b,0}(n) \mid n\}.$$

Note that  $\mathcal{N}_b(x) \subseteq \mathcal{N}_{b,0}(x)$  and that  $n \in \mathcal{N}_b(x)$  implies that all the base  $b$  digits of  $n$  are nonzero. Furthermore,  $\mathcal{N}_2(x) = \{2^k - 1 : k \geq 1\}$  and  $\mathcal{N}_{2,0}(x) = \mathbb{N}$ . Hence, in what follows, we will focus only on the case  $b \geq 3$ .

De Koninck and Luca [5] (see also [6] for the correction of a numerical error in [5]) studied  $\mathcal{N}_{10}(x)$  and  $\mathcal{N}_{10,0}(x)$ . They proved the following bounds.

**Theorem 1.1.** *We have*

$$x^{0.122} < \#\mathcal{N}_{10}(x) < x^{0.863}$$

and

$$x^{0.495} < \#\mathcal{N}_{10,0}(x) < x^{0.901}$$

for all sufficiently large  $x$ .

In this paper, we prove some bounds for the cardinalities of  $\mathcal{N}_b(x)$  and  $\mathcal{N}_{b,0}(x)$ . In particular, for  $b = 10$ , we get the following improvement of three of the bounds of Theorem 1.1.

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**Theorem 1.2.** *We have*

$$\#\mathcal{N}_{10}(x) < x^{0.717}$$

and

$$x^{0.526} < \#\mathcal{N}_{10,0}(x) < x^{0.787}$$

for all sufficiently large  $x$ .

**Notation.** We use the Landau–Bachmann “little oh” notation  $o$ , as well as the Vinogradov symbol  $\ll$ . We omit the dependence on  $b$  of the implied constants. We write  $P(n)$  for the greatest prime factor of an integer  $n > 1$ . As usual,  $\pi(x)$  denotes the number of prime numbers not exceeding  $x$ . We write  $\nu_p$  for the  $p$ -adic valuation.

## 2. UPPER BOUNDS

For every  $s \geq 0$ , let us define

$$\zeta_b(s) := \sum_{d=1}^{b-1} \frac{1}{d^s}.$$

We give the following upper bounds for  $\#\mathcal{N}_{b,0}(x)$  and  $\#\mathcal{N}_b(x)$ .

**Theorem 2.1.** *Let  $b \geq 3$  be an integer. We have*

$$\#\mathcal{N}_{b,0}(x) < x^{\eta_{b,0}+o(1)},$$

as  $x \rightarrow +\infty$ , where

$$\eta_{b,0} := 1 + \frac{1}{(1+s_{b,0}) \log b} \log \left( \frac{1 + \zeta_b(s_{b,0})}{b} \right) \in ]0, 1[$$

and  $s_{b,0}$  is the unique solution of the equation

$$(1) \quad \frac{(1+s)\zeta'_b(s)}{1+\zeta_b(s)} - \log \left( \frac{1+\zeta_b(s)}{b} \right) = 0$$

over the positive real numbers.

**Theorem 2.2.** *Let  $b \geq 3$  be an integer. We have*

$$\#\mathcal{N}_b(x) < x^{\eta_b+o(1)},$$

as  $x \rightarrow +\infty$ , where  $\eta_3 := \log 2 / \log 3$ ,

$$\eta_b := 1 + \frac{1}{(1+s_b) \log b} \log \left( \frac{\zeta_b(s_b)}{b} \right), \quad b \geq 4,$$

and  $s_b$  is the unique solution of the equation

$$(2) \quad \frac{(1+s)\zeta'_b(s)}{\zeta_b(s)} - \log \left( \frac{\zeta_b(s)}{b} \right) = 0$$

over the positive real numbers.

We remark that for  $b = 3$  the bound of Theorem 2.2 is obvious. Indeed, it is an easy consequence of the fact that all the base 3 digits of each  $n \in \mathcal{N}_3(x)$  are equal to 1 or 2. We included it just for completeness.

Using the PARI/GP [13] computer algebra system, the author computed  $s_{10,0} = 1.286985\dots$  and  $s_{10} = 1.392189\dots$ , which in turn give  $\eta_{10,0} = 0.7869364\dots$  and  $\eta_{10} = 0.7167170\dots$ . Hence, the upper bounds of Theorem 1.2 follow.

**Proof of Theorem 2.1.** First, we shall prove that Equation (1) has a unique positive solution. For  $s \geq 0$ , let

$$F_b(s) := \frac{(1+s)\zeta'_b(s)}{1+\zeta_b(s)} - \log\left(\frac{1+\zeta_b(s)}{b}\right).$$

Since  $b \geq 3$ , we have

$$(3) \quad F_b(0) = -\frac{\log((b-1)!)}{b} < 0 \quad \text{and} \quad \lim_{s \rightarrow +\infty} F_b(s) = \log\left(\frac{b}{2}\right) > 0.$$

Furthermore, a bit of computation shows that

$$(4) \quad F'_b(s) = \frac{(1+s)((1+\zeta_b(s))\zeta''_b(s) - (\zeta'_b(s))^2)}{(1+\zeta_b(s))^2} > 0,$$

for all  $s \geq 0$ , since, by Cauchy–Schwarz inequality, we have

$$(5) \quad (\zeta'_b(s))^2 = \left(-\sum_{d=1}^{b-1} (\log d)d^{-s}\right)^2 < \left(\sum_{d=1}^{b-1} d^{-s}\right) \left(\sum_{d=1}^{b-1} (\log d)^2 d^{-s}\right) = \zeta_b(s)\zeta''_b(s).$$

At this point, by (3) and (4), it follows that Equation (1) has a unique positive solution.

Let us assume  $x \geq 1$  sufficiently large, and let  $\alpha \in ]0, 1[$  be a constant (depending on  $b$ ) to be determined later. Also, let  $P_b$  be the greatest prime number less than  $b$ , and define the set

$$\mathcal{N}'_b(x) := \{n \leq x : d \mid n \text{ for some } d > x^\alpha \text{ with } P(d) \leq P_b\}.$$

Suppose  $n \in \mathcal{N}'_b(x)$ . Then there exists  $d > x^\alpha$  with  $P(d) \leq P_b$  such that  $d \mid n$ . Clearly, for any fixed  $d$ , there are at most  $x/d$  possible values for  $n$ . Moreover, setting

$$\mathcal{S}(t) := \{d \leq t : P(d) \leq P_b\},$$

it follows easily that  $\#\mathcal{S}(t) \ll (\log t)^{\pi(P_b)}$  for all  $t > 2$ . Therefore, we have

$$\#\mathcal{N}'_b(x) \leq \sum_{x^\alpha < d \leq x} \frac{x}{d} = x \left( \frac{\#\mathcal{S}(t)}{t} \Big|_{t=x^\alpha}^x + \int_{t=x^\alpha}^x \frac{\#\mathcal{S}(t)}{t^2} dt \right) \ll (\log x)^{\pi(P_b)} (1 + x^{1-\alpha}),$$

and consequently

$$(6) \quad \#\mathcal{N}'_b(x) < x^{1-\alpha+o(1)},$$

as  $x \rightarrow +\infty$ .

Now suppose  $n \in \mathcal{N}''_{b,0}(x) := \mathcal{N}_{b,0}(x) \setminus \mathcal{N}'_b(x)$ . Put  $N := \lfloor \log x / \log b \rfloor + 1$ , so that  $n$  has at most  $N$  base  $b$  digits. For each  $d \in \{1, \dots, b-1\}$ , let  $N_d$  be the number of base  $b$  digits of  $n$  which are equal to  $d$ . Also, let  $N_0 := N - (N_1 + \dots + N_{b-1})$ . Hence,  $N_0, \dots, N_{b-1}$  are nonnegative integers such that  $N_0 + \dots + N_{b-1} = N$ . Furthermore,

$$p_{b,0}(n) = 1^{N_1} \dots (b-1)^{N_{b-1}} \leq x^\alpha < b^{\alpha N}.$$

Let  $\beta > 0$  be a constant (depending on  $b$ ) to be determined later. For fixed  $N_0, \dots, N_{b-1}$ , by elementary combinatorics, the number of possible values for  $n$  is at most

$$\frac{N!}{N_0! \dots N_{b-1}!}.$$

Hence, summing over all possible values for  $N_0, \dots, N_{b-1}$ , we get

$$\begin{aligned} \#\mathcal{N}''_{b,0}(x) &\leq \sum_{\substack{N_0 + \dots + N_{b-1} = N \\ 1^{N_1} \dots (b-1)^{N_{b-1}} \leq b^{\alpha N}}} \frac{N!}{N_0! \dots N_{b-1}!} \\ &\leq \sum_{N_0 + \dots + N_{b-1} = N} \frac{N!}{N_0! \dots N_{b-1}!} \left( \frac{b^{\alpha N}}{1^{N_1} \dots (b-1)^{N_{b-1}}} \right)^\beta \\ &= \left( b^{\alpha \beta} (1 + \zeta_b(\beta)) \right)^N, \end{aligned}$$

where we employed the multinomial theorem. Therefore, since  $N \leq \log x / \log b + 1$ , we have

$$(7) \quad \#\mathcal{N}_{b,0}''(x) < x^{\gamma+o(1)},$$

as  $x \rightarrow +\infty$ , where

$$(8) \quad \gamma := \alpha\beta + \frac{\log(1 + \zeta_b(\beta))}{\log b}.$$

At this point, in light of (6) and (7), we shall choose  $\alpha$  and  $\beta$  so that  $\max\{1 - \alpha, \gamma\}$  is minimal. It is easy to see that this requires  $1 - \alpha = \gamma$ , which in turn gives

$$\alpha = -\frac{1}{(1 + \beta)\log b} \log\left(\frac{1 + \zeta_b(\beta)}{b}\right).$$

Note that this choice indeed satisfies  $\alpha \in ]0, 1[$ , as required in our previous arguments. Hence, we have to choose  $\beta$  in order to minimize

$$\gamma = 1 + \frac{1}{(1 + \beta)\log b} \log\left(\frac{1 + \zeta_b(\beta)}{b}\right).$$

Since

$$\frac{\partial\gamma}{\partial\beta} = \frac{F_b(\beta)}{(1 + \beta)^2 \log b},$$

by the previous considerations on  $F_b(s)$ , we get that  $\gamma$  is minimal for  $\beta = s_{b,0}$ . Thus, we make this choice for  $\beta$ , so that  $1 - \alpha = \gamma = \eta_{b,0}$ . Finally, putting together (6) and (7), we obtain

$$\#\mathcal{N}_{b,0}(x) < x^{1-\alpha+o(1)} + x^{\gamma+o(1)} < x^{\eta_{b,0}+o(1)}$$

as  $x \rightarrow +\infty$ . The proof is complete.

**Proof of Theorem 2.2.** The proof of Theorem 2.2 proceeds similarly to the one of Theorem 2.1. We highlight just the main differences. First, we shall prove that, for  $b \geq 4$ , Equation (2) has a unique positive solution. For  $s \geq 0$ , define

$$G_b(s) := \frac{(1 + s)\zeta_b'(s)}{\zeta_b(s)} - \log\left(\frac{\zeta_b(s)}{b}\right).$$

Since  $b \geq 4$ , we have

$$(9) \quad G_b(0) = -\log\left(\left(1 - \frac{1}{b}\right)(b-1)!^{1/(b-1)}\right) < 0 \quad \text{and} \quad \lim_{s \rightarrow +\infty} G_b(s) = \log b > 0.$$

Furthermore, a bit of computation shows that

$$(10) \quad G_b'(s) = \frac{(1 + s)(\zeta_b(s)\zeta_b''(s) - (\zeta_b'(s))^2)}{(\zeta_b(s))^2} > 0,$$

for all  $s \geq 0$ , since (5). Therefore, by (9) and (10), Equation (2) has a unique positive solution. Note also that  $G_3(0) > 0$ , so that  $G_3(s) > 0$  for all  $s \geq 0$ .

Let  $\alpha \in ]0, 1[$  be a constant (depending on  $b$ ) to be determined later, and define  $\mathcal{N}_b'(x)$  as in the proof of Theorem 2.1. Hence, by the previous arguments, the bound (6) holds.

Suppose  $n \in \mathcal{N}_b''(x) := \mathcal{N}_b(x) \setminus \mathcal{N}_b'(x)$ . This time, put  $N := \lfloor \log n / \log b \rfloor + 1$  (instead of  $N := \lfloor \log x / \log b \rfloor + 1$ ), so that  $n$  has exactly  $N$  base  $b$  digits. For each  $d \in \{1, \dots, b-1\}$ , let  $N_d$  be the number of base  $b$  digits of  $n$  which are equal to  $d$ . Note that, since  $p_b(n) \mid n$ , we have  $p_b(n) \neq 0$ , that is, all the base  $b$  digits of  $n$  are nonzero. Hence,  $N_1, \dots, N_{b-1}$  are nonnegative integers such that  $N_1 + \dots + N_{b-1} = N$ . Furthermore,

$$p_b(n) = 1^{N_1} \dots (b-1)^{N_{b-1}} \leq x^\alpha < b^{\alpha N}.$$

Let  $\beta > 0$  be a constant (depending on  $b$ ) to be determined later. Summing over all possible values for  $N_1, \dots, N_{b-1}$  and  $N$ , we get

$$\begin{aligned} \#\mathcal{N}_b''(x) &\leq \sum_{N=1}^{\lfloor \log x / \log b \rfloor + 1} \sum_{\substack{N_1 + \dots + N_{b-1} = N \\ 1^{N_1} \dots (b-1)^{N_{b-1}} \leq b^{\alpha N}}} \frac{N!}{N_1! \dots N_{b-1}!} \\ &\leq \sum_{N=1}^{\lfloor \log x / \log b \rfloor + 1} \sum_{N_0 + \dots + N_{b-1} = N} \frac{N!}{N_1! \dots N_{b-1}!} \left( \frac{b^{\alpha N}}{1^{N_1} \dots (b-1)^{N_{b-1}}} \right)^\beta \\ &= \sum_{N=1}^{\lfloor \log x / \log b \rfloor + 1} (b^{\alpha\beta} \zeta_b(\beta))^N \ll (b^{\alpha\beta} \zeta_b(\beta))^{\log x / \log b}, \end{aligned}$$

and consequently

$$(11) \quad \#\mathcal{N}_b''(x) < x^{\delta + o(1)},$$

as  $x \rightarrow +\infty$ , where

$$(12) \quad \delta := \alpha\beta + \frac{\log \zeta_b(\beta)}{\log b}.$$

At this point, in light of (6) and (11), we shall choose  $\alpha$  and  $\beta$  so that  $\max\{1 - \alpha, \delta\}$  is minimal. This requires  $1 - \alpha = \delta$ , which in turn yields

$$\alpha = -\frac{1}{(1 + \beta) \log b} \log \left( \frac{\zeta_b(\beta)}{b} \right).$$

Note that this choice indeed satisfies  $\alpha \in ]0, 1[$ , as required in our previous arguments. Hence, we have to minimize

$$\delta = 1 + \frac{1}{(1 + \beta) \log b} \log \left( \frac{\zeta_b(\beta)}{b} \right).$$

We have

$$\frac{\partial \delta}{\partial \beta} = \frac{G_b(\beta)}{(1 + \beta)^2 \log b}.$$

Hence, by the previous considerations on  $G_b(s)$ , for  $b = 3$  we have to choose  $\beta = 0$ , while if  $b \geq 4$  we have to choose  $\beta = s_b$ . Making this choice, we get  $1 - \alpha = \delta = \eta_b$ . Finally, putting together (6) and (11), we obtain

$$\#\mathcal{N}_b(x) < x^{1 - \alpha + o(1)} + x^{\delta + o(1)} < x^{\eta_b + o(1)}$$

as  $x \rightarrow +\infty$ . The proof is complete.

### 3. LOWER BOUND

**Theorem 3.1.** *Let  $b \geq 3$  be an integer. We have*

$$(13) \quad \#\mathcal{N}_{b,0}(x) > x^{\rho_{b,0} + o(1)},$$

as  $x \rightarrow +\infty$ , where

$$(14) \quad \rho_{b,0} := \sup_{\alpha_0, \dots, \alpha_{b-1}} \frac{\left( \sum_{d=1}^{b-1} \alpha_d \right) \log \left( \sum_{d=1}^{b-1} \alpha_d \right) - \sum_{d=1}^{b-1} \alpha_d \log \alpha_d}{\left( 1 + \sum_{d=1}^{b-1} \alpha_d \right) \log b}$$

with  $\alpha_0, \dots, \alpha_{b-1} \geq 0$  satisfying the conditions

$$(15) \quad \begin{cases} \alpha_d = 0 & \text{if } d > 1 \text{ and } p \mid d, p \nmid b \text{ for some prime } p, \\ \sum_{d=2}^{b-1} \alpha_d \nu_p(d) \leq 1 & \text{for all primes } p \mid b, \end{cases}$$

and with the convention  $0 \cdot \log 0 := 0$ .

We remark that if  $b$  is a prime number then the bound of Theorem 3.1 is obvious. Indeed, the primality of  $b$  implies  $\alpha_d = 0$  for each  $d \in \{2, \dots, b-1\}$ , so that

$$\rho_{b,0} = \sup_{\alpha_0, \alpha_1 \geq 0} \frac{(\alpha_0 + \alpha_1) \log(\alpha_0 + \alpha_1) - \alpha_0 \log \alpha_0 - \alpha_1 \log \alpha_1}{(1 + \alpha_0 + \alpha_1) \log b} = \frac{\log 2}{\log b},$$

and the bound is

$$(16) \quad \#\mathcal{N}_{b,0}(x) > x^{\log 2 / \log b + o(1)},$$

as  $x \rightarrow +\infty$ . However, the bound (16) follows just by considering that  $\mathcal{N}_{b,0}(x)$  contains all positive integers having their base  $b$  digits in  $\{0, 1\}$ .

If  $b$  is not a prime number, then Theorem 3.1 gives a better bound than (16). In particular, for  $b = 10$ , conditions (15) become

$$(17) \quad \begin{cases} \alpha_3 = \alpha_6 = \alpha_7 = \alpha_9 = 0, \\ \alpha_2 + 2\alpha_4 + 3\alpha_8 \leq 1, \\ \alpha_5 \leq 1, \end{cases}$$

and the right-hand side of (14) can be maximized under the constrains given by (17) using the method of Lagrange multipliers. This gives  $\rho_{10,0} > 0.526$ , for the choice

$$\alpha_0 = \alpha_1 = 1.331, \quad \alpha_2 = 0.476, \quad \alpha_4 = 0.170, \quad \alpha_5 = 1, \quad \alpha_8 = 0.060.$$

Hence, the lower bound for  $\#\mathcal{N}_{10,0}(x)$  of Theorem 1.2 follows.

**3.1. Proof of Theorem 3.1.** Let us assume  $x \geq 1$  sufficiently large, and let  $\alpha_0, \dots, \alpha_{b-1} \geq 0$  be constants (depending on  $b$ ) to be determined later. Define

$$s := \left\lfloor \frac{\log x}{(1 + \alpha_0 + \dots + \alpha_{b-1}) \log b} \right\rfloor.$$

Also, let  $N_d := \lfloor \alpha_d s \rfloor$  for each  $d \in \{0, \dots, b-1\}$ , and put  $N := N_0 + \dots + N_{b-1}$ .

Now suppose  $m$  is a positive integer with at most  $N$  base  $b$  digits, and such that exactly  $N_d$  of its base  $b$  digits are equal to  $d$ , for each  $d \in \{1, \dots, b-1\}$ . Moreover, put  $n := b^s m$ . Clearly,  $n \leq b^{s+N} \leq x$  and  $b^s \mid n$ . Then, imposing the conditions (15), we get that

$$p_{b,0}(n) = 1^{N_1} \dots (b-1)^{N_{b-1}} \mid b^s \mid n,$$

so that  $n \in \mathcal{N}_{b,0}(x)$ . By elementary combinatorics and by using Stirling's formula, the number of possible values for  $m$  is

$$\begin{aligned} \frac{N!}{N_0! \dots N_{b-1}!} &= \frac{(\lfloor \alpha_0 s \rfloor + \dots + \lfloor \alpha_{b-1} s \rfloor)!}{\lfloor \alpha_0 s \rfloor! \dots \lfloor \alpha_{b-1} s \rfloor!} \\ &= \exp \left( s \left( \left( \sum_{d=1}^{b-1} \alpha_d \right) \log \left( \sum_{d=1}^{b-1} \alpha_d \right) - \sum_{d=1}^{b-1} \alpha_d \log \alpha_d + o(1) \right) \right), \end{aligned}$$

as  $s \rightarrow +\infty$ . Hence, lower bound (13) follows. The proof is complete.

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