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(Article begins on next page)
ON NUMBERS DIVISIBLE BY THE PRODUCT OF THEIR NONZERO BASE $b$ DIGITS

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Abstract. For each integer $b \geq 3$ and every $x \geq 1$, let $\mathcal{N}_{b,0}(x)$ be the set of positive integers $n \leq x$ which are divisible by the product of their nonzero base $b$ digits. We prove bounds of the form $x^{\rho_{b,0}+o(1)} < \#\mathcal{N}_{b,0}(x) < x^{\rho_{b,0}+o(1)}$, as $x \to +\infty$, where $\rho_{b,0}$ and $\rho_{b,0}$ are constants in $[0,1]$ depending only on $b$. In particular, we show that $x^{0.520} < \#\mathcal{N}_{10,0}(x) < x^{0.787}$, for all sufficiently large $x$. This improves the bounds $x^{0.495} < \#\mathcal{N}_{10,0}(x) < x^{0.901}$, which were proved by De Koninck and Luca.

1. Introduction

Let $b \geq 2$ be an integer. Then, every positive integer $n$ has a unique representation as

$$n = \sum_{j=0}^{\ell} d_j b^j, \quad d_0, \ldots, d_\ell \in \{0, \ldots, b-1\}, \quad d_\ell \neq 0,$$

where $d_0, \ldots, d_\ell$ are the base $b$ digits of $n$. Positive integers whose base $b$ digits obey certain restrictions have been investigated by several authors. For instance, an asymptotic formula for the counting function of $b$-Niven numbers, that is, positive integers divisible by the sum of their base $b$ digits, has been proved by De Koninck, Doyon, and Kátai [4], and (independently) by Mauduit, Pomerance, and Sárközy [9]. Also, arithmetic properties of integers with a fixed sum of their base $b$ digits have been studied by Luca [8], Mauduit and Sárközy [10]. Moreover, prime numbers with specific restrictions on their base $b$ digits have been investigated by Bourgain [1, 2] and Maynard [11, 12] (see [3, 7] for similar works on almost primes and squarefree numbers).

Let $p_b(n)$ be the product of the base $b$ digits of $n$, and let $p_{b,0}(n)$ be the product of the nonzero base $b$ digits of $n$. For all $x \geq 1$, define the sets

$$\mathcal{N}_b(x) := \{n \leq x : p_b(n) \mid n\} \quad \text{and} \quad \mathcal{N}_{b,0}(x) := \{n \leq x : p_{b,0}(n) \mid n\}.$$

Note that $\mathcal{N}_b(x) \subseteq \mathcal{N}_{b,0}(x)$ and that $n \in \mathcal{N}_b(x)$ implies that all the base $b$ digits of $n$ are nonzero. Furthermore, $\mathcal{N}_2(x) = \{2^k - 1 : k \geq 1\}$ and $\mathcal{N}_{2,0}(x) = \mathbb{N}$. Hence, in what follows, we will focus only on the case $b \geq 3$.

De Koninck and Luca [5] (see also [6] for the correction of a numerical error in [5]) studied $\mathcal{N}_{10}(x)$ and $\mathcal{N}_{10,0}(x)$. They proved the following bounds.

Theorem 1.1. We have

$$x^{0.122} < \#\mathcal{N}_{10}(x) < x^{0.863}$$

and

$$x^{0.495} < \#\mathcal{N}_{10,0}(x) < x^{0.901}$$

for all sufficiently large $x$.

In this paper, we prove some bounds for the cardinalities of $\mathcal{N}_b(x)$ and $\mathcal{N}_{b,0}(x)$. In particular, for $b = 10$, we get the following improvement of three of the bounds of Theorem 1.1.

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Theorem 1.2. We have
\[ \#N_{10}(x) < x^{0.717} \]
and
\[ x^{0.526} < \#N_{10,0}(x) < x^{0.787} \]
for all sufficiently large \( x \).

Notation. We use the Landau–Bachmann “little oh” notation \( o \), as well as the Vinogradov symbol \( \ll \). We omit the dependence on \( b \) of the implied constants. We write \( P(n) \) for the greatest prime factor of an integer \( n > 1 \). As usual, \( \pi(x) \) denotes the number of prime numbers not exceeding \( x \). We write \( \nu_p \) for the \( p \)-adic valuation.

2. Upper bounds

For every \( s \geq 0 \), let us define
\[ \zeta_b(s) := \sum_{d=1}^{b-1} \frac{1}{d^s}. \]
We give the following upper bounds for \( \#N_{b,0}(x) \) and \( \#N_b(x) \).

Theorem 2.1. Let \( b \geq 3 \) be an integer. We have
\[ \#N_{b,0}(x) < x^{\eta_{b,0} + o(1)}, \]
as \( x \to +\infty \), where
\[ \eta_{b,0} := 1 + \frac{1}{(1 + s_{b,0}) \log b} \log \left( \frac{1 + \zeta_b(s_{b,0})}{b} \right) \in ]0, 1[ \]
and \( s_{b,0} \) is the unique solution of the equation
\[ (1 + s)\zeta'_b(s) \frac{1}{1 + \zeta_b(s)} - \log \left( \frac{1 + \zeta_b(s)}{b} \right) = 0 \]
over the positive real numbers.

Theorem 2.2. Let \( b \geq 3 \) be an integer. We have
\[ \#N_b(x) < x^{\eta_b + o(1)}, \]
as \( x \to +\infty \), where \( \eta_3 := \log 2 / \log 3 \),
\[ \eta_b := 1 + \frac{1}{(1 + s_b) \log b} \log \left( \frac{\zeta_b(s_b)}{b} \right), \quad b \geq 4, \]
and \( s_b \) is the unique solution of the equation
\[ (1 + s)\zeta'_b(s) \frac{1}{\zeta_b(s)} - \log \left( \frac{\zeta_b(s)}{b} \right) = 0 \]
over the positive real numbers.

We remark that for \( b = 3 \) the bound of Theorem 2.2 is obvious. Indeed, it is an easy consequence of the fact that all the base 3 digits of each \( n \in \mathcal{N}_3(x) \) are equal to 1 or 2. We included it just for completeness.

Using the PARI/GP [13] computer algebra system, the author computed \( s_{10,0} = 1.286985 \ldots \) and \( s_{10} = 1.392189 \ldots \), which in turn give \( \eta_{10,0} = 0.7869364 \ldots \) and \( \eta_{10} = 0.7167170 \ldots \). Hence, the upper bounds of Theorem 1.2 follow.
Proof of Theorem 2.1. First, we shall prove that Equation (1) has a unique positive solution.
For $s \geq 0$, let
\[ F_b(s) := \frac{(1 + s)\zeta_b'(s)}{1 + \zeta_b(s)} - \log \left( \frac{1 + \zeta_b(s)}{b} \right). \]
Since $b \geq 3$, we have
\[ F_b(0) = -\frac{\log((b - 1)!)}{b} < 0 \quad \text{and} \quad \lim_{s \to +\infty} F_b(s) = \log \left( \frac{b}{2} \right) > 0. \]
Furthermore, a bit of computation shows that
\[ F_b'(s) = \frac{(1 + s) \left( (1 + \zeta_b(s)) \zeta_b''(s) - (\zeta_b'(s))^2 \right)}{(1 + \zeta_b(s))^2} > 0, \]
for all $s \geq 0$, since, by Cauchy–Schwarz inequality, we have
\[ (\zeta_b'(s))^2 = \left( -\sum_{d=1}^{b-1} (\log d)d^{-s} \right)^2 < \left( \sum_{d=1}^{b-1} d^{-s} \right) \left( \sum_{d=1}^{b-1} (\log d)^2 d^{-s} \right) = \zeta_b(s)\zeta_b''(s). \]
At this point, by (3) and (4), it follows that Equation (1) has a unique positive solution.
Let us assume $x \geq 1$ sufficiently large, and let $\alpha \in [0, 1]$ be a constant (depending on $b$) to be determined later. Also, let $P_b$ be the greatest prime number less than $b$, and define the set
\[ N_b'(x) := \{ n \leq x : d \mid n \text{ for some } d > x^\alpha \text{ with } P(d) \leq P_b \}. \]
Suppose $n \in N_b'(x)$. Then there exists $d > x^\alpha$ with $P(d) \leq P_b$ such that $d \mid n$. Clearly, for any fixed $d$, there are at most $x/d$ possible values for $n$. Moreover, setting
\[ S(t) := \{ d \leq t : P(d) \leq P_b \}, \]
it follows easily that $\#S(t) \ll (\log t)^{\pi(P_b)}$ for all $t > 2$. Therefore, we have
\[ \#N_b'(x) \leq \sum_{x^\alpha < d \leq x} \frac{x}{d} = x \left( \#S(t) \int_{t=x^\alpha}^{x} \frac{\#S(t)}{t^2} dt \right) \ll (\log x)^{\pi(P_b)} (1 + x^{1-\alpha}), \]
and consequently
\[ \#N_b'(x) < x^{1-\alpha+o(1)}, \]
as $x \to +\infty$.
Now suppose $n \in N_b''(x) := N_{b,0}(x) \setminus N_b'(x)$. Put $N := \lfloor \log x / \log b \rfloor + 1$, so that $n$ has at most $N$ base $b$ digits. For each $d \in \{1, \ldots, b-1\}$, let $N_d$ be the number of base $b$ digits of $n$ which are equal to $d$. Also, let $N_0 := N - (N_1 + \cdots + N_{b-1})$. Hence, $N_0, \ldots, N_{b-1}$ are nonnegative integers such that $N_0 + \cdots + N_{b-1} = N$. Furthermore,
\[ p_{b,0}(n) = 1^{N_0} \cdots (b-1)^{N_{b-1}} \leq x^\alpha < b^{\alpha N}. \]
Let $\beta > 0$ be a constant (depending on $b$) to be determined later. For fixed $N_0, \ldots, N_{b-1}$, by elementary combinatorics, the number of possible values for $n$ is at most
\[ \frac{N!}{N_0! \cdots N_{b-1}!}. \]
Hence, summing over all possible values for $N_0, \ldots, N_{b-1}$, we get
\[ \#N_{b,0}(x) \leq \sum_{N_0 + \cdots + N_{b-1} = N} \frac{N!}{N_0! \cdots N_{b-1}!} \leq \sum_{N_0 + \cdots + N_{b-1} = N} \frac{N!}{N_0! \cdots N_{b-1}!} \left( \frac{b^{\alpha N}}{1^{N_1} \cdots (b-1)^{N_{b-1}}} \right)^\beta = \left( b^{\alpha \beta} (1 + \zeta_b(\beta)) \right)^N, \]
where we employed the multinomial theorem. Therefore, since \( N \leq \log x / \log b + 1 \), we have
\[
\#N^{(\alpha)}_b(x) < x^{\gamma + o(1)},
\]
as \( x \to +\infty \), where
\[
\gamma := \alpha \beta + \frac{\log(1 + \zeta_b(\beta))}{\log b}.
\]

Furthermore, a bit of computation shows that
\[
\zeta_b(\beta) = \frac{F_b(\beta)}{(1 + \beta)^2 \log b},
\]
by the previous considerations on \( F_b(s) \), we get that \( \gamma \) is minimal for \( \beta = s_{b,0} \). Thus, we make this choice for \( \beta \), so that \( 1 - \alpha = \gamma = \eta_{b,0} \). Finally, putting together (6) and (7), we obtain
\[
\#N_{b,0}(x) < x^{1-\alpha + o(1)} + x^{\gamma + o(1)} < x^{\eta_{b,0} + o(1)}
\]
as \( x \to +\infty \). The proof is complete.

**Proof of Theorem 2.2.** The proof of Theorem 2.2 proceeds similarly to the one of Theorem 2.1. We highlight just the main differences. First, we shall prove that, for \( b \geq 4 \), Equation (2) has a unique positive solution. For \( s \geq 0 \), define
\[
G_b(s) := \frac{(1 + s)\zeta_b'(s)}{\zeta_b(s)} - \log \left( \frac{\zeta_b(s)}{b} \right).
\]
Since \( b \geq 4 \), we have
\[
G_b(0) = - \log \left( \frac{1}{b} \left( 1 - \frac{1}{b} \right) (b - 1)!^{1/(b - 1)} \right) < 0 \quad \text{and} \quad \lim_{s \to +\infty} G_b(s) = \log b > 0.
\]
Furthermore, a bit of computation shows that
\[
G_b'(s) = \frac{(1 + s)(\zeta_b(s)\zeta_b''(s) - (\zeta_b'(s))^2)}{(\zeta_b(s))^2} > 0,
\]
for all \( s \geq 0 \), since (5). Therefore, by (9) and (10), Equation (2) has a unique positive solution. Note also that \( G_3(0) > 0 \), so that \( G_3(s) > 0 \) for all \( s \geq 0 \).

Let \( \alpha \in ]0, 1[ \) be a constant (depending on \( b \)) to be determined later, and define \( N_b'(x) \) as in the proof of Theorem 2.1. Hence, by the previous arguments, the bound (6) holds.

Suppose \( n \in N^{(\alpha)}_b(x) := N_b(x) \setminus N'_b(x) \). This time, put \( N := \lceil \log n / \log b \rceil + 1 \) (instead of \( N := \lceil \log x / \log b \rceil + 1 \)), so that \( n \) has exactly \( N \) base \( b \) digits. For each \( d \in \{1, \ldots, b - 1\} \), let \( N_d \) be the number of base \( b \) digits of \( n \) which are equal to \( d \). Note that, since \( p_b(n) \mid n \), we have \( p_b(n) \neq 0 \), that is, all the base \( b \) digits of \( n \) are nonzero. Hence, \( N_1, \ldots, N_{b-1} \) are nonnegative integers such that \( N_1 + \cdots + N_{b-1} = N \). Furthermore,
\[
p_b(n) = 1^{N_1} \cdots (b - 1)^{N_{b-1}} \leq x^{\alpha} < b^{\alpha N}.
\]
Let $\beta > 0$ be a constant (depending on $b$) to be determined later. Summing over all possible values for $N_1, \ldots, N_{b-1}$ and $N$, we get

$$\# N'_b(x) \leq \sum_{N = 1}^{\lfloor \log x / \log b \rfloor} \sum_{N_1 + \cdots + N_{b-1} = N} \frac{N!}{N_1! \cdots N_{b-1}!} \left( \frac{b^{\alpha N}}{1^{N_1} \cdots (b-1)^{N_{b-1}}} \right)^{\beta}$$

and consequently

$$\# N'_b(x) < x^{\delta + o(1)},$$

as $x \to +\infty$, where

$$\delta := \alpha \beta + \log \zeta_b(\beta) .$$

At this point, in light of (6) and (11), we shall choose $\alpha$ and $\beta$ so that $\max\{1 - \alpha, \delta\}$ is minimal. This requires $1 - \alpha = \delta$, which in turn yields

$$\alpha = -\frac{1}{(1 + \beta) \log b} \log \left( \frac{\zeta_b(\beta)}{b} \right).$$

Note that this choice indeed satisfies $\alpha \in ]0, 1[$, as required in our previous arguments. Hence, we have to minimize

$$\delta = 1 + \frac{1}{(1 + \beta) \log b} \log \left( \frac{\zeta_b(\beta)}{b} \right).$$

We have

$$\frac{\partial \delta}{\partial \beta} = \frac{G_b(\beta)}{(1 + \beta)^2 \log b}.$$

Hence, by the previous considerations on $G_b(s)$, for $b = 3$ we have to choose $\beta = 0$, while if $b \geq 4$ we have to choose $\beta = s_b$. Making this choice, we get $1 - \alpha = \delta = \eta_b$. Finally, putting together (6) and (11), we obtain

$$\# N'_b(x) < x^{1-o(1)} + x^{\delta + o(1)} < x^{\eta_b+o(1)}$$

as $x \to +\infty$. The proof is complete.

3. LOWER BOUND

**Theorem 3.1.** Let $b \geq 3$ be an integer. We have

$$\# N_{b,0}(x) > x^{\rho_{b,0}+o(1)},$$

as $x \to +\infty$, where

$$\rho_{b,0} := \sup_{\alpha_0, \ldots, \alpha_{b-1}} \left( \sum_{d=1}^{b-1} \alpha_d \log \left( \sum_{d=1}^{b-1} \alpha_d \right) - \sum_{d=1}^{b-1} \alpha_d \log \alpha_d \right) \left( 1 + \sum_{d=1}^{b-1} \alpha_d \right) \log b$$

with $\alpha_0, \ldots, \alpha_{b-1} \geq 0$ satisfying the conditions

$$\begin{cases}
\alpha_d = 0 & \text{if } d > 1 \text{ and } p \mid d, \ p \nmid b \text{ for some prime } p, \\
\sum_{d=1}^{b-1} \alpha_d \nu_p(d) \leq 1 & \text{for all primes } p \mid b,
\end{cases}$$

and with the convention $0 \cdot \log 0 := 0$. 

We remark that if \( b \) is a prime number then the bound of Theorem 3.1 is obvious. Indeed, the primality of \( b \) implies \( \alpha_d = 0 \) for each \( d \in \{2, \ldots, b-1\} \), so that

\[
\rho_{b,0} = \sup_{\alpha_0, \alpha_1 \geq 0} \frac{(\alpha_0 + \alpha_1) \log(\alpha_0 + \alpha_1) - \alpha_0 \log \alpha_0 - \alpha_1 \log \alpha_1}{(1 + \alpha_0 + \alpha_1) \log b} = \frac{\log 2}{\log b},
\]

and the bound is

\[
\#N_{b,0}(x) > x^{\log 2/\log b + o(1)},
\]
as \( x \to +\infty \). However, the bound (16) follows just by considering that \( N_{b,0}(x) \) contains all positive integers having their base \( b \) digits in \( \{0,1\} \).

If \( b \) is not a prime number, then Theorem 3.1 gives a better bound than (16). In particular, for \( b = 10 \), conditions (15) become

\[
\begin{align*}
\alpha_3 &= \alpha_6 = \alpha_7 = \alpha_9 = 0, \\
\alpha_2 + 2\alpha_4 + 3\alpha_8 &\leq 1, \\
\alpha_5 &\leq 1,
\end{align*}
\]

and the right-hand side of (14) can be maximized under the constrains given by (17) using the method of Lagrange multipliers. This gives \( \rho_{10,0} > 0.526 \), for the choice

\[
\alpha_0 = \alpha_1 = 1.331, \quad \alpha_2 = 0.476, \quad \alpha_4 = 0.170, \quad \alpha_5 = 1, \quad \alpha_8 = 0.060.
\]

Hence, the lower bound for \( \#N_{10,0}(x) \) of Theorem 1.2 follows.

3.1. **Proof of Theorem 3.1.** Let us assume \( x \geq 1 \) sufficiently large, and let \( \alpha_0, \ldots, \alpha_{b-1} \geq 0 \) be constants (depending on \( b \)) to be determined later. Define

\[
s := \left\lfloor \frac{\log x}{(1 + \alpha_0 + \cdots + \alpha_{b-1}) \log b} \right\rfloor.
\]

Also, let \( N_d := \lfloor \alpha_d s \rfloor \) for each \( d \in \{0, \ldots, b-1\} \), and put \( N := N_0 + \cdots + N_{b-1} \).

Now suppose \( m \) is a positive integer with at most \( N \) base \( b \) digits, and such that exactly \( N_d \) of its base \( b \) digits are equal to \( d \), for each \( d \in \{1, \ldots, b-1\} \). Moreover, put \( n := b^s m \). Clearly, \( n \leq b^s+N \leq x \) and \( b^s \mid n \). Then, imposing the conditions (15), we get that

\[
p_{b,0}(n) = 1^{N_1} \cdots (b-1)^{N_{b-1}} \mid b^s \mid n,
\]

so that \( n \in N_{b,0}(x) \). By elementary combinatorics and by using Stirling’s formula, the number of possible values for \( m \) is

\[
\frac{N!}{N_0! \cdots N_{b-1}!} = \frac{(\lfloor \alpha_0 s \rfloor + \cdots + \lfloor \alpha_{b-1} s \rfloor)!}{\lfloor \alpha_0 s \rfloor! \cdots \lfloor \alpha_{b-1} s \rfloor!} = \exp \left( s \left( \sum_{d=1}^{b-1} \alpha_d \log \left( \sum_{d=1}^{b-1} \alpha_d \right) - \sum_{d=1}^{b-1} \alpha_d \log \alpha_d + o(1) \right) \right),
\]
as \( s \to +\infty \). Hence, lower bound (13) follows. The proof is complete.

**References**


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