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## ERROR ESTIMATES FOR THE EULER DISCRETIZATION OF AN OPTIMAL CONTROL PROBLEM WITH FIRST-ORDER STATE CONSTRAINTS\*

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**Abstract.** We propose some error estimates for the discrete solution of an optimal control problem with first-order state constraints, where the trajectories are approximated with a classical Euler scheme. We obtain order 1 approximation results in the  $L^\infty$  norm (as opposed to the order 2/3 results obtained in the literature). We assume either a strong second-order optimality condition or a weaker formulation in the case where the state constraint is scalar and satisfies some hypotheses for junction points, and where the time step is constant. Our technique is based on some homotopy path of discrete optimal control problems that we study using perturbation analysis of nonlinear programming problems.

**Key words.** optimal control, nonlinear systems, state constraints, Euler discretization, rate of convergence

**AMS subject classifications.** 49M25, 65L10, 65L70, 65K10

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**1. Introduction and discussion of literature.** Numerical methods for the resolution of an optimal control problem are based on a finite-dimensional approximation, generally obtained through a discretization of the trajectory and a piecewise constant or polynomial control. Obtaining error estimates for such approximations is a key point in this issue.

Such problems began to be analyzed in the 1970s. They dealt with convergence of a discrete optimal control solution; see for example [17, 11, 12, 30]. Other results of convergence, provided using modern variational techniques, are also found in [34]; a survey of the results in this area is [13].

In this paper we focus on the case of *pure* state constraints, a case which presents some special difficulties. In particular, it is known that when the constraint qualification holds and the Lagrangian verifies a local condition of coercivity, the discrete problem obtained with an Euler scheme has a solution for a sufficiently fine mesh, and the corresponding Lagrange multipliers are at distance  $O(\bar{h})$  in the  $L^2$  norm, where  $\bar{h}$  is the maximal discretization step, from the continuous solution/multiplier. This important result is due to [16].

The second-order optimality conditions involve neighborhoods in the  $L^\infty$  norm but give only growth estimates in the  $L^2$  norm. This is the so-called “*two-norm discrepancy*” [26]. So, it is not surprising to get error estimates in the  $L^2$  norm. Yet it is useful to recover  $L^\infty$  estimates. For problems without state constraints, it has been shown that standard numerical techniques such as the Euler discretization [14, Thm. 6] or a discrete shooting formulation [27] provide an approximation of the

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solution of the problem where its  $L^\infty$  error can be estimated by a term of order  $\bar{h}$ . For state constrained problems, the situation is more involved. One possibility is to work in a nonlinear space of Lipschitz continuous functions with bounded Lipschitz constants. In this setting, the  $L^2$  convergence implies  $L^\infty$  convergence. This is the method proposed in [16]. Here the authors develop a Lipschitz stability result in  $L^2$  for a perturbation of the linearized Euler discretization, which leads to an optimal error estimate  $O(\bar{h})$  in  $L^2$ . Using a reverse Hölder inequality [15, Lem. 3.1] for Lipschitz functions, this implies an  $O(\bar{h}^{2/3})$  estimate in  $L^\infty$ . In the same work the authors observe that such an estimate is not optimal (cf. [16, sect. 10]).

In this paper, we obtain a tighter  $O(\bar{h})$  error estimate for the  $L^\infty$  norm, assuming either (i) a strong second-order optimality condition, similar to the one in [16] (but we allow a variable time step, whereas in [16] the time step was constant), or (ii) a weaker second-order optimality condition, in the case when the state constraint is scalar, structural hypotheses on arcs and junction points are assumed, and the time step is constant (the precise statement of these hypotheses is in section 2.5).

In this second case our hypotheses allow us to obtain the stability of the extremals (of the continuous problem) under a small perturbation; see [4]. We obtain a similar result for the discretized problem. By contrast, for a vector state constraint we are not aware of such stability results, even in the continuous case. This suggests that it may not be easy to obtain the stability of the extremals after discretization without a strong second-order optimality condition. This is an interesting open question that we leave for future work, as well as the case of higher order state constraints.

**1.1. Structure of the paper.** In section 2 we introduce the problem and the assumptions adopted in the paper, and we state our main result (Theorem 2.5): an  $O(\bar{h})$  uniform error estimate for the control, state, costate, and multiplier. Section 3 presents the homotopy path on which our analysis is based. For each parameter  $\theta \in [0, 1]$  of the path, we define a perturbation ( $\mathcal{P}^\theta$ ) of the discretized optimal control problem and we construct a path of solutions of the corresponding optimality system. For  $\theta = 0$ , ( $\mathcal{P}^\theta$ ) coincides with the discretized optimal control problem, and for  $\theta = 1$  a solution of the (perturbed) discrete optimality system is obtained from the solution of the continuous time problem. Through the study of the regularity of each problem ( $\mathcal{P}^\theta$ ) (section 4) and checking that, under appropriate hypotheses, the homotopy path has bounded derivatives (in a sense clarified in section 5), we can establish the announced convergence estimates for the discrete problem. More precisely, due to some coercivity properties of the Hessian of the Lagrangian, we first obtain a bound in the  $L^2$  norm from which respective estimates in the  $L^\infty$  norm follow. We use there the fact that the state constraint is of first order. Section 6 is dedicated to a simple numerical test. The numerical test is in accordance with our theoretical result and it confirms the tightness of the estimate. Appendices A–D contain some complementary results not essential for the comprehension of the core of the paper.

**1.2. Notations.** By  $\mathbb{R}^n$  we denote the  $n$ -dimensional Euclidean space. Its dual (whose elements are row vectors) is denoted by  $\mathbb{R}^{n*}$ . By  $\nabla$ ,  $\nabla_u$ , etc., we denote the gradient or partial gradient with respect to  $u$ , which are column vectors, in by contrast to the derivatives denoted by, for example,  $Dg(x)$  or  $g'(x)$  depending on the context, which are identified as row vectors if  $g$  is scalar valued. The Lagrange multipliers, including costate variables, are considered as dual elements and are represented by row vectors.

By  $C([0, T])$  we denote the space of real continuous functions over  $[0, T]$ , endowed with the supremum norm. It is known that its topological dual can be identified

with the space  $\mathcal{M}[0, T]$  of regular, finite Borel measures over  $[0, T]$ . Let  $BV([0, T])$  denote the space of bounded variation functions over  $[0, T]$ , and let  $BV_T([0, T])$  be the subspace of such functions with value 0 at time  $T$ . Any continuous linear form on  $C([0, T])$  is of the form  $f \mapsto \int_0^T f(t)d\mu(t)$  with  $\mu \in BV_T([0, T])$ . By  $W^{1,s}(0, T; \mathbb{R}^n)$  with  $s \in [1, \infty]$ , we denote the space of function in  $L^s(0, T; \mathbb{R}^n)$ , with weak derivatives in  $L^s(0, T; \mathbb{R}^n)$ . Moreover,  $H^1(0, T; \mathbb{R}^n) := W^{1,2}(0, T; \mathbb{R}^n)$ .

In the analysis we will define  $\bar{h}$  as the greatest time step of the discretized problem, and  $\theta \in [0, 1]$  as a homotopy parameter. When writing expressions such as  $O(1)$  or  $O(\bar{h})$ , we mean that the corresponding constants are uniform over  $\theta$  for  $\bar{h}$  small enough.

**2. The continuous problem and its discretization.** We consider the following pure state constrained optimal control problem:

$$(2.1) \quad (\mathcal{P}) \begin{cases} \text{Minimize } \phi(y(T)); & \text{subject to} \\ \dot{y}(t) = f(u(t), y(t)) & \text{for a.a. } t \in [0, T]; \\ y(0) = y_0; \\ g_i(y(t)) \leq 0, & t \in [0, T], \quad i = 1, \dots, r, \end{cases}$$

where the initial condition  $y_0$  belongs to  $\mathbb{R}^n$ , the control  $u(t)$  and the state  $y(t)$  belong to the spaces  $\mathcal{U} := L^\infty(0, T; \mathbb{R}^m)$  and  $\mathcal{Y} := W^{1,\infty}(0, T; \mathbb{R}^n)$ , respectively, and  $g_i$  is the  $i$ th component of the vector  $g$ . Moreover, we assume the following.

**(A0)** The mappings  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $f : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  are of class  $C^2$  with locally Lipschitz second-order continuous derivatives and  $g : \mathbb{R}^n \rightarrow \mathbb{R}^r$  is of class  $C^3$  with locally Lipschitz third derivatives. In addition, the initial condition  $y_0 \in \mathbb{R}^n$  satisfies  $g_i(y_0) < 0$ ,  $i = 1, \dots, r$ .

A *trajectory* of  $(\mathcal{P})$  is an element  $(u, y)$  of  $\mathcal{U} \times \mathcal{Y}$ , the solution of the state equation in (2.1). It is said to be *feasible* if it satisfies the state constraint, and then we say that  $u$  is a feasible control. We say that the feasible trajectory  $(\tilde{u}, \tilde{y})$  is a *local solution* of  $(\mathcal{P})$  if it minimizes  $\phi(\cdot)$  over the set of feasible trajectories  $(u, y)$  satisfying  $\|u - \tilde{u}\|_\infty \leq \delta$  for some  $\delta > 0$ . We assume the following.

**(A1)** The *nominal* trajectory  $(\bar{u}, \bar{y})$  is a local solution of  $(\mathcal{P})$  in  $\mathcal{U} \times \mathcal{Y}$ , and  $\bar{u}$  is a continuous function of time.

The hypothesis of continuity of the control may seem restrictive. However, it happens that, even for unconstrained problems, the analysis of second-order optimality conditions is quite involved when the control is discontinuous; see, for instance, [29, sect. 14].

The *first-order time derivative* of the state constraint is the function

$$(2.2) \quad g^{(1)} : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^r, (u, y) \rightarrow g'(y)f(u, y).$$

Note that  $g^{(1)}(\bar{u}(t), \bar{y}(t))$  is the time derivative of  $g(\bar{y}(t))$ . Denote the set of *active constraints* at time  $t$  by

$$\mathcal{A}(t) := \{i = 1, \dots, r \mid g_i(\bar{y}(t)) = 0\}.$$

We say that the trajectory  $(\bar{u}, \bar{y})$  has *regular first-order state constraints* if the following holds.

**(A2)** There exists  $\alpha_g > 0$  such that, for all  $t \in [0, T]$  and  $\lambda \in \mathbb{R}^{r*}$  verifying  $\lambda_i = 0$  if  $i \notin \mathcal{A}(t)$ , the following holds:

$$|\lambda| \leq \alpha_g \left| \sum_{i \in \mathcal{A}(t)} \lambda_i \nabla_u g_i^{(1)}(\bar{u}(t), \bar{y}(t)) \right|.$$

The *Hamiltonian* function, where  $p \in \mathbb{R}^{n^*}$  and  $(u, y) \in \mathbb{R}^m \times \mathbb{R}^n$ , is defined by

$$H[p](u, y) := pf(u, y).$$

With this classical notation we view the Hamiltonian as a function of  $(u, y)$ , parametrized by  $p$ , so that, for example,  $DH[p](u, y)$  denotes the derivative of the Hamiltonian with respect to  $(u, y)$ .

For  $i = 1$  to  $r$ , we define the *contact set* for the  $i$ th constraint by

$$I_i := \{t \in [0, T]; g_i(\bar{y}(t)) = 0\}.$$

We say also that the  $i$ -constraint is *active* at time  $t$  if  $t \in I_i$ ; otherwise the constraint is inactive. A maximal open interval  $(a, b)$  of  $I_i$  (resp., of  $[0, T] \setminus I_i$ ) is called a *boundary arc* (resp., *interior arc*). The left and right endpoints of a boundary arc are called *entry* and *exit* points, respectively. We call the union of entry and exit points *junction points*.

**2.1. Optimality conditions.** We next introduce first-order extremals.

**DEFINITION 2.1.** *A trajectory  $(\bar{u}, \bar{y})$  is a first-order extremal of  $(\mathcal{P})$  if there exist  $\bar{\eta} \in BV_T([0, T], \mathbb{R}^r)$  and  $\bar{p} \in BV([0, T], \mathbb{R}^{n^*})$  such that*

$$(2.3) \quad \dot{\bar{y}}(t) = f(\bar{u}(t), \bar{y}(t)) \text{ a.e. on } [0, T], \quad \bar{y}(0) = y_0,$$

$$(2.4) \quad -d\bar{p}(t) = \bar{p}(t)f_y(\bar{u}(t), \bar{y}(t))dt + \sum_{i=1}^r g'_i(\bar{y}(t))d\bar{\eta}_i(t), \quad \bar{p}(T) = \phi'(\bar{y}(T)),$$

$$(2.5) \quad 0 = H_u[\bar{p}(t)](\bar{u}(t), \bar{y}(t)) = \bar{p}(t)f_u(\bar{u}(t), \bar{y}(t)) \text{ for a.a. } t \in [0, T],$$

$$(2.6) \quad 0 \geq g_i(\bar{y}(t)), \quad d\bar{\eta}_i \geq 0, \quad \int_0^T g_i(\bar{y}(t))d\bar{\eta}_i(t) = 0, \quad i = 1, \dots, r.$$

We say that  $\bar{\eta}$  is the *Lagrange multiplier* associated with the state constraint and that  $\bar{p}$  is the corresponding *costate*.

**THEOREM 2.2.** *Let **(A0)**–**(A2)** hold. Then a local solution  $(\bar{u}, \bar{y})$  of  $(\mathcal{P})$  is a first-order extremal, with unique associated costate and Lagrange multiplier  $(\bar{p}, \bar{\eta})$ .*

*Proof.* see, for instance, [6, Thm. 2.5]. □

The *linearized state equation* at  $(\bar{u}, \bar{y})$  is, for  $v$  in  $\mathcal{V} := L^2(0, T; \mathbb{R}^m)$ ,

$$(2.7) \quad \dot{z}(t) = f'(\bar{u}(t), \bar{y}(t))(v(t), z(t)); \quad z(0) = 0.$$

Its solution in  $\mathcal{Z} := H^1(0, T; \mathbb{R}^n)$  is denoted by  $z[v]$ .

**2.2. A key result.** The next assumption is quite common in these problems and it plays a crucial role in the analysis. We assume that problem  $(\mathcal{P})$  has a local solution  $(\bar{u}, \bar{y})$  with associated multipliers  $\bar{p}$  and  $\bar{\eta}$  satisfying the following condition.

**(A3)** (*Strengthened Legendre–Clebsch condition*) There exists  $\alpha > 0$  such that

$$(2.8) \quad H_{uu}[\bar{p}(t)](\bar{u}(t), \bar{y}(t))(v)^2 \geq \alpha|v|^2 \text{ for all } v \in \mathbb{R}^m \text{ for a.a. } t \in [0, T].$$

Here and below we use the compact notation  $(v)^2$  instead of  $(v, v)$ . We recall that the continuity of the control was stated in **(A1)**.

Observe that, when  $\bar{u}$  and  $\bar{\eta}$  are Lipschitz continuous, denoting by  $\bar{v}(t)$  the derivative of  $\bar{\eta}$ , the costate equation (2.4) can be written in the form

$$(2.9) \quad -\dot{\bar{p}}(t) = \bar{p}(t)f_y(\bar{u}(t), \bar{y}(t)) + \sum_{i=1}^r \bar{v}_i(t)g'_i(\bar{y}(t)) \text{ a.e. on } (0, T); \quad \bar{p}(T) = \phi'(\bar{y}(T)).$$

LEMMA 2.3. *Let (A0)–(A3) hold. Then both  $\bar{u}$  and  $\bar{\eta}$  are Lipschitz continuous and (2.9) holds.*

*Proof.* The result is proved in [18, Thm. 4.2] for the case of a convex problem, using a lemma on “compatible pairs.” It was generalized in [1], using the same lemma, to nonconvex problems with state constraints of any order (see also [21] for the case of a second-order state constraint).  $\square$

In the rest of the paper we assume (A0)–(A3) as a standing hypothesis.

**2.3. Alternative formulation.** We recall the definition of the spaces  $\mathcal{V} := L^2(0, T; \mathbb{R}^m)$  and  $\mathcal{Z} := H^1(0, T; \mathbb{R}^n)$ , respectively. We use also the notations

$$H[t] := H[\bar{p}(t)](\bar{u}(t), \bar{y}(t)); \quad g[t] := g(\bar{y}(t)), \quad f[t] := f(\bar{u}(t), \bar{y}(t));$$

with a similar convention for their partial derivatives, for example,  $H_u[t] := H_u[\bar{p}(t)](\bar{u}(t), \bar{y}(t))$ . In addition, we denote by  $D^2H[t]$  the Hessian of  $H[\bar{p}(t)](\bar{u}(t), \bar{y}(t))$  with respect to  $(u, y)$ . More generally, writing  $[t]$  as the argument of a function means that this function is to be evaluated over the nominal trajectory  $(\bar{u}, \bar{y})$  and, where it is necessary, over the associated multipliers  $(\bar{\eta}, \bar{p})$ . For simplicity of notation, we write in what follows  $D^2H[t](v, z)^2$  instead of  $D^2H[t]((v, z), (v, z))$ .

Let us define the quadratic form over  $\mathcal{V} \times \mathcal{Z}$ , where  $z = z[v]$ :

$$(2.10) \quad \Omega(v) := \int_0^T \left( D^2H[t](v(t), z(t))^2 + \sum_{i=1}^r \bar{v}_i(t) g_i''[t](z(t))^2 \right) dt + \phi''(\bar{y}(T))(z(T))^2,$$

and the set  $C(\bar{u})$  of *strict critical directions* is defined as those  $v \in \mathcal{V}$  such that, for  $z = z[v]$ ,

$$(2.11) \quad \dot{z} = f_u(\bar{u}, \bar{y})v + f_y(\bar{u}, \bar{y})z \text{ on } [0, T]; \quad z(0) = 0,$$

$$(2.12) \quad g_i'(\bar{y}(t))z(t) = 0, \quad t \in I_i, \quad i = 1, \dots, r,$$

$$(2.13) \quad \phi'(\bar{y}(T))z(T) = 0.$$

We discuss in Appendix A the relation of this set with the standard critical cone.

Let us next recall the *alternative formulation* of the optimality conditions, due to [10] and [22], and put on a sound mathematical basis by [25]. (See also [4, 31].) We need this alternative formulation in section 5 and Appendix C. The *alternative Hamiltonian*, where  $g^{(1)}$  is given in (2.2),  $(p^1, \bar{\eta}^1) \in \mathbb{R}^{n^*} \times \mathbb{R}^{r^*}$ , and  $(u, y) \in \mathbb{R}^m \times \mathbb{R}^n$ , is defined by

$$(2.14) \quad \tilde{H}[p^1, \bar{\eta}^1](u, y) := p^1 f(u, y) + \bar{\eta}^1 g^{(1)}(u, y).$$

Now define the *alternative costate and multiplier* of the state constraint:

$$p^1(t) := \bar{p}(t) + \sum_{i=1}^r \bar{\eta}_i(t) g_i'(\bar{y}(t)); \quad \bar{\eta}^1(t) := -\bar{\eta}(t), \quad t \in (0, T).$$

We can check that

$$(2.15) \quad -\dot{p}^1(t) = \tilde{H}_y[p^1(t), \bar{\eta}^1(t)](\bar{u}(t), \bar{y}(t)) \quad \text{a.e. on } (0, T); \quad p^1(T) = \phi'(\bar{y}(T)).$$

At the same time, for any  $u \in \mathbb{R}$ , we have that

$$\tilde{H}[p^1(t), \bar{\eta}^1(t)](u, \bar{y}(t)) = (p^1(t) + \bar{\eta}^1(t) g'(\bar{y}(t))) f(u, \bar{y}(t)) = H[\bar{p}(t)](u, \bar{y}(t)).$$

Consequently, the property of stationarity of the Hamiltonian with respect to the control holds for the original Hamiltonian  $H$  if and only if it holds for the alternative Hamiltonian  $\tilde{H}$ . The corresponding alternative quadratic form, where  $z = z[v]$ , has the following expression:

$$(2.16) \quad \tilde{\Omega}(v) := \int_0^T D^2 \tilde{H}[t](v(t), z(t))^2 dt + \phi''(\bar{y}(T))(z(T))^2.$$

The form above involves the expression of  $D^2 g^{(1)}[t]$ , which is easily checked to be

$$(2.17) \quad D^2 g^{(1)}[t](v, z)^2 = g^{(3)}[t](f[t], z, z) + g'[t]f_{yy}[t](v, z)^2 + 2g''[t](z, f_y[t](v, z)).$$

The next lemma is a variant of some results by Bonnans and Hermant [4] and Malanowski and Maurer [24].

LEMMA 2.4. *We have that  $\tilde{\Omega}(v) = \Omega(v)$  for all  $v \in \mathcal{V}$ .*

*Proof.* We give a short, direct proof in the case of a single constraint for convenience. Using (2.17), we get that

$$\begin{aligned} \int_0^T \bar{v}(t)g''[t](z(t))^2 dt &= \int_0^T g''[t](z(t))^2 d\bar{\eta}(t) = \int_0^T D(g''[t](z(t))^2) \bar{\eta}^1(t) dt \\ &= \int_0^T \left( g^{(3)}(\bar{y}(t))(f[t], z(t), z(t)) + 2g''[t](z(t), \dot{z}(t)) \right) \bar{\eta}^1(t) dt \\ &= \int_0^T \left( D^2 g^{(1)}[t](v(t), z(t))^2 - g'[t]f''[t](v(t), z(t))^2 \right) \bar{\eta}^1(t) dt. \end{aligned}$$

Substituting  $\bar{p}(t) = p^1(t) + \bar{\eta}^1 g'[t]$  in the expression of  $\Omega(\cdot)$ , we obtain that

$$\begin{aligned} \Omega(v) &= \int_0^T \left( (p^1(t) + \bar{\eta}^1(t)g'[t])f''[t](v(t), z(t))^2 + \bar{v}(t)g''[t](z(t))^2 \right) dt \\ &\quad + \phi''(\bar{y}(T))(z(T))^2 \\ &= \int_0^T \left( p^1(t)f''[t](v(t), z(t))^2 + \bar{\eta}^1(t)D^2 g^{(1)}[t](v(t), z(t))^2 \right) dt \\ &\quad + \phi''(\bar{y}(T))(z(T))^2, \end{aligned}$$

as it was to be proved.  $\square$

**2.4. Discrete version.** We introduce now the Euler discretization of the optimal control problem (2.1). Given some nonzero  $N \in \mathbb{N}$  and a collection of positive time steps  $h_k$ ,  $k = 0$  to  $N - 1$ , such that  $\sum_{k=0}^{N-1} h_k = T$ , we set

$$t_k := \sum_{i=0}^{k-1} h_i, \quad k = 0, \dots, N; \quad \bar{h} = \max_{k=0, \dots, N-1} h_k,$$

and we consider the discretized problem

$$(2.18) \quad (\mathcal{P}_d) \begin{cases} \text{Minimize } \phi(y_N); & \text{subject to} \\ y_{k+1} = y_k + h_k f(u_k, y_k) & \text{for } k = 0, \dots, N-1; \\ y_0 = y_0; & \\ g(y_k) \leq 0 & \text{for } k = 1, \dots, N. \end{cases}$$

We define, in analogy with the continuous formulation, by  $\mathcal{U}^N := (\mathbb{R}^m)^N$  the space of discrete control variables (we use the same notation for the other functional spaces). The associated Lagrangian function (with a proper scaling of the state constraint) is

$$\phi(y_N) + \sum_{k=0}^{N-1} p_{k+1}(y_k + h_k f(u_k, y_k) - y_{k+1}) + \sum_{k=1}^N h_k \nu_k g(y_k),$$

where  $\nu_k g(y_k) := \sum_{i=1}^r \nu_{k,i} g_i(y_k)$ . The first-order optimality conditions in qualified form (that is, as discussed in the previous section with the multiplier of the cost function equal to 1; see, for example, [8, sect. 3.1]) for this finite-dimensional optimization problem with finitely many equalities and inequalities, are

$$\begin{aligned} p_k &= p_{k+1} + h_k p_{k+1} f_y(u_k, y_k) + h_k \nu_k g'(y_k), \quad k = 0, \dots, N-1, \\ p_N &= \phi'(y_N) + h_N \nu_N g'(y_N), \\ 0 &= H_u[p_{k+1}](u_k, y_k), \quad k = 0, \dots, N-1, \\ g_i(y_k) &\leq 0, \quad \nu_{k,i} \geq 0; \quad \nu_{k,i} g_i(y_k) = 0, \quad i = 1, \dots, r, \quad k = 0, \dots, N. \end{aligned} \tag{2.19}$$

Analogously to the continuous case, we define also the “integrated” multiplier for both the normal and the alternative formulation:

$$\eta_k := - \sum_{j=k}^N h_j \nu_j, \quad \bar{\eta} := -\eta, \tag{2.20}$$

so that  $h_k \nu_k = \eta_{k+1} - \eta_k = \bar{\eta}_k - \bar{\eta}_{k+1}$  for  $k = 0, \dots, N$ . In the case when the discretization step is *constant* we have  $h_0 = h_1 = \dots = h_{N-1}$ .

**2.5. Main result.** As mentioned before, we present some results for two different cases of interest. We need to preserve the coercivity of the Hessian of the Lagrangian of the discretized problem over some subspace; this can be stated as a hypothesis, as in [16], or obtained under hypotheses on the structure of the times at which a scalar state constraint is active. So we will assume that one of the following two assumptions holds where  $\Omega(\cdot)$  is defined in (2.10).

(A4) There exists a constant  $\alpha > 0$  such that

$$\Omega(v) \geq \alpha \int_0^T |v(t)|^2 dt \quad \text{for all } v \in \mathcal{V},$$

and all discrete steps are of the same order, i.e.,

$$\max_k (h_k/h_{k-1} + h_{k-1}/h_k) = O(1). \tag{2.21}$$

The condition on  $\Omega$  is known to be a sufficient condition for local optimality in  $\mathcal{U}$ ; see [28, Thm. 2.4].

(A5) (*Scalar constraint and finite structure*) Assume that  $r = 1$ , the discretization step is constant, the set  $I$  is a finite union of boundary arcs, the density  $\nu$  is uniformly positive over the boundary arcs, and there exists a constant  $\alpha > 0$  such that

$$\Omega(v) \geq \alpha \int_0^T |v(t)|^2 dt \quad \text{whenever } v \in C(\bar{u}). \tag{2.22}$$



We recall that the set of strict critical directions  $C(\bar{u})$  was defined in (2.11)–(2.13). Condition **(A5)** is known to be a sufficient condition for local optimality in  $\mathcal{U}$ ; see [6, Thm. 6.1(ii)]. We say that  $t \in [0, T]$  is a *touch point* for the  $i$ th state constraint if it is an isolated element of  $I_i$ . Note that **(A5)** excludes touch points.

**THEOREM 2.5.** *Let either **(A4)** or **(A5)** hold. Then there exists  $c_E > 0$  such that the discrete optimal control problem  $(\mathcal{P}_d)$  has a local solution  $(u^h, y^h)$  with associated multipliers  $(p^h, \eta^h)$  such that, for  $h$  small enough,*

$$(2.23) \quad \max_k (|y_k^h - \bar{y}(t_k)| + |u_k^h - \bar{u}(t_k)| + |p_k^h - \bar{p}(t_k)| + |\eta_k^h - \bar{\eta}(t_k)|) \leq c_E \bar{h}.$$

The rest of the paper is devoted to the proof of this result; for that purpose we need to introduce a homotopy path.

**3. Homotopy path.** We consider a family  $(\mathcal{P}^\theta)$ , parametrized by  $\theta \in [0, 1]$ , of finite-dimensional optimization problems, which can be viewed as a perturbation of the discrete optimization problem  $(\mathcal{P}_d)$ :

$$(3.1) \quad (\mathcal{P}^\theta) \begin{cases} \text{Minimize } \phi(y_N^\theta) + \theta \sum_{k=0}^{N-1} h_k^2 (\delta_k^p y_k^\theta + \delta_k^u u_k^\theta); & \text{subject to} \\ y_{k+1}^\theta = y_k^\theta + h_k f(u_k^\theta, y_k^\theta) + \theta h_k^2 \delta_k^y & \text{for } k = 0, \dots, N-1; \\ g_i(y_k^\theta) \leq \theta h_k^2 \delta_{k,i}^g & \text{for } k = 1, \dots, N, \\ y_0^\theta = y_0, & i = 1, \dots, r. \end{cases}$$

The given perturbation terms  $(\delta_k^p, \delta_k^u, \delta_k^y, \delta_k^g) \in \mathbb{R}^{n^*} \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^r$  will be defined later. For  $\theta = 0$ ,  $(\mathcal{P}^\theta)$  reduces to the discrete problem  $(\mathcal{P}_d)$ . Roughly speaking, we choose the perturbation terms in such a way that the values of the solution  $(\bar{u}, \bar{y})$  and the multipliers of the original problem  $(\mathcal{P})$  satisfy the optimality system of  $(\mathcal{P}^\theta)$  at discretization times  $t_k$  when  $\theta = 1$ . The expression of the optimality system of  $(\mathcal{P}^\theta)$  is

$$(3.2) \quad \begin{cases} p_k^\theta = p_{k+1}^\theta + h_k p_{k+1}^\theta f_y(u_k^\theta, y_k^\theta) + h_k \nu_k^\theta g'(y_k^\theta) + \theta h_k^2 \delta_k^p, \\ p_N^\theta = \phi'(y_N^\theta) + h_N \nu_N^\theta g'(y_N^\theta), \\ 0 = H_u[p_{k+1}^\theta](u_k^\theta, y_k^\theta) + \theta h_k \delta_k^u \end{cases}$$

for  $k = 1$  to  $N - 1$  with the complementarity conditions  $i = 1, \dots, r$ :

$$(3.3) \quad g_i(y_k^\theta) - \theta h_k^2 \delta_{k,i}^g \leq 0, \quad \nu_{k,i}^\theta \geq 0, \quad \nu_{k,i}^\theta (g_i(y_k^\theta) - \theta h_k^2 \delta_{k,i}^g) = 0, \quad k = 1, \dots, N.$$

Let us set

$$(3.4) \quad f^k := f(u_k^\theta, y_k^\theta); \quad H^k := H[p_{k+1}^\theta](u_k^\theta, y_k^\theta).$$

For future reference, we note that the expression of the linearization of the costate equation (3.2) is, denoting by  $(v^\theta, z^\theta, q^\theta, \delta \nu^\theta)$  the variables corresponding to the variations of  $(u^\theta, y^\theta, p^\theta, \nu^\theta)$

$$(3.5) \quad \begin{aligned} q_k^\theta &= q_{k+1}^\theta + h_k q_{k+1}^\theta f_y^k + h_k (v_k^\theta)^T H_{u_y}^k + h_k (z_k^\theta)^T H_{y_y}^k \\ &\quad + h_k \nu_k^\theta (z_k^\theta)^T g''(y_k^\theta) + h_k \delta \nu_k^\theta g'(y_k^\theta) - \theta h_k^2 \delta_k^p, \\ q_N^\theta &= (z_N^\theta)^T \phi''(y_N^\theta) + h_N \nu_N^\theta (z_N^\theta)^T g''(y_N^\theta) + h_N \delta \nu_N^\theta g'(y_N^\theta). \end{aligned}$$

The corresponding approximation of  $\bar{\eta}$  is (see (2.20))

$$(3.6) \quad \eta_k^\theta := - \sum_{j=k}^N h_j \nu_j^\theta, \quad k = 0, \dots, N.$$

Next define

$$(3.7) \quad \begin{cases} \hat{u}_k := \bar{u}(t_k), & \hat{y}_k := \bar{y}(t_k), & k = 0, \dots, N-1, \\ \hat{p}_k := \bar{p}(t_k), & \hat{\nu}_k := \int_{t_k}^{t_{k+1}} \bar{\nu}(t) dt, & k = 1, \dots, N, \end{cases}$$

and accordingly

$$\hat{\eta}_k := \eta(t_{k+1}) = \sum_{j=k}^N h_j \hat{\nu}_j, \quad k = 1, \dots, N.$$

For  $\theta = 1$  we define  $u_k^\theta$  and the associated state and multipliers by

$$(3.8) \quad u_k^1 = \hat{u}_k, \quad y_k^1 = \hat{y}_k, \quad p_k^1 = \hat{p}_k, \quad \eta_k^1 = \hat{\eta}_k \quad \text{for } k = 1, \dots, N.$$

We next define the perturbation terms  $(\delta_k^u, \delta_k^y, \delta_k^p, \delta_k^g)$  by giving to them a value such that the terms above are a solution of the discrete optimality system (3.2)–(3.3) for  $\theta = 1$ . Then  $\delta^y$  is determined by the discrete state equation,  $\delta^u$  and  $\delta^p$  are determined by (3.2)–(3.3), and we choose

$$(3.9) \quad \delta_{k,i}^g = \begin{cases} g_i(y_k^\theta)/h_k^2 & \text{if } \nu_{k,i}^\theta > 0, \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 3.1. *We have that*

$$(3.10) \quad \|\delta^y\|_\infty + \|\delta^u\|_\infty + \|\delta^p\|_\infty + \|\delta^g\|_\infty = O(1).$$

*Proof.* If  $\nu_{k,i} \neq 0$ , then  $\max\{g_i(\bar{y}(t)), t \in [t_k, t_{k+1}]\} = 0$ . Since  $u$  is Lipschitz continuous by Lemma 2.3,  $t \rightarrow g(\bar{y}(t))$  has a.e. a bounded second derivative. So there exists a  $c > 0$  independent on  $\bar{h}$  such that  $g_i(\bar{y}(t_k)) \geq -c\bar{h}^2$  for all  $t \in [t_k, t_{k+1}]$ , so that  $\|\delta^g\|_\infty = O(1)$ .

Next, if  $w : [0, T] \rightarrow \mathbb{R}$  is  $C^1$  with a Lipschitz continuous derivative of constant  $L$ , then by a first-order Taylor expansion, we have that

$$|w(t+h) - w(t) - w'(t)h| \leq \frac{1}{2}Lh^2.$$

By Lemma 2.3, the control is Lipschitz and, therefore,  $y(t)$  is also Lipschitz continuous (with respect to  $\theta$ ); we deduce that  $\|\delta^y\|_\infty = O(1)$ . For the costate equation, we have that

$$\bar{p}(t_{k+1}) = \bar{p}(t_k) - \int_{t_k}^{t_{k+1}} \bar{p}(t) f_y[t] dt - \int_{t_k}^{t_{k+1}} \bar{\nu}(t) g'[t] dt.$$

Now since  $\bar{u}$ ,  $\bar{y}$ , and  $\bar{p}$  are Lipschitz continuous,

$$\left| \int_{t_k}^{t_{k+1}} \bar{p}(t) f_y(\bar{u}(t), \bar{y}(t)) dt - \int_{t_k}^{t_{k+1}} \bar{p}(t_{k+1}) f_y(\bar{u}(t_k), \bar{y}(t_k)) dt \right| = O(h_k^2),$$

and (in the parentheses below we recognize the expression of  $\nu_k$ )

$$\left| \int_{t_k}^{t_{k+1}} \bar{\nu}(t) g'[t] dt - \left( \int_{t_k}^{t_{k+1}} \bar{\nu}(t) dt \right) g'(\bar{y}(t_k)) \right| = O(h_k^2).$$

It follows that  $\|\delta^p\|_\infty = O(1)$ . Since  $p$  is Lipschitz continuous and  $H_u[\bar{p}(t_k)](\bar{u}(t_k), \bar{y}(t_k)) = 0$ , we deduce that  $\|\delta^u\|_\infty = O(1)$ . The conclusion follows.  $\square$

In the rest of the paper some of the estimates presented are valid in a special neighborhood of the continuous solution. To state them rigorously, we need the following definition: given  $\varepsilon > 0$  and  $\theta \in [0, 1]$ , we say that a solution  $X^\theta := (u^\theta, y^\theta, p^\theta, \eta^\theta)$  of the optimality system (3.2) is an  $\varepsilon$ -neighboring solution, or an  $\varepsilon$ -n solution for short, if we have that

$$(3.11) \quad \max_k (|y_k^\theta - \bar{y}(t_k)| + |u_k^\theta - \bar{u}(t_k)| + |p_k^\theta - \bar{p}(t_k)| + |\eta_k^\theta - \bar{\eta}(t_k)|) \leq \varepsilon,$$

and we define

$$\theta_m := \inf\{\theta \in [0, 1]; (3.2) \text{ has an } \varepsilon\text{-n solution}\}.$$

When  $\theta = 1$ , the left-hand side of (3.11) has the value 0, and therefore  $\theta_m$  is well defined with a value in  $[0, 1]$ .

Through the structure of the homotopy path problem we are now able to prove the bounds shown in Theorem 2.5. In the following we analyze some technical points. In particular, we use the fact that the  $\varepsilon$ -n solution of  $(\mathcal{P}^\theta)$  is uniformly Lipschitz continuous (the result is found in section 4) and that such a solution is in an  $\varepsilon$ -neighborhood of the solution of  $(\mathcal{P})$ ,  $X^1 = (\bar{u}, \bar{y}, \bar{p}, \bar{\eta})$ ; this point is discussed in section 5.

*Proof of Theorem 2.5.* We prove in section 4 that, if  $\bar{h}$  is small enough, then for  $\theta \in [\theta_m, 1]$ ,  $(u^\theta, y^\theta)$  is uniquely defined and has unique associated multipliers  $(p^\theta, \eta^\theta)$ , and setting  $X^\theta := (u^\theta, y^\theta, p^\theta, \eta^\theta)$  (see section 5),  $\theta \rightarrow X^\theta$  has a Lipschitz constant of order  $\bar{h}$  in the uniform norm. It follows that, for a fixed  $\varepsilon > 0$ ,  $\|X^\theta - X^1\|_\infty < \varepsilon$  when  $\bar{h}$  is small enough, which gives a contradiction if  $\theta_m > 0$ . Therefore,  $X^0$  is well defined and  $\|X^1 - X^0\|_\infty = O(\bar{h})$ , as was to be shown.  $\square$

**4. Regularity of the  $\varepsilon$ -n solutions.** In this section we present some regularity results for the  $\varepsilon$ -n solutions of the homotopy path problem.

Given an  $\varepsilon$ -n solution  $X^\theta$  of  $(\mathcal{P}^\theta)$ , we prove the uniqueness and the uniform Lipschitz continuity of  $X^\theta$ . In the following, when there is no possibility of confusion, we drop  $\theta$  as the upper index in the notation for better readability, keeping the complete notation in the statements of the main propositions. We need to define

$$(4.1) \quad \begin{aligned} C_k^1 &:= \frac{p_k}{h_k} (f_u(u_{k-1}, y_k) - f_u(u_{k-1}, y_{k-1})), \\ C_k^2 &:= p_{k+1} f_y(u_k, y_k) f_u(u_k, y_k), \\ \Delta_k^u &:= h_k \delta_k^u - h_{k-1} \delta_{k-1}^u \\ &= (\hat{p}_k - \hat{p}_{k+1}) f_u(\hat{u}_k, \hat{y}_k) - (\hat{p}_{k-1} - \hat{p}_k) f_u(\hat{u}_{k-1}, \hat{y}_{k-1}), \\ C_k^3 &:= C_k^1 - C_k^2 - \theta h_k \delta_k^p f_u(u_k, y_k) + \theta \Delta_k^u / h_k. \end{aligned}$$

We may see the  $C_k^i = C_k^i(\theta)$  as a function of  $\theta$ ; we call variation of these amounts w.r.t.  $\theta$  the amount  $|C_k^i(\theta) - C_k^i(1)|$ . Observe that, since  $y_k$  is uniformly Lipschitz continuous with respect to  $\theta$ , we have that

$$(4.2) \quad (i) \ C_k^1 = \frac{h_{k-1}}{h_k} H_{uy}[p_k](u_{k-1}, y_k) f(u_{k-1}, y_{k-1}) + O(\bar{h}); \quad (ii) \ |C_k^3| = O(1).$$

LEMMA 4.1.

(i) Let  $X^\theta$  be an  $\varepsilon$ -n solution of  $(\mathcal{P}^\theta)$ . Then there exists  $c_{\mathcal{H}} > 0$  not depending on  $\bar{h}$  or  $\theta$  such that, if  $\varepsilon > 0$  and  $\bar{h}$  are small enough, then

$$(4.3) \quad \sum_{j=1}^r \nu_{k,j}^\theta \nabla_u g_j^{(1)}(u_k^\theta, y_k^\theta) = \mathcal{H}_k^\theta \frac{(u_k^\theta - u_{k-1}^\theta)}{h_k} + C_k^3,$$

where  $\mathcal{H}_k^\theta$  satisfies

$$(4.4) \quad |\mathcal{H}_k^\theta - H_{uu}[p_k](u_k, y_k)| \leq c_{\mathcal{H}}\varepsilon, \quad k = 0, \dots, N - 1.$$

(ii) Let  $\varepsilon' > 0$ . If in addition, the time step is constant,  $t_k$  belongs to a boundary arc  $(t_a, t_b)$ , and  $t_{k_a} + \varepsilon' < t_{k-1} < t_k < t_{k_b} - \varepsilon'$ , then the variation of  $C_k^3$  (with respect to  $\theta$ ) along the homotopy path is of order  $\varepsilon$ .

*Proof.*

(i) Note that  $h_k \delta_k^u = (\hat{p}_k - \hat{p}_{k+1})f_u(\hat{u}_k, \hat{y}_k)$ . By the optimality condition (3.2), we have that

$$(4.5) \quad \begin{aligned} 0 &= H_u[p_{k+1}](u_k, y_k) - H_u[p_k](u_{k-1}, y_{k-1}) + \theta \Delta_k^u \\ &= p_{k+1}f_u(u_k, y_k) - p_kf_u(u_{k-1}, y_{k-1}) + \theta \Delta_k^u \\ &= (p_{k+1} - p_k)f_u(u_k, y_k) + p_k[f_u(u_k, y_k) - f_u(u_{k-1}, y_{k-1})] + \theta \Delta_k^u \\ &= (p_{k+1} - p_k)f_u(u_k, y_k) + p_k[f_u(u_k, y_k) - f_u(u_{k-1}, y_k)] + h_k C_k^1 + \theta \Delta_k^u. \end{aligned}$$

By the mean-value theorem, we have that

$$p_k(f_u(u_k, y_k) - f_u(u_{k-1}, y_k)) = \mathcal{H}_k(u_k - u_{k-1}),$$

where

$$(4.6) \quad \mathcal{H}_k := p_k \int_0^1 f_{uu}(u_{k-1} + \sigma(u_{k-1} - u_k), y_k) d\sigma,$$

so that (4.4) holds. We conclude by combining (4.5) and the discrete costate equation in (3.2), where  $\nabla_u g_i^{(1)}(u_k, y_k) = g'_i(y_k)f_u(u_k, y_k)$ .

(ii) It is easily checked that  $C_k^1$  and  $C_k^2$  have variations of order  $\varepsilon$ , as well as (since it is of the order of  $\bar{h}$ )  $\theta h_k \delta_k^p f_u(u_k, y_k)$ . Since the time step is constant, we have that, by (4.1),  $\Delta_k^u/h_k = \delta_k^u - \delta_{k-1}^u$  is of the order of  $\bar{h}$  over the interior of a boundary arc. The conclusion follows.  $\square$

Let us define

$$(4.7) \quad \begin{aligned} \Delta_g^{k,i} &:= \frac{g_i(y_{k+1}) - g_i(y_k)}{h_k} - \frac{g_i(y_k) - g_i(y_{k-1})}{h_{k-1}}, \\ \Xi_k^\theta &:= g'_i(y_k)(f(u_{k-1}, y_k) - f(u_{k-1}, y_{k-1})) \\ &\quad + \theta g'_i(y_k)(h_k \delta_k^y - h_{k-1} \delta_{k-1}^y) + \frac{1}{2} h_k g''_i(y_k) f(u_k, y_k)^2 \\ &\quad + \frac{1}{2} h_{k-1} g''_i(y_k) f(u_{k-1}, y_{k-1})^2. \end{aligned}$$

LEMMA 4.2. *We have that*

(a) *the following relation holds*

$$(4.8) \quad \Delta_g^{k,i} = g'_i(y_k)f_u(y_k, u_k)(u_k - u_{k-1}) + \Xi_k^\theta + O(\bar{h}^2) + O(\varepsilon|u_k - u_{k-1}|);$$

(b) *if in addition the time step is constant, then*

$$(4.9) \quad \|\Xi^\theta - \Xi^1\|_\infty = O(\bar{h}\varepsilon).$$

*Proof.* Use

$$\begin{aligned}
 g_i(y_{k+1}) - g_i(y_k) &= g'_i(y_k)(y_{k+1} - y_k) + \frac{1}{2}g''_i(y_k)(y_{k+1} - y_k)^2 + O(h_k^3) \\
 &= h_k g'_i(y_k) f(u_k, y_k) \\
 &\quad + \frac{1}{2}h_k^2 (2\theta g'(y_k)\delta_k^y + g''_i(y_k)f(u_k, y_k)^2) + O(h_k^3) \\
 g_i(y_{k-1}) - g_i(y_k) &= g'_i(y_k)(y_{k-1} - y_k) + \frac{1}{2}g''_i(y_k)(y_{k-1} - y_k)^2 + O(h_k^3) \\
 &= -h_{k-1}g'_i(y_k)f(u_{k-1}, y_{k-1}) \\
 &\quad + \frac{1}{2}h_{k-1}^2 (-2\theta g'_i(y_k)\delta_{k-1}^y + g''_i(y_k)f(u_{k-1}, y_{k-1})^2) \\
 &\quad + O(h_{k-1}^3).
 \end{aligned}$$

We obtain (a) by dividing these relations by  $h_k$  and  $h_{k-1}$ , respectively, adding them, and observing that

$$\begin{aligned}
 f(u_k, y_k) - f(u_{k-1}, y_{k-1}) &= f(u_k, y_k) - f(u_{k-1}, y_k) + f(u_{k-1}, y_k) - f(u_{k-1}, y_{k-1}) \\
 &= f_u(u_k, y_k)(u_k - u_{k-1}) + (f(u_{k-1}, y_k) - f(u_{k-1}, y_{k-1})) + O(\varepsilon|u_k - u_{k-1}|).
 \end{aligned}$$

Since  $|y_k - y_{k-1}| = O(h_{k-1})$ , the point (b) follows using that  $h_k = h_{k-1}$  and  $|\delta_k^y - \delta_{k-1}^y| = O(\bar{h})$ .  $\square$

Now we are ready to obtain the uniform Lipschitz estimates of the variables of the perturbed problem. A similar result, for the case of a linear quadratic optimal control problem, was obtained in [15]. Let us set

$$(4.10) \quad w_k = \nu_k \nabla_u g^{(1)}(u_k, y_k) = \sum_{i=1}^r \nu_{k,i} \nabla_u g_i^{(1)}(u_k, y_k).$$

We denote by  $\text{Lip}(u^\theta)$ ,  $\text{Lip}(p^\theta)$  the Lipschitz constants of  $u^\theta$ ,  $p^\theta$ , defined as follows for the control:

$$\text{Lip}(u^\theta) := \max\{|u_{k+1}^\theta - u_k^\theta|/h_k; 0 \leq k < N\},$$

and similarly for the costate.

LEMMA 4.3. *We have that*

$$(4.11) \quad w_k^T \mathcal{H}_k^{-1} w_k = \frac{1}{h_k} w_k \cdot (u_k - u_{k-1}) + w_k^T \mathcal{H}_k^{-1} C_k^3,$$

as well as  $\text{Lip}(u^\theta) + \text{Lip}(p^\theta) + \|\nu^\theta\|_\infty = O(1)$ .

*Proof.* By the Legendre–Clebsch condition **(A3)**, for small enough  $\varepsilon > 0$ ,  $\mathcal{H}_k$  is uniformly invertible. Computing the scalar product of both sides of (4.3) by  $w_k^T \mathcal{H}_k^{-1}$ , we obtain (4.11), and deduce with **(A2)** that, for some  $c_1 > 0$ ,  $c_2 > 0$ , provided  $\bar{h}$  is small enough,

$$(4.12) \quad c_1 |\nu_k|^2 \leq w_k^T \mathcal{H}_k^{-1} w_k \leq \frac{1}{h_k} w_k \cdot (u_k - u_{k-1}) + c_2 |\nu_k|.$$

By (4.8),

$$(4.13) \quad w_k \cdot (u_k - u_{k-1}) = \sum_{i=1}^r \nu_k g'_i(y_k) f_u(y_k, u_k)(u_k - u_{k-1}) = \sum_{i=1}^r \nu_k \Delta_g^{k,i} + O(\bar{h}|\nu_k|),$$

so that with the previous display

$$(4.14) \quad c_1 |\nu_k|^2 \leq \frac{1}{\bar{h}} \sum_{i=1}^r \nu_k \Delta_g^{k,i} + O(|\nu_k|).$$

When  $\nu_{k,i} \neq 0$ , we have that  $g_{k,i}^\theta := g_i(y_k) - \theta h_k^2 \delta_{k,i}^g$  reaches a local maximum and therefore

$$0 \geq \frac{g_{k+1,i}^\theta - g_{k,i}^\theta}{h_k} - \frac{g_{k,i}^\theta - g_{k-1,i}^\theta}{h_{k-1}},$$

which amounts to

$$(4.15) \quad \Delta_g^{k,i} \leq \theta \left( \frac{h_{k+1}^2 \delta_{k+1,i}^g - h_k^2 \delta_{k,i}^g}{h_k} - \frac{h_k^2 \delta_{k,i}^g - h_{k-1}^2 \delta_{k-1,i}^g}{h_{k-1}} \right) \leq O(\bar{h}).$$

Combining with (4.14) it follows that  $\|\nu_k\|_\infty = O(1)$ . Since  $|C_k^3| = O(1)$ , as already noticed, the Lipschitz estimate for the control follows from (4.3) that  $|u_k - u_{k-1}|/h_k = O(|\nu_k| + 1)$ . By (3.2), the discretized costate is also Lipschitz continuous with respect to  $\theta$ .  $\square$

**5. Sensitivity analysis.**

**5.1. Characterization of derivatives with respect to  $\theta$ .** In this section we complete the proof of Theorem 2.5, showing that an  $\varepsilon$ -n solution  $(u^\theta, y^\theta)$  of the perturbed problem  $(\mathcal{P}^\theta)$  is in an  $L^\infty$  neighborhood of  $(\bar{u}, \bar{y})$ , the local solution of the problem  $(\mathcal{P})$ . The strategy consists in establishing that the path  $X^\theta := (u^\theta, y^\theta, p^\theta, \eta^\theta)$  is Lipschitz continuous, and then showing that it has the derivative, (from the left with respect to  $\theta$ ),  $\delta X^\theta := (v^\theta, z^\theta, q^\theta, \delta \eta^\theta)$ , satisfying  $\|\delta X^\theta\|_\infty = O(\bar{h})$ . This will imply that the Lipschitz constant of  $X^\theta$  (in the  $L^\infty$  norm) is of order  $O(\bar{h})$ . We recall (compare to (3.6)) that  $\nu_k^\theta := \eta_{k+1}^\theta - \eta_k^\theta$ ,  $k = 0$  to  $N$ . Consider the following quadratic programming problem:

$$(5.1) \quad (QP) \quad \begin{cases} \min_v \frac{1}{2} \Omega^\theta(v, z) - \theta \sum_{k=0}^{N-1} h_k^2 (\delta_k^p z_k + \delta_k^u v_k) & \text{subject to } z = z^\theta[v] \text{ and} \\ g'_i(y_k) z_k = -\theta h_k^2 \delta_{k,i}^g, & k \in I_+^{i,\theta}, \\ g'_i(y_k) z_k \leq -\theta h_k^2 \delta_{k,i}^g, & k \in I_0^{i,\theta}. \end{cases}$$

Here  $v \in \mathcal{V}^N$ ,  $z^\theta[v] \in \mathcal{Z}^N$  is defined as the unique solution of the state equation of the linearized problem

$$(5.2) \quad z_{k+1}^\theta = z_k^\theta + h_k f_y(u_k, y_k)(v_k, z_k^\theta) - \theta h_k^2 \delta_k^y, \quad k = 0, \dots, N-1, \quad i = 1, \dots, r,$$

and the set of constraints  $I_+^{i,\theta}$  and  $I_0^{i,\theta}$  are defined as the inequality constraints of problem  $(\mathcal{P}^\theta)$  that are active at  $y^\theta$ , i.e.,

$$(5.3) \quad \begin{cases} I_+^{i,\theta} := \{k = 0, \dots, N; \nu_{k,i}^\theta > 0\}, \\ I_0^{i,\theta} := \{k = 0, \dots, N; \nu_{k,i}^\theta = g_i(y_k^\theta) = 0\}. \end{cases}$$

Finally, the Hessian of the Lagrangian of the discretized problem is, setting  $H_k^\theta := p_{k+1}^\theta f(u_k^\theta, y_k^\theta)$  and for  $z^\theta = z^\theta[v]$ ,

$$(5.4) \quad \Omega^\theta(v, z) := \sum_{k=0}^{N-1} h_k D^2 H_k^\theta(v_k, z_k)^2 + \sum_{k=0}^N h_k \nu_k^\theta D^2 g_k^\theta(z_k)^2 + \phi''(y_N^\theta)(z_N)^2.$$

PROPOSITION 5.1. *Let either (A4) or (A5) hold. Then problem (QP) has a unique solution  $\delta X^\theta := (v^\theta, z^\theta, q^\theta, \delta\eta^\theta)$ , which is equal to the left derivative of  $\theta \mapsto X^\theta$ .*

*Proof.* We apply Jittorntrum’s result [23, Thm. 4]. It states that, for a nonlinear programming problem, the directional derivative (in our case simply the left derivative) of the solution is obtained by solving a quadratic problem whose cost function is the Hessian of the Lagrangian (with respect to both the optimization parameters and the perturbation in the desired direction) under the constraint of linearization of all active constraints. We recall that in this case the inequality constraints associated with a nonzero multiplier are changed into equalities. This theorem has two hypotheses: (i) the surjectivity of the derivative of active constraints for the nonlinear programming problem (which gives the existence and uniqueness of a Lagrange multiplier), and (ii) the positivity of the Hessian of the Lagrangian (with respect to both the optimization parameters) over the *extended critical cone*. The latter is the set of directions in the kernel of the linearization of the constraints associated with a positive Lagrange multiplier. Since the quadratic problem corresponds to (QP) in our setting, we just have to check both hypotheses.

We prove (i) in Proposition 5.2, and (ii) when (A5) holds in Lemma D.2. Finally, assume that (A4) holds. Then (ii) follows from Lemma C.1 in the Appendix C.  $\square$

Let us introduce the following *discrete alternative formulation*: we underline the analogy with the alternative formulation recalled in section 2.3. We first define the set of inequality constraints that are active at the solution of (QP):

$$(5.5) \quad I^{i,\theta} := I_+^{i,\theta} \cup \{k \in I_0^{i,\theta}; g'_i(y_k)z_k^\theta = 0\}.$$

For  $i = 1, \dots, r$ , denote by  $k[i, 1] < \dots < k[i, M_i^\theta]$  the elements of  $I^{i,\theta}$ , set  $k[i, 0] = 0$ , and for  $j = 0, \dots, M_i^\theta - 1$ ,

$$(5.6) \quad \begin{cases} \Delta t_{i,j} & := t_{k[i,j+1]} - t_{k[i,j]}, \\ b_{i,j}^E & := -\theta(h_{k[i,j+1]}^2 \delta_{k[i,j+1]}^g - h_{k[i,j]}^2 \delta_{k[i,j]}^g) / \Delta t_{i,j}. \end{cases}$$

Set

$$(5.7) \quad G_k(v_k, z_k) := \frac{g'(y_{k+1}) - g'(y_k)}{h_k} z_k + g'(y_{k+1}) (f'(u_k, y_k)(v_k, z_k) - \theta h_k \delta_k^y).$$

If  $z = z^\theta[v]$ , then

$$(5.8) \quad G_k(v_k, z_k) = \frac{g'(y_{k+1})z_{k+1} - g'(y_k)z_k}{h_k}.$$

Since  $z_0 = 0$ , it follows that

$$(5.9) \quad g'(y_k^\theta)z_k = \sum_{q=0}^{k-1} h_q G_q(v_q, z_q).$$

So the solution of (QP) satisfies the following equality constraints, denoting by  $G_{k,i}$  the  $i$ th component of  $G_k$ :

$$(5.10) \quad \sum_{q=0}^{k[i,j]-1} h_q G_{i,q}(v_q, z_q) = -\theta h_k^2 \delta_{k[i,j],i}^g, \quad i = 1, \dots, r, \quad j = 1, \dots, M_i^\theta - 1.$$

An equivalent set of equality constraints is

$$(5.11) \quad b_{i,j}^E - \frac{1}{\Delta t_{i,j}} \sum_{k=k[i,j]}^{k[i,j+1]-1} h_k G_{k,i}(v_k, z_k) = 0, \quad i = 1, \dots, r, \quad j = 0, \dots, M_i^\theta - 1.$$

The corresponding quadratic problem is

$$(5.12) \quad (QP_E) \begin{cases} \min_v \frac{1}{2} \Omega^\theta(v, z) - \theta \sum_{k=0}^{N-1} h_k^2 (\delta_k^p z_k + \delta_k^u v_k) \\ \text{subject to } z = z^\theta[v] \text{ and (5.11).} \end{cases}$$

We call  $v^\theta$  the solution of this problem and  $\delta\bar{\eta}^\theta$  the multiplier associated with constraints (5.11) and  $\hat{q}$  the costate. The Lagrangian of  $(QP_E)$  is by the definition

$$(5.13) \quad \begin{aligned} \Omega^\theta(v, z) - \theta \sum_{k=0}^{N-1} h_k^2 (\delta_k^p z_k + \delta_k^u v_k) + \sum_{k=0}^{N-1} \hat{q}_{k+1} (h_k f'_k(v_k, z_k) + z_k - z_{k+1} - \theta h_k^2 \delta_k^y) \\ + \sum_{i,j} \Delta t_{i,j} \delta\bar{\eta}_{i,j}^\theta \left( b_{i,j}^E - \sum_{k=k[i,j]}^{k[i,j+1]-1} h_k G_k(v_k, z_k) / \Delta t_{i,j} \right). \end{aligned}$$

The optimality conditions of  $(QP_E)$  have the following form. The costate equation is

$$(5.14) \quad \begin{aligned} \hat{q}_k &= \hat{q}_{k+1} + h_k \hat{q}_{k+1} f_y^k + h_k (v_k)^T H_{uy}^k + h_k (z_k)^T H_{yy}^k + h_k \nu_k (z_k)^T g''(y_k) \\ &\quad - \sum_{i=1}^r \delta\bar{\eta}_{i,j[i,k]} (g'(y_{k+1}) - g'(y_k) + h_k g'(y_{k+1}) f_y(u_k, y_k)) - \theta h_k^2 \delta_k^p; \\ \hat{q}_N &= (z_N)^T \phi''(y_N) + h_N \nu_N (z_N)^T g''(y_N). \end{aligned}$$

Given  $k \leq M_i$ , set

$$(5.15) \quad j[i, k] := \min\{j \in I^{i,\theta}; \quad j \geq k + 1\}.$$

Expressing the stationarity of the Lagrangian with respect to  $v$ , we get that

$$(5.16) \quad (v_k)^T H_{uu}^k + (z_k)^T H_{uy}^k + \left( \hat{q}_{k+1} - \sum_{i=1}^r \delta\bar{\eta}_{i,j[i,k]} g'(y_{k+1}) \right) f_u^k = 0.$$

This suggests defining

$$(5.17) \quad \begin{aligned} \tilde{q}_{k+1} &:= \hat{q}_{k+1} - \sum_{i=1}^r \delta\bar{\eta}_{i,j[i,k]} g'_i(y_{k+1}), \quad k = 0, \dots, N - 1, \\ \delta\bar{\nu}_k &:= \delta\bar{\eta}_{i,j[i,k]} - \delta\bar{\eta}_{i,j[i,k-1]}, \quad k = 0, \dots, N. \end{aligned}$$

Then  $\tilde{q}$  is a solution of

$$(5.18) \quad \begin{aligned} \tilde{q}_k &= \tilde{q}_{k+1} + h_k \tilde{q}_{k+1} f_y^k + h_k (v_k)^T H_{uy}^k + h_k (z_k)^T H_{yy}^k + h_k (z_k)^T g''(y_k) \\ &\quad + h_k \delta\bar{\nu}_k g'(y_k) - \theta h_k^2 \delta_k^p, \\ \tilde{q}_N &= (z_N)^T \phi''(y_N) + h_N \nu_N (z_N)^T g''(y_N) + h_N \delta\bar{\nu}_N g'(y_N). \end{aligned}$$

In addition, we observe that if  $\delta\bar{\nu}_k \neq 0$ , then  $\delta\bar{\eta}_{i,j[i,k]} > \delta\bar{\eta}_{i,j[i,k-1]}$  by (5.17), and therefore the  $i$ th state constraint is active at step  $k$ . Since  $(QP)$  has a unique multiplier, we deduce that  $\tilde{q}$  is equal to the solution  $q$  of (3.5), and

$$(5.19) \quad \delta\bar{\nu}_k = \delta\nu_k; \quad \delta\bar{\eta}_{i,j[i,k]} = \delta\eta_{i,k+1}.$$



**5.2. Uniform surjectivity.** We write here the *linear mappings* involved in the quadratic problem  $(QP_E)$ , starting with

$$(5.20) \quad z_{k+1}^L = z_k^L + h_k f_y(u_k, y_k)(v_k, z_k^L), \quad k = 0, \dots, N-1; \quad z_0^L = 0.$$

For any  $v$  in the space  $\mathcal{V}^N$ , (5.20) has a unique solution denoted by  $z^L[v]$ . Let  $\xi_k$  be the solution of  $\xi_0 = 0$  and

$$(5.21) \quad \xi_{k+1} = \xi_k + h_k f_y^k \xi_k - \theta h_k^2 \delta_y^k; \quad k = 0, \dots, N-1; \quad \xi_0 = 0.$$

We have that

$$(5.22) \quad z_k = z_k^L + \xi_k; \quad k = 0, \dots, N, \quad \|\xi\|_\infty = O(\bar{h}).$$

We set

$$(5.23) \quad \begin{aligned} \mathcal{G}_{i,j}^L(v) &:= (g'_i(y_{k[i,j+1]})z_{k[i,j+1]}^L[v] - g'_i(y_{k[i,j]})z_{k[i,j]}^L[v]) / \Delta t_{i,j}, \\ j &= 0, \dots, M_i^\theta - 1; \quad i = 1, \dots, r. \end{aligned}$$

The linear (homogeneous) equations corresponding to those of  $(QP_E)$  are therefore  $\mathcal{G}^L(v) = 0$ . Consider the following perturbation of the right-hand side of these equations, where  $\bar{b}$  is an arbitrary term:

$$(5.24) \quad \mathcal{G}^L(v) = \bar{b}.$$

We use the following norm for  $s \in [1, \infty)$ :

$$(5.25) \quad \|\bar{b}\|_s^s := \sum_{i=1}^r \sum_{j=0}^{M_i^\theta - 1} \Delta t_{i,j} |\bar{b}_{i,j}|^s.$$

These norms can be identified with the usual  $L^s$  norms on  $[0, T]$  for piecewise constant functions and therefore we have the usual Cauchy–Schwarz and Hölder inequalities, in particular

$$(5.26) \quad \|\bar{b}\|_1 \leq \sqrt{rT} \|\bar{b}\|_\infty$$

**PROPOSITION 5.2.** *There exist constants  $C_s, s \in [1, \infty]$ , such that the linear equation  $\mathcal{G}^L(v) = \bar{b}$  has, for small enough  $\bar{h}$ , a solution  $v$  verifying*

$$(5.27) \quad \|v\|_s \leq C_s \|\bar{b}\|_s \quad \text{for each } s \in [1, \infty].$$

Before proving Proposition 5.2, we introduce some notations. For  $t \in [0, T]$  and  $\varepsilon_0 > 0$ , we define the set of  $\varepsilon_0$  *active constraints* as

$$(5.28) \quad \mathcal{A}_{\varepsilon_0}(t) := \{1 \leq i \leq r; |t' - t| \leq \varepsilon_0 \text{ for some } t' \text{ such that } i \in \mathcal{A}(t')\}.$$

Since the control is continuous by **(A1)**, and the first-order state constraint satisfies **(A2)**, we have that, for  $\varepsilon_0 > 0$  small enough,

$$(5.29) \quad \left| \sum_{i \in \mathcal{A}_{\varepsilon_0}(t)} \lambda_i \nabla_u g_i^{(1)}(\bar{u}, \bar{y}) \right| \geq \frac{1}{2} \alpha |\lambda|, \quad \text{if } \lambda_i = 0 \text{ when } i \notin \mathcal{A}_{\varepsilon_0}(t), \text{ for all } t \in [0, T].$$

For  $i \in \{1, \dots, r\}$ , we denote the set of  $\varepsilon_0$  active constraints by  $J_{\varepsilon_0}^{i,\theta}$ ,  $i = 1, \dots, r$ . This is a union of closed balls (in  $[0, T]$ ) of radius  $\varepsilon_0$ . Since every connected component has length at least  $2\varepsilon_0$ ,  $J_{\varepsilon_0}^{i,\theta}$  is a finite union of closed intervals.

We next define the one-time-step analogue of  $\mathcal{G}^L$ :

$$(5.30) \quad G_{k,i}^L(v_k, z_k) := \frac{g'_i(y_{k+1}) - g'_i(y_k)}{h_k} z_k[v] + g'_i(y_{k+1})(f'(u_k, y_k)(v_k, z_k[v])).$$

Note that  $G_{k,i}^L(v_k, z_k) = (g'_i(y_{k+1})z_{k+1}[v] - g'_i(y_k)z_k[v])/h_k$ .

*Proof of Proposition 5.2.* The idea is to compute, for each  $k$ ,  $v_k$  as the minimum norm solution of the linear equations

$$(5.31) \quad G_{k,i}^L(v_k, z_k^L) = \tilde{b}_{k,i}, \quad i \in \mathcal{A}_\varepsilon(t_k),$$

where the variable size vector  $\tilde{b}_{k,i}$  for  $i$  in  $\mathcal{A}_\varepsilon(t_k)$  will be defined later, and then to set

$$(5.32) \quad \tilde{b}_{k,i} := G_{k,i}^L(v_k, z_k^K) \text{ for } i \notin \mathcal{A}_\varepsilon(t_k).$$

Thanks to the expression of  $G_{k,i}^L$  and (5.29) setting  $z^L = z^L[v]$ , we have that

$$(5.33) \quad |v_k| \leq c_1 \left( |z_k^L| + \sum_{i \in \mathcal{A}_\varepsilon(t_k)} |\tilde{b}_{k,i}| \right).$$

Here the  $c_i$  are positive constants not depending on  $v$  or  $k$ . It follows with (5.30) that

$$(5.34) \quad |\tilde{b}_k| \leq c_2 (|v_k| + |z_k^L[v]|) \leq c_3 \left( |z_k^L| + \sum_{i \in \mathcal{A}_\varepsilon(t_k)} |\tilde{b}_{k,i}| \right).$$

So, by (5.20)

$$(5.35) \quad |z_{k+1}^L| \leq (1 + c_4 h_k) |z_k^L| + c_5 h_k |v_k| \leq (1 + c_6 h_k) |z_k^L| + c_7 h_k \sum_{i \in \mathcal{A}_\varepsilon(t_k)} |\tilde{b}_{k,i}|.$$

By the discrete Gronwall's lemma it follows that

$$(5.36) \quad \|z^L\|_\infty \leq c_8 \sum_{k=0}^{N-1} h_k \sum_{i \in \mathcal{A}_\varepsilon(t_k)} |\tilde{b}_{k,i}|,$$

and therefore, with (5.33),

$$(5.37) \quad \|v\|_s^s \leq c_9 \sum_{k=0}^{N-1} h_k \sum_{i \in \mathcal{A}_\varepsilon(t_k)} |\tilde{b}_{k,i}|^s,$$

and in the right-hand side we recognize the  $L^s$  norm of the  $\varepsilon_0$  active components of  $\tilde{b}$ . We next end the proof by fixing the  $\tilde{b}_k$  in such a way that

$$(5.38) \quad G_{i,j}^L(v) = \bar{b}_{i,j}; \quad j = 0, \dots, M_i^\theta - 1; \quad \|\tilde{b}\|_s = O(\|\bar{b}\|_s).$$

We obtain the second relation by induction over  $k$ : we prove that there exists  $c > 0$  such that

$$(5.39) \quad \sum_{\ell \leq k} |\tilde{b}_\ell|^s \leq c \sum_{i; k[i,j] \leq k} |\bar{b}_{k,i}|^s.$$

We distinguish two cases.

(a) If  $\{t_{k[i,j]}, \dots, t_{k[i,j+1]}\} \in J_{\varepsilon_0}^{i,\theta}$ , then take

$$(5.40) \quad \tilde{b}_{k,i} = \bar{b}_{k[i,j+1]}, \quad k = k[i,j] + 1, \dots, k[i,j + 1].$$

(b) If  $\{t_{k[i,j]}, t_{k[i,j+1]}\} \notin J_{\varepsilon_0}^{i,\theta}$ , let  $k'$  be the smallest index in  $k[i,j], \dots, k[i,j + 1]$  such that  $k \in J_{\varepsilon_0}^{i,\theta}$  whenever  $k' \leq k \leq k[i,j + 1]$ ; then

$$(5.41) \quad t_{k'} + \frac{1}{2}\varepsilon_0 \leq t_{k[i,j+1]}.$$

We choose

$$(5.42) \quad \tilde{b}_{k,i} = \begin{cases} \bar{b}_{i,j}, & k = k[i,j] + 1, \dots, k', \\ \gamma, & k = k' + 1, \dots, k[i,j + 1] \end{cases}$$

for some  $\gamma$  such that

$$(5.43) \quad \sum_{k=k[i,j]+1}^{k'} h_k \tilde{b}_{k,i} + \gamma(t_{k[i,j+1]} - t_{k'}) = (t_{k[i,j+1]} - t_{k[i,j]})\bar{b}_{k[i,j+1]},$$

so that the first relation in (5.38) holds. At the same time, since  $\frac{1}{2}\varepsilon_0 \leq t_{k[i,j+1]} - t_{k'}$ , we have that

$$(5.44) \quad \begin{aligned} \gamma &\leq \frac{2}{\varepsilon_0} \left| (t_{k[i,j+1]} - t_{k[i,j]})\bar{b}_{k[i,j+1]} - \sum_{k=k[i,j]+1}^{k'} h_k \tilde{b}_{k,i} \right| \\ &\leq \frac{2}{\varepsilon_0} \left( T|\bar{b}_{k[i,j+1]}| + \sum_{k \leq k'} h_k |\tilde{b}_{k,i}| \right). \end{aligned}$$

We use the induction hypothesis (5.39) in order to estimate  $\sum_{k \leq k'} h_k |\tilde{b}_{k,i}|$ . The conclusion follows.  $\square$

From the same argument of the previous result we can deduce also an estimate for the control of a feasible trajectory.

Note that

$$(5.45) \quad G_{k,i}(v_k, z_k) = G_{k,i}^L(v_k) + O(\bar{h}), \quad i = 1, \dots, r.$$

**COROLLARY 5.3.** *Problem  $(QP_E)$  has a feasible point  $\tilde{v}$  such that  $\|\tilde{v}\|_\infty = O(\bar{h})$ .*

*Proof.* In view of the proposition above, it is enough to check that we can write the active constraints of  $(QP_E)$  in the form

$$(5.46) \quad G_{k,i}^L(v) = \bar{b}_{k,i}; \quad \|\bar{b}\|_\infty = O(\bar{h}).$$

Remember that  $z[v] = z^L[v] + \xi_k[v]$  with  $\|\xi[v]\|_\infty = O(\|v\|_1)$ ; therefore

$$(5.47) \quad \begin{aligned} \bar{b}_{i,j} = & -\theta \left( \frac{g'_i(y_{k[i,j+1]}) - g'_i(y_{k[i,j]})}{\Delta t_{i,j}} + g'_i(y_{k[i,j+1]}) f'_y(u_{k[i,j]}, y_{k[i,j]}) \right) \xi_{k[i,j]} \\ & - \theta (h_{k+1}^2 \delta_{k+1}^g - h_k^2 \delta_k^g) / h_k, \end{aligned}$$

which is of the desired form. □

**COROLLARY 5.4.** *Let either (A4) or (A5) hold. Then  $(QP_E)$  has a unique solution  $v^\theta$  associated with a unique alternative multiplier  $\delta\bar{\eta}^\theta$ , which satisfy*

$$(5.48) \quad \|v^\theta\|_2 + \|\delta\bar{\eta}^\theta\|_2 = O(\bar{h}^2).$$

*Proof.* The existence and uniqueness properties were obtained in Proposition 5.1, and (5.48) follows from Lemma B.1, whose hypotheses hold in view of Proposition 5.2 and Lemma C.1. □

**5.3. Estimate of the derivatives with respect to  $\theta$ .** We arrive, finally, at the main result of the section. We recall that  $\delta X^\theta := (v^\theta, z^\theta, q^\theta, \delta\eta^\theta)$  is the derivative (from the left with respect to  $\theta$ ) of  $X^\theta := (u^\theta, y^\theta, p^\theta, \eta^\theta)$ .

**PROPOSITION 5.5.** *Let either (A4) or (A5) hold. Then, for fixed  $C > 0$ ,*

$$\|v^\theta\|_\infty + \|z^\theta\|_\infty + \|q^\theta\|_\infty + \|\delta\eta^\theta\|_\infty \leq C\bar{h}.$$

*Proof.* Applying Lemma B.1 to  $(QP_E)$ , where  $X$  and  $Y$  have norms defined by (5.27) (where  $s = 2$ ) and (5.25), and keeping in mind that, by the definition of the Lagrangian, we have identified the image space (of  $b^E$ ) with its dual, we get that

$$(5.49) \quad \|v^\theta\|_2 + \sum_{i,j} \Delta t_{i,j} |\delta\bar{\eta}_{i,j}^\theta|^2 \leq c_1 \bar{h}.$$

Fix  $\varepsilon_\eta > 0$ , not depending on  $\bar{h}$ . Then

$$(5.50) \quad \text{If } \Delta t_{i,j} > \varepsilon_\eta, \text{ then } |\delta\bar{\eta}_{i,j}^\theta| \leq c_1 \varepsilon_\eta^{-1/2} \bar{h}.$$

So, as far as  $\delta\bar{\eta}^\theta$  is concerned, it remains to obtain a uniform estimate when  $\Delta t_{i,j} \leq \varepsilon_\eta$ . It easily follows from (5.49), the state equation (5.2), and the costate equations (5.14) that

$$(5.51) \quad \|z^\theta\|_\infty + \|q^\theta\|_\infty \leq c_2 \bar{h}.$$

By (5.6),  $|\bar{b}_{i,j}| = O(\bar{h})$ . Evaluating the contribution of the term containing  $z_k$  and observing  $\sum_{k=k[i,j]+1}^{k[i,j+1]} h_k = \Delta t_{i,j}$ , we have

$$(5.52) \quad \frac{1}{\Delta t_{i,j}} \sum_{k=k_i+1}^{k_{i+1}} h_k \nabla_u \hat{g}_{k,i}^{(1)} v_k = O(\bar{h}).$$

Eliminating  $v_k$  in (5.16), we get

$$(5.53) \quad \frac{1}{\Delta t_{i,j}} \sum_{k=k[i,j]+1}^{k[i,j+1]} h_k \nabla_u \hat{g}_{k,i}^{(1)} (H_{uu}^k)^{-1} \sum_{i'=1}^r (\nabla_u \hat{g}_{i',k}^{(1)})^T \delta\bar{\eta}_{i',j[i',k]}^\theta = O(\bar{h}).$$

Multiplying by  $\delta\bar{\eta}_{i,j[i,k]}^\theta$  on the left, summing over  $i$  and  $j \in I_{i',k}$ , and setting

$$(5.54) \quad w_k := \sum_{i=1}^r (\nabla_u \hat{g}_{k,i}^{(1)})^T \delta\bar{\eta}_{i,j[i,k]}^\theta,$$

we get

$$(5.55) \quad \frac{1}{\Delta t_{i,j}} \sum_{i=1}^r \sum_{k=k[i,j]+1}^{k[i,j+1]} h_k w_k^T (H_{uu}^k)^{-1} w_k = O(\bar{h}) |\delta\bar{\eta}^\theta|.$$

Denote by  $\hat{\eta}$  the vector of components  $\delta\bar{\eta}_{i,j[i,k]}^\theta$  for  $i = 1$  to  $r$ . For small enough  $\varepsilon$  depending on  $\hat{\eta}$ , by **(A2)**–**(A3)**, we have that

$$(5.56) \quad |\hat{\eta}|^2 \leq \alpha_g^2 |w_k|^2 \leq \alpha^{-1} \alpha_g^2 w_k^T (H_{uu}^k)^{-1} w_k,$$

and therefore, since  $\Delta t_{i,j} = \sum_{k=k[i,j]+1}^{k[i,j+1]} h_k$ ,

$$(5.57) \quad |\hat{\eta}|^2 \leq \frac{\alpha^{-1} \alpha_g^2}{\Delta t_{i,j}} \sum_{k=k[i,j]+1}^{k[i,j+1]} w_k^T (H_{uu}^k)^{-1} w_k = O(\bar{h}) |\hat{\eta}|.$$

Therefore, we get with (5.19) that

$$(5.58) \quad |\delta\bar{\eta}_{i,j[i,k]}^\theta| \leq O(\bar{h}).$$

The corresponding estimates for  $v^k$  and  $\delta\eta^\theta$  follow from (5.16) and (5.19). □

**6. Example.** We present an academic example which is a variant of the application discussed in [20]. Let us consider the following optimal control problem, for some  $\bar{\varepsilon} > 0$ , where  $g := 9.8$ :

$$\begin{aligned} & \min \int_0^1 \left( \frac{1}{2} u^2(t) + gy(t) \right) dt + (y(1) - c)^2 / \bar{\varepsilon}, \\ & \text{subject to } \dot{y}(t) = u(t), \quad y(0) = c, \quad y(t) \geq 0. \end{aligned}$$

The solution of this problem can be seen as the minimum energy state of a system composed by an elastic line of uniformly distributed mass in a constant gravity field with the presence of a lower constraint. We can add a state variable, say  $\tilde{y}$ , with zero initial condition and derivative equal to the integrand of the integral cost, and reformulate the cost as  $\tilde{y}(1) + (y(1) - c)^2 / \bar{\varepsilon}$ , in order to comply with the format of the theoretical results. We obtain the problem (for a fixed parameter  $\bar{\varepsilon} > 0$ )

$$\begin{aligned} & \min \phi(y(1), \tilde{y}(1)) := \tilde{y}(1) + (y(1) - c)^2 / \bar{\varepsilon} \\ & \text{subject to } \begin{cases} \dot{y}(t) = u(t), \\ \dot{\tilde{y}}(t) = \frac{1}{2} u^2(t) + gy(t), \end{cases} \quad \begin{cases} y(0) = c, \\ \tilde{y}(0) = 0, \end{cases} \quad y(t) \geq 0. \end{aligned}$$

It is known that the costate associated with  $\tilde{y}$  has value 1 (observe that this problem is qualified) and that the costate associated with  $y$  and the measure associated with the state constraints are invariant under this reformulation.

The exact solution for this problem is (assuming  $\bar{\varepsilon} < 2c/g$ )

$$u(t) = \begin{cases} g(t - t_{en}), & t \in [0, t_{en}), \\ 0, & t \in [t_{en}, t_{ex}), \\ g(t - t_{ex}), & t \in [t_{ex}, 1], \end{cases}$$

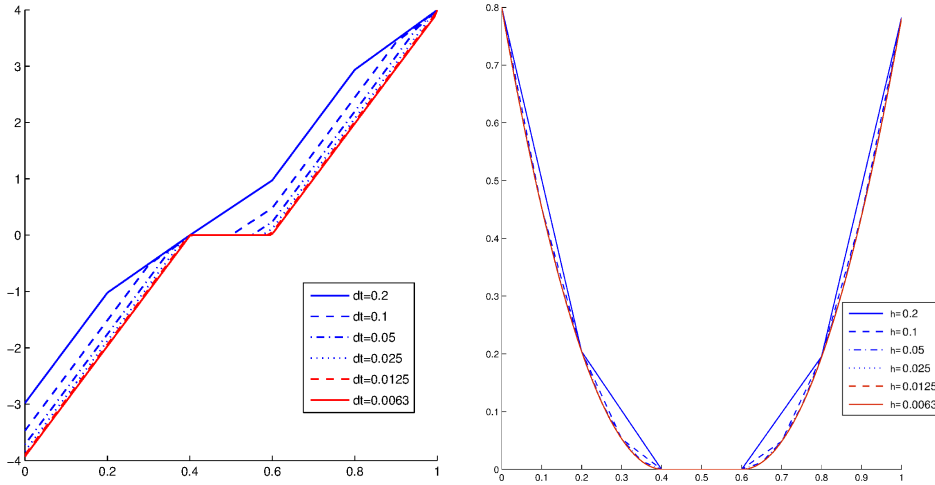


FIG. 1. Control and state with various choices of  $h$ . We can experimentally observe the convergence of the numerical approximation, the regularity in the junction points, and the stability of the boundary arcs.

TABLE 1  
Experimental error at various discretization steps.

$h$	$\ y^h - \bar{y}\ _\infty$	$Ord(L^\infty)$	$\ u^h - \bar{u}\ _\infty$	$Ord(L^\infty)$
<b>0.2</b>	0.0513		0.9996	
<b>0.1</b>	0.0139	1.87	0.4996	1
<b>0.05</b>	0.0044	1.65	0.2546	0.97
<b>0.025</b>	0.0020	1.18	0.1311	0.96
<b>0.0125</b>	0.0012	1	0.0668	0.97
<b>0.0063</b>	0.0006	1	0.0334	1
<b>0.0032</b>	0.0003	1	0.0172	0.96

where  $t_{en} = \sqrt{2c/g}$  and  $t_{ex} = 1 - \sqrt{2c/g - \bar{\varepsilon}}$ . The optimal trajectory is

$$y(t) = \begin{cases} g(t^2/2 - t_{en}t) + c, & t \in [0, t_{en}), \\ 0, & t \in [t_{en}, t_{ex}), \\ g(t - t_{ex})^2/2, & t \in [t_{ex}, 1], \end{cases}$$

which means that  $y(1) = c - g\bar{\varepsilon}/2$ . If  $\bar{\varepsilon} \geq 2c/g$ , then the solution is as above for  $t \in [0, t_{en})$  and simply equal to zero otherwise.

We solved the discrete solution using an SQP (sequential quadratic programming) algorithm, a popular iterative method for nonlinear optimization [2, sect. 15]; in Figure 1 and Table 1 are shown the results at various constant discrete steps  $h$  for  $c = 0.8$  and  $\bar{\varepsilon} = 0.01$ . This test confirms the convergence results stated previously. The small discrepancy on the order of convergence of the control  $u$  comes from some numerical error introduced in the optimization of the discrete system.

**Appendix A. About strict critical cones.** The strict critical cone  $C(\bar{u})$  defined in (2.11)–(2.13) can be compared to the (standard) critical cone  $\hat{C}(\bar{u})$  defined as the set of  $v \in \mathcal{V}$  such that  $z := z[v] \in \mathcal{Z}$  satisfies

(A.1) 
$$g'_i(\bar{y}(t))z(t) \leq 0, \quad t \in I_i,$$

(A.2) 
$$\phi'(\bar{y}(T))z(T) \leq 0.$$

LEMMA A.1. *We have that  $C(\bar{u}) \subset \hat{C}(\bar{u})$ . If  $(\bar{u}, \bar{y})$  is a stationary point with associated multiplier  $(p, \eta)$  such that the support of  $\eta_i$  is equal to the contact set  $I_i$ , for  $i = 1$  to  $r$ , then  $C(\bar{u}) = \hat{C}(\bar{u})$ .*

*Proof.* That  $C(\bar{u}) \subset \hat{C}(\bar{u})$  follows from the definition of these sets. Let  $(p, \eta)$  be as in the lemma. For the sake of simplicity of notations we prove the second statement for the case of a scalar state constraint. The Lagrangian function for the continuous problem is

$$(A.3) \quad L(u, y, p, \eta) := \phi(y(T)) + \int_0^T p(t) \cdot (f(u(t), y(t)) - \dot{y}(t)) dt + \int_0^T g(y(t)) d\eta(t).$$

Let  $v$  be a critical direction and set  $z := z[v]$ . Then, by the definition of a critical direction,

$$(A.4) \quad DL(u, y, p, \eta)(v, z) = \phi'(y(T))z(T) + \int_0^T g'(y(t))z(t) d\eta(t).$$

On the other hand, we can observe that the stationarity conditions imply the terms  $D_u L(u, y, p, \eta) = 0$  and  $D_y L(u, y, p, \eta) = 0$ , then

$$(A.5) \quad \phi'(y(T))z(T) + \int_0^T g'(y(t))z(t) d\eta(t) = 0.$$

Since  $\phi'(y(T))z(T) \leq 0$  and  $g'(y(t))z(t) d\eta(t) \leq 0$  a.e., this implies  $\phi'(y(T))z(T) = 0$  and  $g'(y(t))z(t) d\eta(t) = 0$  a.e. The conclusion follows.  $\square$

**Appendix B. Coercive quadratic programs.** We recall a classical consequence of the coercivity of the cost function of an equality constrained quadratic problem over its feasible set. To keep the notation as simple as possible, we formulate the problem in an abstract way. The result will be stated with the same notation as in Corollary 5.4. Given two Hilbert spaces  $X$  and  $Y$ , identified with their dual, consider the optimization problem

$$(B.1) \quad \min_{x \in X} (c, x)_X + \frac{1}{2} (Hx, x)_X \quad \text{subject to} \quad Ax = b \text{ in } Y,$$

where  $(\cdot, \cdot)_X$  denotes the scalar product in  $X$  (and  $\|\cdot\| := (\cdot, \cdot)_X$ ) with a similar convention for  $Y$ ,  $H : X \rightarrow X$  is symmetric,  $A \in L(X, Y)$ ,  $c \in X$ , and  $b \in Y$ . The Lagrangian of the problem is

$$(B.2) \quad (c, x)_X + \frac{1}{2} (Hx, x)_X + (\lambda, Ax - b)_Y.$$

The associated optimality conditions are

$$(B.3) \quad c + Hx + A^T \lambda = 0; \quad Ax = b.$$

The next lemma is classical (see [9, Thm. 1. and Prop. 1.1]) but it is worth giving a short proof of it.

LEMMA B.1. *Let  $\alpha > 0$  and  $c_A > 0$  be such that*

- (i) *(Coercivity)  $\alpha \|x\|^2 \leq (Hx, x)_X$  for all  $x \in \text{Ker } A$ ;*
- (ii) *(Strong surjectivity) for any  $b' \in Y$ , there exists  $x' \in X$  such that  $Ax = b'$  and  $\|x'\| \leq c_A \|b'\|$ .*

Then there exists  $\kappa > 0$ , a function of  $\alpha$  and  $c_A$ , such that (B.1) has a unique solution  $\bar{x}$  and an associated Lagrange multiplier  $\lambda$  such that

$$(B.4) \quad \|\bar{x}\| + \|\lambda\| \leq \kappa(\|b\| + \|c\|).$$

*Proof.* The fact that (B.1) has a unique solution  $\bar{x}$  is consequence of the coercivity (which in the case of a quadratic cost implies strong convexity over the feasible set since the latter is a vector subspace). The uniqueness of the Lagrange multiplier  $\lambda$  is a consequence of the surjectivity of  $A$ , implied by the strong surjectivity.

The latter also implies the existence of  $x^0$  such that  $Ax^0 = b$  and  $\|x^0\| \leq c_A\|b\|$ . Then  $\delta x := \bar{x} - x^0$  is a solution of

$$(B.5) \quad H\delta x + A^T\lambda = -c - Hx^0; \quad A\delta x = 0.$$

Therefore, since  $\delta x \in \text{Ker } A$ ,

$$\alpha\|\delta x\|^2 \leq \delta x^T H\delta x = -(\delta x, c + Hx^0)_X + (A\delta x, \lambda)$$

so that  $\|\delta x\| \leq (\|c\| + \|Hx^0\|)/\alpha$ . Since  $\|x^0\| \leq c_A\|b\|$ , we deduce that

$$(B.6) \quad \|\bar{x}\| \leq \|\delta x\| + \|x^0\| \leq c_A\|b\| + \frac{1}{\alpha}(\|c\| + \|H\|c_A\|b\|).$$

By the surjectivity hypothesis,

$$(B.7) \quad \|\lambda\| \leq \frac{1}{c_A}\|A^T\lambda\| \leq \frac{1}{c_A}(\|c\| + \|H\|\|x\|).$$

The conclusion follows. □

**Appendix C. Stability of the Hessian of the Lagrangian.** Given  $v \in \mathcal{V}^N$ , we denote by  $\bar{v}$  the associated corresponding piecewise constant element defined by

$$(C.1) \quad \bar{v}(t) = v_k, \quad t \in (t_k, t_{k+1}) \quad \text{for all } k = 0 \text{ to } N - 1.$$

Note that  $v$  and  $\bar{v}$  have the same  $L^s$  norm,  $s \in [1, \infty]$ . Setting  $\bar{z} := z[\bar{v}]$  (the solution of the state equation (2.7) for the linearized, continuous in time problem), we easily check that

$$(C.2) \quad \|z^\theta - \bar{z}\|_\infty = \|v\|_\infty O(\varepsilon + \bar{h}).$$

We apply the previous result to  $(QP_E)$  using the following lemma.

LEMMA C.1. *We have that, for any  $v$  in  $\mathcal{U}^N$ ,*

$$|\Omega^\theta(v, z^L[v]) - \Omega(\bar{v})| = O(\varepsilon\|\bar{v}\|)^2.$$

*Proof.* Since, by the definition (see (2.20)),  $h_k^\theta \nu_k^\theta = \bar{\eta}_k^\theta - \bar{\eta}_{k+1}^\theta$ , it follows that

$$\begin{aligned} \Delta &:= \sum_{k=0}^N h_k \nu_k^\theta D^2 g_k^\theta(z_k)^2 = \sum_{k=0}^N (\bar{\eta}_k^\theta - \bar{\eta}_{k+1}^\theta) D^2 g_k^\theta(z_k)^2 \\ &= \sum_{k=1}^N \bar{\eta}_k^\theta (D^2 g_k^\theta(z_k)^2 - D^2 g_{k-1}^\theta(z_{k-1})^2) = \Delta_1 + \Delta_2, \end{aligned}$$



where, by a Taylor expansion of  $D^2g^\theta$ , we define, for some  $\hat{y}_k \in [y_{k-1}, y_k]$ ,

$$\begin{aligned} \Delta_1 &:= \sum_{k=1}^N \bar{\eta}_k (D^2g_k^\theta(z_k)^2 - D^2g_{k-1}^\theta(z_k)^2) = \sum_{k=1}^N h_k \bar{\eta}_k D^3g^\theta(\hat{y}_k)(f_{k-1}^\theta, z_k, z_k), \\ \Delta_2 &:= \sum_{k=1}^N \bar{\eta}_k^\theta (D^2g_{k-1}^\theta(z_k)^2 - D^2g_{k-1}^\theta(z_{k-1})^2) \\ &= \sum_{k=1}^N h_{k-1} \eta_k^\theta D^2g_{k-1}^\theta(z_k + z_{k-1}, Df_{k-1}^\theta(v_{k-1}, z_{k-1})), \end{aligned}$$

where we used the identity  $A(b, b) - A(a, a) = A(a + b, a - b)$  for any symmetric bilinear form  $A$  and the linearized state equation. We deduce that, for  $\bar{h}$  small enough,

$$\Delta_1 = \int_0^T g^{(3)}(\bar{y}(t))(f[t], \bar{z}(t), \bar{z}(t)) \bar{\eta}(t) dt + O(\varepsilon \|\bar{v}\|)$$

and

$$\Delta_2 = 2 \int_0^T g''(\bar{y}(t))(\bar{z}(t), f_y[t](\bar{z}(t), \bar{v}(t)) \bar{\eta}(t) dt + O(\varepsilon \|\bar{v}\|).$$

Using the identity (2.17), we can claim that  $|\Omega^\theta(v, z) - \tilde{\Omega}(\bar{v})| = O(\varepsilon \|\bar{v}\|)^2$ . We conclude with Lemma 2.4. □

**Appendix D. Analysis of assumption (A5).** As shown before, a key point of the theory is the coercivity of the Hessian of the Lagrangian of  $(QP)$  over the kernel of equality constraints. Under the assumptions (A5), we can obtain it directly by showing the stability of the boundary arcs.

A main point is contained in the following lemma.

LEMMA D.1. *Let (A5) hold. Given a boundary arc  $(t_a, t_b)$ , let  $k_a$  and  $k_b$  be defined as*

$$(D.1) \quad k_a \text{ (resp., } k_b\text{): first index (resp., last index) for which } t_k > t_a \text{ (resp., } t_k < t_b\text{)}.$$

*Let  $\varepsilon' > 0$ . Reducing  $\varepsilon > 0$  small enough, we have that when  $\bar{h}$  is small enough, the following holds.*

$$\nu_k^\theta > 0 \text{ for all } 0 \leq k \leq N \text{ such that } t_{k_a} + \varepsilon' < t_k < t_{k_b} - \varepsilon'.$$

*Proof.* The argument has two steps.

(a) By the definition  $\|\eta^\theta - \bar{\eta}\|_\infty < \varepsilon$ , and by (A5),  $\bar{\eta}$  has a uniformly positive derivative over  $(t_{k_a} + \varepsilon' < t_k < t_{k_b} - \varepsilon')$  minorized by  $c_1 > 0$ . We have that, for  $k_a < k < k_b$ ,

$$(D.2) \quad \eta_k^\theta \geq \bar{\eta}(t_k) - \varepsilon \geq \bar{\eta}(t_{k_a}) + c_1(t_k - t_{k_a}) - \varepsilon \geq \eta_{k_a}^\theta + c_1(t_k - t_{k_a}) - 2\varepsilon,$$

that is,

$$(D.3) \quad c_1(t_k - t_{k_a}) - 2\varepsilon \leq \eta_k^\theta - \eta_{k_a}^\theta.$$

Therefore, if  $t_k - t_{k_a} > 2\varepsilon/c_1$ , then the above right-hand side must be positive, proving that the constraint is active for some  $k$  such that  $t_k \leq t_{k_a} + 2\varepsilon/c_1$ .

- (b) If the conclusion does not hold, by step (a),  $g(y_k^\theta)$  would have a local minimum (possibly with zero value) for some  $k_a < k < k_b$ , such that  $\nu_k^\theta = 0$ . Then  $\Delta_g^k$  defined in (4.7) is nonnegative. Multiplying (4.3) by  $\nabla_u g^{(1)}(u_k, y_k)^T (\mathcal{H}_k^\theta)^{-1}$  on the left and using Lemma 4.1(ii) and (4.9), we get

$$(D.4) \quad -\frac{\Delta_g^k}{h_k} + \nu_k (\nabla_u g^{(1)}(u_k, y_k))^T \mathcal{H}_k^{-1} \nabla_u g^{(1)}(u_k, y_k) = \hat{\Xi}_k^\theta,$$

where  $\hat{\Xi}_k^\theta$  is such that  $\|\hat{\Xi}_k^\theta - \hat{\Xi}^1\|_\infty = O(\varepsilon)$ . We have that the left-hand side of (D.4) is greater than a positive constant independent on  $\bar{h}$ , since, for  $\theta = 1$ , for some  $K > 0$ ,  $\Delta_g^k = 0$ ,  $\nu_k > K$ , and  $|\nabla_u g^{(1)}| > K$ , i.e., for some  $C > 0$ ,

$$(D.5) \quad -\frac{\Delta_g^k}{h_k} + \nu_k (\nabla_u g^{(1)}(u_k, y_k))^T \mathcal{H}_k^{-1} \nabla_u g^{(1)}(u_k, y_k) \geq C.$$

This relation is still valid for  $\varepsilon$  and  $\bar{h}$  small enough, for all  $\theta \in [\theta_m, 1]$ , in view of the continuity of the right-hand side of (D.4). However, it cannot hold when  $\nu_k = 0$  and  $\Delta_g^k \geq 0$ , so we get the desired contradiction.  $\square$

Here we prove the coercivity of  $\Omega^\theta$  over the kernel of equality constraints of  $(QP)$ . Note that such a property is naturally preserved passing to the alternative formulation  $\tilde{\Omega}^\theta$  as shown, for the continuous case in section 2.

LEMMA D.2. *Let (A5) hold. Then  $v \mapsto \Omega^\theta(v, Z^L[v])$  is uniformly (over  $\bar{h}$  small enough) coercive over the kernel of equality constraints of  $(QP)$ .*

*Proof.* We first examine the continuous problem and prove that  $\Omega$  is, for  $\varepsilon > 0$  small enough, coercive for  $\bar{h}$  sufficiently small over the following enlargement of the critical cone:

$$(D.6) \quad C_\varepsilon := \{v \in \mathcal{V} \mid g'(\bar{y}(t))z[v](t) = 0 \text{ for all } t \in [t_a + \varepsilon, t_b - \varepsilon]\}.$$

This holds because otherwise we would have a sequence  $\varepsilon_q \downarrow 0$  and  $v^q$  in  $C_{\varepsilon_q}$  of the unit norm such that  $\Omega(v^q) \leq o(1)$ . Extracting a subsequence if necessary, assume that  $v^q$  weakly converges to  $\bar{v}$  in  $\mathcal{V}$ . Thanks to the Legendre condition we have that  $\Omega$  is a Legendre form and therefore

$$(D.7) \quad \Omega(\bar{v}) \leq \liminf_{q \rightarrow 0} \Omega(v^q) \leq 0.$$

At the same time, by standard compactness arguments,  $g'(\bar{y})\bar{z} = 0$  when the constraint is active (where  $\bar{z}$  is the state variable of the linearized problem associated with  $\bar{v}$ ), and so  $\bar{v}$  is a critical direction. By (A5),  $\Omega(\bar{v}) \leq 0$  implies that  $\bar{v} = 0$ . But then  $\Omega(\bar{v}) = \lim_q \Omega(v^q)$ , so  $v^q$  (of the unit norm) strongly converges to  $\bar{v} = 0$ , which gives the desired contradiction.

Now let  $v$  belong to the feasible domain of  $(QP)$ . By Lemma D.1, we know that  $v$  belongs to the set

$$(D.8) \quad \{v \in \mathcal{V}^N; |g'(y_k^\theta)z_k| \leq \varepsilon, 0 \leq k \leq N, \text{ such that } t_{k_a} + \varepsilon < t_k < t_{k_b} - \varepsilon\}.$$

Let  $\bar{v}$  be the associated element of  $\mathcal{V}$  and  $\bar{z}$  the corresponding state. Given  $\varepsilon > 0$ , it is easily checked that  $\bar{v} \in C_\varepsilon$  when  $\bar{h}$  is small enough, and so, by step (a),  $\Omega(\bar{v}) \geq \frac{1}{2}\alpha\|\bar{v}\|^2$ . We conclude with Lemma C.1.  $\square$

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## REFERENCES

- [1] J. F. BONNANS, *Lipschitz solutions of optimal control problems with state constraints of arbitrary order*, Ann. Acad. Rom. Sci. Ser. Math. Appl., 2 (2010), pp. 78–98.
- [2] J. F. BONNANS, J. C. GILBERT, C. LEMARECHAL, AND C. A. SAGASTIZBAL, *Numerical Optimization: Theoretical and Practical Aspects*, Springer, Berlin, 2006.
- [3] J. F. BONNANS AND A. HERMANT, *Well-posedness of the shooting algorithm for state constrained optimal control problems with a single constraint and control*, SIAM J. Control Optim., 46 (2007), pp. 1398–1430.
- [4] J. F. BONNANS AND A. HERMANT, *Stability and sensitivity analysis for optimal control problems with a first-order state constraint and application to continuation methods*, ESAIM Control Optim. Calc. Var., 14 (2008), pp. 825–863.
- [5] J. F. BONNANS AND A. HERMANT, *Revisiting the analysis of optimal control problems with several state constraints*, Control Cybernet., 38 (2009), pp. 1021–1052.
- [6] J. F. BONNANS AND A. HERMANT, *Second-order analysis for optimal control problems with pure state constraints and mixed control-state constraints*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 26 (2009), pp. 561–598.
- [7] J. F. BONNANS AND A. HERMANT, *No-gap second-order optimality conditions for optimal control problems with a single state constraint and control*, Math. Program, 117 (2009), pp. 21–50.
- [8] J. F. BONNANS AND A. SHAPIRO, *Perturbation Analysis of Optimization Problems*, Springer Series in Operations Research, Springer, New York, 2000.
- [9] F. BREZZI, *On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers*, RAIRO Oper. Res., 8.2 (1974), pp. 129–151.
- [10] A. E. BRYSON, W. F. DENHAM, AND S. E. DREYFUS, *Optimal programming problems with inequality constraints*, AIAA J., 1 (1963), pp. 2544–2550.
- [11] B. M. BUDAK, E. M. BERKOVICH, AND E. N. SOLOV’EVA, *Difference approximations in optimal control problems*, SIAM J. Control, 7 (1969), pp. 18–31.
- [12] J. CULLUM, *Finite-dimensional approximations of state-constrained continuous optimal control problems*, SIAM J. Control, 10 (1972), pp. 649–670.
- [13] A. L. DONTCHEV, *Discrete approximations in optimal control*, in Nonsmooth Analysis and Geometric Methods in Deterministic Optimal Control, IMA Vols. Math. Appl. 78, Springer, New York, 1996, pp. 59–80.
- [14] A. L. DONTCHEV AND W. W. HAGER, *Lipschitzian stability in nonlinear control and optimization*, SIAM J. Control Optim., 31 (1993), pp. 569–603.
- [15] A. L. DONTCHEV AND W. W. HAGER, *Lipschitzian stability for state constrained nonlinear optimal control*, SIAM J. Control Optim., 36 (1998), pp. 698–718.
- [16] A. L. DONTCHEV AND W. W. HAGER, *The Euler approximation in state constrained optimal control*, Math. Comp., 70 (2001), pp. 173–203.
- [17] W. W. HAGER, *The Ritz–Trefftz method for state and control constrained optimal control problems*, SIAM J. Numer. Anal., 12 (1975), pp. 854–867.
- [18] W. W. HAGER, *Lipschitz continuity for constrained processes*, SIAM J. Control Optim., 17 (1979), pp. 321–338.
- [19] W. W. HAGER, *Runge–Kutta methods in optimal control and the transformed adjoint system*, Numer. Math., 87 (2000), pp. 247–282.
- [20] W. W. HAGER AND G. D. IANULESCU, *Dual approximations in optimal control*, SIAM J. Control Optim., 22 (1984), pp. 423–465.
- [21] A. HERMANT, *Stability analysis of optimal control problems with a second-order state constraint*, SIAM J. Optim., 20 (2009), pp. 104–129.
- [22] D. H. JACOBSON, M. M. LELE, AND J. L. SPEYER, *New necessary conditions of optimality for control problems with state-variable inequality constraints*, J. Math. Anal. Appl., 35 (1971), pp. 255–284.
- [23] K. JITTORNTUM, *Solution point differentiability without strict complementarity in nonlinear programming*, in Sensitivity, Stability and Parametric Analysis, Mathematical Programming Studies 21, Springer, Berlin, 1984, pp. 127–138.
- [24] K. MALANOWSKI AND H. MAURER, *Sensitivity analysis for state constrained optimal control problems*, Discrete Contin. Dyn. Syst., 4 (1998), pp. 241–272.
- [25] H. MAURER, *On the minimum principle for optimal control problems with state constraints*, Technical report, Schriftenreihe des Rechenzentrum 41, Universität Münster, 1979.

- [26] H. MAURER, *First and second order sufficient optimality conditions in mathematical programming and optimal control*, in Mathematical Programming at Obwegach, Mathematical Programming Studies 14, Springer, Berlin, 1981, pp. 163–177.
- [27] H. MAURER AND H. J. PESCH, *Solution differentiability for nonlinear parametric control problems*, SIAM J. Control Optim., 32 (1994), pp. 1542–1554.
- [28] H. MAURER AND J. ZOWE, *First and second-order necessary and sufficient optimality conditions for infinite-dimensional programming problems*, Math. Program., 16 (1979), pp. 98–110.
- [29] A. A. MILYUTIN AND N. P. OSMOLOVSKII, *Calculus of Variations and Optimal Control*, American Mathematical Society, Providence, RI, 1998.
- [30] B. SH. MORDUKHOVICH, *On difference approximations of optimal control systems*, J. Appl. Math. Mech., 42 (1978), pp. 452–461.
- [31] L. W. NEUSTADT, *A general theory of extremals*, J. Comput. Syst. Sci. Int., 3 (1969), pp. 57–92.
- [32] S. M. ROBINSON, *First order conditions for general nonlinear optimization*, SIAM J. Appl. Math., 30 (1976), pp. 597–607.
- [33] S. M. ROBINSON, *Strongly regular generalized equations*, Math. Oper. Res., 5 (1980), pp. 43–62.
- [34] A. SCHWARTZ AND E. POLAK, *Consistent approximations for optimal control problems based on Runge–Kutta integration*, SIAM J. Control Optim., 34 (1996), pp. 1235–1269.
- [35] R. VINTER, *Optimal Control*, Birkhäuser, Boston, MA, 2000.
- [36] J. ZOWE AND S. KURCYUSZ, *Regularity and stability for the mathematical programming problem in Banach spaces*, Appl. Math. Optim., 5 (1979), pp. 49–62.