An approximation scheme for an eikonal equation with discontinuous coefficient

AN APPROXIMATION SCHEME FOR AN EIKONAL EQUATION WITH DISCONTINUOUS COEFFICIENT *

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Abstract. We consider the stationary Hamilton-Jacobi equation

$$\sum_{i,j=1}^{N} b_{ij}(x) u_{x_i} u_{x_j} = [f(x)]^2 , \quad \text{in } \Omega,$$

where Ω is an open set of $\mathbb{R}^n$, b can vanish at some points and the right-hand side f is strictly positive and is allowed to be discontinuous. More precisely, we consider special class of discontinuities for which the notion of viscosity solution is well-suited. We propose a semi–Lagrangian scheme for the numerical approximation of the viscosity solution in the sense of Ishii and we study its properties. We also prove an *a-priori* error estimate for the scheme in L^1. The last section contains some applications to control and image processing problems.

Key words. Hamilton-Jacobi equation, discontinuous Hamiltonian, viscosity solutions, semi–Lagrangian schemes, *a-priori* error estimates.

AMS subject classifications. 35F30, 35R05, 65N15

1. Introduction. In this paper we study the following boundary value problem. Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain with a Lipschitz boundary $\partial\Omega$, we consider the Dirichlet problem

$$\begin{cases} \displaystyle\sum_{i,j=1}^{N} b_{ij}(x) u_{x_i} u_{x_j} = [f(x)]^2 , & \text{for } x \in \Omega, \\ u(x) = g(x), & \text{for } x \in \partial\Omega; \end{cases} \tag{1.1}$$

where f and g are given functions whose regularity will be specified later. However, the main focus of this paper is on the case where f is Borel measurable and possibly discontinuous.

In the most classical case, the matrix (b_{ij}) is the identity matrix and f is a positive function, so the partial differential equation in (1.1) reduces to

$$|Du(x)| = f(x), \quad \text{in } \Omega, \tag{1.2}$$

which is the classical form of an Eikonal equation.

This equation arises in the study of many problems, e.g. in geometrical optics, computer vision, control theory and robotic navigation. In geometrical optics, to describe the propagation of light the Eikonal equation appears in the form

$$\sum_{i,j=1}^{N} b_{ij}(x) u_{x_i} u_{x_j}(x) = c(x), \quad \text{in } \Omega, \tag{1.3}$$

where c has the meaning of the refraction index of the media crossed by the light rays. Typically, the refraction law applies across surfaces of discontinuity of c.

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Another example is offered by a classical problem in computer vision, the Shape-from-Shading model. In this classical inverse problem we want to reconstruct a graph surface $z = u(x)$ corresponding to a single given gray-level image. Indicating with $I : \Omega \to [0,1]$ the light intensity (brightness function), in the simplest case where the light source is on the vertical axis and all the rays are parallel (see the survey paper [14] for the classical assumptions and various approaches to this problem) the equation describing the problem is

$$\sqrt{1 + |Du(x)|^2} = \frac{1}{I(x)}, \quad \text{in } \Omega \tag{1.4}$$

which can be easily written as an eikonal equation. Note that in this application I is usually discontinuous when the object has edges because I depends on the angle of reflection of the rays hitting the surface, so it depends on the normal to the surface.

Another motivation to deal with discontinuous Hamiltonians comes directly from control theory. In this framework discontinuous functions can be used to represent targets (for example using f as a characteristic function) and/or state constraints (using f as an indicator function) [7]. Clearly, the well-posedness of (1.1) in the case of continuous f follows from the theory of viscosity solutions for HJ equations, the interested reader can find the details in [3] and [2] where there are summarized well-known results by Crandall, Lions, Ishii and other authors. It is interesting to point out that, when the Hamiltonian is discontinuous, the knowledge of f at every point will not guarantee the well-posedness of the problem even in the framework of viscosity solutions. In fact, for equation (1.1) it can be easily observed that, even when f is defined point wise and has appropriate discontinuities, the value function for the corresponding control problem will not satisfy the equation in the viscosity sense. In order to define viscosity solutions for this case, we use appropriate semicontinuous envelopes of f, following some techniques and ideas introduced by Ishii in [20].

It is worth to mention that the notion of viscosity solution in the case of discontinuous Hamiltonian has been proposed by Ishii in [20] where some existence and regularity results are illustrated. Other results of well-posedness of Hamilton-Jacobi equations in presence of discontinuous coefficients have been presented by various authors (see [6, 18, 4, 12]) and in the specific case of the Eikonal equation [34, 25].

Our primary goal is to prove convergence for a semi-Lagrangian scheme which has been shown to be rather effective in the approximation of Hamilton–Jacobi equations. The results which have been proved for this type of schemes work for convex and non convex Hamiltonians but use the uniform continuity of the Hamiltonian. Moreover, the typical convergence result is given for the L^∞ norm which is rather natural when dealing with classical viscosity solutions (see e.g. the result by Crandall and Lions [9], Barles and Souganidis [5] and the monograph by Falcone and Ferretti [17]). For classical viscosity solutions, at our knowledge, the only two convergence results in $L^1(\Omega)$ has been proved by Lin and Tadmor [32, 23] for a central finite difference scheme and by Bokanowsky et al. [7] in dimension one. We also have to mention the level set approach for discontinuous solutions proposed by Tsai et al. [33]. Although classical schemes tailored for the the approximation of regular cases with convex Hamiltonians can give reasonable results also for some discontinuous Hamiltonians, it is interesting to have a theoretical framework guaranteeing convergence. Deckelnick and Elliott [13] have studied a problem where the solution is still Lipschitz continuous although the Hamiltonian is discontinuous. In particular, they have proposed a finite difference scheme for the approximation of (1.2) and their scheme is very similar to a finite difference schemes usually applied for regular Hamiltonians. Their result is important also because they are able to prove an *a-priori* error estimate still in $L^\infty(\Omega)$.

Although our work has been also inspired by their results, we use different techniques and our analysis is devoted to a scheme of semi–Lagrangian type (*SL-scheme*). The benefits of a SL-scheme with respect to a finite difference scheme are a better ability to follow information driven by the characteristics, the fact that one can use a larger time-step in evolutive problems still having stability and the fact that SL-schemes do not require a structured grid. These features give us a faster and more accurate approximation in many cases as it has been reported in the literature (see e.g. [16, 11] or Appendix A of [2]). It is also important to note that we prove an *a-priori* error estimate which improves the result in [13] because we consider a more general case (1.1) where also discontinuous viscosity solutions can appear.

This paper is organized as follows. In Section 2 we recall some definitions and theoretical results available for viscosity solutions and discontinuous Hamiltonian. Section 3 is devoted to the presentation of the scheme and to the proof of some properties which will be used in the proof of convergence. In Section 4 we prove convergence and establish an *a-priori* error estimate giving the rate of convergence in the L^1 norm. Finally, in Section 5 we present our numerical experiments dealing with control and image processing problems.

2. The model problem and previous results. We present, for readers convenience, some results of well-posedness mainly taken from a work of Soravia [30]. We also introduce our assumptions, which are summarized below.

The boundary data

$$g : \partial\Omega \to [0, +\infty) \text{ is continuous,} \tag{2.1}$$

the matrix of the coefficients can be written as

$$(b_{ij}) = (\sigma_{ik}) \cdot (\sigma^T_{kj}) \tag{2.2}$$

where $i, j = 1, \ldots, N$ and $k = 1, \ldots, M$ and $(M \leq N)$. Then (b_{ij}) is a symmetric, positive semidefinite and possibly degenerate matrix,

$$\sigma(\cdot) \equiv (\sigma_{ik})_{i=1,\ldots N;\ k=1,\ldots M} : \overline{\Omega} \to \mathbb{R}^{NM} \text{ is Lipschitz continuous.} \tag{2.3}$$

we will denote by L_σ its Lipschitz constant. Moreover, the function $f : \mathbb{R}^N \to [\rho, +\infty)$, $\rho > 0$ is Borel measurable and possibly discontinuous.

Let us denote by $\sigma_k : \Omega \to \mathbb{R}^N$, $k = 1, \ldots M$ the columns of the matrix $(\sigma_{ik})_{i,k}$. We can give an optimal control interpretation of (1.1), rewriting the differential operator in the following form

$$\sum_{i,j=1}^{N} b_{ij}(x) p_i p_j = \sum_{k=1}^{M} (p \cdot \sigma_k(x))^2 = |p\sigma(x)|^2, \tag{2.4}$$

i.e the σ_k are the vector fields of the dynamics. We define the nonnegative constant

$$M_\sigma = \max_i \sum_k ||\sigma_{ik}||_\infty. \tag{2.5}$$

In this way the Eikonal equation (1.1) becomes, for $a = (a_1, \ldots a_M) \in \mathbb{R}^M$, the following Bellman equation

$$\max_{|a|\leq 1} \left\{ -Du(x) \cdot \sum_{k=1}^{M} a_k \sigma_k(x) \right\} = f(x) \tag{2.6}$$

associated to the symmetric controlled dynamics

$$\dot{y} = \sum_{k=1}^{M} a^k \sigma_k(y), \quad y(0) = x, \tag{2.7}$$

where the measurable functions $a : [0, +\infty) \to \{a \in \mathbb{R}^M : |a| \leq 1\}$ are the controls. We will denote in the sequel by $y_x(\cdot) := y_x(\cdot, a)$ the solutions of (2.7). In this system, optimal trajectories are the geodesics associated to the metric defined by the matrix (b_{ij}). Note that they are straight lines when (b_{ij}) is the identity matrix. A solution of the equation (2.6) corresponds to the value function of a minimum time problem with running cost, i.e. it can be written as the minimum of the following functional

$$J(x, a(\cdot)) = \int_0^{\tau_x} f(y(t))dt + g(y(\tau_x)) \tag{2.8}$$

where $\tau_x(a(\cdot)) = \inf\{t : y_x(t, a) \notin \Omega\}$ is the first time of arrival on $\partial\Omega$. (See [15] for details).

Let us introduce the concept of *discontinuous viscosity solution* for (1.1) introduced by Ishii in [20]. Let f be bounded in Ω and let

$$f_*(x) = \lim_{r \to 0^+} \inf\{f(y) : |y - x| \leq r\} \tag{2.9}$$

$$f^*(x) = \lim_{r \to 0^+} \sup\{f(y) : |y - x| \leq r\} \tag{2.10}$$

f_* and f^* are respectively the lower semicontinuous and the upper semicontinuous envelope of f.

Definition 2.1.

i) A lower semicontinuous function $u : \Omega \to \mathbb{R}$ is a viscosity supersolution of the equation (1.1) if for every $\phi \in C^1(\Omega)$, and $x \in \Omega$ point of minimum of the function $(u - \phi)$, we have

$$\sum_{i,j=1}^{N} b_{ij}(x)\phi_{x_i}(x)\phi_{x_j}(x) \geq [f_*(x)]^2 ,$$

ii) An upper semicontinuous function $u : \Omega \to \mathbb{R}$ is a viscosity subsolution of the equation (1.1) if for every $\phi \in C^1(\Omega)$, and $x \in \Omega$ point of maximum of the function $(u - \phi)$, we have

$$\sum_{i,j=1}^{N} b_{ij}(x)\phi_{x_i}(x)\phi_{x_j}(x) \leq [f^*(x)]^2 .$$

A function u is a discontinuous viscosity solution of (1.1) if u^* is a subsolution and u_* is a supersolution according to i) and ii).

We remind also that the Dirichlet condition is satisfied in the following weaker sense

Definition 2.2. *An upper semicontinuous function $u : \overline{\Omega} \to \mathbb{R}$, subsolution of (1.1), satisfies the Dirichlet type boundary condition in the viscosity sense if for all $\phi \in C^1(\Omega)$ and $x \in \partial\Omega$, $x \in \overline{\Omega}$ point of maximum of the function $(u - \phi)$ such that $u(x) > g(x)$, then we have*

$$\sum_{i,j=1}^{N} b_{ij}(x)\phi_{x_i}\phi_{x_j} \leq [f^*(x)]^2 .$$

Lower semicontinuous functions that satisfy a Dirichlet type boundary condition are defined accordingly. In order to see how easily uniqueness can fail without proper assumptions on f, now that we accepted that envelopes of function should be used let us consider the 1D equation

$$|u'(x)| = f(x), \quad x \in [-2, 2], \quad u(-2) = u(2) = 0, \tag{2.11}$$

with the choice $f(x) = 2\chi_{\mathbf{Q}}$, where $\chi_{\mathbf{Q}}$ is the characteristic function of the set of rational numbers $\mathbb{Q}$. Then one easily checks that both $u_1 \equiv 0$ and $u_2 = 2 - 2|x|$ are viscosity solutions. It is clear, that in general we do not have uniqueness of the discontinuous viscosity solution. We add a key assumption on the coefficient f.

Assumption A1. Let us assume that there exist $\eta > 0$ and $K \geq 0$ such that for every $x \in \Omega$ there is a direction $n = n_x \in B(0, 1)$, (where $B(0, 1)$ is the N-dimensional unite ball) with

$$f(y + rd) - f(y) \leq Kr, \tag{2.12}$$

for every $y \in \Omega$, $d \in B(0, 1)$, $r > 0$ with $|y - x| < \eta$, $|d - n| < \eta$ and $y + rd \in \Omega$.

Under Assumption (A1) the following comparison theorem holds. This result, under some more general hypotheses, is presented in [30].

Theorem 2.3. *Let Ω be an open domain with Lipschitz boundary. Assume (2.1), (2.2), (2.3) and Assumption (A1). Let $u, v : \overline{\Omega} \to \mathbb{R}$ be respectively an upper and a lower–semicontinuous function, bounded from below, respectively a subsolution and a supersolution of*

$$\sum_{i,j=1}^{N} b_{i,j}(x) u_{x_i} u_{x_j} = [f(x)]^2, \quad x \in \Omega$$

Let us assume that v restricted to $\partial\Omega$ is continuous and that u satisfies the Dirichlet type boundary condition. Suppose moreover that u or v is Lipschitz continuous. Then $u \leq v$ in $\overline{\Omega}$.

From this result, it follows directly that we have uniqueness of a continuous solution.

Corollary 2.4. *Assume (2.1), (2.2), (2.3) and (A1). Let $u : \overline{\Omega} \to \mathbb{R}$ be a continuous, bounded viscosity solution of the problem (1.1). Then u is unique in the class of discontinuous solutions of the corresponding Dirichlet type problem.*

Example 1 (Soravia [29]). *This example shows that discontinuous solutions may exists without any contradiction with the previous result. This is due to the fact that Corollary 2.4 does not cover all possible situations. Let us consider the Dirichlet problem*

$$\begin{cases} x^2 \left(u_x(x,y)\right)^2 + \left(u_y(x,y)\right)^2 = [f(x,y)]^2 & (-1,1) \times (-1,1) \\ u(\pm 1, y) = u(x, \pm 1) = 0 & x, y \in [-1, 1] \end{cases} \tag{2.13}$$

where $f(x, y) = 2$, for $x > 0$, and $f(x, y) = 1$ for $x \leq 0$. In this case we have that

$$b_{i,j} = \begin{pmatrix} x^2 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma(x) = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix},$$

therefore the Bellman's equation in this case is

$$\max_{|a| \leq 1} \left\{ -Du(x, y) \cdot a_1 (x, 0)^T - Du(x, y) \cdot a_2 (0, 1)^T \right\} = f(x, y). \tag{2.14}$$

It is easy to verify that the piecewise continuous function,

$$u(x,y) = \begin{cases} 2(1-|y|) & \text{for } x \geq 0, |y| > 1+\ln x, \\ -2ln(x) & \text{for } x > 0, |y| \leq 1+\ln x, \\ \frac{u(-x,y)}{2} & \text{for } x < 0, \end{cases} \tag{2.15}$$

is a viscosity solution of the problem. We know, as indirect implication of Corollary 2.4 that there is no continuous solution. We note that all the class of functions with values in $x = 0$ between $1 - |y|$ and $2(1 - |y|)$ are discontinuous viscosity solutions. However, we have that all discontinuous solutions have u as upper semicontinuous envelope.

As shown in Example 1, in general we do not have existence of a continuous solution and, in general, we can not expect a unique solution. However, restricting ourselves to a special class of solutions, essentially the case presented in the previous example, we can preserve the accuracy of numerical approximations and we can also get an error estimate, as we will see in the sequel.

Since the presence of discontinuities is due to the degeneracy of the coefficient σ, we need some additional hypotheses to handle this case. In this case, however, the assumption will be given on the interface of degeneracy of σ.

From here we will restrict ourselves to the case $N = 2$.

Let us denote by $\ell(C)$ the length of a curve C and assume the existence of a regular curve Σ_0 which splits the domain Ω in two non degenerating parts. We denote by $\eta(x) = (\eta_1(x), \eta_2(x))$ the usual unit normal to Σ_0 on the point $x \in \Sigma_0$.

Assumption A2. There exists a curve $\Sigma_0 \subset \Omega$ such that, for the points $x \in \Sigma_0$ we have

$$\eta_1(x)\sigma_1(x) + \eta_2(x)\sigma_2(x) = 0.$$

Moreover, the following conditions have to be satisfied:

1. $p_1(x)\sigma_1(x) + p_2(x)\sigma_2(x) \neq 0$ for every $(p_1, p_2) \in B(0,1)$ and $x \notin \Sigma_0$;
2. $\ell(\Sigma_0) < +\infty$.
3. Let $\Omega = \Omega_1 \cup \Omega_2 \cup \Sigma_0$, where, in each subset Ω_j there is not degeneracy of σ, we have $\overline{\Omega}_j \cap \partial\Omega \neq \emptyset$ for $j = \{1,2\}$.

We conclude this section with the following result, which can be derived by adapting the classical proof by Ishii [21]:

Theorem 2.5. *Let Ω be an open domain with Lipschitz boundary. Assume (2.1), (2.2), (2.3), and Assumptions (A1) and (A2). Let $u : \overline{\Omega} \to \mathbb{R}$ be a bounded viscosity solution of the problem (1.1). It is Lipschitz continuous in every set Ω_1 and Ω_2.*

Proof. Take a parameter $\delta > 0$, and define the set

$$\Sigma_\delta := \left\{ x \in \Omega | \overline{B(x,\delta)} \cap \Sigma_0 \neq \emptyset \right\}; \tag{2.16}$$

we want to study the regularity of the viscosity solution in the set $\overline{\Omega_1 \setminus \Sigma_\delta} = \overline{\Omega}_1^\delta$.

In order to describe our boundary assumptions on $\overline{\Omega}_1^\delta \cap \Sigma_\delta$ let us define $L : \overline{\Omega}_1^\delta \times \overline{\Omega}_1^\delta \to \mathbb{R}$ by

$$L(x,y) := \inf \left\{ \int_0^1 N(f^*(\gamma(t)), \gamma'(t))dt | \gamma \in W^{1,\infty}((0,1), \overline{\Omega}_1^\delta) \right. \\ \left. \text{with } \gamma(0) = x,\ \gamma(1) = y \right\}, \tag{2.17}$$

where, $W^{1,\infty}$ is the usual space of continuous functions with bounded first derivative, and

$$N(r,\zeta) := \sup\left\{-(\zeta,p) | \max_{|a|\le 1}\left\{-p\cdot\sum_{k=1}^{M} a^k\sigma_k(x) = r\right\}\right\}. \tag{2.18}$$

Then we extend the boundary condition to $\overline{\Omega}_1^\delta \cap \Sigma_\delta$ in the following way:

$$g(x) = \inf_{y\in\partial\overline{\Omega}_1^\delta\setminus\partial\Sigma_\delta}\{g(y)+L(x,y)\} \qquad \text{for } x\in\overline{\Omega}_1^\delta\cap\Sigma_\delta. \tag{2.19}$$

We can claim now, that there exists a viscosity solution $u_1^\delta \in C^{0,1}(\overline{\Omega}_1^\delta)$ of (1.1) with the Dirichlet conditions introduced above. This is proved in Ishii [21].

We do the same on the set Ω_2, getting the function $u_2^\delta \in C^{0,1}(\overline{\Omega}_2^\delta)$. Now the class of functions

$$u^\delta(x) := \begin{cases} u_1^\delta & \text{when } x\in\overline{\Omega}_1^\delta, \\ u_2^\delta & \text{when } x\in\overline{\Omega}_2^\delta, \end{cases} \tag{2.20}$$

in a viscosity solution of (1.1) in $\overline{\Omega}_1^\delta \cup \overline{\Omega}_2^\delta$. For the arbitrariness of δ and defining u on the discontinuity as said previously we get the thesis. □

Which value the solution can assume in Σ_0? As shown in Example 2 and in accordance with the definition of discontinuous viscosity solutions, we can choose for $x\in\Sigma_0$ every value between u_* and u^*.

We can observe that in this class we can also include an easier case. If we consider $\sigma(x) = c(x)\mathbb{I}$, with $\mathbb{I}$ identity matrix of dimension $N\times N$, $c(x):\Omega\to\mathbb{R}$ where $c(x)\ge 0$ for all $x\in\Omega$ (i.e. c can vanish at some points) we obtain a slightly famous case for applicative reasons (*isotropic growing interface*). In particular, in this case we will define $\Sigma_0 := \{x\in\Omega | c(x) = 0\}$ and the previous hypothesis on the nature of Σ_0 reduces to

$$\ell(\Sigma_0) < +\infty \text{ and } \Omega_j\cap\partial\Omega \ne \emptyset \text{ for } j=\{1,2\}.$$

3. The semi-Lagrangian approximation scheme and its properties. We construct a semi-Lagrangian approximation scheme for the equation (1.1) following the approach [16] .

Introducing the Kruzkov's change of variable, as, for example in [2], $v(x) = 1-e^{-u(x)}$ and using (2.6) and (2.4) the problem (1.1) becomes

$$\begin{cases} |Dv(x)\cdot\sigma(x)| = f(x)(1-v(x)) & \text{for } x\in\Omega, \\ v(x) = 1-e^{-g(x)} & \text{for } x\in\partial\Omega. \end{cases} \tag{3.1}$$

To come back to the original unknown u we can use the inverse transform, i.e. $u(x) = -\ln(1-v(x))$.

Let us to observe that since $u(x)\ge 0$, we have $0\le v(x) < 1$. We can write the previous equation in the equivalent way

$$\begin{cases} v(x) + \frac{1}{f(x)}|Dv(x)\cdot\sigma(x)| = 1 & \text{for } x\in\Omega, \\ v(x) = 1-e^{-g(x)} & \text{for } x\in\partial\Omega. \end{cases} \tag{3.2}$$

We want to build a discrete approximation of (3.2), which can be written in the following form

$$v(x) + \sup_{a\in B(0,1)} \left\{ \frac{\sum_k a^k \sigma_k(x)}{f(x)} \cdot Dv(x) \right\} = 1. \tag{3.3}$$

where the relation with an optimal control problem is more clear. In fact, v can be interpreted as the value function of an optimization problem of constant running cost and discount factor equal to one. The dynamics will be given by $a \cdot \sigma(x)/f(x)$.

We discretize the left-hand side term of (3.3) as a directional derivative obtaining the following discrete problem:

$$\begin{cases} v_h(x) = \frac{1}{1+h} \inf\limits_{a\in B(0,1)} \left\{ v_h\left(x - \frac{h}{f(x)} \sum_k a^k \sigma_k(x)\right)\right\} + \frac{h}{1+h} & \text{for } x \in \Omega, \\ v_h(x) = 1 - e^{-g(x)} & \text{for } x \in \partial\Omega, \end{cases} \tag{3.4}$$

where h is a positive real number which can be interpreted as a time step Δt for the discretization of the dynamics. We will assume (to simplify the presentation) that $x - \frac{h}{f(x)} \sum_k a^k \sigma_k(x) \in \overline{\Omega}$ for every $a \in B(0,1)$.

Note that for $x \in \overline{\Omega}$ and a direction $d \in \partial B(0,1)$, we always can find an $a \in B(0,1)$ such that $\frac{a}{|a|} = d$ and $x - \frac{h}{f(x)} \sum_k a^k \sigma_k(x) \in \overline{\Omega}$, (see Figure 3.1) because Ω is an open set and we can choose $a = 0$ to stay at x.

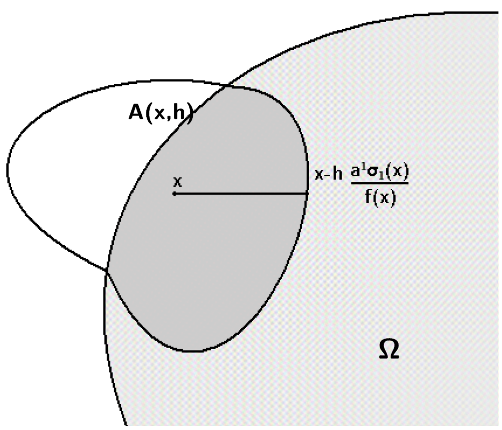

FIG. 3.1. *The set* $A(x,h) := \left\{x - \frac{h}{f(x)} \sum_k a^k \sigma_k(x); a \in B(0,1)\right\}$ *in dimension 2. In dark grey* $\Omega \cap A(x,h)$

Let us introduce a space discretization of (3.4) to get a fully discrete scheme. We construct a regular triangulation of Ω made by a family of simplices S_j, such that $\overline{\Omega} = \cup_j S_j$, denoting x_m, $m = 1, ..., L$, the nodes of the triangulation, by

$$\Delta x := \max_j diam(S_j) \tag{3.5}$$

the size of the mesh ($diam(B)$ denotes the diameter of the set B) and by G the set of the knots of the grid.

We look for a solution of

$$\begin{cases} W(x_m) = \frac{1}{1+h} \min\limits_{a\in B(0,1)} I[W](x_m - \frac{h}{f(x_m)} \sum\limits_k a^k \sigma_k(x_m)) + \frac{h}{1+h}, & \text{for } x_m \in G \\ W(x_m) = 1 - e^{-g(x_m)}, & \text{for } x_m \in G \cap \partial\Omega \end{cases} \tag{3.6}$$

where $I[W](x)$ is a linear interpolation of W at the point x, in the space of piecewise linear functions on $\overline{\Omega}$

$$\mathcal{W}^{\Delta x} := \left\{ w : \overline{\Omega} \to \mathbb{R} | w \in C(\Omega) \text{ and } Dw(x) = c_j \text{ for any } x \in S_j \right\}.$$

THEOREM 3.1. *Let $x_m - \frac{h}{f(x_m)} \sum_k a^k \sigma_k(x_m) \in \overline{\Omega}$, for every $x_m \in G$ and $a \in B(0,1)$, then there exists a unique solution W of (3.6) in $\mathcal{W}^{\Delta x}$*

Proof. By our assumption, starting from any $x_m \in G$ we will reach points which still belong to Ω. So, for every $w \in \mathcal{W}^{\Delta x}$ we have

$$w\left(x_m - \frac{h}{f(x_m)} \sum_k a^k \sigma_k(x_m)\right) = \sum_{j=1}^{L} \lambda_{mj}(a) w(x_j),$$

where $\lambda_{mj}(a)$ are the coefficients of the convex combination representing the point $x_m - \frac{h}{f(x_m)} \sum_k a^k \sigma_k(x_m)$, and L is the number of nodes of G, i.e.

$$x_m - \frac{h}{f(x_m)} \sum_k a^k \sigma_k(x_m) = \sum_{j=1}^{L} \lambda_{mj}(a) x_j, \tag{3.7}$$

where

$$0 \le \lambda_{mj}(a) \le 1 \quad \text{and} \quad \sum_{j=1}^{L} \lambda_{mj}(a) = 1 \quad \text{for any } a \in B(0,1). \tag{3.8}$$

Then (3.6) is equivalent to the following fixed point problem in finite dimension

$$W = T(W),$$

where the map $T : \mathbb{R}^L \to \mathbb{R}^L$ is defined componentwise as

$$(T(W))_m := \left[\frac{1}{1+h} \min_{a\in B(0,1)} \Lambda(a) W + \frac{h}{1+h} \right]_m, \quad \text{with } m \in 1, ..., L; \tag{3.9}$$

$W_m \equiv W(x_m)$ and $\Lambda(a)$ is the $L \times L$ matrix of the coefficients λ_{mj} satisfying (3.7), (3.8) for $m, j \in 1, ..., L$.

T is a contraction mapping. In fact, let $\overline{a}$ be a control giving the minimum in $T(V)_m$, we have

$$\begin{aligned} [T(W) - T(V)]_m &\le \frac{1}{1+h} [\Lambda(\overline{a})(W - V)]_m \\ &\le \frac{1}{1+h} \max_{m,j} |\lambda_{mj}(a)| ||W - V||_\infty \le \frac{1}{1+h} ||W - V||_\infty. \end{aligned} \tag{3.10}$$

Switching the role of W and V we can conclude that

$$\|T(W) - T(V)\|_\infty \le \frac{1}{1+h} \|W - V\|_\infty \tag{3.11}$$

□

3.1. Properties of the scheme. It is rather important to note that the scheme(3.6)some useful proprieties which guarantee the convergence: consistency and monotonicity.

Consistency

From (3.6), we obtain

$$W(x_m) - \frac{1}{h} \min_{a\in B(0,1)} \left\{ -W(x_m) + I[W](x_m - \frac{h}{f(x_m)} \sum_k a^k \sigma_k(x_m)) \right\} = 1. \quad (3.12)$$

We can see that the term which we want to minimize is a first order approximation of the directional derivative

$$- \min_{a\in B(0,1)} \left\{ DW \cdot \sum_k a^k \sigma_k(x) \right\} + o(h) = 1 - W(x_m), \quad (3.13)$$

Then it is easy to check that the local error is $O(h + \Delta x)$.

Monotonicity. Let us denote by $T[W]$ the right-hand side of (3.6). Since we use a piecewise linear interpolation we have that

$$U \leq W \text{ implies } T[U](x) \leq T[W](x).$$

Convergence of the iterative sequence

$$W^n = T(W^{n-1}), \quad (3.14)$$

is guaranteed by the fact that T is a contraction mapping in $\mathbb{R}^L$. The sequence will converge to W, for every choice of the initial condition $W^0 \in \mathbb{R}^L$ but it will monotone non decreasing if we start from the set of subsolutions, i.e. from a point belonging to the set $\mathcal{S} = \{V : V \leq T[V]\}$. This remark can also be exploited to accelerate convergence (see [15] for more details). Note that the following estimate holds true:

$$||W^n - W||_\infty \leq \left(\frac{1}{1+h} \right)^n ||W^0 - W||_\infty. \quad (3.15)$$

The proof is easy and will be left to the reader.

4. An a-priori estimate in $L^1(\Omega)$. In this section we present our main result. Using the $L^1(\Omega)$ norm we can extend the convergence result also to the class of of discontinuous value functions.In the sequel $v(x)$ will be the viscosity solution of (3.2) whereas $W(x)$ will represent the solution (3.6) extended to Ω by linear interpolation.

THEOREM 4.1. *Let the hypotheses (2.1), (2.2), (2.3), (A1) and (A2) hold true. Moreover, let the discretization steps satisfy the condition*

$$\frac{h}{\Delta x} < \frac{\rho}{M_\sigma} \quad (4.1)$$

where ρ is the lower bound for the values of f and M_σ is defined in (2.5). Then, there exist two positive constants C, C' independent from h and Δx, such that

$$||v(x) - W(x)||_{L^1(\Omega)} \leq C\sqrt{h} + C'\Delta x. \quad (4.2)$$

Proof. We start defining the set $\Sigma_{\Delta x}$

$$\Sigma_{\Delta x} := \{x \in \Omega | B(x, \Delta x) \cap \Sigma_0 \neq \emptyset\}.$$

First we observe that

$$||v(x) - W(x)||_{L^1(\Omega)} \leq \int_{\Omega\setminus\Sigma_{\Delta x}} |v(x) - W(x)|dx + \int_{\Sigma_{\Delta x}} |v(x) - W(x)|dx$$
$$\leq \sum_j \int_{\Omega_j} |v(x) - W(x)|dx + \int_{\Sigma_{\Delta x}} |v(x) - W(x)|dx, \quad (4.3)$$

where $\Omega := \cap_j \Omega_j$ is the partition of Ω generated from Σ_0 as stated in the definition of the set Σ_0.

From the Kruzkov's transform we know that $|v(x) - W(x)| \leq 2$ for all $x \in \Omega$. We can show this just considering that $|v(x)| \leq 1$ for every point $x \in \Omega$; we can get $||W(x)||_\infty \leq 1$ from the definition (3.6) just observing that for any $x_m \in G$,

$$W(x_m) \leq \frac{1}{1+h}||W||_\infty + \frac{h}{1+h}.$$

Now, by the assumptions on the set Σ_0 we get, for a fixed $C' > 0$,

$$\int_{\Sigma_{\Delta x}} |v(x) - W(x)|dx \leq 2\int_{\Sigma_{\Delta x}} dx \leq 2\ell(\Sigma_0)\Delta x \leq C'\Delta x. \quad (4.4)$$

To prove the statement, we need an estimate for the terms $\int_{\Omega_j} |v(x) - W(x)|dx$ for every choice of j. With this aim, we remind that, for Theorem 2.5, both $v(x)$ and $W(x)$ are Lipschitz continuous, so we can use a modification of the classical argument based on the duplication of variables (similar arguments can be found on [31, 13] and [30]).

We are focusing on the problem (3.6) restricted on the region $\widehat{\Omega}_j := \Omega_j \setminus \Sigma_{\Delta x}$ with some compatible Dirichlet conditions on $\Omega_j \cap \partial\Omega$. We do not have any Dirichlet conditions on $\partial\widehat{\Omega}_j \cap \partial\Sigma_{\Delta x}$, so we extend the boundary conditions as in (2.19). Inside the region $\widehat{\Omega}_j$ the solution $v(x)$ is Lipschitz continuous by Theorem 2.5.

Define $G_j := G \cup \Omega_j$. Let us choose a point

$$\widehat{x} = \arg\max_{x\in G_j} |v(x) - W(x)|, \quad (4.5)$$

and assume that $v(\widehat{x}) \geq W(\widehat{x})$. The opposite case can be treated similarly.
Case (a): $dist(\widehat{x}, \partial\widehat{\Omega}_j) \leq \sqrt{h}$.
In this case the Dirichlet conditions and the Lipschitz continuity of v and W imply that

$$\max_{x\in G_j} |v(x) - W(x)| = v(\widehat{x}) - W(\widehat{x}) \leq C\sqrt{h}. \quad (4.6)$$

Case (b): $dist(\widehat{x}, \partial\widehat{\Omega}_j) > \sqrt{h}$.
We define the auxiliary function

$$\psi(x, y) := v(x) - W(y) - L_1\frac{|x - y - \sqrt{h}\eta|^2}{2\sqrt{h}} - L_2\sqrt{h}|y - \widehat{x}|^2, \quad \text{for } (x, y) \in \Omega_j \times G_j. \quad (4.7)$$

Where η is the inward normal to Ω_j like stated in previous assumptions and L_1 and L_2 two positive constants.

It is not hard to check that the boundedness of Ω_j and the continuity of ψ, imply the existence of some $(\overline{x},\overline{y})$ (depending on h) such that

$$\psi(\overline{x},\overline{y}) \geq \psi(x,y) \quad \text{for all } (x,y) \in \widehat{\Omega}_j \times G_j. \tag{4.8}$$

Since $dist(\widehat{x},\partial\widehat{\Omega}_j) > \sqrt{h}$, clearly $\widehat{x}+\sqrt{h}\eta \in \widehat{\Omega}_j$ and therefore

$$\psi(\overline{x},\overline{y}) \geq \psi(\widehat{x}+\sqrt{h}\eta,\widehat{x}), \tag{4.9}$$

or equivalently

$$v(\overline{x}) - W(\overline{y}) - \frac{L_1}{\sqrt{h}}|\overline{x}-\overline{y}-\sqrt{h}\eta|^2 - L_2\sqrt{h}|\overline{y}-\widehat{x}|^2 \geq v(\widehat{x}-\sqrt{h}\eta) - W(\widehat{x}). \tag{4.10}$$

(4.10) implies

$$\begin{aligned}
\frac{L_1}{\sqrt{h}}|\overline{x}-\overline{y}-\sqrt{h}\eta|^2 + L_2\sqrt{h}|\overline{y}-\widehat{x}|^2 &\leq v(\overline{x}) - v(\widehat{x}-\sqrt{h}\eta) + W(\widehat{x}) - W(\overline{y}) \\
&\leq v(\overline{x}) - v(\overline{y}) + [(v(\overline{y}-W(\overline{y})) - (v(\widehat{x}-W(\widehat{x}))] + v(\widehat{x}) - v(\widehat{x}-\sqrt{h}\eta) \\
&\leq L_v|\overline{x}-\overline{y}| + \sqrt{h}\,L_v \leq L_v|\overline{x}-\overline{y}-\sqrt{h}\eta| + 2\sqrt{h}L_v \\
&\leq \left(\sqrt{\frac{L_1}{\sqrt{h}}}|\overline{x}-\overline{y}-\sqrt{h}\eta|\right)\left(\sqrt{\frac{\sqrt{h}}{L_1}}L_v\right) + 2\sqrt{h}L_v \\
&\leq \frac{L_1}{2\sqrt{h}}\,|\overline{x}-\overline{y}-\sqrt{h}\eta|^2 + \frac{\sqrt{h}}{2L_1}\,L_v^2 + 2\sqrt{h}\,L_v,
\end{aligned} \tag{4.11}$$

where L_v is the Lipschitz constant of v, and therefore we can conclude

$$\frac{1}{h}|\overline{x}-\overline{y}-\sqrt{h}\eta|^2 \leq \frac{1}{{L_1}^2}L_v^2 + \frac{4}{L_1}L_v < \left(\frac{\epsilon}{2+\epsilon}\right)^2, \tag{4.12}$$

$$|\overline{y}-\widehat{x}|^2 \leq \frac{1}{2L_1L_2}L_v^2 + \frac{2}{L_2}L_v < \epsilon^2, \tag{4.13}$$

for a $\epsilon > 0$, provided L_1, L_2 are sufficiently large.

We firstly consider the case $(\overline{x},\overline{y}) \in \widehat{\Omega}_j \times G_j$, i.e. both the points are not on the boundary of the respective set.

By (3.4) we have, for a $x \in G_j$

$$W\left(x - h\frac{\sum \tilde{a}^k\sigma_k(x)}{f(x)}\right) = W(x) + hW(x) - h, \tag{4.14}$$

for some $\tilde{a} = \tilde{a}(x)$. This equation is verified a.e. and the point $x - h\frac{\sum \tilde{a}^k\sigma_k(x)}{f(x)} \in \Omega_j$ by the definition of the admissible choice of $\overline{a}$ and the hypotheses on the discretization steps. Since the map

$$x \mapsto v(x) - \left[W(\overline{y}) + L_1\frac{|x-\overline{y}-\sqrt{h}\eta|^2}{2\sqrt{h}} + L_2\sqrt{h}|\overline{y}-\widehat{x}|^2\right], \tag{4.15}$$

has a maximum at $\overline{x}$, by (3.2) we obtain

$$-L_1\frac{|(\overline{x}-\overline{y}-\sqrt{h}\eta)\cdot\sigma(\overline{x})|}{\sqrt{h}} \leq f_*(\overline{x}) - f_*(\overline{x})v(\overline{x}), \tag{4.16}$$

and then

$$v(\overline{x}) \leq 1 + \frac{L_1}{f_*(\overline{x})}\frac{|(\overline{x}-\overline{y}-\sqrt{h}\eta)\cdot\sigma(\overline{x})|}{\sqrt{h}} \leq 1 + \frac{L_1}{\sqrt{h}}(\overline{x}-\overline{y}-\sqrt{h}\eta)\cdot\frac{\sum \overline{a}^k\sigma_k(\overline{x})}{f_*(\overline{x})}. \quad (4.17)$$

The inequality $\psi(\overline{x},\overline{y}) \geq \psi\left(\overline{x}, \overline{y} - \frac{h}{f(\overline{y})}\sum \tilde{a}^k\sigma_k(\overline{y})\right)$ gives

$$\begin{aligned} &-W(\overline{y}) - L_1\frac{|\overline{x}-\overline{y}-\sqrt{h}\eta|^2}{2\sqrt{h}} - L_2\sqrt{h}|\overline{y}-\widehat{x}|^2 \geq -W\left(\overline{y} - \frac{h}{f(\overline{y})}\sum \tilde{a}^k\sigma_k(\overline{y})\right) \\ &\quad - L_1\frac{\left|\overline{x}-h\overline{y}-\sqrt{h}\eta - \frac{\sum \tilde{a}^k\sigma_k(\overline{y})}{f(\overline{y})}\right|^2}{2\sqrt{h}} - L_2\sqrt{h}\left|\overline{y}-\widehat{x}-h\frac{\sum \tilde{a}^k\sigma_k(\overline{y})}{f(\overline{y})}\right|^2, \quad (4.18)\end{aligned}$$

and then

$$\begin{aligned} &W\left(\overline{y} - \frac{h}{f(\overline{y})}\sum \tilde{a}^k\sigma_k(\overline{y})\right) \\ &\quad \geq W(\overline{y}) - \frac{L_1}{2\sqrt{h}}\left[\left|\overline{x}-\overline{y}-\sqrt{h}\eta\right|^2 - \left|\overline{x}-\overline{y}-\sqrt{h}\eta - \frac{\sum \tilde{a}^k\sigma_k(\overline{y})}{f(\overline{y})}\right|^2\right] \\ &\qquad + L_2\sqrt{h}\left[|\overline{y}-\widehat{x}|^2 - \left|\overline{y}-\widehat{x} - \frac{\sum \tilde{a}^k\sigma_k(\overline{y})}{f(\overline{y})}\right|^2\right]. \quad (4.19)\end{aligned}$$

Substituting the left hand side term with (4.14) and using the fact that for every $a,b,c \in \mathbb{R}^n$ we can prove that $|a-b|^2 - |a-b-hc|^2 = 2h(a-b)\cdot c - h^2|c|^2$, we get

$$\begin{aligned} W(\overline{y}) \geq 1 &+ \frac{L_1}{2\sqrt{h^3}}\left[2h(\overline{x}-\overline{y}-\sqrt{h}\eta)\cdot\frac{\sum \tilde{a}^k\sigma_k(\overline{y})}{f(\overline{y})} - h^2\left|\frac{\sum \tilde{a}^k\sigma_k(\overline{y})}{f(\overline{y})}\right|^2\right] \\ &+ \frac{L_2}{2\sqrt{h}}\left[2h(\overline{y}-\widehat{x})\cdot\frac{\sum \tilde{a}^k\sigma_k(\overline{y})}{f(\overline{y})} - h^2\left|\frac{\sum \tilde{a}^k\sigma_k(\overline{y})}{f(\overline{y})}\right|^2\right]. \quad (4.20)\end{aligned}$$

Now, subtracting (4.20) from (4.17) and using the estimates (4.12) and (4.13)

$$\begin{aligned} v(\overline{x}) - W(\overline{y}) &\leq \left(\frac{L_1}{2}\sqrt{h} + \frac{L_2}{2}\sqrt{h^3}\right)\left|\frac{\sum \tilde{a}^k\sigma_k(\overline{y})}{f(\overline{y})}\right|^2 - \frac{L_1}{\sqrt{h}}(\overline{x}-\overline{y}-\sqrt{h}\eta) \\ &\cdot\left(\frac{\sum \tilde{a}^k\sigma_k(\overline{y})}{f(\overline{y})} - \frac{\sum \overline{a}^k\sigma_k(\overline{x})}{f_*(\overline{x})}\right) - L_2\sqrt{h}(\overline{y}-\widehat{x})\cdot\frac{\sum \overline{a}^k\sigma_k(\overline{x})}{f(\overline{x})} \\ &\leq \left(\frac{L_1}{2}\sqrt{h} + \frac{L_2}{2}\sqrt{h^3}\right)\left|\frac{\sum \tilde{a}^k\sigma_k(\overline{y})}{f(\overline{y})}\right|^2 \\ &- L_1\frac{\epsilon}{2+\epsilon}\left|\frac{\sum \tilde{a}^k\sigma_k(\overline{y})}{f(\overline{y})} - \frac{\sum \overline{a}^k\sigma_k(\overline{x})}{f_*(\overline{x})}\right| - L_2\sqrt{h}\epsilon\left|\frac{\sum \overline{a}^k\sigma_k(\overline{x})}{f(\overline{x})}\right|. \quad (4.21)\end{aligned}$$

Finally, choosing $\epsilon = \sqrt{h}$ by the boundedness of f and σ, we obtain

$$v(\overline{x}) - W(\overline{y}) \leq C\sqrt{h}, \quad (4.22)$$

where C is a suitable positive constants. Then, the inequality $\psi(\overline{x},\overline{y}) \geq \psi(x,x)$ gives

$$v(x) - W(x) \leq v(\overline{x}) - W(\overline{y}) \leq C\sqrt{h}, \tag{4.23}$$

for all $x \in \widehat{\Omega}_j$. Let us now consider consider the case when $\overline{y} \in \partial G_j$ or $\overline{x} \in \partial\widehat{\Omega}_j$. If $\overline{y} \in \partial G_j$. the Dirichlet conditions imply that $v(\overline{y}) = W(\overline{y})$ and we have

$$\begin{aligned} v(\widehat{x}) - W(\widehat{x}) &\leq v(\widehat{x} - \sqrt{h}\eta) - v(\widehat{x}) + v(\overline{y}) - v(\overline{x}) \\ &\leq L_v(\sqrt{h} + |\overline{x} - \overline{y}|) \leq L_v(2\sqrt{h} + |\overline{x} - \overline{y} - \sqrt{h}\eta|) \leq C\sqrt{h}. \end{aligned} \tag{4.24}$$

In a similar way we can treat the case $\overline{x} \in \partial\widehat{\Omega}_j$.

To prove the inequality $W(x) - v(x) \leq C\sqrt{h}$ it is enough to interchange the roles of v and W in the auxiliary function ψ.

We add this estimate in (4.3), getting the thesis

$$||v(x) - W(x)||_{L^1} \leq C\sqrt{h} + C'\Delta x. \tag{4.25}$$

□

Remark 1. *It is important to note that the second term in the estimate* (4.25) *is due to the presence of space discontinuities whereas the order* 1/2 *in* h *is typical of the methods based on dynamic programming (see e.g. [2]). Most likely, in order to obtain high-order estimates in our case, one should introduce adaptivity in space, in order to deal with the discontinuities, and adaptivity in the polynomial approximation, to obtain better estimates in the regularity regions. This is an interesting program which we will try to develop in a future work.*

5. Numerical experiments and applications. In this section we present some results for (1.1) on series of benchmarks coming from front propagation, control theory and image processing. In all these examples the discontinuity of the coefficients appears in a natural way and has a natural interpretation with respect to the model.

5.1. Test 1: a front propagation problem. Front propagation problems arise in many different fields of mathematics. A typical approach is to use the Hamilton-Jacobi framework to solve them via the level-set method, as in [26] or by the stationary version of the same problem [15]. Our first test can be interpreted as a front propagation in a discontinuous media. In this model, the level sets of the value function have the meaning of the regions with the same time of arrival of the front.

Let $\Omega := (-1,1) \times (0,2)$ and $f : \Omega \to \mathbb{R}$ be defined by

$$f(x_1,x_2) := \begin{cases} 1 & \text{for } x_1 < 0, \\ 3/4 & \text{for } x_1 = 0 \\ 1/2 & \text{otherwise.} \end{cases} \tag{5.1}$$

It is not difficult to see that f satisfies conditions (2.12). We can verify that the function

$$u(x_1,x_2) := \begin{cases} \frac{1}{2}x_2, & \text{for } x_1 \geq 0, \\ \\ -\frac{\sqrt{3}}{2}x_1 + \frac{1}{2}x_2, & \text{for } -\frac{1}{\sqrt{3}}x_2 \leq x_1 \leq 0, \\ \\ x_2, & \text{for } x_1 < -\frac{1}{\sqrt{3}}x_2. \end{cases} \tag{5.2}$$

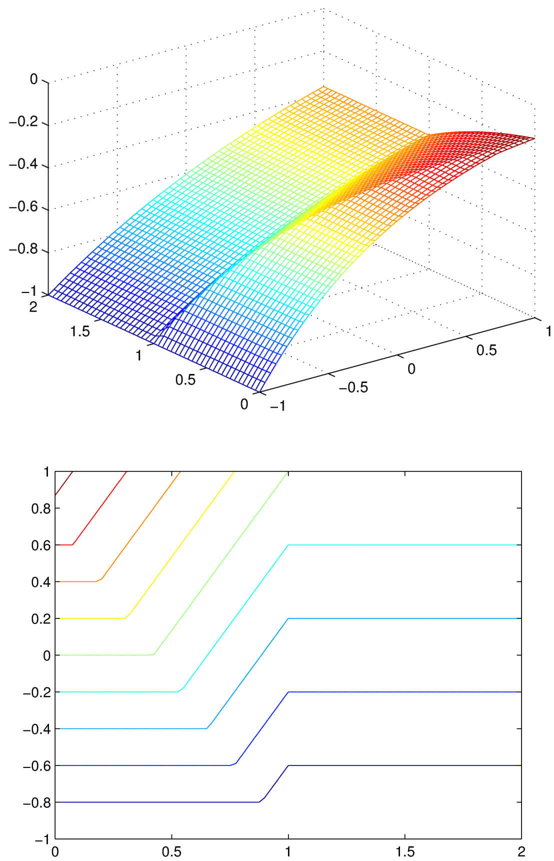

Fig. 5.1. *Test 1.*

$\Delta x = h$	$\|\|\cdot\|\|_\infty$	$Ord(L^\infty)$	$\|\|\cdot\|\|_1$	$Ord(L^1)$
0.1	1.734e-1		8.112e-2	
0.05	8.039e-2	1.1095	3.261e-2	1.3148
0.025	4.359e-2	0.8830	1.616e-2	1.0178
0.0125	2.255e-2	0.9509	7.985e-3	1.0271

Table 5.1
Test 1: experimental error.

is a viscosity solution of $|Du| = f(x)$ in the sense of our definition. Moreover, we take $g := u_{|\partial\Omega}$. We show in the Table 5.1 and in Figure 5.1 our results. As one can see the L^∞ error is not decreasing for a decreasing sequence of Δx whereas the L^1 error is decreasing.

We also show, in Table 5.2 a comparison with the FD methods proposed in [13]. They proposed two techniques: in the first there is a regularization of the Hamiltonian with a viscosity term ($DF - reg$), in the second one ($DF - FS$), better results are obtained, but numerically there are more difficulties; the authors solve them using $FastSweeping$ (see [35]) as acceleration technique and they archive very good results.

$\Delta x = h$	our method	Ord	DF-reg	Ord	DF-FS	Ord
0.1	1.734e-1		1.243e-1		5.590e-2	
0.05	8.039e-2	1.1095	7.229e-2	0.78	2.795e-2	1.00
0.025	4.359e-2	0.8830	4.085e-2	0.82	1.397e-2	1.00
0.0125	2.255e-2	0.9509	2.266e-2	0.85	3.493e-3	1.00

TABLE 5.2
Test 1: comparison between different numerical methods (uniform norm).

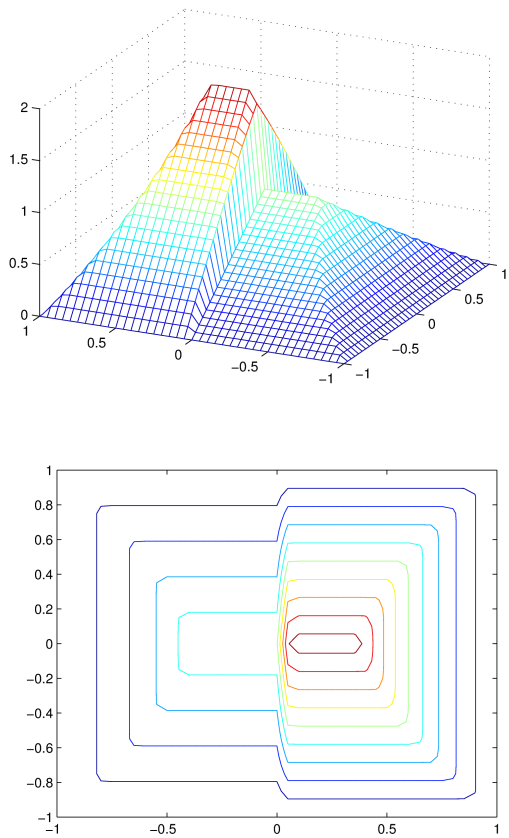

FIG. 5.2. *Test 2.*

Our technique has, in this test, a performance similar to $DF - reg$, In our scheme, the interpolation operator (in this case bilinear) helps adding a regularization. In general, our methods has better performances with respect to FD techniques on more complicated cases, in particular in the problems where characteristics are not straight lines.

5.2. Test 2: a control problem with a discontinuous value function. In this test we present a case where a continuous solution does not exist. In this case it is evident that a convergence in uniform norm will not be possible.

$\Delta x = h$	$\|\| \cdot \|\|_\infty$	$Ord(L_\infty)$	$\|\| \cdot \|\|_1$	$Ord(L_1)$
0.2	1.0884		0.4498	
0.1	1.0469	-	0.2444	0.88
0.05	1.0242	-	0.1270	0.9444
0.025	1.0123	-	0.0628	0.9708
0.0125	1.0062	-	0.0327	0.9867
0.00625	1.0031	-	0.0221	0.5652

Table 5.3
Test 2: experimental error.

Let us get back to our previous Example 1. As already said, let $\Omega := [-1,1]^2$ we want to solve

$$\begin{cases} x^2\,(u_x(x,y))^2 + (u_y(x,y))^2 = [f(x,y)]^2 & \text{for } (x,y) \in (-1,1)\times(-1,1) \\ u(\pm 1, y) = u(x, \pm 1) = 0 & \text{for } x, y \in [-1,1], \end{cases} \tag{5.3}$$

with $f(x,y) = 2$, for $x > 0$, and $f(x,y) = 1$ for $x \leq 0$. The correct viscosity solution is

$$u(x,y) = \begin{cases} 2(1-|y|) & \text{for } x > 0, |y| > 1 + \ln x, \\ -2ln(x) & \text{for } x > 0, |y| \leq 1 + \ln x, \\ \frac{u(-x,y)}{2} & \text{for } x \leq 0. \end{cases} \tag{5.4}$$

We show in Figure 5.2 our results. In this case the convergence in the uniform norm fails. Convergence in the integral norm $L^1(\Omega)$ as proved in Section 4 is confirmed by Table 5.3.

5.3. Test 3: Shape-from-Shading with discontinuous brightness. The Shape-from-Shading problem consists in reconstructing the three-dimensional shape of a scene from the brightness variation (shading) in a greylevel photograph of that scene. The study of the Shape-from-Shading problem started in the 70s (see [19] and references therein) and since then a huge number of papers have appeared on this subject (see e.g. [14]. More recently, the mathematical community was interested in Shape-from-Shading since its formulation is based on a first order partial differential equation of Hamilton-Jacobi type (see the survey [28, 27]).

The equation related to this problem is the following: for a brightness function (Sfs-data) $I(x,y) : \mathbb{R}^2 \supset \Omega \to [0,1]$, in the case of vertical light source, to reconstruct the unknown surface, we need to solve

$$|Du(x,y)| = \left(\sqrt{\frac{1}{I(x,y)^2} - 1} \right), \quad (x,y) \in \Omega. \tag{5.5}$$

Points (x,y) where I is maximal (i.e. equal to 1) correspond to the particular situation when the light direction and n are parallel. These points are usually called "singular points" and, if they exist, equation (5.5) is said to be degenerate. The notion of singular points is strictly related to that of concave/convex ambiguity, we refer to [24, 22] for details on this point.

It is important to note that, whatever the final equation is, in order to compute a solution we will have to impose some boundary conditions on $\partial\Omega$ and/or inside

Ω. A natural choice is to consider Dirichlet type boundary conditions in order to take into account at least two different possibilities. The first corresponds to the assumption that the surface is standing on a flat background, i.e. we set $u(x, y) = 0$ for $(x, y) \in \partial\Omega$. The second possibility occurs when the height of the surface on the boundary (*silhouette*) is known: $u(x, y) = g(x, y)$ for $(x, y) \in \partial\Omega$. The above boundary conditions are widely used in the literature although they are often unrealistic since they assume a previous knowledge of the surface.

Let us focus on two important points:

- We note that a digital image is always a discontinuous datum. Is is a piecewise constant function with a fixed measure of his domain of regularity (pixel). So this is the interest of our analysis for discontinuous cases of f.
- In the case of maximal gray tone ($I(x) = 1$), we are violating the positiveness of f. We overcome this difficulty, as suggest in [10]. We regularize the problem making a truncation of f. It is possible to show that this regularized problem goes to the maximal subsolution of the problem with $\epsilon \to 0^+$. And that this particular solution is the correct one from the applicative point of view.

We consider, now a test with a precise discontinuity on I, and we will discuss some issue about this case.

We firstly consider a simple problem in 1D to point out an aspect of the model. Let the function I be

$$I = \begin{cases} \sqrt{1-x^2} & \text{if } -1 \le x \le 0.2 \\ \frac{\sqrt{2}}{2} & \text{if } 0.2 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{5.6}$$

we can see that we have a discontinuity on $x = 0.2$; despite this, because of the non degeneracy of the dynamics, the solution will be continuous. For this reason we can see that changing the boundary condition of the problem, the solution will be the maximal Lipschitz solution that verifies continuously the boundary condition.

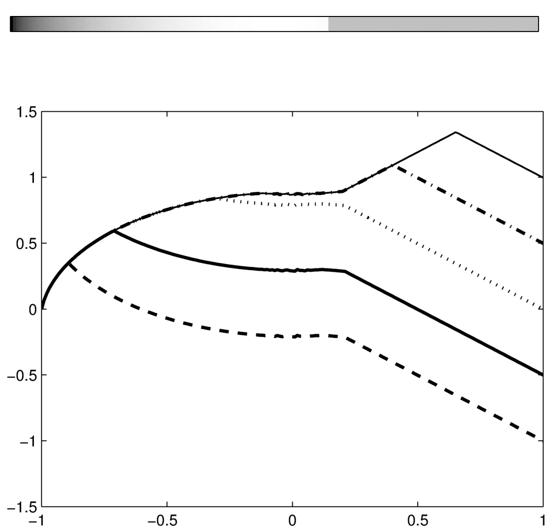

FIG. 5.3. *Sfs-data and solutions with* $u(-1) = 0$*,* $u(1) \in \{-1, -0.5, 0, 0.5, 1\}$.

To see this we have solved this simple one-dimensional problem with various Dirichlet condition, in particular we require $u(-1) = 0$, and $u(1) = \{-1, 0.5, 0, 0.5, 1\}$. With $\Delta x = 0.01$ and $\Delta t = 0.002$, we obtain the results shown in Figure 5.3.

We can realize, in this way, an intrinsic limit of the model. It can not represent an object with discontinuities. We make another example that is more complicated and more close to a real application.

Fig. 5.4. *Basilica of Saint Paul Outside the Walls: satellite image and simplified Sfs-data.*

We consider a simplified sfs-datum for the Basilica of Saint Paul Outside the Walls in Rome, as shown in Figure 5.4. We have not the correct boundary value on the silhouette of the image and on the discontinuities, so we impose simply $u \equiv 0$ on the boundary. Computing the equation with $\Delta t = 0.001$ we get the solution described on Figure 5.5.

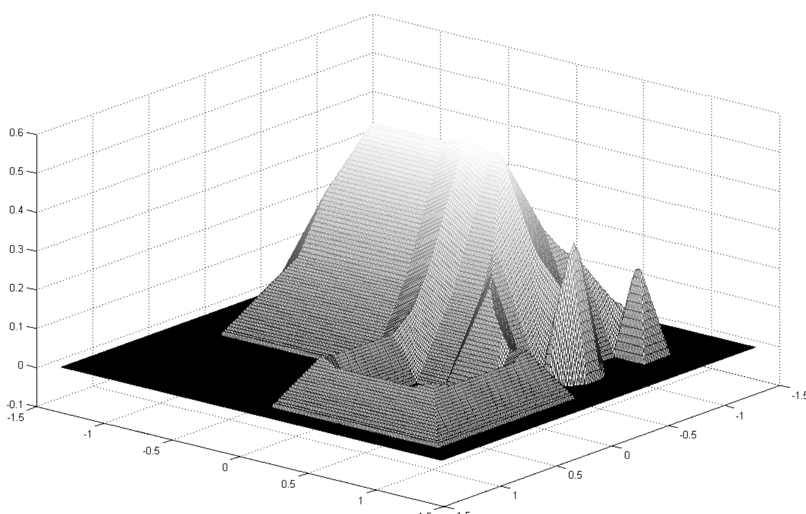

Fig. 5.5. *Test 3: reconstructed shape without boundary data.*

Test	$\|\cdot\|_\infty$	$\|\cdot\|_1$
w/o correct boundary data	1.7831	1.5818
w boundary data	0.8705	0.5617
w boundary + disc. detect.	0.7901	0.3062

TABLE 5.4
Test 3: Comparison between various methods

We can see that, although the main features of the shape as the slope of the roofs, the points of maximum are well reconstructed. Despite it, the shape which we get is not so close to our expectations. We can try to get better results adding the correct height of the surface along the silhouette as discussed above and, in this case, we get the solution shown on Figure 5.6. We can notice a more convincing shape, but also in this case it is quite not satisfactory. For example we have that the correct boundary conditions we imposed are not attained, and we create some discontinuity on some parts of them. This is due to the fact that they can be not compatible with the statement of the problem. Essentially the limit which we can see, as described above, is that we cannot have discontinuity on the viscosity solution (Theorem 2.5).

We propose a different model for this problem, which allows discontinuous solutions. At this point we do not care about the physical interpretation of it, instead we are trying to find a solution closer to the correct solution. We want to solve the equation

$$\begin{cases} \max\limits_{|a|\le 1}\left\{-Du(x)\cdot \sum\limits_{k=1}^{2} a^k\sigma_k(x,y)\right\} = \sqrt{\frac{1}{I^2(x,y)}-1} & \text{for } x\in\Omega, \\ u(x)=g(x) & \text{for } x\in\partial\Omega, \end{cases} \tag{5.7}$$

with the map $\sigma:\Omega\to\mathbb{R}^{2,2}$ is

$$\sigma(x,y)=\begin{pmatrix} (1+|I(x-h,y)-I(x+h,y)|)^{-p} & 0 \\ 0 & (1+|I(x,y-h)-I(x,y+h)|)^{-p} \end{pmatrix}, \tag{5.8}$$

where $p\in\mathbb{R}$ is a tuning parameter. Obviously this choice of the anisotropic evaluator σ is a bit trivial. This pick is done for the sake of simplicity. More complicated proposal can be found for example in [1].

In this way we use the results about the degeneracy of the dynamics permitting to the viscosity solution to be discontinuous. Of course this is, in some sense, the opposite situation with respect to the classical formulation: in this case every non smooth point of the surface is interpreted as discontinuity and we try to reconstruct it using the data coming from the silhouette.

The results are shown in Figure 5.7 and in Table 5.4 we can see an accuracy comparison of the various procedure.

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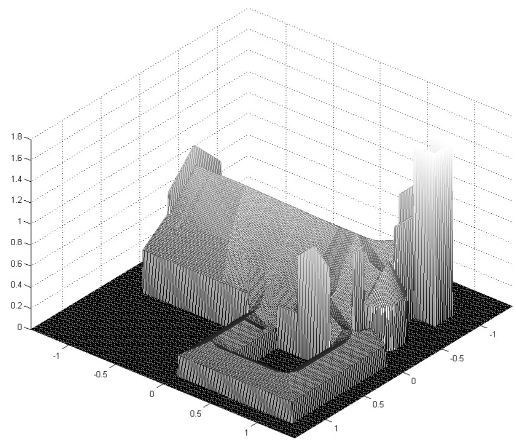

Fig. 5.6. *Test 3: Dirichlet condition on the silhouette.*

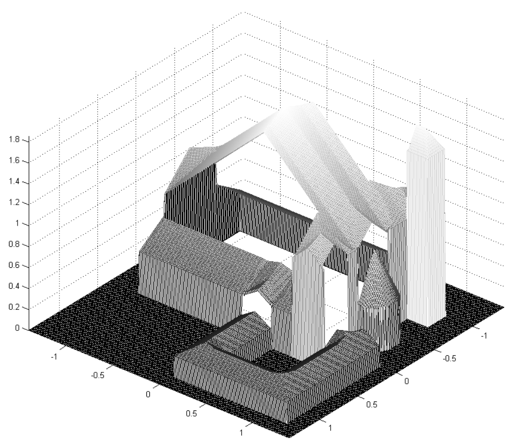

Fig. 5.7. *Test 3: Dirichlet condition and discontinuous dynamics.*

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