Optimal Route Planning for Sailing Boats: A Hybrid Formulation

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Optimal Route Planning for Sailing Boats: A Hybrid Formulation

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Abstract

We present an optimal hybrid control approach to the problem of stochastic route planning for sailing boats, especially in short course fleet races, in which minimum average time is an effective performance index. We show that the hybrid setting is a natural way of taking into account tacking/gybing maneuvers and other discrete control actions, and provide a practical example of a hybrid model for this problem. Moreover, we carry out a numerical validation of the approach and show that results are in good agreement with theoretical and practical knowledge.

Keywords Hybrid systems · Optimal control · Route planning

Mathematics Subject Classification 93E20 · 65C20 · 65N06

1 Introduction

In the last decades, the sport of sailing has experienced an increasing impact of new technologies and notably of scientific computing. Among all computational problems relevant for sailing, we focus here on *route planning* and *race strategy*, i.e., the optimization of the yacht route.

In the most typical and basic form, the route planning problem requires reaching a windward mark in minimum time within a variable wind field. According to the

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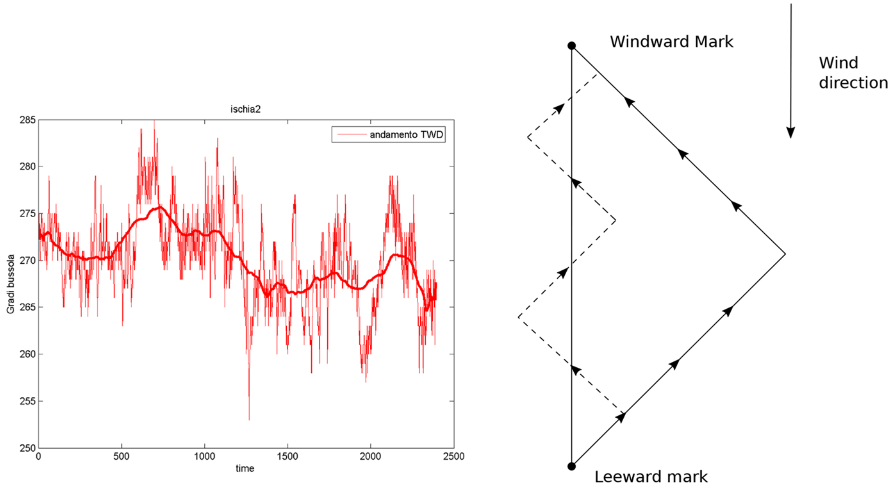


Fig. 1 (left) Evolution of the wind observed during a race (thickened plot the average over 180 s) (left). Upwind tacking: to go from the Leeward to the Windward mark it is not possible to point directly to the target (right)

previous literature and experimental evidence, this problem is approached from a stochastic viewpoint, in which the wind field is modeled as having both a deterministic and a stochastic component [1]. Even having a reliable forecast of its evolution, the wind field is affected by random fluctuations (Fig. 1, left) that represent a crucial part of the problem.

Among the various tools, used to tackle the route planning problem, we will concentrate on dynamic programming, which is the basis for some state-of-the-art techniques. A discrete Markov chain approximation of the stochastic route planning problem has been proposed, for example, in [2], based on an assumption of Markovian behavior for the wind model, and similar ideas are also developed in [3–5]. The common feature of these techniques is to be conceived from the very start in discrete form. On the other hand, a continuous stochastic control model has been developed in [6], in which the optimal strategy, at least in a simplified setting, is characterized via a *switching curve*, resulting from the solution of a free boundary problem. We also quote [7,8], which address the modeling and racing strategy for the specific case of match races, but also provide an excellent review of the relevant literature.

The velocity of a sailing boat is usually characterized via the so-called *polar plot* (see Sect. 3), which shows, at fixed wind speed, the boat velocity as a function of the angle between the boat direction and the wind direction (true wind angle or TWA) and speed (true wind speed or TWS). In every polar plot, the boat speed goes to zero, when pointing in the direction of the wind; this implies that, when there is a change of direction while passing through the zero TWA (i.e., when *tacking*), the boat slows down. This delay is a crucial point in the model: Avoiding to take it into account may result in unrealistic optimal paths, possibly heading directly against the wind via an infinitely fast switching ("chattering") between a positive and a negative angle.

A very natural form of taking into account the delay effects of tacking, without increasing the dimension of the problem, is to use the framework of *hybrid systems*. This notion has been proposed in the 1990s to treat control systems operated by both continuous and discrete control actions. Among the extensive literature on hybrid control, we quote here the general approach proposed in [9], along with the study in [10,11], aimed at treating optimal hybrid control problems in the framework of viscosity solutions, for, respectively, the deterministic and the stochastic case (see also [12]). In the present work, this specific framework is used to obtain a sound theoretical background [13] for the convergence of monotone schemes. Previous works in this direction are [14,15], both of which, however, treat the deterministic case. A stochastic route planning problem, close to the hybrid formulation although with a slightly different model, has been considered in [16].

In the problem under consideration, we will regard as continuous control the choice of a particular route on a fixed tack and as discrete control a change in the dynamics (including tacking and any other discrete action like a change in the sail configuration). The dynamic programming approach, in the relatively low dimension of this problem, gives some clear advantages:

- we obtain a *static feedback control*, which is computed once and for all for a given target and wind model;
- costs associated with discrete controls (tacking, changes in configuration) may be easily taken into account;
- we can deal with the presence of a *stochastic term* in the system, modeling the variations in the wind with respect to the expected data;
- the presence of *state constraints* (coasts, obstacles), as well as current and tides, can be handled in a relatively straightforward way.

At a comparison with the other approaches proposed, the one pursued in this work is much in the same spirit as [6]. We characterize the optimal solution through a continuous Bellman equation, which provides a solid background for fairly general formulations. Opposite to [6], however, we handle the problem by numerical techniques. Convergence of the fully discrete approximations to the underlying continuous problem (see Sect. 4) ensures that the approximate optimal strategy has a relatively minor dependence on the discretization.

The paper is structured as follows: in Sect. 2, we introduce the mathematical framework of stochastic hybrid systems. In Sect. 3, we discuss in detail the modeling of the route planning problem, while Sect. 4 describes and analyzes the numerical framework. Last, in Sect. 5, we perform various tests to validate the approach and show the effectiveness of the technique in different scenarios of application.

2 Hybrid Control Framework for the Route Planning Problem

We start the section by introducing a motivating example, which will be generalized later.

2.1 A Basic Example

In the simplest situation, the route planning problem may be formulated as follows. A boat must reach in minimum time a windward target (e.g., a mark) in a wind of constant strength and random angle θ with zero mean value, centered at the vertical direction as in Fig. 1, which evolves as

$$\theta(t) = \bar{\sigma} W_t, \tag{1}$$

with W_t a standard Brownian process. On the $x_1 - x_2$ plane, the boat moves at an angle with the wind (say, $\pm\pi/4$) corresponding to the highest windward component of the speed, i.e., it moves starting from $X(0)$ and according to

$$\begin{aligned} \dot{X}_1(t) &= \bar{r} \sin(-\theta(t) \pm \pi/4) \\ \dot{X}_2(t) &= \bar{r} \cos(\theta(t) \pm \pi/4), \end{aligned} \tag{2}$$

where the plus sign applies on the starboard tack (i.e., with a TWA between 180° and 360°), the minus sign applies on the port tack (i.e., with a TWA between 0° and 180°) and the magnitude of the speed is $\bar{r}$. Such a "composite" dynamics is controlled by changes in tack, which imply a loss of speed, to be taken into account via a switching cost $\bar{C}$. In a more formal way, we can associate with the two dynamics an additional function $Q(t) \in \{1, 2\}$ which indicates the active dynamics. If, for example, the port tack is coded with the value $Q = 1$ and the starboard tack with $Q = 2$, then (2) can be rewritten as

$$\begin{aligned} \dot{X}_1(t) &= \bar{r} \sin\left(-\theta + (-1)^{Q(t)}\pi/4\right) \\ \dot{X}_2(t) &= \bar{r} \cos\left(\theta + (-1)^{Q(t)}\pi/4\right). \end{aligned} \tag{3}$$

Since the optimal solution depends not only on the position $X(0)$, but also on $Q(0)$, the function Q should be regarded as part of the state of the system.

To sum up, the problem is to drive to a target set a stochastic system (boat) which is controllable via discrete changes in the dynamics (tacking), so as to minimize the sum of travel time and tacking delays. Note that, in a more general framework, we can also consider continuous adjustments in the direction via a conventional control function $u(t)$, as well as changes in the configuration of the sails via discrete switches in $Q(t)$—the former extension will be given in Sect. 3, whereas the latter will not be pursued here.

We will now describe how this problem admits a natural description in terms of hybrid control.

2.2 Hybrid Stochastic Minimum Time Problem

A *hybrid control system* is a system that, in addition to the usual continuous control, can undergo discrete control action, like switching between different dynamics or jumping from one state to another, in a discontinuous way.

Among the various mathematical formulations describing a hybrid system, we refer here to a simplified version of the one proposed in [11]. (Similar formulations for the deterministic case have been proposed in [9,10], while a stochastic hybrid problem similar to the one under consideration has been studied in [17].) Let $\mathcal{I} = \{1, 2, \ldots, N_{\mathcal{I}}\}$ be finite, and consider the controlled system (X, Q) described by:

$$\begin{aligned} &\mathrm{d}X(t) = f(X(t), Q(t), u(t))\mathrm{d}t + \sigma(X(t), Q(t))\, \mathrm{d}W_t, \\ &X(0) = x,\ \ Q(0^+) = q, \end{aligned} \tag{4}$$

where $x, X \in \mathbb{R}^d$, $q, Q \in \mathcal{I}$ and $\mathrm{d}W_t$ is the differential of a d-dimensional standard Brownian process. Here, $X(t)$ and $Q(t)$ denote, respectively, the continuous and the discrete component of the state at time t, and, in order to end up with a stationary dynamic programming equation, we are assuming that f depends on t only via X, Q and u. The function $f : \mathbb{R}^d \times \mathcal{I} \times U \to \mathbb{R}^d$ represents the continuous dynamics, for a set of controls given by:

$$\mathcal{U} = \{u :]0, \infty[\to U \mid u \text{ measurable},\ U \text{ compact}\},$$

and we assume both f and σ to be globally bounded and uniformly Lipschitz continuous w.r.t. x.

The term $Q(t)$ models possible switches between the various dynamics of the system and takes values in a set of piecewise constant functions $\mathcal{Q}$:

$$\mathcal{Q} = \left\{ Q(\cdot) :]0, \infty[\to \mathcal{I} \mid Q(t) = \sum_{i \in \mathbb{N}} w_i \chi_{t_i}(t) \right\},$$

where $\chi_{t_i}(t) = 1$ if $t \in [t_i, t_{i+1}[$ and 0 otherwise, $\{t_i\}_{i \in \mathbb{N}}$ are the (ordered) switching times and $\{w_i\}_{i \in \mathbb{N}}$ are values in $\mathcal{I}$. With respect to the more general setting, we are assuming some simplifications, and in particular:

- The discrete control is in the form of a *switching*, i.e., it can only change the discrete component of the state $Q(t)$, leaving $X(t)$ unchanged;
- A controlled switching can occur in the whole state set $\mathbb{R}^d \times \mathcal{I}$ and, for simplicity, no "autonomous" switching (e.g., because of obstacles or state constraints) is included in the model.

The trajectory starts from $(x, q) \in \mathbb{R}^d \times \mathcal{I}$. The choice of the control strategy defined as $\mathcal{S} := \big(u, \{t_i\}, \{Q(t_i^+)\}\big)$ has the objective of minimizing the following cost functional of discounted minimum time type:

$$J(x, q; \mathcal{S}) := \mathbb{E}\left(\int_0^{\tau_{x,q}} \mathrm{e}^{-\lambda t} \mathrm{d}t + \sum_{i=0}^{N} C\left(X(t_i), Q(t_i^-), Q(t_i^+)\right) \mathrm{e}^{-\lambda t_i} \right), \tag{5}$$

where $\mathbb{E}$ denotes the expected value, $\tau_{x,q}$ is the first time of arrival in a given compact set $\mathcal{T} \subset \mathbb{R}^d$, i.e.,

$$\tau_{x,q} := \min_{t\in[0,+\infty[} \{t \ : \ X(t) \in \mathcal{T}\},$$

$\lambda > 0$ is the discount factor, and $C : \mathbb{R}^d \times \mathcal{I} \times \mathcal{I} \to \mathbb{R}_+$ is the switching cost between the dynamics, which is assumed to have a strictly positive infimum, to be bounded and Lipschitz continuous w.r.t. x and to satisfy the condition

$$C(x, q_1, q_2) < C(x, q_1, q_3) + C(x, q_2, q_2), \tag{6}$$

for any triple of indices q_1, q_2 and q_3.

The value function v of the problem is defined, for $\mathcal{S} \in \mathcal{U} \times \mathbb{R}_+^{\mathbb{N}} \times \mathcal{I}^{\mathbb{N}}$, as:

$$v(x, q) := \inf_{\mathcal{S}} J(x, q; \mathcal{S}), \tag{7}$$

and is characterized via a suitable Hamilton–Jacobi–Bellman (HJB) equation. Continuity of the value function, which allows to apply the framework of viscosity solutions, is a delicate matter in deterministic hybrid control problems (we refer the reader to [10] for a precise set of assumptions), whereas in the stochastic case the literature reports somewhat weaker assumptions (see [11]).

We recall some basic analytical results about the value function (7). Using a suitable generalization of the dynamic programming principle, it is possible to prove that the value function of the problem solves a Bellman equation in a quasi-variational inequality form. More precisely, defining for $x, p \in \mathbb{R}^d$ and $q \in \mathcal{I}$ the Hamiltonian function by

$$H(x, q, p) := \sup_{u\in U} \{-f(x, q, u) \cdot p - 1\} \tag{8}$$

and the controlled switching operator $\mathcal{N}$ by:

$$\mathcal{N}\phi(x, q) := \inf_{q'\in\mathcal{I}} \{\phi(x, q') + C(x, q, q')\},$$

we have a Bellman equation of the following form:

$$\max\left(v - \mathcal{N}v, \frac{1}{2}\operatorname{tr}\left(\sigma\sigma^t D^2 v\right) + \lambda v + H(x, q, Dv)\right) = 0, \tag{9}$$

defined on $(\mathbb{R}^d \setminus \mathcal{T}) \times \mathcal{I}$, i.e., a system of quasi-variational inequalities, complemented with the boundary condition

$$v(x, q) = 0 \quad (x \in \partial\mathcal{T}).$$

In what follows, we will assume that the problem is posed on a set Ω of the state space and that either $\Omega = \mathbb{R}^d$, or a stopping cost is defined on the boundary of Ω (in the form of a weak Dirichlet condition, see [18]), so as to enforce state constraints by penalization, and work on a finite computational domain. We will give suitable examples in the numerical test section.

In (9), we can identify two separate Bellman operators, which provide, respectively, the best possible switching and the best possible continuous control. The argument attaining the maximum in (9) represents the overall optimal control strategy.

3 Practical Hybrid Models for Route Planning

In order to apply the techniques introduced above to solve route planning problems, we outline in this section some general ideas toward a formal modeling of the problem.

In general, we expect that a reliable modeling should use at least a three-dimensional state space, i.e., $d \geq 3$. In this setting, two components x_1 and x_2 of the state space represent the position of the boat, while the components x_3 to x_d account for the evolution of the wind.

We start by describing the motion of the boat as resulting from both the wind vector field and the boat characteristics. We will make the standing assumption that the control u denotes the (unsigned) angle between the boat direction and the wind, so that $u(t) \in U = [0, \pi]$, and that the stochastic component of the dynamics appears only in the wind evolution (described by the wind speed s and direction θ). Keeping the conventions of (3), the motion of the boat is then described by

$$\begin{aligned} \dot{X}_1(t) &= r(s(X(t), t), u(t)) \sin\left(-\theta(X(t), t) + (-1)^{Q(t)} u(t)\right) \\ \dot{X}_2(t) &= r(s(X(t), t), u(t)) \cos\left(\theta(X(t), t) + (-1)^{Q(t)} u(t)\right). \end{aligned} \tag{10}$$

The function $r : \mathbb{R}_+ \times [0, \pi] \to \mathbb{R}_+$ models the *polar plot* of the boat and provides the boat speed as a function of the angle u of the trajectory w.r.t. the wind and of the wind speed s. Figure 2 summarizes the geometric setting.

The polar plot is related to the technical characteristics of the craft and is determined via experimental measures. It differs from one boat to another, but we can typically assume it to be Lipschitz continuous w.r.t. both variables s and u, to be independent of both time and position and to satisfy the condition $r(s, 0) = 0$, which means that the boat always has zero speed when "pointing directly against the wind."

In (10), the wind is characterized by two functions: direction θ and speed s. These function may depend on position and time, but we assume that they evolve in time according to a lumped parameter model, i.e., the system of stochastic differential equations (SDEs)

$$\begin{aligned} \mathrm{d}s(x, t) &= g_1(x, s(x, t), \theta(x, t))\mathrm{d}t + g_2(x, s(x, t), \theta(x, t))\mathrm{d}W_t^{(1)} \\ \mathrm{d}\theta(x, t) &= h_1(x, s(x, t), \theta(x, t))\mathrm{d}t + h_2(x, s(x, t), \theta(x, t))\mathrm{d}W_t^{(2)}, \end{aligned} \tag{11}$$

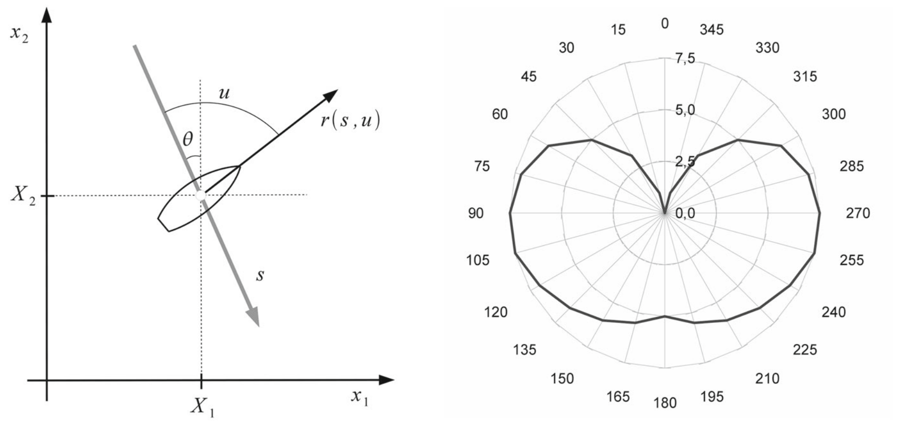


Fig. 2 (Left) Geometric setting: Wind (gray arrow), boat speed (black arrow) and control u (left). Polar plot of the boat speed as a function of the TWA, with fixed TWS (right)

in which x is considered as a parameter, $\mathrm{d}W_t^{(i)}$ $(i = 1, 2)$ denotes the differential of a standard Brownian process, and, depending on the complexity of the model, both s and θ could be multi-dimensional.

3.1 A Detailed Example

We will discuss now in detail a simplified model, still containing the basic features of the full problem, at least in an upwind leg of a race. We assume that $d = 3$, the third state variable being the wind direction, $x_3 = \theta$, whereas the wind speed s is kept constant w.r.t. position and time, $s(x, t) \equiv \bar{s}$. With the notations used in (3), we have $Q \in \mathcal{I} = \{1, 2\}$, and we can write

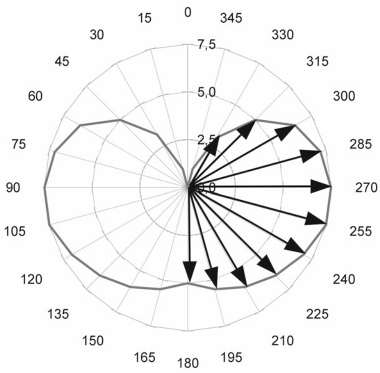
$$
\begin{aligned}
\dot{X}_1(t) &= r(\bar{s}, u) \sin\left(-\theta + (-1)^{Q(t)} u\right) \\
\dot{X}_2(t) &= r(\bar{s}, u) \cos\left(\theta + (-1)^{Q(t)} u\right).
\end{aligned} \tag{12}
$$

Assuming further that the control u is frozen at the most efficient angle u^* w.r.t. wind, for example, $u \equiv u^* = \pi/4$ and that tacking is the only control action at all (see Fig. 3), then, for $\bar{r} = r(\bar{s}, u^*)$, we recover dynamics (3).

The evolution of the wind direction is given by the one-dimensional SDE

$$
\mathrm{d}X_3 = a(X_3)\mathrm{d}t + \bar{\sigma}\mathrm{d}W_t. \tag{13}
$$

In particular, taking $a(X_3) = \bar{a}$ (constant) would model a situation in which the wind direction comes from the superposition of a constant (clockwise or anticlockwise) drift with a random fluctuation. [In particular, for $\bar{a} = 0$ we obtain (1).] Dynamics (12) and (13) correspond then to choosing



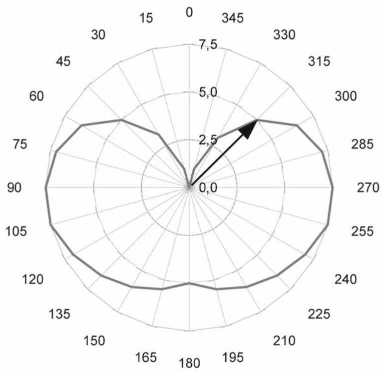


Fig. 3 One of the two dynamics of the system ($q = 1$), superposed on the polar plot of the speed (left) and simplified dynamics (right) based on the angle of largest windward component of the speed

$$\sigma = (0, 0, \bar{\sigma})^T,$$

and the diffusion term in the HJB Eq. (9) degenerates and acts only along the third dimension. Note that, roughly speaking, assuming a one-dimensional model for θ parallels the assumption of discrete Markov process without memory as in [2].

3.1.1 Resulting Hamilton–Jacobi–Bellman Equation

We collect now all the information in an explicit HJB equation. Concerning the deterministic component of the dynamics, we have, substituting (13) into (12):

$$\begin{aligned} f_1(x, q, u) &= r(\bar{s}, u) \sin\left(-x_3 + (-1)^q u\right) \\ f_2(x, q, u) &= r(\bar{s}, u) \cos\left(x_3 + (-1)^q u\right) \\ f_3(x, q, u) &= a(x_3), \end{aligned}$$

which results in a Hamiltonian function of the form

$$\begin{aligned} H(x, q, p) = \sup_{u \in U} \Big\{ - r(\bar{s}, u) \Big[&\sin\left(-x_3 + (-1)^q u\right) p_1 \\ &+ \cos\left(x_3 + (-1)^q u\right) p_2 \Big] \Big\} - a(x_3) p_3 - 1. \end{aligned} \tag{14}$$

The term related to the stochastic component of the evolution reads in turn:

$$\frac{1}{2} \operatorname{tr}\left(\sigma \sigma^t D^2 v\right) = \frac{\bar{\sigma}^2}{2} v_{x_3 x_3}.$$

Last, we assume a constant switching cost $\bar{C} > 0$. Defining $\hat{q} = 2$ if $q = 1$, $\hat{q} = 1$ if $q = 2$, we have that, if optimal, the only possible switch may be from q to $\hat{q}$. The switching operator $\mathcal{N}$ can therefore be simplified as:

$$\mathcal{N}\phi(x,q) = \phi(x,\hat{q}) + \bar{C}. \tag{15}$$

Then, the Bellman equation takes the form of the following system of two quasi-variational inequalities (for $x \in \mathbb{R}^3$, $q = 1, 2$):

$$\max\left(v(x,q) - v(x,\hat{q}) - \bar{C}\,,\ \frac{\bar{\sigma}^2}{2} v_{x_3x_3}(x,q) + \lambda v(x,q) + H(x,q,Dv)\right) = 0, \tag{16}$$

in which H is defined by (14) and the $\sup_u$ is clearly dropped if $u \equiv u^*$.

3.2 Possible Generalizations of the Model

While the previous model seems a reasonably simple way of taking into account the major processes involved in the route planning problem, we sketch some related models which lead to a different definition of the state space.

- *x-dependence of the wind direction/speed* Note that it is possible to introduce in (12) and (13) an x-dependence without increasing the conceptual difficulty of the problem. This change only results in making the Hamiltonian depend on x_1 and x_2 and has no consequences on the definition of the state space.
- *1D (possibly stochastic) evolution for wind speed* In this case, dynamics (12) for the wind would be replaced by

$$\begin{aligned} \mathrm{d}\theta &= a(\theta,s)\mathrm{d}t + \bar{\sigma}_1 \mathrm{d}W_t^{(1)} \\ \mathrm{d}s &= b(\theta,s)\mathrm{d}t + \bar{\sigma}_2 \mathrm{d}W_t^{(2)}. \end{aligned}$$

  Then, we would define $x_4 = s$ and obtain a state space of dimension $d = 4$. In the relevant literature, this model is usually simplified taking $b(\theta,s) \equiv 0$.
- *Different configurations of the sails* A change in configuration should appear as a discrete control. If N_c different configurations of the sails are possible, then every sail configuration should be replicated on both tacks, and we would obtain $N_\mathcal{I} = 2N_\mathrm{c}$, with fixed dimension d. In this case, the polar of the vessel would depend on Q, i.e., we would have $r = r(s, Q, u)$.
- *Long course racing* When long course racing is concerned, a local wind model like (13) is no longer possible. In this case, the deterministic component in the wind model might be dependent on time (and could be taken, e.g., from suitable weather forecasts like GRIB data), and in the simplest case, the problem could be modeled via two state variables plus time. Accordingly, the HJB equation would be recast in the time-dependent form, with $d = 2$ and $N_\mathcal{I} = 2N_\mathrm{c}$.

4 Numerical Solution via Monotone Schemes

In order to set up a numerical approximation for (9), we construct a discrete grid of nodes (x_j, q) in the state space with discretization parameters Δx and, possibly, Δt.

(Time discretization should be understood in the "time marching" sense; see [18].) In what follows, we denote the discretization steps in compact form by $\Delta = (\Delta t, \Delta x)$ and the approximate value function by V^Δ.

Following [14], we write the scheme at (x_j, q) in fixed point form as

$$V^\Delta(x_j, q) = \min\left(NV^\Delta(x_j, q), \Sigma\left(x_j, q, V^\Delta\right)\right). \tag{17}$$

In (17), the numerical operator Σ is related to the continuous control, or, in other terms, to the approximation of the Hamiltonian function (8). As it has been done in (15), the discrete switch operator N is computed as

$$NV^\Delta(x_j, q) := V^\Delta(x_j, \hat{q}) + C(x_j, q, \hat{q}), \tag{18}$$

and, in fact, this corresponds to the exact definition.

4.1 Theoretical Analysis for Monotone Schemes

The theoretical analysis of monotone schemes for HJB equations arising in stochastic hybrid control problems can be carried out using (with some adaptations) the arguments in [14], which will be sketched here. We start by proving that the value iteration for (17), i.e.,

$$V^\Delta_{k+1}(x_j, q) = \min\left(NV^\Delta_k(x_j, q), \Sigma\left(x_j, q, V^\Delta_k\right)\right) \tag{19}$$

is convergent. To this end, we denote by $S(V^\Delta_k)$ the vector (indexed by j) of the values at the right-hand side of (19).

1. *If the scheme Σ is monotone, then the mapping $S(\cdot)$ is also monotone.*
   Assume that $U \geq W$ element by element. Since, as proved in [14], the operator N is monotone, we only have to use the property that the min of two monotone operators is monotone and therefore that $S(U) \geq S(W)$.
2. *The sequence V^Δ_k is monotone decreasing if V^Δ_0 is chosen so that $V^\Delta_0 \geq S(V^\Delta_0)$.*
   In fact, this condition implies that $V^\Delta_1 \leq V^\Delta_0$, so that by monotonicity of S we have $S(V^\Delta_1) \leq S(V^\Delta_0)$, that is, $V^\Delta_2 \leq V^\Delta_1$, and, inductively, $V^\Delta_{k+1} \leq V^\Delta_k$. Note that this also entails that the scheme is stable in the ∞-norm.
3. *If, for any admissible Δ, V^Δ_k is positive, then V^Δ_k converges to a fixed point V^Δ solution of* (17).
   We recall that, under the assumptions made on the cost functional, positivity of solutions is a natural assumption, which is typically satisfied by monotone schemes (upwind, Lax–Friedrichs, semi-Lagrangian) under stability conditions. Convergence to a fixed point follows then from monotonicity and boundedness for all elements of the vectors V^Δ_k.

Last, once ensured that (17) has a solution (which can be obtained by value iteration), the convergence of V^Δ to the value function v is ensured by the Barles–Souganidis theorem [13], provided the scheme Σ is consistent with the Hamiltonian function (8), since a comparison principle holds for the continuous problem (see [11]). We have therefore the following.

Theorem 4.1 *Let the basic assumptions of Sect. 2 hold. Let moreover V_0^Δ be chosen so that $V_0^\Delta \geq S(V_0^\Delta)$. If the operator Σ is monotone and such that $S(V) \geq 0$ for any $V \geq 0$, then the value iteration (19) converges to V^Δ solution of (17). Moreover, if the scheme $V = \Sigma(V)$ is also consistent with the equation*

$$\frac{1}{2}\operatorname{tr}\left(\sigma\sigma^t D^2 v\right) + \lambda v + H(x, q, Dv) = 0,$$

then $V^\Delta(x_j, q) \to v(x_j, q)$ for $\Delta \to (0, 0)$.

4.2 Example: A Semi-Lagrangian Scheme

A viable technique for solving (9) is a semi-Lagrangian scheme, obtained by adapting the scheme proposed in [19]. The main advantage of such approach is unconditional stability of the scheme with respect to the discretization parameters, still keeping monotonicity.

The scheme requires extending the node values to all $x \in \mathbb{R}^d$ using an interpolation $\mathbb{I}$. We denote by $\mathbb{I}\left[V^\Delta\right](x, q)$ the interpolation of the values $V^\Delta(x_j, q)$ computed at (x, q). With this notation, a standard semi-Lagrangian discretization of the Hamiltonian and the diffusive term in (16) is given by

$$\Sigma\left(x_j, q, V^\Delta\right) = \Delta t + \mathrm{e}^{-\lambda\Delta t}\min_{u\in U}\left\{\frac{1}{2}\left(\mathbb{I}\left[V^\Delta\right](\xi_j^+, q) + \mathbb{I}\left[V^\Delta\right](\xi_j^-, q)\right)\right\}, \tag{20}$$

where $\xi_j^\pm := x_j + \Delta t\ f(x_j, q, u) \pm \sqrt{\Delta t}\,\bar{\sigma} e_3$, with $e_3 = (0, 0, 1)^T$. Then, the full scheme is obtained by substituting (18)–(20) in (17). Note that, in the basic setting, the use of a positive discount factor λ ensures convergence of the value iteration. However, this solver may show a very slow convergence, and more efficient techniques include:

- *Fast solvers* Since the control problem is in the form of a target problem, a careful discretization preserving *causality* would allow for the use of fast-marching/fast-sweeping solvers. We refer the reader to the discussion and references in [18].
- *Modified policy iteration* In this technique, an inexact policy iteration is implemented by alternating iterations in which a new feedback policy is computed to iterations of plain linear advection. Again, we do not provide details and rather refer the reader to [15] and the references therein, for a detailed study of the hybrid case.
- *Parallel computing* In order to apply domain decomposition techniques, the hyperbolic nature of the problem requires some care to optimize the communication between the treads. Recent works on this subject are [20–22]. In particular, the *obstacle problem* treated in [22] has a strong link with our case.

5 Numerical Tests

We provide in this section a numerical validation of the technique under consideration, using some typical situations in race strategy, and computing the value functions via the semi-Lagrangian scheme (20). According to the general features of dynamic programming techniques, the optimal control is computed in feedback form: In particular, the optimal choice for u at a point (x_j, q) is the value achieving the min in (20), while the need for a switch is indicated by the situation in which

$$\min\left(NV^{\Delta}(x_j, q), \Sigma\left(x_j, q, V^{\Delta}\right)\right) = NV^{\Delta}(x_j, q) = V^{\Delta}(x_j, \hat{q}) + \bar{C}, \qquad (21)$$

moreover, in this case, the optimal switching is toward the dynamics $\hat{q}$. Then, the numerical tests have been carried out in two phases. In the first (offline) phase, the value function of the problem has been computed, and both the optimal feedback u and the optimal switch have been suitably stored in memory. In the second (real-time) phase, a sample trajectory of the Brownian process is generated, with the increments ΔW of the process simulated via a pseudorandom Gaussian number generator, and a standard stochastic Euler scheme (see [23]) is used to approximate the state Eq. (12), applying the approximate optimal control/switching computed in the offline phase.

5.1 Test 1

As a first test, we choose the very basic case of a constant average direction of wind ($\bar{a} = 0$), with $(x_1, x_2, x_3) \in [-1.4, 1.4] \times [0, 2] \times [-1, 1]$ taking as target (windward mark) a disk centered in $(0, 1.8)$ of radius 0.04. The goal of this test is to observe the effects of variations in the diffusion coefficient $\bar{\sigma}$.

State constraints have been implemented by penalization of the boundary value, introducing a stopping cost $\bar{b} = 100$, with $\lambda = 10^{-6}$, and tacking cost $C(x, 1, 2) = C(x, 2, 1) \equiv 2$. We have adopted both the complete and the simplified dynamics described in Sect. 3.1. In the first case, the vessel speed around the optimal angle $u^* = \pi/4$ is given by

$$r(s, u) \equiv r(u) = 0.05\left((\pi/4)^2 - (u - \pi/4)^2\right), \qquad (22)$$

whereas in the simplified case we choose a constant vessel speed $r \equiv 0.05$. The approximation is computed with uniform steps $\Delta x = 0.02$ and $\Delta t = 0.1$.

The test is performed for $\bar{\sigma} = 0, 0.02, 0.1$. In each case, the starting point of the trajectory is set at the points $x = (-0.7, 0, 0), (0, 0, 0), (0.7, 0, 0)$, for $q = 1, 2$. The results are shown in Fig. 4.

In the case $\bar{\sigma} = 0$, the problem is deterministic, and the best strategy is to minimize the number of switches, by reaching the lay lines and switching just once until the target. In practice (see the upper plots of Fig. 4), the solution computed with the simplified dynamics shows a second (and, possibly, a third) switch closer to the target. This effect is caused by the combination of numerical viscosity (see [18]) and reduced controllability. In fact, with the parameters chosen, the scheme has a numerical viscosity of the order of

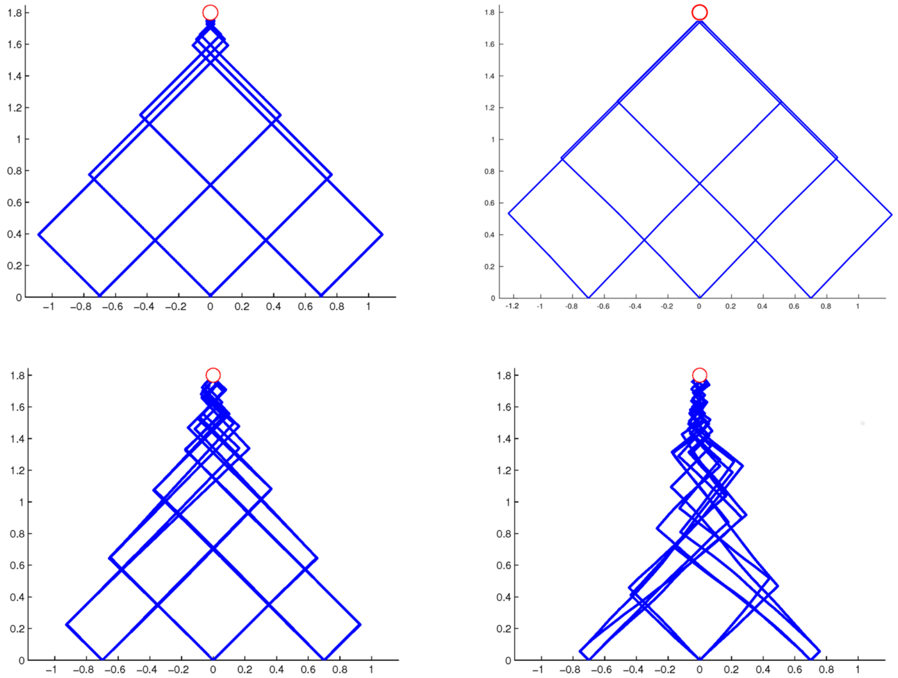


Fig. 4 Test 1, sample optimal trajectories for various starting points and different diffusion coefficients. The red circle identifies the target (above/left $\bar{\sigma} = 0$, simplified dynamics; above/right $\bar{\sigma} = 0$, complete dynamics; bottom/left $\bar{\sigma} = 0.02$; bottom/right $\bar{\sigma} = 0.1$)

$$\frac{\Delta x^2}{8\Delta t} = 5 \cdot 10^{-4},$$

which corresponds to a (fictitious) diffusion term $\bar{\sigma} \approx 0.03$. Thus, the switching strategy is computed *as if it were related to a stochastic problem*, and if the only possible control action is switching, then the optimal control implements a more conservative strategy, in order to avoid to miss the target at all. This problem does not occur in the complete dynamics, in which the route can be adjusted continuously without tacking. For larger diffusion terms, the numerical viscosity becomes secondary, and the two models provide comparable results. (We have avoided duplicating the plots.)

The number of switches in the dynamics increases at the increase in $\bar{\sigma}$, i.e., in case of a larger diffusion coefficient for the wind direction, the optimal dynamics prefers to pay more times the cost of switching to remain in the center of the domain. The trajectories tend then to cluster inside a cone (termed *tacking triangle*) that progressively shrinks for increasing $\bar{\sigma}$. This effect is well recognized in empirically based sailing tactics theory [24] as well as in [3].

The generation of the tacking triangle results from a strategy known as *tacking on a lift*: When the wind direction shows both clockwise and anticlockwise variations, the former should be preferably used to tack to starboard, while the latter should be

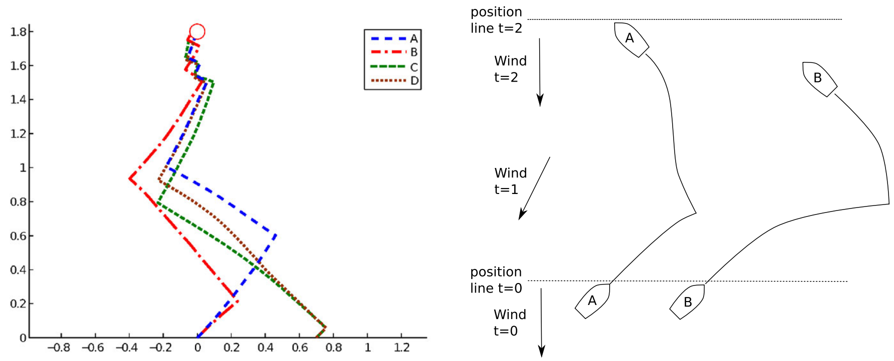


Fig. 5 Test 1: (left) automatic evaluation of the optimal tacking in the presence of wind perturbations and (right) a classical strategy of "tacking on a lift"

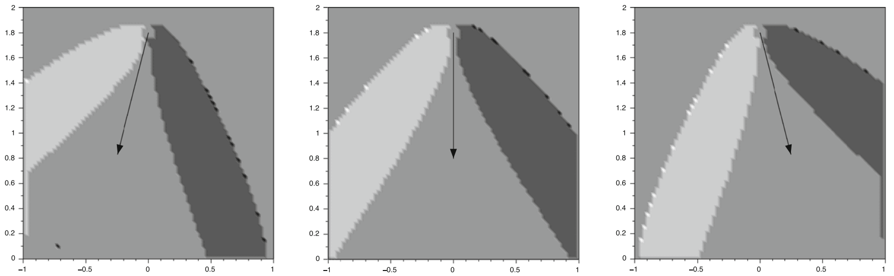


Fig. 6 Test 1: Three sections of the switching sets computed for $\theta = x_3 = -0.25, 0, 0.25$, compared with the corresponding directions of the wind (arrows)

preferably used to tack to port. This is schematically shown in Fig. 5 (right), in which A tacks when the wind rotates, and gains on B that does it later.

The model under consideration correctly reproduces this strategy (see Fig. 5, left). Figure 6 shows three section of the switching sets, computed as in (21), for the simplified dynamics and $\bar{\sigma} = 0.1$. Here, the dark gray corresponds to a switching toward the starboard tack, the light gray to a switching to port and the average gray to no switching. The sections are at, respectively, $\theta = x_3 = -0.25$, $\theta = x_3 = 0$ and $\theta = x_3 = 0.25$, and arrows show the corresponding wind directions. The "no tack" region is split into two parts: The upper one is made of points from which the target is unreachable, whereas the lower one is the actual tacking region. Note that if, for example, a boat is on the port tack and the wind rotates strongly clockwise, the dark gray region may hit the boat position and thus cause a tack to starboard, as required by the correct strategy.

5.1.1 Test 2

In the second typical scenario, we impose a nonzero value to the drift $\bar{a}$ in (11) to obtain an average anticlockwise rotation. In this case, we adopt the complete dynamics. Using the same setting of Test 1, we compare the optimal trajectories obtained for various

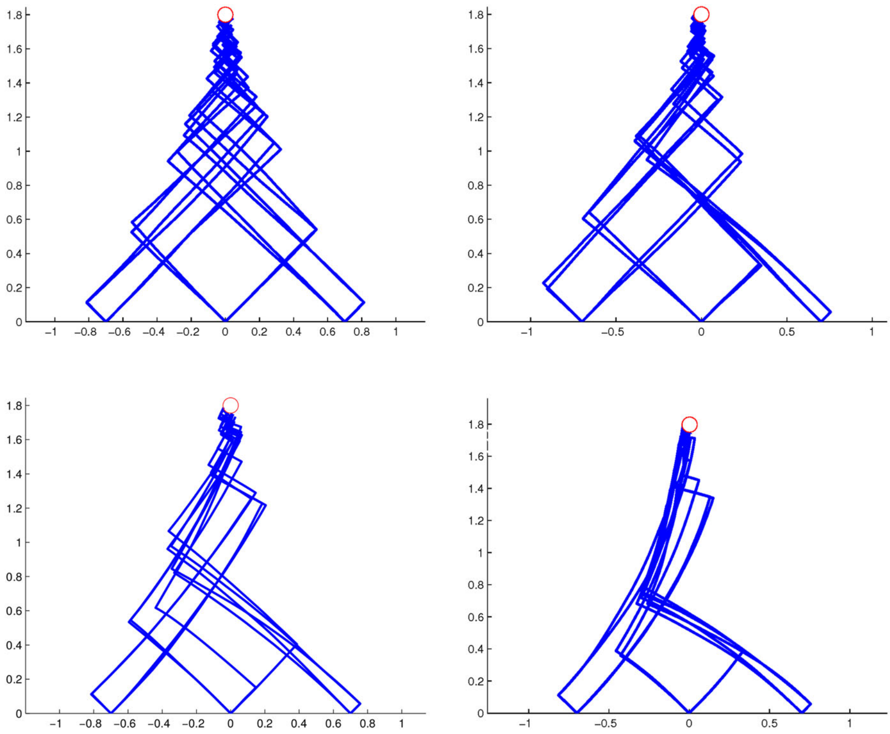


Fig. 7 Test 2: some sample optimal trajectories for various starting points and different drift values. The diffusion coefficient is fixed $\bar{\sigma} = 0.05$, variable drift (above/left $a = 0$, above/right $a = 0.05$, bottom/left $a = 0.15$, bottom/right $a = 0.3$)

values of the drift $\bar{a}$, for a constant value of diffusion $\bar{\sigma} = 0.05$. As in the previous test, we consider trajectories starting from the points $(-0.7, 0, 0)$, $(0, 0, 0)$, $(0.7, 0, 0)$ and $q = 1, 2$. The results are shown in Fig. 7.

Observe that, as $\bar{a} > 0$, the approximatex symmetry of optimal trajectories in the previous test is progressively lost. In practice, if the average rotation of the wind is anticlockwise, then the best strategy is to occupy the left region of the domain (i.e., for x_1 small or negative), and this behavior is enhanced by higher values of $\bar{a}$ and lower values of $\bar{\sigma}$.

This resulting solution also includes the "tacking on a lift" strategy, which also holds in the situation of a variable average wind direction with stochastic variations. The computed optimal solutions correctly blend the two strategies.

6 Conclusions

We have shown how a hybrid control framework can be an efficient tool in route planning problems of minimum time type for a single sailing boat, and lend itself to a very general setting of the problem. A natural development of the model is to include the presence of other competitors, which would cause a more complicated (non-convex) structure of the HJB equations associated with the problem, and result in a higher dimension of the state space. However, in the case of two players with opposite goals (a *match race*, see [7]), the problem becomes a differential game of pursuit–evasion type and allows for a simplified treatment. The study of the corresponding techniques for this case is the object of an ongoing work [25].

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