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IMPROVED MULTIPOLAR POINCARÉ-HARDY INEQUALITIES ON CARTAN-HADAMARD MANIFOLDS

ELVISE BERCHIO, DEBDIP GANGULY, AND GABRIELE GRILLO

Abstract. We prove a family of improved multipolar Poincaré-Hardy inequalities on Cartan-Hadamard manifolds. For suitable configurations of poles, these inequalities yield an improved multipolar Hardy inequality and an improved multipolar Poincaré inequality such that the critical unipolar singular mass is reached at any pole.

1. Introduction

This paper aims at proving new multipolar Hardy inequalities on negatively curved manifolds. To introduce the subject, let us recall the simplest form of the unipolar Hardy inequality on Riemannian manifolds, which is due to Carron [9]. If $(\mathcal{M}, g)$ is an $N \geq 3$ dimensional Cartan-Hadamard manifold, $\mathrm{d}(.,.)$ is the geodesic distance and $x_0 \in \mathcal{M}$, the following inequality holds

$$\int_{\mathcal{M}} |\nabla_g u|^2 \, \mathrm{d}v_g \geq \frac{(N-2)^2}{4} \int_{\mathcal{M}} \frac{u^2}{\mathrm{d}(x,x_0)^2} \, \mathrm{d}v_g \quad \forall \, u \in C_c^\infty(\mathcal{M}) \,. \tag{1.1}$$

Here ∇_g is the Riemannian gradient, $\mathrm{d}v_g$ the Riemannian volume.

When $\mathcal{M} = \mathbb{R}^N$, the Euclidean space, or $\mathcal{M} = \mathbb{H}^N$, the hyperbolic space, the constant $\frac{(N-2)^2}{4}$ is optimal and never attained. After the seminal work [9] there have been several attempt to improve (1.1) in terms of adding remainder terms, see for instance [2, 16, 17].

Motivated by applications to various fields of research, several authors have turned their attention to the multipolar version of (1.1). In $\mathbb{R}^N$ a relevant contribution in this direction is the one of Bossi-Dolbeault-Esteban in [7]. Letting $y_1, \dots y_M$ be $M \geq 2$ distinct points in $\mathbb{R}^N$ and $d = \min_{i \neq j} \frac{|y_i - y_j|}{2}$, they showed that

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, \mathrm{d}x + \frac{4\pi^2 + (M+1)(N-2)^2}{4d^2} \int_{\mathbb{R}^N} u^2 \, \mathrm{d}x \geq \frac{(N-2)^2}{4} \sum_{i=1}^M \int_{\mathbb{R}^N} \frac{u^2}{|x-y_i|^2} \mathrm{d}x, \tag{1.2}$$

for any $u \in H^1(\mathbb{R}^N)$. In few words, their result shows that the unipolar critical mass, $\frac{(N-2)^2}{4} \frac{1}{|x-y_i|^2}$, is reached at any singular pole y_i, modulo adding a lower order L^2-term on the left hand side. Trying to remove the above L^2-correction term, the authors in [7] also proved the following inequality:

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, \mathrm{d}x \geq \frac{(N-2)^2}{4M} \sum_{i=1}^M \int_{\mathbb{R}^N} \frac{u^2}{|x-y_i|^2} \, \mathrm{d}x \, + \, \frac{(N-2)^2}{4M^2} \sum_{1 \leq i < j \leq M} \int_{\mathbb{R}^N} \frac{|y_i - y_j|^2}{|x-y_i|^2 |x-y_j|^2} u^2 \, \mathrm{d}x \,, \tag{1.3}$$

Key words and phrases. Hyperbolic space, multipolar Hardy inequality, Poincaré inequality.

for any $u \in H^1(\mathbb{R}^N)$. Here the contribution at each singularity y_i is

$$\frac{(N-2)^2}{4}\frac{2M-1}{M^2}\frac{1}{|x-y_i|^2}\,.$$

Hence, a constant strictly less than $\frac{(N-2)^2}{4}$ has to be taken in order to remove the positive correction on the left hand side of (1.2). In fact, a remarkable result of Felli, Marchini and Terracini in [12] says that there exists no configuration of the poles such that a multipolar Hardy inequality holds with any of the Hardy terms contributing with the best unipolar constant $(N-2)^2/4$, or even with the sum of such constants being equal to $(N-2)^2/4$.

The optimality issues were not addresses in [7] while in [8] the following *optimal* inequality was proved

$$\int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x \geq \frac{(N-2)^2}{M^2} \sum_{1\leq i<j\leq M} \int_{\mathbb{R}^N} \frac{|y_i-y_j|^2}{|x-y_i|^2|x-y_j|^2} u^2\,dx \tag{1.4}$$

for any $u \in H^1(\mathbb{R}^N)$. Since the potential in (1.4) behaves asymptotically near y_i as

$$\frac{(N-2)^2}{4}\frac{4M-4}{M^2}\frac{1}{|x-y_i|^2}\,,$$

for $M \geq 2$ it provides a larger weight near poles than that in (1.3). On the other hand, as $x \to +\infty$ the decay of the potential in (1.4) is faster than that in (1.3), hence the two weights are not globally comparable. An inequality similar to (1.3) and (1.4) is also proved in [1] using different techniques. It is worth recalling a further multipolar Hardy inequality from [10] :

$$\int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x \geq \frac{(N-2)^2}{(M+1)^2}\left[\sum_{i=1}^M \int_{\mathbb{R}^N} \frac{u^2}{|x-y_i|^2}\,\mathrm{d}x + \sum_{1\leq i<j\leq M} \int_{\mathbb{R}^N} \frac{|y_i-y_j|^2}{|x-y_i|^2|x-y_j|^2} u^2\,\mathrm{d}x\right] \tag{1.5}$$

for any $u \in H^1(\mathbb{R}^N)$. The potential in (1.5) is smaller near poles than those listed above, indeed it behaves like

$$\frac{(N-2)^2}{4}\,\frac{4M}{(M+1)^2}\,\frac{1}{|x-y_i|^2}\,,$$

nevertheless the resulting operator turns out to be *critical*, namely inequality (1.5) cannot be further improved. It should be said that in [10] the authors proved also the criticality of the operator associated to (1.4).

In view of (1.1), it is quite natural to generalize the above results to the non Euclidean setting. To our best knowledge, the first result in this direction is that given in [11] where the authors, following the super solutions approach of [8] to get (1.4), for any $(\mathcal{M}, g)$ N-dimensionale complete Riemannian manifold proved that

$$\begin{aligned}\int_{\mathcal{M}} |\nabla_g u|^2 \,\mathrm{d}v_g \geq \frac{(N-2)^2}{M^2} \sum_{1\leq i<j\leq M} \int_{\mathcal{M}} \left|\frac{\nabla_g(\mathrm{d}(x,y_i))}{\mathrm{d}(x,y_i)} - \frac{\nabla_g(\mathrm{d}(x,y_j))}{\mathrm{d}(x,y_j)}\right|^2 u^2\,\mathrm{d}v_g \\ + \frac{(N-2)(N-1)}{M}\sum_{k=1}^M \int_{\mathcal{M}} \frac{\mathrm{d}^2(x,y_k)\Delta_g(\mathrm{d}^2(x,y_k)) - (N-1)}{\mathrm{d}^2(x,y_k)}\,u^2\,\mathrm{d}v_g\end{aligned} \tag{1.6}$$

for all $u \in C_c^\infty(\mathcal{M})$. Here Δ_g is the Laplace-Beltrami operator.

If $\mathcal{M}$ has asymptotically non-negative Ricci curvature, the operator associated to (1.6) is also proved to be critical in the above mentioned sense. Furthermore, on Cartan-Hadamard

manifolds the last term of (1.6) is nonnegative, while it vanishes when $\mathcal{M} = \mathbb{R}^N$ and one recovers exactly inequality (1.4). In any case, the total mass at each pole never reaches the optimal unipolar mass of (1.1). Motivated by this remark, in the present paper we combine the localizing approach of [7] with some optimal Poincaré-Hardy unipolar inequalities from [6], and we investigate whether the contribution of the curvature helps in removing the l.h.s. correction term of (1.2); namely in getting a multipolar Hardy inequality with optimal unipolar critical mass reached at each pole.

The proposed goal is reached on Cartan-Hadamard manifolds having sectional curvatures bounded from above by a strictly negative constant, provided the distance between poles satisfies a suitable lower bound. Remainder terms for the multipolar Hardy inequality are also provided. See Corollary 4.3 for the precise statement and the discussion just below for a comparison between our inequality and inequality (1.6). The proof of our multipolar Hardy inequality follows as a corollary of a more general family of improved multipolar Poincaré-Hardy inequalities, see Theorem 4.2, having its own interest and from which we also derive a multipolar Hardy improvement for the (non optimal) Poincaré inequality such that the critical unipolar singular mass is reached at any pole, see Corollary 4.4.

The paper is organized as follows. In Section 2, for the sake of clarity, we state our results on the prototype Cartan-Hadamard manifold, namely on the hyperbolic space $\mathbb{H}^N$. In Section 3 we prove all the statements in the hyperbolic setting. Finally, in Section 4 we state our results on general Cartan-Hadamard manifolds and in Section 5 we discuss the criticality issue for a further multipolar Hardy inequality in $\mathbb{H}^N$ that we derive from the general results of [10].

2. Main results in the hyperbolic setting.

We start by recalling a unipolar Poincaré-Hardy inequality from [6] that will play a key role in our proofs. Let $\lambda_1(\mathbb{H}^N)$ denote the bottom of the L^2 spectrum of $-\Delta_{\mathbb{H}^N}$. For all $\lambda \le \lambda_1(\mathbb{H}^N) = \left(\frac{N-1}{2}\right)^2$, consider the potential

$$V_{\lambda,x_0}(x) := H_N(\lambda)\,\frac{1}{\mathrm{d}^2(x,x_0)} + C_N(\lambda)\frac{1}{\sinh^2(\mathrm{d}(x,x_0))} + D_N(\lambda) g(\mathrm{d}(x,x_0))\,, \tag{2.1}$$

where the function g is defined by

$$g(r) := \frac{r\coth r - 1}{r^2} > 0 \quad \text{for } r>0\,.$$

Hence, g is strictly decreasing and satisfies

$$g(r) \sim \frac{1}{3} \ \text{ as } \ r\to 0^+ \quad \text{ and } \quad g(r)\sim \frac{1}{r} \ \text{ as } \ r\to+\infty\,.$$

Furthermore, the constants H_N, C_N, D_N are given explicitly by

$$\begin{aligned} H_N(\lambda) :=& \frac{(\gamma_N(\lambda)+1)^2}{4}\,, \quad D_N(\lambda) := \frac{\gamma_N(\lambda)(\gamma_N(\lambda)+1)}{2} \\ C_N(\lambda) :=& \frac{(N-1+\gamma_N(\lambda))(N-3-\gamma_N(\lambda))}{4}\,, \end{aligned} \tag{2.2}$$

with $\gamma_N(\lambda) := \sqrt{(N-1)^2-4\lambda}$. It is worth noting that the constant $H_N(\lambda)$ cannot exceed the Hardy constant and its maximum value is achieved for $\gamma_N(N-2) = N-3$, namely for $\lambda = N-2$. We may finally state:

Proposition 2.1. [6, Theorem 2.1] *Let $N \geq 3$ and $x_0 \in \mathbb{H}^N$. For all $N-2 \leq \lambda \leq \left(\frac{N-1}{2}\right)^2$ and all $u \in C_c^\infty(\mathbb{H}^N)$ there holds*

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} - \lambda \int_{\mathbb{H}^N} u^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \int_{\mathbb{H}^N} V_{\lambda,x_0}(x)\, u^2 \, \mathrm{d}v_{\mathbb{H}^N}\,, \tag{2.3}$$

where V_{λ,x_0} is as defined in (2.1). Besides, the operator $-\Delta_{\mathbb{H}^N} - \lambda - V_{\lambda,x_0}$ is critical in $\mathbb{H}^N$ in the sense that the inequality

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} - \lambda \int_{\mathbb{H}^N} u^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \int_{\mathbb{H}^N} V(x) u^2 \, \mathrm{d}v_{\mathbb{H}^N} \quad \forall u \in C_c^\infty(\mathbb{H}^N)$$

is not valid for all $u \in C_c^\infty(\mathbb{H}^N)$ given any $V \gneqq V_\lambda$.

We note that Proposition 2.1 holds with the same statement also for $\lambda < N-2$ but in that case, $\gamma_N(N-2) > N-3$ and the constant $C_N(\lambda)$ becomes negative. Hence, even if V_{λ,x_0} is still a positive function, the singularity at x_0 splits into a positive and a negative part. For this reason, in what follows, in order to avoid too many distinctions, we will always assume $\lambda \geq N-2$. We refer the interested reader to [3, 4, 5, 6, 14, 15] for more details and for related Poincaré-Hardy inequalities in the higher order case or in the L^p-setting.

The main idea behind the proof of Proposition 2.1 is to find a suitable radial supersolution. A supersolution technique has been used in the multipolar case in [10], in the Euclidean case, and in [11] in the Riemannian case. Some results along this line will also be proved here in Section 5, but here, in order to get locally stronger inequalities, we adapt instead the approach of [7]. Namely, we first use the so-called "IMS" truncation method (see [13]) to localize to Hardy potentials with one singularity, then we refine the estimates by means of inequality (2.3) which allows us to exploit some positive terms on the r.h.s. not usable in the flat case.

Let $M \geq 2$ and $y_1, \dots y_M \in \mathbb{H}^N$, we consider the multipolar analogous to (2.1), namely

$$V_{\lambda,y_1,\dots y_M}(x) := \sum_{k=1}^{M} V_{\lambda,y_k}(x)\,, \tag{2.4}$$

where the functions $V_{\lambda,y_k}(x)$ are as in (2.1). Now we set

$$d := \min_{1\leq j\neq k\leq M} \frac{\mathrm{d}(y_j,y_k)}{2} \tag{2.5}$$

and we state our main theorem

Theorem 2.2. *Let $N \geq 3$ and $M \geq 2$. Furthermore, let $y_1, \dots y_M \in \mathbb{H}^N$ and let $d > 0$ be defined in (2.5). For all $N-2 \leq \lambda \leq \left(\frac{N-1}{2}\right)^2$, the following inequality holds*

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} + \left(\frac{\pi^2}{d^2} + (M+1)K_N(\lambda,d) - \lambda\right) \int_{\mathbb{H}^N} u^2 \mathrm{d}v_{\mathbb{H}^N} \geq \int_{\mathbb{H}^N} V_{\lambda,y_1,\dots y_M}(x)\, u^2 \mathrm{d}v_{\mathbb{H}^N} \tag{2.6}$$

for all $u \in C_c^\infty(\mathbb{H}^N)$, where $V_{\lambda,y_1,\dots y_M}$ is as given in (2.4) while

$$K_N(\lambda,d) := H_N(\lambda)\,\frac{1}{d^2} + C_N(\lambda)\frac{1}{\sinh^2(d/2)} + D_N(\lambda) g(d/2)\,, \tag{2.7}$$

with the constants H_N, C_N, D_N as in (2.2) and the function g as in (2.1).

A few special cases have to be singled out. In fact, for $\lambda = N-2$ one has $H_N(N-2) = \frac{(N-2)^2}{4}$, $C_N(N-2) = 0$, $D_N(N-2) = \frac{(N-3)(N-2)}{2}$, and Theorem 2.2 provides an interesting inequality that we state separately here below. Let $\bar{d} = \bar{d}(M,N) > 0$ be defined implicitly as follows

$$\frac{\pi^2}{\bar{d}^2} + (M+1)\left(\frac{(N-2)^2}{4}\frac{1}{\bar{d}^2} + \frac{(N-3)(N-2)}{2} g(\bar{d}/2)\right) = N-2\,. \tag{2.8}$$

Then we have the following result:

Corollary 2.3. *Let $N \geq 3$ and $M \geq 2$. Furthermore, let $y_1, \dots y_M \in \mathbb{H}^N$, $d > 0$ as defined in (2.5) and $\bar{d} = \bar{d}(M,N) > 0$ as defined in (2.8). If $d \geq \bar{d}$, the following inequality holds*

$$\begin{aligned}\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \frac{(N-2)^2}{4} \sum_{k=1}^M \int_{\mathbb{H}^N} \frac{u^2}{\mathrm{d}^2(x,y_k)} \, \mathrm{d}v_{\mathbb{H}^N} \\ + \frac{(N-2)(N-3)}{2} \sum_{k=1}^M \int_{\mathbb{H}^N} g(\mathrm{d}(x,y_k))\, u^2 \, \mathrm{d}v_{\mathbb{H}^N}\end{aligned} \tag{2.9}$$

for all $u \in C_c^\infty(\mathbb{H}^N)$ with the function g as in (2.1).

The interest of inequality (2.9) is clearly related to the fact that one recovers the unipolar Hardy potential with best constant at each pole, and further positive remainder term on the right hand side, the resulting inequality being locally stronger than (1.6). This turns out to be true under the assumption $d \geq \bar{d}$, namely the poles cannot be too close to each other. With a similar assumption, exploiting inequality (2.6) with $\lambda = \left(\frac{N-1}{2}\right)^2$, one can also derive a multipolar improvement of a Poincaré inequality with constant arbitrarily close to the optimal one. More precisely, for $N-2 \leq \lambda < \left(\frac{N-1}{2}\right)^2$, let $\bar{d}_\lambda = \bar{d}_\lambda(M,N) > 0$ be defined implicitly as follows

$$\frac{\pi^2}{\bar{d}_\lambda} + (M+1)\left(\frac{1}{4}\frac{1}{\bar{d}_\lambda^2} + \frac{(N-1)(N-3)}{4}\,\frac{1}{\sinh^2(\bar{d}_\lambda/2)}\right) = \left(\frac{N-1}{2}\right)^2 - \lambda\,. \tag{2.10}$$

Then the following result holds:

Corollary 2.4. *Let $N \geq 3$ and $M \geq 2$. Furthermore, let $y_1, \dots y_M \in \mathbb{H}^N$, $d > 0$ as defined in (2.5) and $\bar{d}_\lambda = \bar{d}_\lambda(M,N) > 0$ as defined in (2.10). Let $N-2 \leq \lambda < \left(\frac{N-1}{2}\right)^2$, if $d \geq \bar{d}_\lambda$, the following inequality holds*

$$\begin{aligned}\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \lambda \int_{\mathbb{H}^N} u^2 \, \mathrm{d}v_{\mathbb{H}^N} + \frac{1}{4} \sum_{k=1}^M \int_{\mathbb{H}^N} \frac{u^2}{\mathrm{d}^2(x,y_k)} \, \mathrm{d}v_{\mathbb{H}^N} \\ + \frac{(N-1)(N-3)}{4} \sum_{k=1}^M \int_{\mathbb{H}^N} \frac{u^2}{\sinh^2(\mathrm{d}(x,y_k))} \, \mathrm{d}v_{\mathbb{H}^N}\end{aligned} \tag{2.11}$$

for all $u \in C_c^\infty(\mathbb{H}^N)$.

We note that the Hardy improvement of (2.11) is locally optimal for all λ in the sense that

$$\frac{1}{4}\frac{1}{\mathrm{d}^2(x,y_k)} + \frac{(N-1)(N-3)}{4}\frac{1}{\sinh^2(\mathrm{d}(x,y_k))} \sim \frac{(N-2)^2}{4}\frac{1}{\mathrm{d}^2(x,y_k)} \quad \text{as } x \to y_k\,,$$

namely the unipolar critical mass is reached at any pole. It is also to be noted that the constant $1/4$ in the unipolar inequality

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \frac{(N-1)^2}{4} \int_{\mathbb{H}^N} u^2 \, \mathrm{d}v_{\mathbb{H}^N} + \frac{1}{4} \int_{\mathbb{H}^N} \frac{u^2}{\mathrm{d}^2(x, x_0)} \, \mathrm{d}v_{\mathbb{H}^N}$$

is known to be sharp by the results of [4].

3. Proof of Theorem 2.2

3.1. **Technical lemmas.** We adapt the approach of [7, Section 1] to our framework. Consider a partition of unity in $\mathbb{H}^N$, namely a finite set $\{J_k\}_{k=1}^{M+1}$ of real valued functions $J_k \in W^{1,\infty}(\mathbb{H}^N)$ such that $\sum_{k=1}^{M+1} J_k^2 = 1$ a.e. in $\mathbb{H}^N$. Hence, $J_{M+1} := \sqrt{1 - \sum_{k=1}^{M} J_k^2}$ and, by writing Riemannian gradient in local coordinates, i.e., $(\nabla_{\mathbb{H}^N})^j = g^{ij}\partial_i$, one has $\sum_{k=1}^{M+1} J_k (\nabla_{\mathbb{H}^N})^j J_k = 0$ for every $j = 1, ..., N$. Furthermore, if one requires that

$$\mathrm{Int}(\mathrm{supp}(J_k)) \cap \mathrm{Int}(\mathrm{supp}(J_l)) = \emptyset \quad \text{for any } k, l = 1, \dots, M, k \neq l \,, \tag{3.1}$$

it follows that

$$\sum_{k=1}^{M+1} |\nabla_{\mathbb{H}^N} J_k|^2 = \sum_{k=1}^{M} \frac{|\nabla_{\mathbb{H}^N} J_k|^2}{1 - J_k^2} \quad \text{a.e. in } \mathbb{H}^N. \tag{3.2}$$

First we state

Lemma 3.1. *Let $\{J_k\}_{k=1}^{M+1}$ be a partition of unity satisfying condition (3.1). For any $u \in H^1(\mathbb{H}^N)$ and potential $V \in L^1_{loc}(\mathbb{H}^N)$, there holds*

$$\langle Lu, u\rangle := \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} - \int_{\mathbb{H}^N} V(x) u^2 \, \mathrm{d}v_{\mathbb{H}^N} = \sum_{k=1}^{M} \langle L(J_k u), (J_k u)\rangle + R_M \,,$$

with

$$\begin{aligned} R_M = &\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} (J_{M+1} u)|^2 - \int_{\mathbb{H}^N \setminus \Omega} V(x) \, u^2 \, \mathrm{d}v_{\mathbb{H}^N} \\ &- \sum_{k=1}^{M} \int_{\Omega_k} \left(\frac{|\nabla_{\mathbb{H}^N} J_k|^2}{1 - J_k^2(x)} + (1 - J_k^2(x)) V(x) \right) u^2 \, \mathrm{d}v_{\mathbb{H}^N} \,, \end{aligned}$$

where $\Omega_k := \mathrm{Int}(\mathrm{supp}(J_k))$ and $\Omega := \cup_{k=1}^{M} \mathrm{supp}(J_k)$.

Proof. Using the fact that

$$\begin{gathered} |\nabla_{\mathbb{H}^N} (J_k u)|^2 = |u(\nabla_{\mathbb{H}^N} J_k) + J_k (\nabla_{\mathbb{H}^N} u)|^2 = \\ u^2 |\nabla_{\mathbb{H}^N} J_k|^2 + J_k^2 |\nabla_{\mathbb{H}^N} u|^2 + (J_k \nabla_{\mathbb{H}^N} J_k \,, \nabla_{\mathbb{H}^N} (u^2))_{\mathbb{H}^N} \end{gathered}$$

and exploiting the properties of the J_k listed above, one has that

$$\begin{aligned} \langle Lu, u\rangle &= \sum_{k=1}^{M+1} \int_{\mathbb{H}^N} \langle L(J_k u), (J_k u)\rangle - \int_{\mathbb{H}^N} \left(\sum_{k=1}^{M+1} |\nabla_{\mathbb{H}^N} J_k|^2 \right) u^2 \, \mathrm{d}v_{\mathbb{H}^N} \\ &= \sum_{k=1}^{M} \langle L(J_k u), (J_k u)\rangle + \int_{\mathbb{H}^N} \left(|\nabla_{\mathbb{H}^N} (J_{M+1} u)|^2 - V(x) (J_{M+1} u)^2 \right) \mathrm{d}v_{\mathbb{H}^N} \\ &\quad - \int_{\mathbb{H}^N} \left(\sum_{k=1}^{M+1} |\nabla_{\mathbb{H}^N} J_k|^2 \right) u^2 \, \mathrm{d}v_{\mathbb{H}^N} =: \sum_{k=1}^{M} \langle L(J_k u), (J_k u)\rangle + R_M \,. \end{aligned}$$

Note that, by (3.2), R_M can rewritten as

$$R_M = \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N}(J_{M+1}u)|^2 - \int_{\mathbb{H}^N} V(x)\left(1-\sum_{k=1}^{M} J_k^2(x)\right) u^2 \,\mathrm{d}v_{\mathbb{H}^N} - \sum_{k=1}^{M}\int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} J_k|^2}{1-J_k^2}\, u^2 \,\mathrm{d}v_{\mathbb{H}^N}$$

and the statement of Lemma 3.1 follows by recalling (3.1) and writing $\mathbb{H}^N = (\mathbb{H}^N \setminus \cup_{k=1}^{M}\mathrm{supp}(J_k)) \cup (\cup_{k=1}^{M}\mathrm{supp}(J_k))$. □

Next we denote by $B(x_0, r)$ the geodesic ball of center $x_0 \in \mathbb{H}^N$ and radius $r > 0$ and we prove:

Lemma 3.2. *Let $y_1, y_2 \in \mathbb{H}^N$ with $0 < \mathrm{d}(y_1, y_2) =: 2d$. There is a partition of unity $\{J_k\}_{k=1}^3$ with $J_i \equiv 1$ on $B(y_i, \frac{d}{2})$ and $J_i \equiv 0$ on $B(y_i, d)^c$, for $i = 1, 2$. Furthermore, for every $N - 2 \le \lambda \le (\frac{N-1}{2})^2$ and for a.e. $x \in B(y_1, d) \cup B(y_2, d)$, there holds*

$$\sum_{k=1}^{2} \frac{|\nabla_{\mathbb{H}^N} J_k(x)|^2}{1-J_k^2(x)} + (1 - J_1^2(x) - J_2^2(x))\, V_{\lambda,y_1,y_2}(x) \le \frac{\pi^2}{d^2} + 2K_N(\lambda, d)\,,$$

where $V_{\lambda,y_1,y_2}(x)$ is as defined in (2.4) and $K_N(\lambda, d)$ is as given in (2.7).

Proof. Let $J : [0, +\infty) \to [0, 1]$ be a continuous map defined as follows

$$J(t) = \begin{cases} 1 & \text{if } 0 \le t \le \frac{1}{2} \\ \sin(\pi t) & \text{if } \frac{1}{2} \le t \le 1 \\ 0 & \text{if } t \ge 1\,. \end{cases}$$

For all $x \in \mathbb{H}^N$, we set $J_1(x) := J\left(\frac{\mathrm{d}(x,y_1)}{d}\right)$, and $J_2(x) := J\left(\frac{\mathrm{d}(x,y_2)}{d}\right)$, and $J_3(x) := \sqrt{1 - J_1^2(x) - J_2^2(x)}$. It is readily seen that $J_i \in W^{1,\infty}(\mathbb{H}^N)$ for $i = 1, ..., 3$. Then, we consider

$$\sup_{x \in B(y_1,d)} \left[\frac{|\nabla_{\mathbb{H}^N} J_1(x)|^2}{1-J_1^2(x)} + (1 - J_1^2(x))\,(V_{\lambda,y_1}(x) + V_{\lambda,y_2}(x))\right].$$

Since for $y_1 \in \mathbb{H}^N$ fixed, we have $|\nabla_{\mathbb{H}^N}\mathrm{d}(x, y_1)|^2 = 1$, for a.e. $x \in B(y_1, d)$ we infer

$$|\nabla_{\mathbb{H}^N} J_1(x)|^2 = \frac{1}{d^2}\left|J'\left(\frac{\mathrm{d}(x,y_1)}{d}\right)\right|^2 |\nabla_{\mathbb{H}^N}\mathrm{d}(x,y_1)|^2 = \frac{1}{d^2}\left|J'(t)\right|^2 \quad \text{with } t := \frac{\mathrm{d}(x,y_1)}{d} \in [0,1)$$

and, using the fact that

$$\mathrm{d}(x, y_2) \ge \mathrm{d}(y_1, y_2) - \mathrm{d}(x, y_1) = (2 - t)d\,,$$

we obtain

$$V_{\lambda,y_1}(x) + V_{\lambda,y_2}(x) \le \frac{H_N(\lambda)}{d^2}\left[\frac{1}{t^2} + \frac{1}{(2-t)^2}\right] + C_N(\lambda)\left[\frac{1}{\sinh^2(td)} + \frac{1}{\sinh^2((2-t)d)}\right] + D_N(\lambda)\left[g(td) + g((2-t)d)\right] =: W_\lambda(t, d)\,.$$

From the above computations we deduce

$$\begin{aligned}
&\sup_{x\in B(y_1,d)}\left[\frac{|\nabla_{\mathbb{H}^N}J_1(x)|^2}{1-J_1^2(x)}+(1-J_1^2(x))\,(V_{\lambda,y_1}(x)+V_{\lambda,y_2}(x))\right]\\
&\le \sup_{t\in(\frac12,1)}\left[\frac{(J'(t))^2}{d^2(1-J^2(t))}+(1-J^2(t))\,W_\lambda(t,d)\right]\\
&\le \frac{\pi^2}{d^2}+\max_{t\in[\frac12,1]}\left[\frac{H_N(\lambda)}{d^2}\left[\frac{1}{t^2}+\frac{1}{(2-t)^2}\right]\cos^2(\pi t)\right]\\
&+C_N(\lambda)\frac{2}{\sinh^2(d/2)}+2D_N(\lambda)\,g(d/2)=\frac{\pi^2}{d^2}+2K_N(\lambda,d)\,,
\end{aligned}$$

where $K_N(\lambda,d)$ is given in (2.7). Similar computations, inverting the roles of y_1 and y_2, give

$$\sup_{x\in B(y_2,d)}\left[\frac{|\nabla_{\mathbb{H}^N}J_2(x)|^2}{1-J_2^2(x)}+(1-J_2^2(x))\,(V_{\lambda,y_1}(x)+V_{\lambda,y_2}(x))\right]\le\frac{\pi^2}{d^2}+2K_N(\lambda,d)\,.$$

This completes the proof. □

3.2. **Hardy inequality with two singularities.** This section is devoted to prove Theorem 2.2 in the case of two singularities. More precisely, we shall prove the following result.

Proposition 3.3. *Let $N\ge 3$, Let $y_1,y_2\in\mathbb{H}^N$ with $0<\mathrm{d}(y_1,y_2)=:2d$. For all $N-2\le\lambda\le\left(\frac{N-1}{2}\right)^2$ the following inequality*

$$\int_{\mathbb{H}^N}|\nabla_{\mathbb{H}^N}u|^2\,\mathrm{d}v_{\mathbb{H}^N}+\left(\frac{\pi^2}{d^2}+3K_N(\lambda,d)-\lambda\right)\int_{\mathbb{H}^N}u^2\,\mathrm{d}v_{\mathbb{H}^N}\ge\int_{\mathbb{H}^N}V_{\lambda,y_1,y_2}(x)\,u^2\,\mathrm{d}v_{\mathbb{H}^N}$$

holds for all $u\in C_c^\infty(\mathbb{H}^N)$, where $K_N(\lambda,d)$ is given in (2.7) and V_{λ,y_1,y_2} is given in (2.4).

Proof. Consider the partition of unity given by Lemma 3.2, from Lemma 3.1 with $V=V_{\lambda,y_1,y_2}$ we have

$$\langle Lu,u\rangle:=\int_{\mathbb{H}^N}|\nabla_{\mathbb{H}^N}u|^2\,\mathrm{d}v_{\mathbb{H}^N}-\int_{\mathbb{H}^N}V_{\lambda,y_1,y_2}(x)\,u^2\,\mathrm{d}v_{\mathbb{H}^N}=\sum_{k=1}^2\langle L(J_ku),(J_ku)\rangle+R_2\,,$$

where

$$\begin{aligned}
R_2=&\int_{\mathbb{H}^N}|\nabla_{\mathbb{H}^N}(J_3u)|^2\,\mathrm{d}v_{\mathbb{H}^N}-\int_{\mathbb{H}^N\setminus\Omega}V_{\lambda,y_1,y_2}(x)\,u^2\,\mathrm{d}v_{\mathbb{H}^N}\\
&-\sum_{k=1}^2\int_{B(y_k,d)}\left(\frac{|\nabla_{\mathbb{H}^N}J_k|^2}{1-J_k^2(x)}+(1-J_k^2(x))V_{\lambda,y_1,y_2}(x)\right)u^2\,\mathrm{d}v_{\mathbb{H}^N}\,,
\end{aligned}$$

and $\Omega=\overline{B(y_1,d)}\cup\overline{B(y_2,d)}$. In particular if $x\in B(y_1,d)$, since $\mathrm{d}(x,y_2)>d$ and recalling (2.1) and (2.4), we have

$$V_{\lambda,y_1,y_2}(x)\le V_{\lambda,y_1}(x)+V_\lambda(d)\,,$$

where

$$V_\lambda(d):=\left(H_N(\lambda)\,\frac{1}{d^2}+C_N(\lambda)\frac{1}{\sinh^2(d)}+D_N(\lambda)g(d)\right)\,. \tag{3.3}$$

A similar argument works if $x \in B(y_2, d)$. Using this fact and exploiting in each $B(y_k, d)$ inequality (2.3) with $x_0 = y_k$, for all $N-2 \leq \lambda \leq \left(\frac{N-1}{2}\right)^2$, we get

$$\sum_{k=1}^{2} \langle L(J_k u), (J_k u)\rangle \geq \lambda \int_\Omega ((J_1 u)^2 + (J_2 u)^2)\, \mathrm{d}v_{\mathbb{H}^N} \, - \, V_\lambda(d) \int_\Omega ((J_1 u)^2 + (J_2 u)^2)\, \mathrm{d}v_{\mathbb{H}^N} \, .$$

Concerning the estimate of R_2, using the Poincaré inequality (without remainder terms), the estimate $V_{\lambda,y_1,y_2}(x) \leq 2V_\lambda(d)$ in $\mathbb{H}^N \setminus \overline{\Omega}$, and Lemma 3.2, we have

$$\begin{aligned}
R_2 &\geq \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N}(J_3 u)|^2 \, \mathrm{d}v_{\mathbb{H}^N} - \int_{\mathbb{H}^N \setminus \overline{\Omega}} V_{\lambda,y_1,y_2}(x)\, (J_3 u)^2 \, \mathrm{d}v_{\mathbb{H}^N} \\
&\quad - \sum_{k=1}^{2} \int_{B(y_k,d)} \left(\frac{\pi^2}{d^2} + 2K_N(\lambda, d) \right) u^2 \, \mathrm{d}v_{\mathbb{H}^N} \\
&\geq \lambda \int_{\mathbb{H}^N} (J_3 u)^2 \, \mathrm{d}v_{\mathbb{H}^N} - 2V_\lambda(d) \int_{\mathbb{H}^N \setminus \overline{\Omega}} (J_3 u)^2 \, \mathrm{d}v_{\mathbb{H}^N} \\
&\quad - \sum_{k=1}^{2} \int_{B(y_k,d)} \left(\frac{\pi^2}{d^2} + 2K_N(\lambda, d) \right) u^2 \, \mathrm{d}v_{\mathbb{H}^N} \\
&\geq \lambda \int_{\mathbb{H}^N} (J_3 u)^2 \, \mathrm{d}v_{\mathbb{H}^N} - \left(\frac{\pi^2}{d^2} + 2K_N(\lambda, d) \right) \int_{\mathbb{H}^N} u^2 \, \mathrm{d}v_{\mathbb{H}^N} \, ,
\end{aligned}$$

where in the last step we exploit the fact that $(J_3 u)^2 = u^2$ in $\mathbb{H}^N \setminus \overline{\Omega}$ and that $K_N(\lambda, d) > V_\lambda(d)$ for all $d > 0$ and $N-2 \leq \lambda \leq \left(\frac{N-1}{2}\right)^2$. Summarising, we get

$$\begin{aligned}
\langle Lu, u\rangle &\geq \lambda \int_\Omega ((J_1 u)^2 + (J_2 u)^2)\, \mathrm{d}v_{\mathbb{H}^N} + \lambda \int_{\mathbb{H}^N} (J_3 u)^2 \, \mathrm{d}v_{\mathbb{H}^N} \\
&\quad - \, V_\lambda(d) \int_\Omega ((J_1 u)^2 + (J_2 u)^2)\, \mathrm{d}v_{\mathbb{H}^N} - \left(\frac{\pi^2}{d^2} + 2K_N(\lambda, d) \right) \int_{\mathbb{H}^N} u^2 \, \mathrm{d}v_{\mathbb{H}^N} \\
&\geq \lambda \int_{\mathbb{H}^N} u^2 \, \mathrm{d}v_{\mathbb{H}^N} - \, K_N(\lambda, d) \int_{\mathbb{H}^N} ((J_1 u)^2 + (J_2 u)^2 + (J_3 u)^2)\, \mathrm{d}v_{\mathbb{H}^N} \\
&\quad - \left(\frac{\pi^2}{d^2} + 2K_N(\lambda, d) \right) \int_{\mathbb{H}^N} u^2 \, \mathrm{d}v_{\mathbb{H}^N} \\
&\geq \lambda \int_{\mathbb{H}^N} u^2 \, \mathrm{d}v_{\mathbb{H}^N} - \left(\frac{\pi^2}{d^2} + 3K_N(\lambda, d) \right) \int_{\mathbb{H}^N} u^2 \, \mathrm{d}v_{\mathbb{H}^N} \, .
\end{aligned}$$

□

3.3. **Proof of Theorem 2.2.** The proof relies on Proposition 3.3 for the case of two singularities. Let $\{J_k\}_{k=1}^{M+1}$ be a partition of unity satisfying condition (3.1) and such that $J_k(x) := J\left(\frac{\mathrm{d}(x,y_k)}{d}\right)$ for all $x \in \mathbb{H}^N$ and $k = 1, ..., M$, with J as defined in the proof of Lemma 3.2. Hence, $\mathrm{Int}(\mathrm{supp}(J_k)) = B(y_k, d)$ for $k = 1, ..., M$. From Lemma 3.1 we know that

$$\langle Lu, u\rangle := \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} - \int_{\mathbb{H}^N} V_{\lambda,y_1,\dots y_M}(x)\, u^2 \, \mathrm{d}v_{\mathbb{H}^N} = \sum_{k=1}^{M} \langle L(J_k u), (J_k u)\rangle + R_M \, ,$$

with

$$R_M = \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N}(J_{M+1}u)|^2 - \int_{\mathbb{H}^N\setminus\Omega} V_{\lambda,y_1,\dots y_M}(x)\, u^2 \,\mathrm{d}v_{\mathbb{H}^N} - \sum_{k=1}^M \int_{B(y_k,d)} \left(\frac{|\nabla_{\mathbb{H}^N} J_k|^2}{1-J_k^2(x)} + (1-J_k^2(x)) V_{\lambda,y_1,\dots y_M}(x) \right) u^2 \,\mathrm{d}v_{\mathbb{H}^N} ,$$

where $\Omega := \cup_{k=1}^M \overline{B(y_k,d)}$. Next, using $\mathrm{d}(x,y_l) > d$ for all $x \in B(y_k,d)$ and all $l \neq k$, we estimate

$$V_{\lambda,y_1,\dots y_M}(x) \le V_{\lambda,y_k}(x) + (M-1)K_N(\lambda,d) \quad \text{ for all } x \in B(y_k,d)$$

and, exploiting in each $B(y_k,d)$ inequality (2.3) with $x_0 = y_k$, for all $N-2 \le \lambda \le \left(\frac{N-1}{2}\right)^2$ we obtain

$$\begin{aligned} &\sum_{k=1}^M \langle L(J_k u), (J_k u)\rangle \\ &= \sum_{k=1}^M \left(\int_{B(y_k,d)} |\nabla_{\mathbb{H}^N} J_k u|^2 \,\mathrm{d}v_{\mathbb{H}^N} - \int_{B(y_k,d)} V_{\lambda,y_1,\dots y_M}(x)(J_k u)^2 \,\mathrm{d}v_{\mathbb{H}^N} \right) \qquad (3.4) \\ &\ge \lambda \int_\Omega \sum_{k=1}^M (J_k u)^2 \,\mathrm{d}v_{\mathbb{H}^N} - (M-1)K_N(\lambda,d) \int_\Omega \sum_{k=1}^M (J_k u)^2 \,\mathrm{d}v_{\mathbb{H}^N} . \end{aligned}$$

Concerning the estimate of R_M, we exploit the Poincaré inequality (without remainder terms) and the fact $\mathrm{d}(x,y_k) \ge d$ for all $x \in \mathbb{H}^N \setminus \Omega$ to deduce that

$$\begin{aligned} R_M &\ge \lambda \int_{\mathbb{H}^N} (J_{M+1}u)^2 - M V_\lambda(d) \int_{\mathbb{H}^N\setminus\Omega} u^2 \,\mathrm{d}v_{\mathbb{H}^N} \\ &\quad - \sum_{k=1}^M \int_{B(y_k,d)} \left(\frac{|\nabla_{\mathbb{H}^N} J_k|^2}{1-J_k^2(x)} + (1-J_k^2(x)) V_{\lambda,y_1,\dots y_M}(x) \right) u^2 \,\mathrm{d}v_{\mathbb{H}^N} , \end{aligned}$$

with $V_\lambda(d)$ as given in (3.3). Consider the last quantity in the r.h.s. above we notice that, since $J_l(x) = 0$ for $x \in B(y_k,d)$ and $l \neq k$, we can write

$$\begin{aligned} &- \int_{B(y_k,d)} \left(\frac{|\nabla_{\mathbb{H}^N} J_k|^2}{1-J_k^2(x)} + (1-J_k^2(x)) V_{\lambda,y_1,\dots y_M}(x) \right) |u|^2 \,\mathrm{d}v_{\mathbb{H}^N} \\ &= - \int_{B(y_k,d)} \left(\frac{|\nabla_{\mathbb{H}^N} J_k|^2}{1-J_k^2(x)} + \frac{|\nabla_{\mathbb{H}^N} J_l|^2}{1-J_l^2(x)} + (1-J_k^2(x)-J_l^2(x)) V_{\lambda,y_1,\dots y_M}(x) \right) u^2 \,\mathrm{d}v_{\mathbb{H}^N} \\ &\ge - \int_{B(y_k,d)} \left(\frac{|\nabla_{\mathbb{H}^N} J_k|^2}{1-J_k^2(x)} + \frac{|\nabla_{\mathbb{H}^N} J_l|^2}{1-J_l^2(x)} + (1-J_k^2(x)-J_l^2(x)) V_{\lambda,y_k,y_l}(x) \right) u^2 \,\mathrm{d}v_{\mathbb{H}^N} \\ &\quad - (M-2)V_\lambda(d) \int_{B(y_k,d)} (1-J_k^2(x))\, u^2 \,\mathrm{d}v_{\mathbb{H}^N} . \end{aligned}$$

Then we apply Lemma 3.2 on $B(y_k,d)$ with $(y_1,y_2) = (y_k,y_l)$, and we get

$$\begin{aligned} &\int_{B(y_k,d)} \left(\frac{|\nabla_{\mathbb{H}^N} J_k|^2}{1-J_k^2(x)} + (1-J_k^2(x)) V_{\lambda,y_1,\dots y_M}(x) \right) u^2 \,\mathrm{d}v_{\mathbb{H}^N} \\ &\ge - \int_{B(y_k,d)} \left(\frac{\pi^2}{d^2} + 2K_N(\lambda,d) \right) u^2 \,\mathrm{d}v_{\mathbb{H}^N} - (M-2)V_\lambda(d) \int_{B(y_k,d)} (1-J_k^2(x))\, u^2 \,\mathrm{d}v_{\mathbb{H}^N} . \end{aligned}$$

Coming back to the estimate of R_M, since $K_N(\lambda,d) > V_\lambda(d)$ for all $d>0$ and $N-2 \le \lambda \le \left(\frac{N-1}{2}\right)^2$, we infer

$$\begin{aligned} R_M &\ge \lambda \int_{\mathbb{H}^N} (J_{M+1}u)^2 - MV_\lambda(d) \int_{\mathbb{H}^N\setminus\Omega} u^2 \,\mathrm{d}v_{\mathbb{H}^N} - \sum_{k=1}^M \int_{B(y_k,d)} \left(\frac{\pi^2}{d^2} + 2K_N(\lambda,d)\right) u^2 \,\mathrm{d}v_{\mathbb{H}^N} \\ &\quad - (M-2)V_\lambda(d) \sum_{k=1}^M \int_{B(y_k,d)} (1-J_k^2(x))\, u^2 \,\mathrm{d}v_{\mathbb{H}^N} \\ &\ge \lambda \int_{\mathbb{H}^N} (J_{M+1}u)^2 - MK_N(\lambda,d) \int_{\mathbb{H}^N\setminus\Omega} u^2 \,\mathrm{d}v_{\mathbb{H}^N} \\ &\quad - \int_{\mathbb{H}^N} \frac{\pi^2}{d^2}\, u^2 \,\mathrm{d}v_{\mathbb{H}^N} - K_N(\lambda,d) \sum_{k=1}^M \int_{B(y_k,d)} (M-(M-2)J_k^2(x))\, u^2 \,\mathrm{d}v_{\mathbb{H}^N}\,. \end{aligned}$$

The latter inequality, together with (3.4) and summing up all terms, yields

$$\begin{aligned} \langle Lu,u\rangle &\ge \lambda \int_{\mathbb{H}^N} \sum_{k=1}^{M+1} (J_k u)^2 \,\mathrm{d}v_{\mathbb{H}^N} - \int_{\mathbb{H}^N} \frac{\pi^2}{d^2}\, u^2 \,\mathrm{d}v_{\mathbb{H}^N} \\ &\quad - K_N(\lambda,d) \sum_{k=1}^M \int_{B(y_k,d)} (M+J_k^2(x))\, u^2 - MK_N(\lambda,d) \int_{\mathbb{H}^N\setminus\Omega} u^2 \,\mathrm{d}v_{\mathbb{H}^N} \\ &\ge \lambda \int_{\mathbb{H}^N} u^2 \,\mathrm{d}v_{\mathbb{H}^N} - \int_{\mathbb{H}^N} \frac{\pi^2}{d^2}\, u^2 \,\mathrm{d}v_{\mathbb{H}^N} \\ &\quad - K_N(\lambda,d)(M+1) \sum_{k=1}^M \int_{\Omega_k} u^2 - MK_N(\lambda,d) \int_{\mathbb{H}^N\setminus\Omega} u^2 \,\mathrm{d}v_{\mathbb{H}^N} \\ &\ge \lambda \int_{\mathbb{H}^N} u^2 \,\mathrm{d}v_{\mathbb{H}^N} - \int_{\mathbb{H}^N} \frac{\pi^2}{d^2}\, u^2 \,\mathrm{d}v_{\mathbb{H}^N} - K_N(\lambda,d)(M+1) \int_{\mathbb{H}^N} u^2 \,\mathrm{d}v_{\mathbb{H}^N}\,, \end{aligned}$$

where we have also exploited $J_k^2(x) \le 1$ for all $x \in \mathbb{H}^N$. The above estimate completes the proof.

4. General manifolds

Let $(\mathcal{M}, g)$ be an $N \ge 3$ dimensional simply connected Riemannian manifold and $\mathrm{d}(.,.)$ be the geodesic distance on $\mathcal{M}$. Furthermore, denote $K_{x_0}(x)$ the sectional curvature on $\mathcal{M}$ at a point $x_0 \in \mathcal{M}$ and direction $x \in \mathcal{M}$. Within this Section we will require that

$$K_{x_0}(x) \le -c \quad \forall x_0, x \in \mathcal{M} \text{ and for some } c>0\,. \tag{4.1}$$

Hence, only Cartan-Hadamard manifolds will be considered. We comment that this assumption can be replaced by a weaker but more cumbersome one, namely by requiring that sectional curvature in each radial direction with respect to the M poles satisfies the same bound.

Next, for $\lambda \le \left(\frac{N-1}{2}\right)^2$, $c>0$ and $x_0 \in \mathbb{H}^N$, we set

$$V_{\lambda,c,x_0}(x) := H_N(\lambda) \frac{1}{\mathrm{d}^2(x,x_0)} + C_N(\lambda) \frac{c}{\sinh^2(\sqrt{c}\mathrm{d}(x,x_0))} + D_N(\lambda)\, g_c(\mathrm{d}(x,x_0))\,, \tag{4.2}$$

where $g_c(r) := \frac{r\sqrt{c}\coth(\sqrt{c}r)-1}{r^2}$ for all $r>0$ and $H_N(\lambda), C_N(\lambda), D_N(\lambda)$ are as given in (2.2). Finally, we recall the counterpart of Proposition 2.1 on $\mathcal{M}$:

Proposition 4.1. [6, Corollary 4.2] *Let $(\mathcal{M},g)$ be an $N\geq 3$ dimensional Cartan-Hadamard manifold satisfying condition (4.1) and let $x_0\in\mathcal{M}$. For all $N-2\leq\lambda\leq\left(\frac{N-1}{2}\right)^2$ the following inequality holds*

$$\int_{\mathcal{M}}|\nabla_g u|^2\,\mathrm{d}v_g-\lambda\,c\int_{\mathcal{M}}u^2\,\mathrm{d}v_g\geq\int_{\mathcal{M}}V_{\lambda,c,x_0}(x)\,u^2\,\mathrm{d}v_g\,,$$

for all $u\in C_c^\infty(\mathcal{M})$, where V_{λ,c,x_0} is as given in (4.2).

By exploiting Proposition 2.1 and proceedings along the lines given in the special case $\mathcal{M}=\mathbb{H}^N$ one proves the following Theorem:

Theorem 4.2. *Let $(\mathcal{M},g)$ be an $N\geq 3$ dimensional Cartan-Hadamard manifold satisfying condition (4.1). Furthermore, let $y_1,\dots y_M\in\mathcal{M}$ and $d>0$ as defined in (2.5). Then, for all $N-2\leq\lambda\leq\left(\frac{N-1}{2}\right)^2$ the following inequality holds*

$$\int_{\mathcal{M}}|\nabla_g u|^2\,\mathrm{d}v_g+\left(\frac{\pi^2}{d^2}+(M+1)K_{N,c}(\lambda,d)-\lambda c\right)\int_{\mathcal{M}}u^2\,\mathrm{d}v_g\geq\int_{\mathcal{M}}V_{\lambda,c,y_1,\dots y_M}(x)\,u^2\,\mathrm{d}v_g$$

for all $u\in C_c^\infty(\mathcal{M})$ with

$$K_{N,c}(\lambda,d):=H_N(\lambda)\frac{1}{d^2}\,+\,C_N(\lambda)\frac{c}{\sinh^2(\sqrt{c}d/2)}+\,D_N(\lambda)\,g_c(d/2)\,,$$

with g_c as given in (4.2) and, $V_{\lambda,c,y_k}(x)$ being as in (4.2),

$$V_{\lambda,c,y_1,\dots y_M}(x)=\sum_{i=1}^M V_{\lambda,c,y_k}(x)\,.$$

Note that for $\lambda=N-2$, $H_N(\lambda)=\frac{(N-2)^2}{4}$ and $C_N(\lambda)=0$. Hence, since $g_c(r)\to 0$ for every $r>0$ as $c\to 0^+$, one recovers the Euclidean inequality (1.2).

From Theorem 4.2 one may derive the counterpart of Corollary 2.3 on $\mathcal{M}$. Let $\bar{d}_c=\bar{d}_c(M,N)>0$ be defined implicitly as follows

$$\frac{\pi^2}{\bar{d}_c^2}+(M+1)\left(\frac{(N-2)^2}{4}\frac{1}{\bar{d}_c^2}+\frac{(N-3)(N-2)}{2}g_c(\bar{d}_c/2)\right)=(N-2)c\,. \tag{4.3}$$

We have

Corollary 4.3. *Let $(\mathcal{M},g)$ be an $N\geq 3$ dimensional Cartan-Hadamard manifold satisfying condition (4.1). Furthermore, let $y_1,\dots y_M\in\mathcal{M}$, $d>0$ as defined in (2.5) and $\bar{d}_c=\bar{d}_c(M,N)>0$ as defined in (4.3). If $d\geq\bar{d}_c$, the following inequality holds*

$$\begin{aligned}\int_{\mathcal{M}}|\nabla_g u|^2\,\mathrm{d}v_g\geq\frac{(N-2)^2}{4}\sum_{k=1}^M\int_{\mathcal{M}}\frac{u^2}{\mathrm{d}^2(x,y_k)}\,\mathrm{d}v_g\\+\frac{(N-2)(N-3)}{2}\sum_{k=1}^M\int_{\mathcal{M}}g_c(\mathrm{d}(x,y_k))\,u^2\,\mathrm{d}v_g\end{aligned} \tag{4.4}$$

for all $u\in C_c^\infty(\mathcal{M})$, where g_c is as given in (4.2).

Inequality (4.4) should be compared with inequality (3.5) in [11] which, under the same assumptions of Corollary 4.3, reads:

$$\begin{aligned}\int_{\mathcal{M}} |\nabla_g u|^2 \,\mathrm{d}v_g \geq \frac{(N-2)^2}{M^2} \sum_{1\leq i<j\leq M} \int_{\mathcal{M}} \left| \frac{\nabla_g(\mathrm{d}(x,y_i))}{\mathrm{d}(x,y_i)} - \frac{\nabla_g(\mathrm{d}(x,y_j))}{\mathrm{d}(x,y_j)} \right|^2 u^2 \,\mathrm{d}v_g \\ + \frac{(N-2)(N-1)}{M} \sum_{k=1}^{M} \int_{\mathcal{M}} g_c(\mathrm{d}(x,y_k))\, u^2 \,\mathrm{d}v_g\end{aligned} \tag{4.5}$$

for all $u \in C_c^\infty(\mathcal{M})$. Hence, the potential in (4.4), compared to that in (4.5), is smaller near poles for $M = 2$ but larger for $M \geq 3$.

We conclude with the analogue of Corollary 2.4 on $\mathcal{M}$, which follows along similar lines. For $N-2 \leq \lambda < \left(\frac{N-1}{2}\right)^2$, let $\bar{d}_{\lambda,c} = \bar{d}_{\lambda,c}(M,N) > 0$ be defined implicitly as follows

$$\frac{\pi^2}{\bar{d}_{\lambda,c}} + (M+1)\left(\frac{1}{4}\frac{1}{\bar{d}^2_{\lambda,c}} + \frac{(N-1)(N-3)}{4}\,\frac{c}{\sinh^2((\sqrt{c}\bar{d}_{\lambda,c})/2)}\right) = \left(\left(\frac{N-1}{2}\right)^2 - \lambda\right) c\,.$$

There holds:

Corollary 4.4. *Let $(\mathcal{M}, g)$ be an $N \geq 3$ dimensional Cartan-Hadamard manifold satisfying condition (4.1). Furthermore, let $y_1, \dots y_M \in \mathcal{M}$ and $d > 0$ as defined in (2.5). Let $N-2 \leq \lambda < \left(\frac{N-1}{2}\right)^2$, if $d \geq \bar{d}_{\lambda,c}$, the following inequality holds*

$$\begin{aligned}\int_{\mathcal{M}} |\nabla_g u|^2 \,\mathrm{d}v_g \geq \lambda c \int_{\mathcal{M}} u^2 \,\mathrm{d}v_g + \frac{1}{4}\sum_{k=1}^{M} \int_{\mathcal{M}} \frac{u^2}{\mathrm{d}^2(x,y_k)} \,\mathrm{d}v_g \\ + \frac{(N-1)(N-3)c}{4} \sum_{k=1}^{M} \int_{\mathcal{M}} \frac{u^2}{\sinh^2((\sqrt{c}\mathrm{d}(x,y_k))/2)}\, \mathrm{d}v_g\end{aligned}$$

for all $u \in C_c^\infty(\mathcal{M})$.

4.1. **Proof of Theorem 4.2.** The proof of Theorem 4.2 follows by repeating the proof of Theorem 2.2 with very few changes, hence we omit it. We limit ourselves to remark that the counterpart of Lemma 3.2 on $\mathcal{M}$ follows by replacing Proposition 2.1 with Proposition 4.1 and by noting that for every $x_0 \in \mathcal{M}$ the eikonal condition holds:

$$|\nabla_g \mathrm{d}(.,x_0)|^2 = 1 \quad \text{a.e. on } \mathcal{M}\,.$$

5. Remarks on the criticality issue

The criticality issue of Proposition 2.1 does not carry over to the multipolar framework of Theorem 2.2. A critical multipolar Hardy-weight can instead be derived from the general results of [10]. For simplicity we focus on the hyperbolic setting. Let $y_1, \dots y_M \in \mathbb{H}^N$ and let $G(x,y_i)$ be the Green function of $-\Delta_{\mathbb{H}^N}$ at y_i, which reads

$$G(x,y_i) = \int_{\mathrm{d}(x,y_i)}^{\infty} (\sinh s)^{-(N-1)}\,\mathrm{d}s\,.$$

Finally, we set

$$W(x) := \left(\frac{1}{M+1}\right)^2 \left[\sum_{i=1}^{M} \left|\frac{|\nabla G(x,y_i)|}{G(x,y_i)}\right|^2 + \sum_{1\leq i<j\leq M} \left|\frac{|\nabla G(x,y_i)|}{G(x,y_i)} - \frac{|\nabla G(x,y_j)|}{G(x,y_j)}\right|^2\right] \tag{5.1}$$

and we state the counterpart of inequality (1.5) in $\mathbb{H}^N$:

Proposition 5.1. *Let $N \geq 3$ and $M \geq 2$. Furthermore, let $y_1, \dots y_M \in \mathbb{H}^N$, $d > 0$ as defined in (2.5) and $W(x)$ as defined in (5.1). Then, the following multipolar inequality holds*

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \int_{\mathbb{H}^N} W(x)\, u^2 \, \mathrm{d}v_{\mathbb{H}^N}$$

for all $u \in C_c^\infty(\mathbb{H}^N)$, and the operator $-\Delta_{\mathbb{H}^N} - W$ is critical in $\mathbb{H}^N \setminus \{y_1, ..., y_M\}$.

It is worth noting that

$$\frac{|\nabla G(x, y_i)|}{G(x, y_i)} \sim \frac{N-2}{\mathrm{d}(x, y_i)} \quad \text{as } x \to y_i \,,$$

while

$$\frac{|\nabla G(x, y_i)|}{G(x, y_i)} = N - 1 + \frac{(N-1)^2}{2(N-2)} \left(\sinh(\mathrm{d}(x, y_i))\right)^{-2} \quad \text{as } x \to +\infty \,.$$

Namely,

$$W(x) \sim M \left(\frac{N-2}{M+1}\right)^2 \frac{1}{\mathrm{d}^2(x, y_i)} \quad \text{as } x \to y_i \quad \text{and} \quad W(x) \sim \frac{M(N-1)^2}{(M+1)^2} \quad \text{as } x \to +\infty \,.$$

Hence, the above weight is smaller than that in (4.5) (obtained in [11]) near poles but larger at infinity. Furthermore, near poles one recovers the same behavior of the Euclidean weight in (1.5).

Proof. The proof of Proposition 5.1 follows from Theorem 12.3 in [10] which in $\mathbb{H}^N$ reads: let P be a subcritical operator, if there exist $u_0, ..., u_M$, $M+1$ positive solutions to the equation $Pu = 0$ in $\mathbb{H}^N \setminus \{y_1, ..., y_M\}$ of minimal growth near y_i and ∞, i.e.

$$\lim_{x \to \infty} \frac{u_i(x)}{u_0(x)} = 0 \text{ for all } i \neq 0 \quad \text{and} \quad \lim_{x \to x_i} \frac{u_j(x)}{u_i(x)} = 0 \text{ for all } j \neq i \,,$$

then

$$W_\alpha(x) = \sum_{i<j} \alpha_i \alpha_j \left| \nabla \left(\log \frac{u_i(x)}{u_j(x)} \right) \right|^2 \quad \text{with } \alpha_j \in (0, 1/2] \text{ and } \sum_{j=0}^{M} \alpha_j = 1 \,,$$

is a critical Hardy-weight for P.

In our case, we have $P = -\Delta_{\mathbb{H}^N}$ and we may take the functions $u_0(x) = 1$ and $u_i(x) = G(x, y_i)$, with $i = 1, ..., M$, which are positive solutions to $-\Delta_{\mathbb{H}^N} u = 0$ in $\mathbb{H}^N \setminus \{y_1, ..., y_M\}$ and satisfy the required growth conditions. Finally, the weight W in (5.1) is obtained by choosing in W_α uniformly $\alpha_j = \frac{1}{M+1}$ for all $j = 0, \dots, M$. □

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