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ABSTRACT
The main objective of this paper is the characterization of the notion of uniform algebraic observability for multi-switching linear systems. In particular, it is shown that a multi-switching linear system is 'generically' uniformly algebraically observable if and only if the number of available measurements is greater than the number of switching signals acting on the systems. This result is established without requiring any knowledge about the (possibly, stochastic) process generating the switching pattern. Moreover, since such systems can be used to model sensor networks, the results given in this paper can 'generically' be used to design simple observers for sensor networks able to cope with sensor failures and data losses.

KEYWORDS
Uniform algebraic observability, multi-switching systems; observer design.

1. Introduction

The fast advancement of reliable wireless communications and of low-cost electronic devices paved the way to the development of *multi-sensor networks*, *i.e.*, huge groups of dedicated sensors that sense, process, record and communicate the physical conditions of a plant (Akyildiz, Su, Sankarasubramaniam, & Cayirci, 2002; Biswas, Jain, Ghosh, & Agrawal, 2006; Sinopoli, Sharp, Schenato, Schaffert, & Sastry, 2003). Applications of such networks span from health care monitoring (Peiris, 2013) and air pollution analysis (Tsujita, Yoshino, Ishida, & Moriizumi, 2005) to environmental monitoring (Longhi & Marrocco, 2017; N. Wang, Zhang, & Wang, 2006) and safeness of constructions (Hubbard et al., 2015). The main advantages of such networks with respect to classical sensing protocols are their ability to cope with sensor failures (Schenato, Sinopoli, Franceschetti, Poolla, & Sastry, 2007) and their scalability (Yick, Mukherjee, & Ghosal, 2008).

Due to the large employment of such sensor networks in practical applications, an extensive research effort has been spent to characterize their properties. In order to capture the behavior of these network in presence of data loss and sensor failures, these systems are usually modeled as plants with non-switching, continuous-time dynamics and with switching outputs (Li, Ugrinovskii, & Orsi, 2007; Ugrinovskii, 2013). In

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particular, several approaches have been proposed to design the switching signal that governs the output response of the system. For instance, in Hespanha, Naghshtabrizi, and Xu (2007); Matei, Martins, and Baras (2009), the switching signal is governed by a Markov chain whose states correspond to the possible communication topologies, whereas in Langbort, Gupta, and Murray (2006); Subbotin and Smith (2009), the switching signal is governed by independent, two-state Markov processes describing the status of the individual links. An alternative method to model sensor networks is given in Manjunatha, Verma, and Srividya (2008), by using a fuzzy logic approach.

Several approaches have been given in the literature to merge data coming from different sensors. For instance, in Subbotin and Smith (2009), the data fusion is carried out by using Kalman filters, in Ugrinovskii (2013), the estimation is carried out by using H_∞ techniques, in Jiao and Wu (2018), the measurements are processed by using a fuzzy model, in X. Wang, Hu, Jia, and Tang (2018), support vector machines are used, whereas, in Garin and Schenato (2010), a linear consensus algorithms is proposed to compute averages of local estimates.

The main objective of this paper is to provide necessary and sufficient conditions ensuring that the uniform algebraic observability is a 'generic' property for multi-switching systems.

In particular, it is shown that if the number of outputs is greater than the number of switching signals, then multi-switching linear systems are 'generically' uniformly observable with respect to the switching signals. On the other hand, it is proved that if the number of outputs is lower than or equal to the number of switching signals, then multi-switching systems are 'generically' not uniformly observable. Furthermore, in the case of uniformly observable systems, algebraic geometry techniques are proposed to compute the matrix relating the time derivatives of the output with the state of the system and it is shown how to use such a matrix to design a state observer. The main interest in this class of systems relies on the fact that they are able to replicate the output response of those switching plants that are usually employed to model multi-sensor networks (see the subsequent Lemmas 1 and 2). Thus, characterizing their observability properties is the first step toward the construction of reliable and simple observers that are able to deal with sensor and link failures.

2. The considered class of multi-switching systems

Notation: The sets of real and integer numbers are denoted $\mathbb{R}$ and $\mathbb{Z}$, respectively; given $a \in \mathbb{R}$ and $b \in \mathbb{Z}$, define $\mathbb{R}_{\geqslant a} := \{s \in \mathbb{R} : s \geqslant a\}$ and $\mathbb{Z}_{\geqslant b} := \{s \in \mathbb{Z} : s \geqslant b\}$; $\mathbb{R}[x]$ denotes the ring of all the polynomials in $x := \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^\top$ with coefficients in $\mathbb{R}$, and $\mathbb{R}(x)$ denotes the field of all the rational functions in x with coefficients in $\mathbb{R}$; I_n is the $n \times n$ identity matrix and $0_{m,n}$ is the $m \times n$ zero-matrix. Given a function $\alpha(\varepsilon)$ of a single variable ε, $\lim_{\varepsilon \searrow \varepsilon^\star} \alpha(\varepsilon)$ denotes the *one-sided* limit of $\alpha(\varepsilon)$ as ε approaches $\varepsilon^\star$ from above; $\lim_{\varepsilon \searrow 0} \alpha(\varepsilon^\star - \varepsilon)$ and $\lim_{\varepsilon \searrow 0} \alpha(\varepsilon^\star + \varepsilon)$ are the *left* and the *right* one-sided limits of $\alpha(\varepsilon)$ at $\varepsilon^\star$, respectively. Given two matrices $C \in \mathbb{R}^{m \times n}$ and $A \in \mathbb{R}^{n \times n}$, the *observability matrix* associated with pair (C, A) is: $\mathbb{O}(C, A) := \begin{bmatrix} C^\top & \vdots & (CA^{n-1})^\top \end{bmatrix}^\top$. Given a vector field $f(x) \in \mathbb{R}^n$ and a function $h(\sigma, x) \in \mathbb{R}$, the *successive directional derivatives* of h along f are defined as: $L_f^0 h = h$ and $L_f^{j+1} h = (\frac{\partial}{\partial x} L_f^j h) f$, $j \in \mathbb{Z}_{\geqslant 0}$.

Consider a plant whose dynamics are described by

$$\frac{\mathrm{d}\, x(t)}{\mathrm{d}t} = A\, x(t), \tag{1a}$$

where $x(t) \in \mathbb{R}^n$ is the *state* vector at time $t \in \mathbb{R}$; $t = 0$ is the initial time. The measured *output* vector $y(t) = \begin{bmatrix} y_1(t) & \dots & y_m(t) \end{bmatrix}^\top \in \mathbb{R}^m$ is taken by an observation network at time $t \in \mathbb{R}$ and is given by

$$y(t) = \tilde{C}_{\sigma(t)}\, x(t), \tag{1b}$$

where $\tilde{C}_j \in \mathbb{R}^{m\times n}$, $j = 0, \dots, S$, and $\sigma : \mathbb{R} \to \{0, \dots, S\}$, $S \in \mathbb{Z}_{\geqslant 1}$, is a *signal* (*i.e*, a function of time) to be specified.

Definition 1. Consider a signal $s : \mathbb{R} \to \mathcal{S}$, where $\mathcal{S}$ is a finite subset of $\mathbb{Z}$. Since the co-domain $\mathcal{S}$ of the signal $s(t)$ is a finite set, if $s(t)$ is a continuous function on $\mathbb{R}$, then $s(t)$ is necessarily constant, *i.e.*, $s(t) = k$, for some $k \in \mathcal{S}$. If $s(t)$ is not continuous, then let $\hat{t} \in \mathbb{R}$ be the time of one of its discontinuities; since $\mathcal{S}$ is a finite set, if one of the two limits $\lim_{\varepsilon\searrow 0} s(\hat{t}+\varepsilon)$ and $\lim_{\varepsilon\searrow 0} s(\hat{t}+\varepsilon)$ exists, then it is necessarily finite; in addition, always because $\mathcal{S}$ is a finite set, a discontinuity can only be of three types:

(1.1) *removable discontinuity* at $t_a \in \mathbb{R}$, *i.e.*, the two limits $\lim_{\varepsilon\searrow 0} s(t_a - \varepsilon)$ and $\lim_{\varepsilon\searrow 0} s(t_a + \varepsilon)$ exist and coincide, but $s(t_a) \neq \lim_{\varepsilon\searrow 0} s(t_a + \varepsilon)$; such a discontinuity can be removed by redefining the signal $s(t)$ so that $s(t_a) = \lim_{\varepsilon\searrow 0} s(t_a + \varepsilon)$;

(1.2) *jump discontinuity* at $t_b \in \mathbb{R}$, *i.e.*, the two limits $\lim_{\varepsilon\searrow 0} s(t_b - \varepsilon)$ and $\lim_{\varepsilon\searrow 0} s(t_b+\varepsilon)$ exist, but they do not coincide (*i.e*, $\lim_{\varepsilon\searrow 0} s(t_b-\varepsilon) \neq \lim_{\varepsilon\searrow 0} s(t_b+\varepsilon)$); in such a case, the time t_b is called a *switching time*; the signal $s(t)$ can always be redefined so that $s(t_b) = \lim_{\varepsilon\searrow 0} s(t_b + \varepsilon)$, whence so to guarantee its right-continuity at the switching times;

(1.3) *essential discontinuity* at $t_c \in \mathbb{R}$, *i.e.*, at least one of the two limits $\lim_{\varepsilon\searrow 0} s(t_c - \varepsilon)$ and $\lim_{\varepsilon\searrow 0} s(t_c + \varepsilon)$ does not exist; in such a case, the time t_c is called a *Zeno switching time*; if $t_c \in \mathbb{R}$ is a Zeno switching time and $\lim_{\varepsilon\searrow 0} s(t_c + \varepsilon)$ exists, then the signal $s(t)$ can always be redefined so that $s(t_c) = \lim_{\varepsilon\searrow 0} s(t_c + \varepsilon)$, whence so to guarantee its right-continuity at such Zeno switching times.

If $s(t)$ has no removable discontinuities, has countable discontinuities (either Zeno or not), and is right-continuous at each discontinuity time, then it is a *switching signal.*

According to Definition 1, the discontinuity times (*i.e.*, either the switching times or the Zeno switching times) of a switching signal are assumed to be countable, whence they can be bi-univocally associated with an integer at subscript and denoted t_i, $i \in \mathbb{Z}$; hence, there is no discontinuity in the open interval

$$T_j := (t_j, \inf\{t_i : t_j < t_i\}), \quad j \in \mathbb{Z}.$$

No minimum (or average) *dwell time* is assumed.

The switching signal $\sigma(t)$ is characterized by the next assumption, which is extended throughout the paper to any other switching signal, without an explicit reference.

Assumption 1. *(1.1) The signal $\sigma : \mathbb{R} \to \{0, \dots, S\}$, $S \in \mathbb{Z}_{\geqslant 1}$, is a switching signal defined according to Definition 1;*

(1.2) the signal $\sigma(t)$ is not measured *and, in particular, the discontinuity times t_i* are not known *before they happen;*

(1.3) the number $S + 1$ of the possible values taken by the switching signal $\sigma(t)$ is not known.

The following lemma shows that the output (1b) of system (1) can be expressed as a linear function of x, with coefficients being polynomials in $\sigma(t)$ of degree S.

Lemma 1 (Menini, Possieri, and Tornambe (2014)). *There exist polynomials* $p_0, \ldots, p_S \in \mathbb{R}[\sigma]$ *having degree* S *such that he output of system* (1) *is given by*

$$y(t) = (p_0(\sigma(t))\, \tilde{C}_0 + \ldots + p_S(\sigma(t))\, \tilde{C}_S)\, x(t). \tag{2}$$

The following lemma shows that the representation (2) can be simplified by introducing additional switching signals (satisfying Definition 1), thus obtaining an expression that is linear in x, with coefficients being polynomials of degree 1 (linear) in such switching signals.

Lemma 2. *There exist* $N \in \mathbb{Z}_{\geqslant 1}$ *switching signals* $\sigma_j : \mathbb{R} \to \{0, \ldots, S_j\}$, $j = 1, \ldots, N$, *and* $C_0, \ldots, C_N \in \mathbb{R}^{m \times n}$ *such that the output of system* (1) *can be expressed as*

$$y(t) = (C_0 + \sigma_1(t) C_1 + \ldots + \sigma_N(t) C_N)\, x(t). \tag{3}$$

Proof. Let $N = S$. Hence, let $S_j = 1$ and define

$$\sigma_j(t) := \begin{cases} 1, & \text{if } \sigma(t) = j, \\ 0, & \text{if } \sigma(t) \neq j, \end{cases} \tag{4}$$

$j = 1, \ldots, N$. Thus, for each t such that $\sigma(t) = 0$, one has

$$C_0 + \sigma_1(t) C_1 + \ldots + \sigma_N(t) C_N = C_0,$$

whereas, for each t such that $\sigma(t) = j$, $j \in \{1, \ldots, S\}$,

$$C_0 + \sigma_1(t) C_1 + \ldots + \sigma_N(t) C_N = C_0 + C_j.$$

Therefore, one has that (3) holds with $C_0 = \tilde{C}_0$ and $C_j = \tilde{C}_j - \tilde{C}_0$, $j = 1, \ldots, N$. $\square$

By Lemmas 1 and 2, the following system is considered in the remainder of the paper (with $t \in \mathbb{R}$) instead of (1):

$$\frac{\mathrm{d}\, x(t)}{\mathrm{d}t} = A\, x(t), \tag{5a}$$
$$y(t) = (C_0 + \sigma_1(t) C_1 + \ldots + \sigma_N(t) C_N)\, x(t), \tag{5b}$$

where $\sigma_j : \mathbb{R} \to \{0, \ldots, S_j\}$, $S_j \in \mathbb{Z}_{\geqslant 1}$, $j = 1, \ldots, N$, are switching signals defined according to Definition 1, and satisfying Assumption 1. At each discontinuity time t_i, at least one of the switching signals $\sigma_j(t_i)$ has a discontinuity. A system in the form (5) is referred to as *multi-switching.*

The main objective of this paper is to give necessary and sufficient conditions ensuring that the uniform algebraic observability is a 'generic' property for the multi-switching system (5) (see the subsequent Definition 2), despite the fact that the switching signals $\sigma_1, \ldots, \sigma_N$ are not measured (whence, their discontinuity times are not known) and their co-domains are not known.

By Assumption 1, the following statements hold:
(i) if $t^\star \in T_i$, then $\frac{\mathrm{d}^k\, y(t)}{\mathrm{d}t^k}$ is continuous at $t = t^\star$ and

$$\frac{\mathrm{d}^k\, y(t)}{\mathrm{d}t^k}\Big|_{t=t^\star} = (C_0 + \textstyle\sum_{j=1}^{N} \sigma_j(t_i) C_j) A^k\, x(t^\star);$$

(ii) if $t^\star = t_i$ and t_i is Zeno, then $\lim_{\varepsilon\searrow 0} \frac{\mathrm{d}^k y(t)}{\mathrm{d}t^k}|_{t=t^\star-\varepsilon}$ does not exists, but $\lim_{\varepsilon\searrow 0} \frac{\mathrm{d}^k y(t)}{\mathrm{d}t^k}|_{t=t^\star-\varepsilon}$ exists,

$$\lim_{\varepsilon\searrow 0} \frac{\mathrm{d}^k y(t)}{\mathrm{d}t^k}|_{t=t^\star+\varepsilon} = (C_0 + \textstyle\sum_{j=1}^N \sigma_j(t_i) C_j) A^k\, x(t^\star). \tag{6}$$

Similarly, if $t^\star = t_i$ and t_i is not Zeno, then $\lim_{\varepsilon\searrow 0} \frac{\mathrm{d}^k y(t)}{\mathrm{d}t^k}|_{t=t^\star+\varepsilon}$ exists and is given by (6), whereas

$$\lim_{\varepsilon\searrow 0} \frac{\mathrm{d}^k y(t)}{\mathrm{d}t^k}|_{t=t^\star-\varepsilon} = (C_0 + \textstyle\sum_{j=1}^N \sigma_j(t_{i-1}) C_j) A^k\, x(t^\star).$$

Therefore, the signal $\frac{\mathrm{d}^k y(t)}{\mathrm{d}t^k}$ need not be defined nor be continuous at the discontinuity times $t = t_i$. Hence, for the above reasons, let $y^{(k)}(t)$ denote the following right-continuous function of time:

$$y^{(k)}(t) := \lim_{\varepsilon\searrow 0} \frac{\mathrm{d}^k y(\tau)}{\mathrm{d}\tau^k}|_{\tau=t+\varepsilon},$$

which is defined at all $t \in \mathbb{R}$, is continuous at each time $t \neq t_i$, and only right-continuous at the discontinuity times $t = t_i$.

Let $p_A(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n$ be the characteristic polynomial of A. Hence, by the Cayley-Hamilton Theorem (Kailath, 1980), $p_A(A) := A^n + a_1 A^{n-1} + \cdots + a_n I_n = 0$.

Lemma 3. *For each $t \in \mathbb{R}$ (including the switching times t_i)*

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_n y^{(0)}(t) = 0.$$

Proof. By construction, one has $y^{(k)}(t) = (C_0 + \sigma_1(t)C_1 + \ldots + \sigma_N(t)C_N)A^k\, x(t)$, for all $t \in \mathbb{R}$. Therefore, it holds that

$$\begin{aligned} y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_n y^{(0)}(t) &= \\ &(C_0 + \sigma_1(t)C_1 + \ldots + \sigma_N(t)C_N) p_A(A) x(t) = 0. \quad \square \end{aligned}$$

According to the above lemma, the signal $y_j^{(n)}(t) + a_1 y_j^{(n-1)}(t) + \cdots + a_n y_j^{(0)}(t)$ is a continuous function of $t \in \mathbb{R}$ (actually, it is constant and equal to 0), although its addends $y^{(n)}(t), a_1 y^{(n-1)}(t), \ldots, a_n y^{(0)}(t)$ are only right-continuous.

It is now possible to formalize the notion of uniform algebraic observability.

Definition 2. System (5) is *uniformly* (with respect $\sigma_1, \ldots, \sigma_N$) *algebraically observable* if there exist $K \in \mathbb{Z}_{\geqslant 0}$ and $F \in \mathbb{R}^{n\times m(K+1)}$ such that, letting

$$\check{x}(t) = F \begin{bmatrix} y^{(0)}(t) \\ \vdots \\ y^{(K)}(t) \end{bmatrix}, \tag{7}$$

one has $x(t) = \check{x}(t)$ for all times $t \in \mathbb{R}$.

The right-hand side of equation (7) is continuous at each $t \in \mathbb{R}$, $t \neq t_i$, and right-continuous at each $t = t_i$, since the functions $y^{(j)}(t)$ are right-continuous. In addition, since $x(t)$ is a function being continuous also at the discontinuity times t_i (at which $y^{(j)}(t)$ is only right-continuous), the condition $x(t) = \check{x}(t)$ for all $t \neq t_i$ implies that the right-hand side of (7) cannot have discontinuities, whence it is a continuous function of time on the whole $\mathbb{R}$, when system (5) is uniformly algebraically observable; clearly, if $y^{(0)}(t), \ldots, y^{(K)}(t)$ are replaced in (7) by some estimates $\hat{y}^{(0)}(t), \ldots, \hat{y}^{(K)}(t)$ for the computation of an estimate $\hat{x}(t)$ of $\check{x}(t)$, even infinitesimal errors in the recognition of some discontinuity time will imply peaking in the resulting estimation errors.

In Definition 2, *observable* means that $x(t)$ can be expressed as a function of $y^{(0)}(t), \ldots, y^{(K)}(t)$, for some K, *algebraic* means that such a function is linear, whereas *uniform* means that such a function is independent of the switching signals.

The objective of the following sections is the study of the uniform algebraic observability of system (5). The two fundamental parameters in such a study are the number m of the entries of the output vector $y(t)$ and the number N of the switching signals σ_j; it will be shown that if $m \geqslant N + 1$, then system (5) is uniformly algebraically observable for 'almost all' $A \in \mathbb{R}^{n \times n}$ and $C_0, \ldots, C_N \in \mathbb{R}^{m \times n}$, whereas if $m \leqslant N$, then system (5) is not uniformly algebraically observable for 'almost all' $A \in \mathbb{R}^{n \times n}$ and $C_0, \ldots, C_N \in \mathbb{R}^{m \times n}$; in particular, it will be shown that both statements hold independently of the dimension n of the state vector and of the sets of the values taken by the switching signals.

3. Review of algebraic geometry tools

Given $p_1, \ldots, p_\ell \in \mathbb{R}[x]$, the set

$$\mathbf{V}(p_1, \ldots, p_\ell) := \{x \in \mathbb{R}^n : p_j(x) = 0, j = 1, \ldots, \ell\}$$

is the *affine variety* of $\mathbb{R}^n$ generated by $p_1, \ldots, p_\ell$, whereas

$$\langle p_1, \ldots, p_\ell \rangle := \{\textstyle\sum_{j=1}^{\ell} q_j p_j : q_j \in \mathbb{R}[x], j = 1, \ldots, \ell\}$$

is the *ideal generated* by $p_1, \ldots, p_\ell$ (the set $\{p_1, \ldots, p_\ell\}$ is a *basis* of such an ideal). An ideal that can be generated by a single polynomial is said to be *principal*.

Let $\mathcal{I}$ be an ideal of $\mathbb{R}[x]$. The set $\mathbf{V}(\mathcal{I}) := \{x \in \mathbb{R}^n : p(x) = 0, \forall p \in \mathcal{I}\}$ is a variety of $\mathbb{R}^n$ and $\mathbf{V}(\mathcal{I}) = \mathbf{V}(p_1, \ldots, p_\ell)$, for any basis $\{p_1, \ldots, p_\ell\}$ of $\mathcal{I}$. Given any subset $\mathcal{S}$ of $\mathbb{R}^n$, $\mathbf{I}(\mathcal{S}) := \{p \in \mathbb{R}[x] : p(x) = 0, \forall x \in \mathcal{S}\}$ is an ideal of $\mathbb{R}[x]$ even if $\mathcal{S}$ is not a variety and $\mathcal{S} \subseteq \mathbf{V}(\mathbf{I}(\mathcal{S}))$. In particular, $\overline{\mathcal{S}} := \mathbf{V}(\mathbf{I}(\mathcal{S}))$ is the smallest variety of $\mathbb{R}^n$ containing $\mathcal{S}$, which is called the *Zariski closure* of $\mathcal{S}$.

Definition 3. Let $\mathcal{W} \subseteq \mathbb{R}^\ell$ be the set of all $\beta \in \mathbb{R}^\ell$ for which some property does not hold; it is said that such a property holds for *'almost all'* $\beta \in \mathbb{R}^\ell$ (briefly, it is *'generic'* in $\mathbb{R}^\ell$) if the Zariski closure $\overline{\mathcal{W}}$ of $\mathcal{W}$ does not coincide with $\mathbb{R}^\ell$.

Assume that a property is 'generic' in $\mathbb{R}^\ell$; if such a property holds at $\beta^\circ \in \mathbb{R}^\ell$, then there exists a neighborhood $\mathcal{B}$ of β° such that the property holds for all $\beta \in \mathcal{B}$; on the other hand, if the property does not hold at $\beta^\circ \in \mathbb{R}^\ell$, then, for any arbitrarily small neighborhood $\mathcal{B}$ of β°, there exists another $\hat{\beta}^\circ \in \mathcal{B}$ such that the property holds at $\hat{\beta}^\circ$.

If $\mathfrak{a} = [\mathfrak{a}_1 \ldots \mathfrak{a}_n]^\top$, $\mathfrak{a}_j \in \mathbb{Z}_{\geqslant 0}$, $j = 1, \ldots, n$, is a *multi-index*, then $x^{\mathfrak{a}} := x_1^{\mathfrak{a}_1} \cdots x_n^{\mathfrak{a}_n}$ is a *monomial* and the *total degree* (briefly, *degree*) of $x^{\mathfrak{a}}$, denoted $\deg(x^{\mathfrak{a}})$, is defined as $\deg(x^{\mathfrak{a}}) := \sum_{i=1}^{n} \mathfrak{a}_i =: |\mathfrak{a}|$. A *monomial order* in $\mathbb{R}[x]$, denoted $\succ$, is a total, well order on the set of monomials $x^{\mathfrak{a}} \in \mathbb{R}[x]$. The monomial order used in this paper is the *Lexicographic* monomial order (briefly, Lex order) (Cox, Little, & O'Shea, 2015). Given any $p \in \mathbb{R}[x]$ and any monomial order $\succ$, the *leading term* $\mathrm{LT}(p) = cx^{\mathfrak{a}}$ is the greatest term in p, $\mathrm{LC}(p) = c$ is the *leading coefficient* and $\mathrm{LM}(p) = x^{\mathfrak{a}}$ is the *leading monomial* of p; hence, $\deg(p) := \deg(\mathrm{LT}(p)) = |\mathfrak{a}|$. A polynomial $r \in \mathbb{R}[x]$ is *reduced* with respect to $\{p_1, \ldots, p_\ell\}$ if either $r = 0$ in $\mathbb{R}[x]$ or no monomial of r is divisible by any $\mathrm{LT}(p_j), j = 1, \ldots, \ell$. Given an ideal $\mathcal{I}$ of $\mathbb{R}[x]$ and a monomial order $\succ$, a finite set $\mathcal{G} = \{g_1, \ldots, g_\ell\} \subseteq \mathcal{I}$ is a *Gröbner basis* of $\mathcal{I}$ if $\langle \mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_\ell)\rangle = \langle \mathrm{LT}(\mathcal{I})\rangle$, where $\mathrm{LT}(\mathcal{I}) = \{cx^{\mathfrak{a}} : \exists f \in \mathcal{I} : \mathrm{LT}(f) = cx^{\mathfrak{a}}\}$. By the Hilbert Basis Theorem, each ideal of $\mathbb{R}[x]$ admits a finite Gröbner basis. A Gröbner basis is *reduced* if $\mathrm{LC}(g_j) = 1$, $j = 1, \ldots, \ell$, and g_j is reduced with respect to $\{g_1, \ldots, g_{j-1}, g_{j+1}, \ldots, g_\ell\}$, $j = 1, \ldots, \ell$. Each ideal $\mathcal{I} \neq \langle \emptyset \rangle$ has a *unique* reduced Gröbner basis, with respect to the chosen monomial order $\succ$ (Cox et al., 2015). Let $\mathcal{I}$ be an ideal of $\mathbb{R}[x_b, x_c]$. The *elimination* ideal of $\mathcal{I}$ that eliminates x_b is $\mathcal{I} \cap \mathbb{R}[x_c]$.

Given two polynomial rings $\mathcal{R}_a$ and $\mathcal{R}_b$, if an element (*e.g.*, a polynomial or an ideal) has been defined taking $\mathcal{R}_a$ as the ambient ring, the *coercion* is that operation for which such an element can be thought of as if it had been defined with $\mathcal{R}_b$ being the ambient ring, when possible and convenient.

4. Some algebraic tools for the observation problem for polynomial systems

In this section, it is shown how the tools reviewed in Section 3 can be used to characterize the observability of non-linear polynomial systems (see also Menini et al., 2014; Menini, Possieri, & Tornambe, 2016; Menini, Possieri, & Tornambe, 2019).

Consider the following polynomial system:

$$\frac{\mathrm{d}\,x(t)}{\mathrm{d}t} = f(x(t)), \qquad y(t) = h(x(t)), \tag{8a}$$

where $x \in \mathbb{R}^n$, $f \in \mathbb{R}^n[x]$ and $h \in \mathbb{R}^m[x]$. Let $y^{(j)}(t) := \frac{\mathrm{d}^j y(t)}{\mathrm{d}t^j}$, $j \in \mathbb{Z}_{\geqslant 0}$ (note that the definition of $y^{(j)}(t)$ given in the following Section 2 coincides with the definition above in absence of discontinuities, as in this section). Roughly speaking, since h and f are polynomial, pair (h, f) is *observable* if the state vector $x(t)$ at time $t \in \mathbb{R}$ can be expressed as a function of $y^{(0)}(t), \ldots, y^{(L)}(t)$, for some sufficiently high $L \in \mathbb{Z}$, $L \geqslant n$ (see Theorem 1 of Inouye, 1977); this is correlated with the injectivity of the following *observation map*:

$$y_{e,L} := \begin{bmatrix} y^{(0)} \\ \vdots \\ y^{(L)} \end{bmatrix} = \begin{bmatrix} h(x) \\ \vdots \\ L_f^L h(x) \end{bmatrix} =: H_L(x).$$

Definition 4. If there exists a neighborhood $\mathcal{B}_{x^\circ}$ of x° and a function $x = F_L(y_{e,L})$ from $\mathcal{B}_{x^\circ}$ to $\mathbb{R}^n$ such that $y_{e,L} = H_L(F_L(y_{e,L}))$, for all $y_{e,L} \in \mathcal{A}_{x^\circ}$, where $\mathcal{A}_{x^\circ}$ is the image of $\mathcal{B}_{x^\circ}$ through $H_L(\cdot)$, then pair (h, f) is *locally observable* about $(x^\circ, H_L(x^\circ))$. If $\mathcal{B}_{x^\circ}$ coincides with the whole $\mathbb{R}^n$, then pair (h, f) is *globally observable.* If $\mathcal{B}_{x^\circ}$ coincides with the whole $\mathbb{R}^n$ and $F_L(y_{e,L})$ is a linear function of $y_{e,L}$, then pair (h, f) is *algebraically observable.*

Assumption 2. *There exists* $L \in \mathbb{Z}_{\geqslant n}$ *and* $x^\circ \in \mathbb{R}^n$ *such that* $\operatorname{rank}(J_L(x^\circ)) = n$, *where* $J_L(x) := \frac{\partial H_L(x)}{\partial x}$.

Assumption 2 guarantees the local observability of pair (h, f), as stated in the following lemma, whose proof is straight-forward by the Implicit Function Theorem (Burckel, 1980), and which holds if one is working over both $\mathbb{R}$ and $\mathbb{C}$.

Lemma 4. *If Assumption 2 holds for some* $L \in \mathbb{Z}_{\geqslant 0}$*, then it holds with* L *substituted by* $L + j$*,* $j \in \mathbb{Z}_{\geqslant 0}$*. Under Assumption 2, pair* (h, f) *is locally observable about* $(x^\circ, H_L(x^\circ))$*.*

Consider the following ideal of $\mathcal{R}_a = \mathbb{R}[x, y_{e,L}]$:

$$\mathcal{I}_L = \langle y_{e,L} - H_L(x) \rangle.$$

The set of all polynomial constraints existing among the entries of $y_{e,L}$ (called *output embeddings*) is given by $\mathcal{J}_L = \mathcal{I}_L \cap \mathbb{R}[y_{e,L}]$. In particular, in order to find if there exists a (possibly, local) left inverse of $x = H_L(y_{e,L})$, first coerce $\mathcal{J}_L$ into $\mathbb{R}[y_{e,L}]$, define the quotient ring $\mathbb{R}[y_{e,L}]/\mathcal{J}_L$, and let $\mathbb{K}\{y_{e,L}\}$ be the field of the rational functions having numerator and denominator in $\mathbb{R}[y_{e,L}]/\mathcal{J}_L$. Hence, fix $\mathcal{R}_b = \mathbb{K}\{y_{e,L}\}[x]$ as ambient ring and coerce $\mathcal{I}_L$ into $\mathcal{R}_b$; finally, consider the ideals $\mathcal{K}_{L,j} = \mathcal{I}_L \cap \mathbb{K}\{y_{e,L}\}[x_j]$, $j = 1, \dots, n$.

Lemma 5. *Under Assumption 2, the ideal* $\mathcal{I}_L$ *thought of as an ideal of* $\mathcal{R}_b$ *is zero-dimensional; in particular, there exists* $q_j \in \mathbb{K}\{y_{e,L}\}[x_j]$*, of degree (as a polynomial in* x_j*) greater than or equal to* 1*, such that* $\mathcal{K}_{L,j} = \langle q_j \rangle$*,* $j = 1, \dots n$*.*

Proof. The ideal $\mathcal{I}_L$ is zero dimensional in $\mathcal{R}_b$. In fact, if $\mathcal{I}_L$ was not zero-dimensional, then for each $x^\circ \in \mathbb{R}^n$, the map $x = F_L(y_{e,L})$ would have an infinite number of complex left inverses, but this would be an absurd by Lemma 4.

Since any ideal of $\mathbb{K}\{y_{e,L}\}[x_j]$ is principal (since x_j is a single variable), there are three possible cases: *(i)* $\mathcal{K}_{L,j} = \langle \emptyset \rangle$, *(ii)* $\mathcal{K}_{L,j} = \langle 1 \rangle$, and *(iii)* $\mathcal{K}_{L,j} = \langle q_j \rangle$, for some non-constant polynomial $q_j \in \mathbb{K}\{y_{e,L}\}[x_j]$. Case *(i)* cannot happen, because otherwise $\mathcal{I}_L$ is not zero-dimensional. Case *(ii)* cannot happen, because in that case for almost all $y_{e,L} \in \mathbf{V}(\mathcal{J}_L)$, the equation $y_{e,L} = H_L(x)$ has no (either real or complex) solutions, but this is false because such an equation has at least one real solution for almost all $y_{e,L} \in \mathbf{V}(\mathcal{J}_L)$. Hence, there is a non-constant $q_j \in \mathbb{K}\{y_{e,L}\}[x_j]$ such that $\mathcal{K}_{L,j} = \langle q_j \rangle$. □

By Lemma 5, for each $j \in \{1, \dots, n\}$, there exists a non-constant polynomial $q_j \in \mathbb{K}\{y_{e,L}\}[x_j]$ such that $\mathcal{K}_{L,j} = \langle q_j \rangle$, which shows that x_j is an algebraic function over $\mathbb{K}\{y_{e,L}\}$. If $\operatorname{degree}(q_j) = 1$, then x_j is a linear function over $\mathbb{K}\{y_{e,L}\}$, whence it is defined for almost all values of $y_{e,L} \in \mathbf{V}(\mathcal{J}_L)$.

Example 1. Let $n = 1$, $m = 1$, $f = x^2 - \frac{1}{2}x$ and $h = x^2$. One has $L_f h = 2x^3 - x^2$. Consider the ambient ring $\mathcal{R}_a = \mathbb{R}[x, y^{(0)}, y^{(1)}]$ and the ideal

$$\mathcal{I}_1 = \langle y^{(0)} - h, y^{(1)} - L_f h \rangle = \langle -x^2 + y^{(0)}, -2x^3 + x^2 + y^{(1)} \rangle.$$

One has $\mathcal{J}_1 = \mathcal{I}_1 \cap \mathbb{R}[y^{(0)}, y^{(1)}] = \langle g_1 \rangle$, where:

$$g_1(y^{(0)}, y^{(1)}) = 4(y^{(0)})^3 - (y^{(0)})^2 - 2y^{(0)}y^{(1)} - (y^{(1)})^2,$$

which shows that $y^{(0)}$ and $y^{(1)}$ are algebraically dependent functions of time t, *i.e.*, $g_1(y^{(0)}(t), y^{(1)}(t)) = 0$, $\forall t \in \mathbb{R}$. In order to take into account such a constraint, instead of working over $\mathcal{R}_a$, one has to work over the ambient ring $\mathcal{R}_b = \mathbb{K}\{y^{(0)}, y^{(1)}\}[x]$, where $\mathbb{K}\{y^{(0)}, y^{(1)}\}$ is the field of rational functions having numerator and denominator in the quotient ring $\mathbb{R}[y^{(0)}, y^{(1)}]/\langle g_1(y^{(0)}, y^{(1)})\rangle$. The ideal $\mathcal{I}_1$ can be coerced into $\mathcal{R}_b$; choosing the Lex order in $\mathbb{R}[y^{(0)}, y^{(1)}]$, with $y^{(0)} \succ y^{(1)}$, the reduced Gröbner basis of $\mathcal{I}_1$ is $\mathcal{G}_1 = \langle x - \frac{y^{(0)}+y^{(1)}}{2y^{(0)}}\rangle$. If $y^{(0)} = 0$, by the constraint $g_1(y^{(0)}, y^{(1)}) = 0$, one has $y^{(1)} = 0$; hence, x can be globally represented by:

$$x = \begin{cases} 0, & \text{if } y^{(0)} = 0, \\ \frac{y^{(0)}+y^{(1)}}{2y^{(0)}}, & \text{if } y^{(0)} \neq 0. \end{cases}$$

Another global expression of the left inverse is $x = (\frac{y^{(0)}+y^{(1)}}{2})^{1/3}$, and it is worth pointing out that $(\frac{y^{(0)}+y^{(1)}}{2y^{(0)}})^3 = \frac{y^{(0)}+y^{(1)}}{2}$ along the variety $\mathbf{V}(g_1)$, since

$$\left(\frac{y^{(0)}+y^{(1)}}{2y^{(0)}}\right)^3 - \frac{y^{(0)}+y^{(1)}}{2} = -\frac{1}{8}\frac{y^{(0)}+y^{(1)}}{(y^{(0)})^3} g_1(y^{(0)}, y^{(1)}) = 0.$$

Pair (h, f) is therefore globally observable, although the map $y = h(x)$ alone is not sufficient for the reconstruction of x.

In the following section, it is shown how the algebraic geometry tools reviewed in Section 3 can be employed to study the uniform observability of a switching linear system under different assumptions on the switching signal.

5. A motivating example

Consider system (1) characterized by $n = 2$, $m = 1$, $N = 1$ and by the following randomly generated matrices:

$$A = \begin{bmatrix} \frac{7}{9} & 1 \\ \frac{3}{5} & 9 \end{bmatrix}, \qquad C_0 = \begin{bmatrix} \frac{2}{5} & \frac{1}{9} \end{bmatrix}, \quad C_1 = \begin{bmatrix} \frac{1}{9} & \frac{4}{9} \end{bmatrix};$$

the characteristic polynomial of matrix A is $p_A(\lambda) = \lambda^2 - \frac{88}{9}\lambda + \frac{32}{5}$. Let $f(x) = A\,x$, $h(\sigma_1, x) = (C_0 + C_1\sigma_1)\,x$, and $y(t) = h(\sigma_1(t), x(t))$, and compute

$$\begin{aligned} h(\sigma_1, x) &= \tfrac{5\sigma_1+18}{45}x_1 + \tfrac{4\sigma_1+1}{9}x_2, \\ L_f h(\sigma_1, x) &= \tfrac{143\sigma_1+153}{405}x_1 + \tfrac{185\sigma_1+63}{45}x_2, \\ L_f^2 h(\sigma_1, x) &= \tfrac{49960\sigma_1+20664}{18225}x_1 + \tfrac{15128\sigma_1+5256}{405}x_2, \\ L_f^3 h(\sigma_1, x) &= \tfrac{4025824\sigma_1+1421856}{164025}x_1 + \tfrac{6176800\sigma_1+2149344}{18225}x_2, \end{aligned}$$

and so forth; hence, one has $y^{(j)}(t) = L_f^j h(\sigma_1(t), x(t))$, $j \in \mathbb{Z}_{\geqslant 0}$, where the switching signal $\sigma_1(t)$ takes values in $\{0, 1\}$.

(i) If the switching signal is measured and the set of the values taken by the switching signal $\sigma_1(t)$ is not known before they are actually taken, then it is convenient to take $\mathcal{R}_a := \mathbb{R}(\sigma_1)[x_1, x_2, y^{(0)}, y^{(1)}, \ldots, y^{(L)}]$ as ambient ring, (where $L \in \mathbb{Z}_{\geqslant 0}$ denotes

the maximum number of time-derivatives that are used for the solution of the observability problem), which means that σ_1 is taken as a known parameter and that all the considered polynomials have coefficients that are rational functions of σ_1. Define the following ideals of $\mathcal{R}_a$:

$$\mathcal{I}_j^a = \langle y^{(0)} - h(\sigma_1, x), \ldots, y^{(j)} - L_f^j h(\sigma_1, x)\rangle,$$

for $j \in \mathbb{Z}_{\geqslant 0}$. The monomials of $\mathcal{R}_a$ are ordered according to the Lex order, with $x_1 \succ x_2 \succ y^{(L)} \succ \ldots \succ y^{(0)}$. The reduced Gröbner basis $\mathcal{G}_1^a$ of $\mathcal{I}_1^a$ is $\mathcal{G}_1^a = \{g_0^a, g_1^a\}$, where:

$$\begin{aligned} g_0^a &= x_1 + \tfrac{(8100\sigma_1+2025)y^{(1)}}{5465\sigma_1^2+29030\sigma_1+9441} + \tfrac{(-74925\sigma_1-25515)y^{(0)}}{5465\sigma_1^2+29030\sigma_1+9441}, \\ g_1^a &= x_2 + \tfrac{(-2025\sigma_1-7290)y^{(1)}}{5465\sigma_1^2+29030\sigma_1+9441} + \tfrac{(6435\sigma_1+6885)y^{(0)}}{5465\sigma_1^2+29030\sigma_1+9441}; \end{aligned}$$

the reduced Gröbner basis $\mathcal{G}_2^a$ of $\mathcal{I}_2^a$ is $\mathcal{G}_2^a = \mathcal{G}_1^a \cup \{g_2^a\}$, where:

$$g_2^a = y^{(2)} - \tfrac{88}{9} y^{(1)} + \tfrac{32}{5} y^{(0)};$$

the reduced Gröbner basis $\mathcal{G}_3^a$ of $\mathcal{I}_3^a$ is $\mathcal{G}_3^a = \mathcal{G}_2^a \cup \{g_3^a\}$, where:

$$g_3^a = y^{(3)} - \tfrac{36128}{405} y^{(1)} + \tfrac{2816}{45} y^{(0)}.$$

In general, one has that the reduced Gröbner basis $\mathcal{G}_{j+1}^a$ of $\mathcal{I}_{j+1}^a$ is $\mathcal{G}_{j+1}^a = \mathcal{G}_j^a \cup \{g_{j+1}^a\}$, where $g_{j+1}^a = y^{(j+1)} + a_j y^{(1)} + b_j y^{(0)}$, for some constants $a_j, b_j \in \mathbb{R}$, for $j \in \mathbb{Z}_{\geqslant 1}$. From the polynomials g_0^a, g_1^a, one deduces that the state variables x_1 and x_2 can be expressed as linear functions of $y^{(0)}$ and $y^{(1)}$ (*i.e.*, of the measured output and of its first time-derivative), with coefficients that are rational functions of σ_1:

$$\begin{aligned} x_1 &= -\tfrac{(8100\sigma_1+2025)y^{(1)}}{5465\sigma_1^2+29030\sigma_1+9441} - \tfrac{(-74925\sigma_1-25515)y^{(0)}}{5465\sigma_1^2+29030\sigma_1+9441}, \\ x_2 &= -\tfrac{(-2025\sigma_1-7290)y^{(1)}}{5465\sigma_1^2+29030\sigma_1+9441} - \tfrac{(6435\sigma_1+6885)y^{(0)}}{5465\sigma_1^2+29030\sigma_1+9441}; \end{aligned}$$

since the other polynomials $g_j^a, j \geqslant 2$, are independent of x_1, x_2 and of the switching signal σ_1, one can conclude that no improvements in the above formulas could be obtained by taking into account the time-derivatives of $y(t)$ of order greater than 1. Since the denominator $5465\sigma_1^2 + 29030\sigma_1 + 9441$ does not vanish for $\sigma_1 \in \{0, 1\}$, the above expressions can used to compute x_1, x_2 from the knowledge of $y^{(0)}, y^{(1)}$, but this can be done only if the switching signal $\sigma_1(t)$ is measured.

It is worth pointing out that, by the formal substitutions $y^{(k)} \to \lambda^k$, $k \in \mathbb{Z}_{\geqslant 0}$, in the polynomial embeddings $g_j^a = 0$, $j \in \mathbb{Z}_{\geqslant 2}$, one obtains the characteristic polynomials associated with them: $p_{g_2^a}(\lambda) = \lambda^2 - \frac{88}{9}\lambda + \frac{32}{5}$ (*i.e.*, the characteristic polynomial $p_A(\lambda)$ of A), $p_{g_2^a}(\lambda) = \lambda^3 - \frac{36128\lambda}{405} + \frac{2816}{45} = \left(\lambda + \frac{88}{9}\right)\left(\lambda^2 - \frac{88\lambda}{9} + \frac{32}{5}\right)$ (*i.e.*, a multiple of $p_A(\lambda)$) and so forth ($p_{g_j^a}(\lambda) = \lambda^{j+1} + a_j\lambda + b_j$ being a multiple of $p_A(\lambda)$). This, in particular, implies that the signals $\alpha_2(t) = y^{(2)}(t) - \frac{88}{9}y^{(1)}(t) + \frac{32}{5}y^{(0)}(t)$ and $\alpha_3(t) = y^{(3)}(t) - \frac{36128}{405}y^{(1)}(t) + \frac{2816}{45}y^{(0)}(t)$ and so forth (obtained from the embeddings with the substitutions $y^{(k)} \to y^{(k)}(t)$, $k \in \mathbb{Z}_{\geqslant 0}$) are continuous and equal to zero for all $t \in \mathbb{R}$, including the switching times t_i.

(ii) If the switching signal is measured and its co-domain is known before they are actually taken, one has (letting $\sigma_1 = 0$ in the expressions computed above)

$$x_1 = \tfrac{225}{1049} y^{(1)} + \tfrac{2835}{1049} y^{(0)}, \qquad\qquad x_2 = \tfrac{810}{1049} y^{(1)} - \tfrac{765}{1049} y^{(0)},$$

and (letting $\sigma_1 = 1$ in the expressions computed above)

$$x_1 = -\tfrac{10125}{43936} y^{(1)} + \tfrac{12555}{5492} y^{(0)}, \qquad\qquad x_2 = \tfrac{9315}{43936} y^{(1)} - \tfrac{1665}{5492} y^{(0)};$$

hence, by multiplying the first equations by $1 - \sigma_1$, the second equations by σ_1, and adding the results, one obtains the following polynomial expressions:

$$\begin{aligned} x_1 &= -\left(\tfrac{225}{1049} + \tfrac{735525\sigma_1}{46088864}\right) y^{(1)} - \left(\tfrac{2399625\sigma_1}{5761108} - \tfrac{2835}{1049}\right) y^{(0)}, \\ x_2 &= \left(\tfrac{810}{1049} - \tfrac{25816725\sigma_1}{46088864}\right) y^{(1)} + \left(\tfrac{2454795\sigma_1}{5761108} - \tfrac{765}{1049}\right) y^{(0)}. \end{aligned}$$

From the analysis carried out in the cases *(i)* and *(ii)*, one concludes that in order to express x_1 and x_2 as linear functions of $y^{(0)}, \ldots, y^{(L)}$, for some $L \in \mathbb{Z}_{>0}$, there are actually two possibilities: the first one, which is rational in σ_1 and holds independently of the values taken by σ_1, and the second one, which is polynomial in σ_1, but obtained (valid) only because one has exploited in such a computation the knowledge of the possible values taken by σ_1. Therefore, there is no formula expressing x_1 and x_2 as linear functions of $y^{(0)}, \ldots, y^{(L)}$, being independent of σ_1.

(iii) One can try to answer the following question: 'is it possible to express pair x_1, x_2 as a function of $y^{(0)}, \ldots, y^{(L)}$, possibly not linear, but independent of σ_1, without knowing the set of the values taken by σ_1?'. To answer such a question, it is convenient to take $\mathcal{R}_b := \mathbb{R}[\sigma_1, x_1, x_2, y^{(0)}, y^{(1)}, \ldots, y^{(L)}]$ as ambient ring, which means that σ_1 is treated, instead of as a known parameter, as a single variable, like all the other variables. The ideals $\mathcal{I}_j^a$, $j \in \mathbb{Z}_{\geqslant 1}$, which have been defined in case *(i)* as subsets of $\mathcal{R}_a$, can be coerced into $\mathcal{R}_b$. The monomials of $\mathcal{R}_b$ are ordered according to the Lex order, with $\sigma_1 \succ x_1 \succ x_2 \succ y^{(L)} \succ \ldots \succ y^{(1)} \succ y^{(0)}$. Define the elimination ideals $\mathcal{J}_j^{b,1} = \mathcal{I}_j^a \cap \mathbb{R}[x_1, y^{(0)}, y^{(1)}, \ldots, y^{(L)}]$. The reduced Gröbner basis $\mathcal{G}_1^{b,1}$ of $\mathcal{J}_1^{b,1}$ is $\mathcal{G}_1^{b,1} = \emptyset$; the reduced Gröbner basis $\mathcal{G}_{j+1}^{b,1}$ of $\mathcal{J}_{j+1}^{b,1}$ is $\mathcal{G}_{j+1}^{b,1} = \mathcal{G}_j^{b,1} \cup \{g_{j+1}^{b,1}\}$, where $g_{j+1}^{b,1} = g_{j+1}^a$, with the polynomials g_{j+1}^a being those computed in the case *(i)*. This allows one to conclude that, if the set of the values taken by σ_1 is not known, it is not possible to express x_1, x_2 as functions of $y^{(0)}, \ldots, y^{(L)}$, being independent of σ_1, also allowing non-linearities in such expressions.

(iv) One can try to answer the following question: 'is it possible to express pair x_1, x_2 as a function of $y^{(0)}, \ldots, y^{(L)}$, possibly not linear, being independent of σ_1, but knowing the set of the values taken by σ_1?'. To answer such a question, it is convenient to take $\mathcal{R}_c := \mathbb{R}[\sigma_1, x_1, x_2, y^{(0)}, y^{(1)}, \ldots, y^{(L)}]/\langle \sigma_1(\sigma_1 - 1)\rangle$ as ambient ring, which means that σ_1 is treated, instead of as a known parameter, as a single variable whose domain is not the whole $\mathbb{R}$ but $\{0, 1\}$ (the roots of $\sigma_1(\sigma_1 - 1)$). The ideals $\mathcal{I}_j^a$, $j \in \mathbb{Z}_{\geqslant 1}$, which have been defined as subsets of $\mathcal{R}_a$, can be coerced into $\mathcal{R}_c$. Fix the Lex order with $\sigma_1 \succ x_1 \succ x_2 \succ y^{(L)} \succ \ldots \succ y^{(1)} \succ y^{(0)}$ and define the elimination ideals $\mathcal{J}_j^{c,1} = \mathcal{I}_j^a \cap \mathbb{R}[x_1, y^{(0)}, y^{(1)}, \ldots, y^{(L)}]$ (respectively, $\mathcal{J}_j^{c,2} = \mathcal{I}_j^a \cap \mathbb{R}[x_2, y^{(0)}, y^{(1)}, \ldots, y^{(L)}]$), which are obtained by eliminating the variables σ_1 and x_2 (respectively, σ_1 and x_1) from $\mathcal{I}_j^a$, $j \in \mathbb{Z}_{\geqslant 1}$; these ideals are principal. The reduced Gröbner basis $\mathcal{G}_1^{c,1}$ of $\mathcal{J}_1^{c,1}$

is $\mathcal{G}_1^{c,1} = \{g_1^{c,1}\}$ (respectively, $\mathcal{G}_1^{c,2}$ of $\mathcal{J}_1^{c,2}$ is $\mathcal{G}_1^{c,2} = \{g_1^{c,2}\}$), where:

$$\begin{aligned} g_1^{b,1} &= 46088864x_1^2 + (20506725y^{(1)} - 229920120y^{(0)})x_1 \\ &\quad + 18225(5y^{(1)} - 63y^{(0)})(25y^{(1)} - 248y^{(0)}), \\ g_1^{b,2} &= 46088864x_2^2 + (47583720y^{(0)} - 45359595y^{(1)})x_2 \\ &\quad + 2025(18y^{(1)} - 17y^{(0)})(207y^{(1)} - 296y^{(0)}); \end{aligned}$$

the reduced Gröbner basis $\mathcal{G}_{j+1}^{c,k}$ of $\mathcal{J}_{j+1}^{c,k}$ is $\mathcal{G}_{j+1}^{c,k} = \mathcal{G}_j^{c,k} \cup \{g_{j+1}^{c,k}\}$, where $g_{j+1}^{c,k} = g_{j+1}^a$, for each $k \in \{1,2\}$; also in this case, since the polynomials $g_j^{c,k}$, $j \in \mathbb{Z}_{\geqslant 2}$, are independent of x_1, x_2 and of the switching signal σ_1, no improvements in the above formula could be obtained by taking into account the time-derivatives of $y(t)$ of order greater than 1. The discriminants of $g_1^{b,1}$ and $g_1^{b,2}$, thought of as polynomials of x_1 and x_2, respectively, are $\Delta_1 = (3269y^{(1)} + 85320y^{(0)})^2$ and $\Delta_2 = (573705y^{(1)} - 436408y^{(0)})^2$, respectively, which imply that pair x_1, x_2 can be *locally* expressed as a function of $y^{(0)}, y^{(1)}$, which is independent of the switching signal σ_1; in particular, one obtains:

$$\begin{aligned} x_1 &= \tfrac{100440y^{(0)} - 10125y^{(1)}}{43936}, & x_1 &= \tfrac{2835y^{(0)} - 225y^{(1)}}{1049}, \\ x_2 &= -\tfrac{765y^{(0)} - 810y^{(1)}}{1049}, & x_2 &= -\tfrac{13320y^{(0)} - 9315y^{(1)}}{43936}. \end{aligned}$$

Such solutions coincide with the ones that can be computed directly from pair (C_0, A) (corresponding to $\sigma_1 = 0$),

$$x_1 = \tfrac{2835y^{(0)} - 225y^{(1)}}{1049}, \qquad x_2 = -\tfrac{765y^{(0)} - 810y^{(1)}}{1049},$$

and from pair $(C_0 + C_1, A)$ (corresponding to $\sigma_1 = 1$)

$$x_1 = \tfrac{100440y^{(0)} - 10125y^{(1)}}{43936}, \qquad x_2 = -\tfrac{13320y^{(0)} - 9315y^{(1)}}{43936}.$$

Such solutions are clearly useless if σ_1 is not measured and the set of the values taken by σ_1 is not known *a priori*, although they are *apparently* independent of σ_1.

(v) To consider the observation problem under the assumption that σ_1 is not measured and its co-domain is not known, consider the additional output $z = (D_0 + \sigma_1 D_1)x$, with the following expressions of D_0 and D_1 being randomly generated

$$D_0 = \begin{bmatrix} \frac{1}{2} & \frac{3}{5} \end{bmatrix}, \qquad D_1 = \begin{bmatrix} \frac{9}{2} & \frac{7}{10} \end{bmatrix}.$$

Hence, let $k(\sigma_1, x) = (D_0 + D_1\sigma_1)\, x$, and compute

$$\begin{aligned} k(\sigma_1, x) &= \tfrac{9\sigma_1 + 1}{2} x_1 + \tfrac{7\sigma_1 + 6}{10} x_2, \\ L_f k(\sigma_1, x) &= \tfrac{1764\sigma_1 + 337}{450} x_1 + \tfrac{108\sigma_1 + 59}{10} x_2, \end{aligned}$$

and so forth; one has $z^{(j)}(t) = L_f^j k(\sigma_1(t), x(t))$, $j \in \mathbb{Z}_{\geqslant 0}$. It is convenient to take $\mathcal{R}_d := \mathbb{R}[\sigma_1, x_1, x_2, y^{(0)}, \ldots, y^{(L)}, z^{(0)}, \ldots, z^{(L)}]$ as ambient ring. Fix the Lex order with $\sigma_1 \succ x_1 \succ x_2 \succ y^{(L)} \succ \ldots \succ y^{(1)} \succ y^{(0)} \succ z^{(L)} \succ \ldots \succ z^{(1)} \succ z^{(0)}$ and define

$$\mathcal{I}_1^d = \langle y^{(0)} - h, y^{(1)} - L_f h, z^{(0)} - k, z^{(1)} - L_f k \rangle \subset \mathcal{R}_d.$$

The reduced Gröbner basis $\mathcal{G}_1^d$ of $\mathcal{I}_1^d \cap \mathbb{R}[x_1, x_2, y^{(0)}, \dots, y^{(L)}, z^{(0)}, \dots, z^{(L)}]$ is $\mathcal{G}_1^d = \{g_1^c, g_2^c, g_3^c\}$,

$$\begin{aligned}
g_1^c &= 1005365203x_1 + 485568675y_1 - 2961637020y^{(0)} \\
&\quad - 69131250z_1 + 95248350z_0, \\
g_2^c &= 1005365203x_2 + 1408481055y_1 - 804144060y^{(0)} \\
&\quad - 526932900z_1 + 368357950z_0, \\
g_3^c &= 856575(y^{(1)})^2 - 8375400y^{(1)}y^{(0)} - 23490y^{(1)}z^{(1)} \\
&\quad - 1298250y^{(1)}z^{(0)} + 5482080(y^{(0)})^2 \\
&\quad + 1527930y^{(0)}z^{(1)} - 150336y^{(0)}z^{(0)} - 60300(z^{(1)})^2 \\
&\quad + 589600z^{(1)}z^{(0)} - 385920(z^{(0)})^2.
\end{aligned}$$

Clearly, g_1^c and g_2^c allow one to express x_1 and x_2, respectively, as linear functions of $y^{(0)}, y^{(1)}, z^{(0)}, z^{(1)}$, being independent of σ_1, which therefore can be employed when σ_1 is not measured and the set of the values taken by σ_1 is not known.

Note that $\mathcal{I}_1^d \cap \mathbb{R}[y^{(0)}, \dots, y^{(L)}, z^{(0)}, \dots, z^{(L)}]$ is *principal* and $\mathcal{I}_1^d \cap \mathbb{R}[y^{(0)}, \dots, y^{(L)}, z^{(0)}, \dots, z^{(L)}] = \langle g_3^c \rangle$. Since g_3^c is not linear, no linear embedding (independent of σ_1) of the considered system is in $\mathcal{I}_1^d$. To have linear embeddings independent of σ_1, define the following ideal of $\mathcal{R}_d$:

$$\begin{aligned}
\mathcal{I}_2^d = \langle y^{(0)} - h(\sigma_1, x), y^{(1)} - L_f h(\sigma_1, x), y^{(2)} - L_f h^2(\sigma_1, x),& \\
z^{(0)} - k(\sigma_1, x), z^{(1)} - L_f k(\sigma_1, x), z^{(2)} - L_f^2 k(\sigma_1, x)\rangle&.
\end{aligned}$$

The two eliminations ideals $\mathcal{I}_2^{d,y} \cap \mathbb{R}[y^{(0)}, \dots, y^{(L)}]$ and $\mathcal{I}_2^{d,z} \cap \mathbb{R}[z^{(0)}, \dots, z^{(L)}]$ are *principal* and generated by

$$p_2^y = y^{(2)} - \tfrac{88}{9}y^{(1)} + \tfrac{32}{5}y^{(0)}, \qquad p_2^z = z^{(2)} - \tfrac{88}{9}z^{(1)} + \tfrac{32}{5}z^{(0)}.$$

Note that the coefficients of these polynomials are those of the characteristic polynomial of the matrix A (see Lemma 3).

6. 'Generic' properties of matrices with entries being rational functions of parameters

Let $\beta = [\ \beta_1 \ \ \dots \ \ \beta_\ell\ ]^\top$ be a vector of real parameters and let $M(\beta)$ be a matrix of dimensions $n \times n$, whose entries are rational functions of β (*i.e.*, $M_{i,j}(\beta) \in \mathbb{R}(\beta)$). The following lemma gives necessary and sufficient conditions for the 'generic' invertibility of the matrix $M(\beta)$.

Lemma 6. *The matrix $M(\hat{\beta})$ is defined and non-singular for 'almost all' specializations $\hat{\beta} \in \mathbb{R}^\ell$ if and only if the rational function* $\det(M(\beta))$ *is not identically zero.*

Proof. Since $M_{j,k} \in \mathbb{R}(\beta)$, there exist co-prime $N_{j,k}, D_{j,k} \in \mathbb{R}[\beta]$ such that $M_{j,k}(\beta) = \frac{N_{j,k}(\beta)}{D_{j,k}(\beta)}$. Define the algebraic variety $\mathcal{V} := \mathbb{V}(D_{1,1}, \dots, D_{1,n}, \dots, D_{n,n})$ of $\mathbb{R}^\ell$, which satisfies $\mathcal{V} \neq \mathbb{R}^\ell$ since $\langle D_{1,1}, \dots, D_{n,n}\rangle \neq \langle \emptyset \rangle$. Clearly, $M(\hat{\beta})$ is defined for all specializations $\hat{\beta} \in \mathbb{R}^\ell \setminus \mathcal{V}$, whence it is defined for 'almost all' specializations $\hat{\beta} \in \mathbb{R}^\ell$, because the Zariski closure of $\mathcal{V}$ coincides with $\mathcal{V}$, whence it is not equal to $\mathbb{R}^\ell$. Since the entries of $M(\beta)$ are in $\mathbb{R}(\beta)$, one has that $\det(M(\beta)) \in \mathbb{R}(\beta)$, whence there exist $N, D \in \mathbb{R}[\beta]$ such that $\det(M(\beta)) = \frac{N(\beta)}{D(\beta)}$. Clearly, $\det(M(\hat{\beta}))$ is defined (*i.e.*, $D(\hat{\beta}) \neq 0$) for all specializations $\hat{\beta} \in \mathbb{R}^\ell \setminus \mathcal{V}$. Define the algebraic variety $\mathcal{W} := \mathcal{V} \cup \mathbb{V}(N)$, where $\mathcal{V} \cup \mathbb{V}(N) \neq \mathbb{R}^\ell$ since $\mathcal{V} \neq \mathbb{R}^\ell$ and $\mathbb{V}(N) \neq \mathbb{R}^\ell$ due to the fact that $\det(M(\beta))$ is not identically zero. Now, $\det(M(\hat{\beta})) \neq 0$ (*i.e.*, $M(\hat{\beta})$ is non-singular) for all specializations $\hat{\beta} \in \mathbb{R}^\ell \setminus \mathcal{W}$, whence the matrix $M(\hat{\beta})$ is defined and non-singular for 'almost all' specializations $\hat{\beta} \in \mathbb{R}^\ell$, because the Zariski closure of $\mathcal{W}$ coincides with $\mathcal{W}$, whence it is not equal to $\mathbb{R}^\ell$. □

Lemma 6 essentially states that $M(\beta)$ is 'generically' invertible if and only if its determinant is not identically zero. The following corollary shows that this condition is equivalent to require that there exists $\hat{\beta}^\circ \in \mathbb{R}^\ell$ such that $\det(M(\hat{\beta}^\circ)) \neq 0$.

Corollary 1. *The matrix $M(\hat{\beta})$ is defined and non-singular for 'almost all' specializations $\hat{\beta} \in \mathbb{R}^\ell$ if and only if there exists $\hat{\beta}^\circ \in \mathbb{R}^\ell$ such that $M(\hat{\beta}^\circ)$ is defined and non-singular.*

Proof. By absurd assume that $\det(M(\beta))$ is identically zero. If there exists a specialization $\hat{\beta}^\circ \in \mathbb{R}^h$ such that $M(\hat{\beta}^\circ)$ is defined and non-singular, then $\det(M(\beta))$ is defined at $\beta = \hat{\beta}^\circ$ and $\det(M(\hat{\beta}^\circ)) \neq 0$, which is a contradiction. □

7. 'Generic' uniform algebraic observability of multi-switching systems

By using the results established in Section 6, the conditions under which system (5) is 'generically' uniformly algebraically observable are studied in this section. Let y_j denote the j-th entry of the output vector y, let $C_\ell^{[j]}$ denote the j-th row of the matrix C_ℓ so that, for $j = 1, \ldots, m$,

$$y_j(t) = (C_0^{[j]} + \sigma_1(t) C_1^{[j]} + \ldots + \sigma_N(t) C_N^{[j]})\, x(t),$$

Let $z_j = [\ y_j^{(0)} \ \ \ldots \ \ y_j^{(n-1)}\ ]^\top$, where $y_j^{(k)}$ is the j-th entry of $y^{(k)}$, $j = 1, \ldots, m$, and let $z = [\ z_1^\top \ \ \ldots \ \ z_m^\top\ ]^\top$. Since

$$y_j^{(k)}(t) = (C_0^{[j]} + \sigma_1(t) C_1^{[j]} + \ldots + \sigma_N(t) C_N^{[j]})\, A^k\, x(t), \tag{9}$$

for all $t \in \mathbb{R}_{\geqslant 0}$ and $k \in \mathbb{Z}_{\geqslant 0}$, one has that

$$z_j(t) = \begin{bmatrix} \mathbb{O}_{j,0} & \mathbb{O}_{j,1} & \ldots & \mathbb{O}_{j,N} \end{bmatrix} \begin{bmatrix} x(t) \\ \sigma_1(t) x(t) \\ \vdots \\ \sigma_N(t) x(t) \end{bmatrix}, \tag{10}$$

where $\mathbb{O}_{j,\ell} = \mathbb{O}(C_\ell^{[j]}, A)$ is the observability matrix of $(C_\ell^{[j]}, A)$, $j = 1, \ldots, m$, $\ell = 0, \ldots, N$. The following theorem provides conditions on the number of the entries of the output vector of system (5) to guarantee its uniform algebraic observability for 'almost all' $A \in \mathbb{R}^{n\times n}$ and $C_0, \ldots, C_N \in \mathbb{R}^{m\times n}$.

Theorem 1. *The following two statements hold independently of the dimension n of the state vector and of the sets of values taken by the switching signals:*

(1.1) if $m \geqslant N+1$, system (5) is uniformly algebraically observable for 'almost all' $A \in \mathbb{R}^{n\times n}$, $C_0, \ldots, C_N \in \mathbb{R}^{m\times n}$;

(1.2) if $m \leqslant N$, system (5) is not uniformly algebraically observable for 'almost all' $A \in \mathbb{R}^{n\times n}$, $C_0, \ldots, C_N \in \mathbb{R}^{m\times n}$.

Proof. Under the assumption that $m \geqslant N+1$, let

$$\Phi := \begin{bmatrix} \mathbb{O}_{1,0} & \mathbb{O}_{1,1} & \cdots & \mathbb{O}_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{O}_{N+1,0} & \mathbb{O}_{m,1} & \cdots & \mathbb{O}_{N+1,N} \end{bmatrix}.$$

First, it is proved that Φ is invertible for 'almost all' $A \in \mathbb{R}^{n\times n}$ and $C_0, \ldots, C_N \in \mathbb{R}^{m\times n}$. Consider the specialization

$$\hat{A}^\circ = \begin{bmatrix} 0_{n-1,1} & I_{n-1} \\ 0_{1,1} & 0_{1,n-1} \end{bmatrix},$$
$$(\hat{C}_\ell^{[j]})^\circ = \begin{cases} [\ 1 \quad 0_{1,n-1}\ ], & \text{if } j = \ell+1, \\ 0_{n,1}, & \text{otherwise.} \end{cases}$$

Hence, since $\mathbb{O}((\hat{C}_\ell^{[j]})^\circ, \hat{A}^\circ) = I_n$, if $j = \ell - 1$, and $\mathbb{O}((\hat{C}_\ell^{[j]})^\circ, \hat{A}^\circ) = 0_{n,n}$, otherwise, one obtains that the specialization Φ° coincides with $I_{n(N+1)}$, whence it is non-singular. Therefore, by Corollary 1, the matrix Φ is invertible for 'almost all' $A \in \mathbb{R}^{n\times n}$ and $C_0, \ldots, C_N \in \mathbb{R}^{m\times n}$. Thus,

$$\begin{bmatrix} x(t) \\ \sigma_1(t)x(t) \\ \vdots \\ \sigma_N(t)x(t) \end{bmatrix} = \Phi^{-1} \begin{bmatrix} z_1 \\ \vdots \\ z_{N+1} \end{bmatrix}, \tag{11}$$

whence, letting $\tilde{F}$ be the first row block of Φ^{-1}, one has that $x = \tilde{F} \begin{bmatrix} z_1^\top & \ldots & z_{N+1}^\top \end{bmatrix}^\top$. Thus, if $m \geqslant N+1$, then system (5) is uniformly algebraically observable, for 'almost all' $A \in \mathbb{R}^{n\times n}$ and $C_0, \ldots, C_N \in \mathbb{R}^{m\times n}$.

On the other hand, assume, by absurd, that $m \leqslant N$ and that system (5) is uniformly algebraically observable, *i.e.*, there exist $K \in \mathbb{Z}_{\geqslant 0}$ and $F \in \mathbb{R}^{n\times m(K+1)}$ such that (7) holds for all $t \in \mathbb{R}_{\geqslant 0}$, for 'almost all' $A \in \mathbb{R}^{n\times n}$ and $C_0, \ldots, C_N \in \mathbb{R}^{m\times n}$. First, notice that, since (7) holds for all the switching patterns of $\sigma_1, \ldots, \sigma_N$, it must hold also for $\sigma_1 = \ldots = \sigma_N = 0$. Thus, since pair (C_0, A) is 'generically' observable by Corollary 1 (see also Kailath, 1980), one has that $K \geqslant n-1$. Furthermore, by (9) and by the Cayley-Hamilton Theorem (Meyer, 2000), one has that $y^{(k)}(t)$ can be expressed as a linear combination of $y^{(0)}, \ldots, y^{(n-1)}$ for all $k \geqslant n$. Therefore, if there exist $K \in \mathbb{Z}_{\geqslant 0}$ and a matrix $F \in \mathbb{R}^{n\times m(K+1)}$ such that (7) holds for all $t \in \mathbb{R}_{\geqslant 0}$, then there exists a matrix $\tilde{F} \in \mathbb{R}^{n\times mn}$ such that, for all $t \in \mathbb{R}_{\geqslant 0}$, one has $x(t) = \tilde{F}z(t)$. Hence, by taking (10) into account, one has that

$$x = \tilde{F} \begin{bmatrix} \mathbb{O}_{1,0} + \sigma_1 \mathbb{O}_{1,1} + \ldots + \sigma_N \mathbb{O}_{1,N} \\ \vdots \\ \mathbb{O}_{m,0} + \sigma_1 \mathbb{O}_{m,1} + \ldots + \sigma_N \mathbb{O}_{m,N} \end{bmatrix} x,$$

for all $x \in \mathbb{R}^n$, $(\sigma_1, \ldots, \sigma_N) \in \{0, \ldots, S_1\} \times \cdots \times \{0, \ldots, S_N\}$. Hence, one has

$$I_n = \tilde{F} \begin{bmatrix} \mathbb{O}_{1,0} \\ \vdots \\ \mathbb{O}_{m,0} \end{bmatrix}, \tag{12a}$$

$$I_n = \tilde{F} \begin{bmatrix} \mathbb{O}_{1,0} + \mathbb{O}_{1,j} \\ \vdots \\ \mathbb{O}_{m,0} + \mathbb{O}_{m,j} \end{bmatrix}, \qquad j = 1, \ldots, N. \tag{12b}$$

Since $m \leqslant N$, the relations given in (12) imply that $\tilde{F}$ is such that (12a) holds and

$$\tilde{F} \begin{bmatrix} \mathbb{O}_{1,1} & \ldots & \mathbb{O}_{1,m} \\ \vdots & \ddots & \vdots \\ \mathbb{O}_{m,1} & \ldots & \mathbb{O}_{m,m} \end{bmatrix} = 0_{n \times n}, \tag{13}$$

which is a contradiction. Therefore, if $m \leqslant N$, then system (5) is not uniformly algebraically observable for 'almost all' $A \in \mathbb{R}^{n \times n}$ and $C_0, \ldots, C_N \in \mathbb{R}^{m \times n}$. $\square$

By Lemma 3, there exists a matrix Q, given by

$$Q = [\ \alpha_n I_m \quad \alpha_{n-1} I_m \quad \cdots \quad \alpha_1 I_m\ ], \tag{14}$$

where $p_A(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n$ is the characteristic polynomial of matrix A, such that

$$y^{(n)} = Q\, y_{e,n-1}, \tag{15}$$

where $y_{e,n-1} = [\ (y^{(0)})^\top \quad \ldots \quad (y^{(n-1)})^\top\ ]^\top$.

In addition, by the proof of Theorem 1, if $m \geqslant N+1$, then, for 'almost all' $A \in \mathbb{R}^{n \times n}$ and $C_0, \ldots, C_N \in \mathbb{R}^{m \times n}$, there exists $F \in \mathbb{R}^{n \times m\, n}$ such that

$$x = F\, y_{e,n-1}. \tag{16}$$

The actual computation of F can be done by geometric arguments. Assume $m \geqslant N+1$, and take $\mathcal{R} = \mathbb{R}[\sigma_1, \ldots, \sigma_N, x, y^{(0)}, \ldots, y^{(n-1)}]$ as ambient ring and fix the Lex order with $\sigma_1 \succ \ldots \succ \sigma_N \succ x_1 \succ \ldots \succ x_n \succ y_1^{(n-1)} \succ \ldots \succ y_1^{(0)} \succ \ldots \succ y_m^{(n-1)} \succ \ldots \succ y_m^{(0)}$.

Define the following ideal $\mathcal{I}_{n-1}$ of $\mathcal{R}$:

$$\mathcal{I}_{n-1} := \langle y_1^{(0)} - (C_0^{[1]} + \sigma_1 C_1^{[1]} + \ldots + \sigma_N C_N^{[1]})\, x, \ldots,$$
$$y_m^{(n-1)} - (C_0^{[m]} + \sigma_1 C_1^{[m]} + \ldots + \sigma_N C_N^{[m]})\, A^{n-1}\, x \rangle.$$

Compute the reduced Gröbner basis $\mathcal{G}_{n-1}$ of the elimination ideal $\mathcal{J}_{n-1} = \mathcal{I}_{n-1} \cap \mathbb{R}[x, y^{(0)}, \dots, y^{(n-1)}]$.

Proposition 1. *Let $m \geqslant N+1$ and let the ideal $\mathcal{J}_{n-1}$ and its reduced Gröbner basis $\mathcal{G}_{n-1}$ be defined as above. For 'almost all' $A \in \mathbb{R}^{n\times n}$ and $C_0, \dots, C_N \in \mathbb{R}^{m\times n}$, the j-th entries of $x - F y_e$ belongs to $\mathcal{G}_{n-1}$ and, in particular, it is the only element of $\mathcal{G}_{n-1}$ depending on x_j, $j \in \{1, \dots, n\}$.*

Proof. The j-th entry of $x - F y_e$ is in $\mathcal{J}_{n-1}$ by construction; in particular, it is the only element of $\mathcal{G}_{n-1}$ depending on x_j due to the elimination properties of the Lex order. □

It is worth noticing that, although the results given in Proposition 1 have been stated in the 'generic' case, they can be actually used to check for uniform algebraic observability of system (5) when $A, C_0, \dots, C_n$ are given matrices. In particular, letting the matrices $A, C_0, \dots, C_n$ be given, system (5) is uniformly algebraically observable if and only if there exist polynomials $q_1, \dots, q_n$ in $\mathcal{G}_{n-1}$ such that $\mathrm{LT}(q_j) = x_j$, $j = 1, \dots, n$. Furthermore, if such polynomials exist, they can be directly used to compute a matrix F such that (7) holds.

7.1. Observer design for multi-switching systems

In order to use (16) to estimate the current state of system (5), the signals $y^{(j)}(t)$ have to be estimated from $y(t)$. Several tools are available in the literature to compute these estimates, such as high-gain observers (Tornambe, 1992), sliding mode differentiators (Shtessel, Edwards, Fridman, & Levant, 2014), and super-twisting algorithms (Moreno & Osorio, 2012).

An alternative approach is to exploit the matrices Q and F to design a state observer for system (5). In particular, letting

$$Q_e := \left[\begin{array}{cc} O_{m(n-1),m} & I_{m(n-1)} \\ \hline \multicolumn{2}{c}{Q} \end{array} \right], \tag{17}$$

the vector embedding $y^{(n)} = Q\, y_{e,n-1}$ admits the following state space representation (with $t \in \mathbb{R}$):

$$\tfrac{\mathrm{d}\,\xi(t)}{\mathrm{d}t} = Q_e\, \xi(t), \qquad y(t) = C_e\, \xi(t), \qquad x(t) = F\, \xi(t), \tag{18}$$

where $C_e = [\; I_m \quad 0_{m\times m} \quad \dots \quad 0_{m\times m} \;]$. Therefore, letting K be a matrix such that $Q_e - K\, C_e$ has all eigenvalues with negative real part (such a matrix exists since pair (C_e, Q_e) is observable), an observer for system (5) that does not require the knowledge of $\sigma_1, \dots, \sigma_N$ can be obtained by the design of a Luenberger observer for (18) and is given by (with $t \in \mathbb{R}$):

$$\tfrac{\mathrm{d}\,\hat{\xi}(t)}{\mathrm{d}t} = Q_e\, \hat{\xi}(t) + K(y(t) - C_e\, \hat{\xi}(t)), \qquad \hat{x}(t) = F\, \hat{\xi}(t), \tag{19}$$

where $\hat{\xi}(t)$ is the estimate of $\xi(t)$ at time t. As remarked above, at each switching time, the observer (19) exhibits a transient behavior, which, however, can be made arbitrarily short by letting the real part of the eigenvalues of $Q_e - K\, C_e$ (which can be assigned arbitrarily) be sufficiently negative.

8. A physically motivated example

Consider the mechanical system depicted in Figure 1, which consists of three bodies having (normalized) mass $\mathcal{M}_1 = 1\,\mathrm{Kg}$, $\mathcal{M}_2 = 1.5\,\mathrm{Kg}$, $\mathcal{M}_3 = 0.5\,\mathrm{Kg}$ respectively, and three springs having (normalized) stiffness $\mathcal{K}_1 = 0.7\,\mathrm{N/m}$, $\mathcal{K}_2 = 0.9\,\mathrm{N/m}$, $\mathcal{K}_3 = 0.8\,\mathrm{N/m}$, respectively.

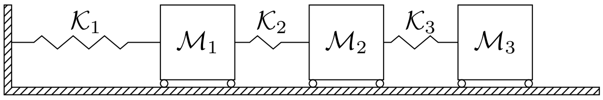

Figure 1. A simple mechanical system.

The dynamics of this plant are given by system (5a), with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{8}{5} & 0 & \frac{9}{10} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{3}{5} & 0 & -\frac{17}{15} & 0 & \frac{8}{15} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{8}{5} & 0 & -\frac{8}{5} & 0 \end{bmatrix}. \tag{20}$$

Let the state of system (20) be measured by a sensor network with $N = 2$ that provides the measurement vector

$$y = \begin{bmatrix} x_1 - \sigma_1 x_3 \\ -x_3 + \sigma_1 x_3 \\ -x_5 + \sigma_2 x_5 \end{bmatrix},$$

where the switching signals σ_1, σ_2 and their co-domain are not known. Note that, since $m = 3$ and $N = 2$, the assumptions of Theorem 1 are met. By using the methods given in Proposition 1, one obtains a matrix F (whose explicit expression has been omitted for space constraints) such that (16) holds. Thus, an observer for system (20) based on the measurement vector y and independent of σ_1, σ_2 is given by (19), where the matrix Q is given by (14) and (17).

A numerical simulation has been carried out to test such an observer, assuming $\hat{y}_e(0) = 0$, $x(0) = [\ 0.1\,\mathrm{m} \quad 0\,\mathrm{m/s} \quad -0.3\,\mathrm{m} \quad 0\,\mathrm{m/s} \quad 0.2\,\mathrm{m} \quad 0\,\mathrm{m/s}\ ]^\top$, letting the switching signal be the one reported in the second plot of Figure 2, and designing K so that the real part of all the eigenvalues of

$$Q_e - K[\ I_3 \quad 0_3 \quad \cdots \quad 0_3\ ]$$

is smaller than -30. Figure 2 depicts the results of such a simulation.

As shown by such a figure, after an initial transient behavior, which occurs at each time instant for which one of the switching signals is not continuous, the observer (19) is able to reconstruct the current state of system (20), without requiring the knowledge of the switching signals σ_1 and σ_2.

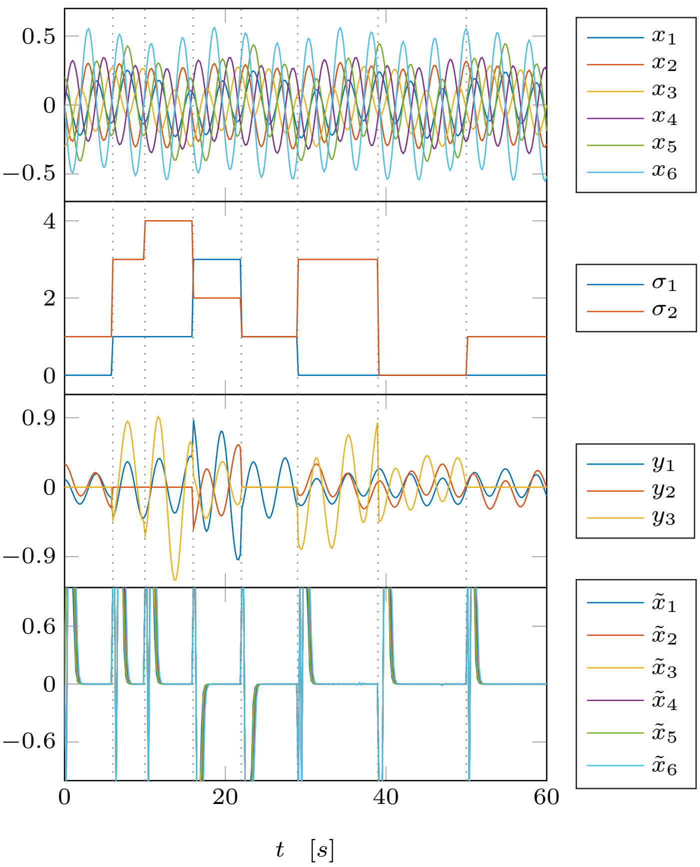

Figure 2. Results of the numerical simulation.

9. Conclusions

In this paper, it has been shown that multi-switching systems are 'generically' uniformly algebraically observable if the number of available measurements is greater than the number of switching signals. Since these systems are able to exactly replicate the output response of those switching plant that are usually employed to model multi-sensor networks (see Lemmas 1 and 2), the results given in this paper allow the construction of reliable and simple observers that do not require any knowledge about the switching signals (*i.e.*, about the failures occurring in the sensor-network).

The results given in this paper can be extended to the case of linear systems having time-varying and switching dynamical matrices. Namely, consider the system

$$\frac{\mathrm{d}\,x(t)}{\mathrm{d}t} = \left(A_0(t) + \sigma_1(t)A_1(t) + \ldots + \sigma_N(t)A_N(t)\right)x(t), \tag{21a}$$

$$y(t) = \left(C_0(t) + \sigma_1(t)C_1(t) + \ldots + \sigma_N(t)C_N(t)\right)x(t), \tag{21b}$$

where $A_0, \ldots, A_N \in \mathbb{R}^{n\times n}[t]$ and $C_0, \ldots, C_N \in \mathbb{R}^{m\times n}[t]$ are time-varying polynomial matrices. Thus, define

$$
\begin{aligned}
h_0(\sigma_1,\ldots,\sigma_N,t,x) &= (C_0(t)+\sigma_1 C_1(t)+\ldots+\sigma_N C_N(t))\,x,\\
h_i(\sigma_1,\ldots,\sigma_N,t,x) &= \frac{\partial h_{i-1}(\sigma_1,\ldots,\sigma_N,t,x)}{\partial t}\\
&\quad + \frac{\partial h_{i-1}(\sigma_1,\ldots,\sigma_N,t,x)}{\partial x}(A_0(t)+\sigma_1 A_1(t)+\ldots+\sigma_N A_N(t))x,
\end{aligned}
$$

for $i = 1,\ldots,L$. Hence, by using the tools detailed in Section 4, define the ideal

$$\mathcal{I}_L := \langle y_0 - h_0(\sigma_1,\ldots,\sigma_N,t,x),\ldots,y_L - h_L(\sigma_1,\ldots,\sigma_N,t,x)\rangle,$$

and let $\mathcal{J}_L = \mathcal{I}_L \cap \mathbb{R}[y_0,\ldots,y_L]$. Hence, coercing $\mathcal{J}_L$ into $\mathbb{R}[y_0,\ldots,y_L]$, define the quotient ring $\mathbb{R}[y_0,\ldots,y_L]/\mathcal{J}_L$, and let $\mathbb{K}\{y_{e,N}\}$ be the field of the rational functions in $\mathbb{R}[y_0,\ldots,y_L]/\mathcal{J}_L$. Thus, fix $\mathcal{R}_b = \mathbb{K}\{y_{e,N}\}[x,t,\sigma_1,\ldots,\sigma_N]$ as ambient ring, coerce $\mathcal{I}_L$ into $\mathcal{R}_b$ and consider the ideals

$$\mathcal{K}_{L,j} = \mathcal{I}_L \cap \mathbb{K}\{y_{e,N}\}[x_j], \quad j = 1,\ldots,n.$$

By the reasoning given in Section 4, these ideals are principal, *i.e.*, there exist non-constant polynomials $q_j \in \mathbb{K}\{y_{e,N}\}[x_j]$ such that $\mathcal{K}_{L,j} = \langle q_j\rangle$, $j = 1,\ldots,n$. If $\text{degree}(q_j) = 1$, then such functions can be readily used to estimate the state of system (21); see Menini et al. (2019) for further details. However, in this case, it is not easy to find a lower bound on L that guarantees 'generic' existence of linear q_j's.

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