Dynamical Social Networks

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Abstract

The classical field of Social Network Analysis (SNA) considers societies and social groups as *networks*, assembled of social actors (individuals, groups or organizations) and relationships among them, or social ties. From the systems and control perspective, a social network may be considered as a complex dynamical system where an actor's attitudes, beliefs and behaviors related to them evolve under influence of the other actors. As a result of these local interactions, complex dynamical behaviors arise that depend on both individual characteristics of actors and the structural properties of a network. This entry provides basic concepts and theoretical tools elaborated to study

dynamical social networks. We focus on (a) structural properties of networks and (b) dynamical processes over them, e.g. dynamics of opinion formation.

Keywords

Social network, dynamical network, centrality measure, opinion formation.

Introduction

Pioneer works of Moreno and Jennings [36] have opened up a new interdisciplinary field of study, which is nowadays called Social Network Analysis (SNA) [20]. SNA has anticipated and inspired, to a great extent, the general theory of complex networks [37] developed in the recent decades. Many important structural properties of networks such as e.g. centrality measures, cliques and communities have arisen as characteristics of social groups and have been studied from a sociological perspective [20]. On a parallel line of research, *dynamics* over social networks have been studied that are usually considered as processes of information diffusion over networks [2; 15] and evolution of individual opinions, attitudes, beliefs and actions related to them under social influence [22; 39; 40].

The theory of dynamical social networks (DSN) combines the two aforementioned lines of research and considers a social network as a dynamical system, aiming at understanding the interplay between the network's structural properties and behaviors of dynamical processes over the network. The ultimate and long-standing goal is to develop the theory of *temporal* social networks, where both vertices (standing for social actors) and edges (standing for social relations) can emerge and disappear.

In this entry, we provide a brief overview of the main concepts and tools, related to the theory of DSN, focusing on its system- and control-theoretic aspects.

Mathematical Representation of a Social Network

Mathematically, a social network is naturally represented by a directed *graph*, that is, a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ constituted by a finite set of *nodes* (or *vertices*) $\mathcal{V}$ and a set of (directed) *arcs* (or *edges*) $\mathcal{E}$ (Fig. 1). The nodes represent social actors (individuals or organizations), and the arcs stand for relations (influence, dependence, similarity etc.).

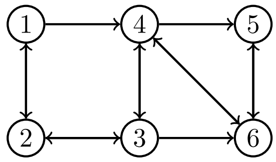

Fig. 1. Example of a directed graph with six nodes: $\mathcal{V} = \{1, \ldots, 6\}$ and $\mathcal{E} = \{(1,2),(1,4),(2,1),(2,3),(3,2),(3,4),(3,6),(4,3),(4,5),(4,6),(5,6),(6,4),(6,5)\}$.

The structure of arcs can be encoded by the adjacency matrix $A = (a_{ij})$, whose entry is $a_{uv} = 1$ if $(u, v) \in \mathcal{E}$ and $a_{uv} = 0$ otherwise. More generally, it is often convenient to assign heterogeneous *weights* to the arcs that can quantify the relations' strengths [31]. In this case, a *weighted* adjacency matrix $W = (w_{ij})$ is introduced, where $w_{uv} \neq 0$ if and only if $(u, v) \in \mathcal{E}$. A triple $(\mathcal{V}, \mathcal{E}, W)$ is referred to as a *weighted* (or valued) graph. In some problems (e.g. structural balance theory [8; 17; 34]), it is convenient to distinguish positive and negative relationships (trust/distrust, friendship/enmity, co-operation/competition etc.) assigning thus positive and negative weights to the arcs; such a graph is said to be *signed*.

Structural properties of social networks

At the dawn of SNA, it was realized that a social group is more than a sum of individual actors and cannot be examined apart from the structure of social relationships, or *ties*.

Numerous characteristics have been proposed in the literature to quantify the structure of social ties and social actors' positions within a network. Many of these characteristics prove to be useful in analysis of general large-scale networks [37].

Density characteristics

Many large-scale social networks[1] are *sparse* in the sense that the number of the (directed) arcs is much smaller than the maximal possible number n^2, where $n = |\mathcal{V}|$ is the number of nodes (also referred to as the network size). A natural coefficient to quantify the network's sparsity property is the *network density* $\rho = |\mathcal{E}|/n^2$.

Another way to quantify the sparsity of the network is to consider the *degree* of a node. If, from sociological perspective, an agent is influenced from few friends, then the in-degree, i.e. the number of edges incoming to a specific node, is low compared to the size of the network. As a consequence, the corresponding row in the adjacency matrix is sparse, i.e. with few non-zero elements. Many real world networks are "scale-free" networks [37] with a degree distribution of the form $p(k) \propto k^{-\gamma}$ with $\gamma \in (2, 3)$ (here $p(k)$ is the proportion of nodes having in-degree $k = 0, 1, 2, \ldots$); it is remarkable that similar distributions were discovered in the early works on sociometry [36].

Many social networks are featured by the presence of few clusters or communities [2; 37]. Social actors within a community are densely connected, whereas relationships between different communities are sparse. These type of networks are described by an influence matrix that can be decomposed as a sum of a low-rank matrix and a sparse matrix. The sparseness, scale-freeness and clustering are common properties that are regularly exploited in data storage [46], reconstruction of influence networks

[1] See e.g. SNAP dataset collection [30] and The Colorado Index of Complex Networks at `https://icon.colorado.edu/\#!/networks`. These social networks can be easily visualized using the software GraphViz [16], or Gephi `https://gephi.org/users/download/`. Another example [5] is the structure of Twitter networks of marketing organizations with a very sparse adjacency matrix.

[41], and identification of communities in large-scale social networks [4]. Sparsity is also related to structural controllability [33]: in some sense, sparse networks are the most difficult to control, whereas dense networks can be controlled via a few driver nodes.

Centrality measures, resilience and structural controllability

The problem of identifying the most "important" (influential) nodes in the social network dates back to first works on sociometry in 1930s. At a "local" level, the influence of a node can be measured by its degree, that is, the number of social actors influenced by the corresponding individuals. However, a person that is influential within his/her small community need not be a global opinion leader. To distinguish globally influential nodes, various *centrality measures* have been introduced [20; 23; 37], among which the most important are *closeness*, *betweenness*, and *eigenvector centralities.*

In a connected network, the closeness centrality relates to how long it will take to spread information from a node u to all other nodes in the network. Mathematically,

$$c_u = \frac{1}{\sum_{v\in\mathcal{V}\setminus\{v\}} d_{uv}}, \quad \forall u \in \mathcal{V}.$$

where d_{uv} is the length of the shortest path between u and v.

Betweenness of node u [19] is defined as

$$b_u = \sum_{j,k\in\mathcal{V}, j\neq k\neq u} \frac{|S_u(j,k)|}{|S(j,k)|}$$

where $S(j,k)$ denotes the set of shortest paths from j to k, and $S_u(j,k)$ the set of shortest paths from j to k that contain the node v. Hence, the more shortest paths pass through node u, the higher is its betweenness. Similarly, the *edge betweenness* can be introduced [37] that measures the number of shortest paths containing an edge. Nodes and edges of high betweenness serve as "bridges" between other nodes.

Eigenvector centrality measures the influence of an individual in a social network by means of the leading eigenvector $\pi^\star$ of a suitable weighted adjacency matrix M, i.e.

the scores of the nodes π_u^* are found from the equations

$$\pi_u^\star = \frac{1}{\lambda}\sum_{v\in\mathcal{V}} M_{uv}\pi_v^\star \tag{1}$$

where λ is the leading eigenvalue of M. This idea leads to Bonacich, Katz and Freeman centrality measures [23] and the PageRank [6]. The main principle of eigenvector centrality is to assign high scores to nodes whose neighbors are highly ranked.

In particular, PageRank is computed as in (1) defining

$$M = (1-m)A + \frac{m}{n}11^\top,$$

where m is a scalar parameter, usually set to 0.15, n is the graph's size and A is the adjacency matrix of a graph renormalized to be row-stochastic (that is, $a_{ij} = 1/d_i$ if i and j are connected, where d_i stands for the degree of node i).

Considering the simple graph in Fig. 1, it should be noticed that different centrality measures may produce very different ranking (see Tables 1,2).

Node	Out-Degree	Closeness	Betweenness	PageRank
1	2	15	0.833	0.061
2	2	11	0.5	0.085
3	3	9	2.166	0.122
4	3	7	3.666	0.214
5	1	9	0	0.214
6	2	7	0.5	0.302

Table 1. Centrality measures of nodes in the graph of Figure 1.

Identification of the most influential nodes may be also considered as an analysis of network's *resilience* properties, that is, its ability to counteract various malicious attacks. This problem is especially important in online social media where attacks on the "most central" nodes allows to disseminate fake news, rumours and other sorts of

Centrality measure	Ranking
Degree	(3,4,1,2,6,5)
Closeness	(1,2,3,5,4,6)
Betweenness	(4,3,1,2,6,5)
PageRank	(6,4,5,3,2,1)

Table 2. Ranking of nodes in the graph of Figure 1 according to different centrality measures.

disinformation. At the same time, users of such media can manipulate their ranks (e.g. by adding fictitious profiles and/or connections), which leads to another important problem: how does the PageRank change under perturbations of a network, e.g. how large values of the rank can one obtain by activating some "hidden" links [12]. As has been shown in [11], families of graphs with "fast-mixing" properties, such as expander graphs, are more resilient to localized perturbations than "slow-mixing" ones that exhibit information diffusion bottlenecks. The mixing-time properties of graphs are related to robustness of stochastic matrices against perturbations [11].

The aforementioned resilience problems are closely related to a problem of *controlling* centrality measures in *weighted graphs*, addressed in [38]. It has been shown that a relatively small "control set" of nodes can produce an *arbitrary* centrality vector by assigning influence weights to their neighbors. To compute the minimal controlling set is a difficult problem, reducing to the NP-hard problem of identifying the minimal *dominating set* in the graph, that is, the set of nodes such that any node is either dominating or adjacent to at least one dominating node (Fig. 2).

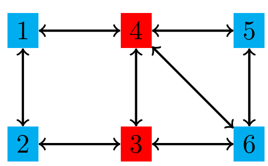

Fig. 2. The minimal dominating set $\mathcal{S} = \{3, 4\}$ of two nodes

Structural balance in signed networks

Structural balance is an important property of a social network with signed weights on the arcs, standing for positive and negative relations among individuals (trust/distrust, cooperation/competition, friendship/enmity etc.). The concept of structural balance originates from the works of Heider [28], further developed by Cartwright and Harary [8]. As Heider argued [28], "an attitude towards an event can alter the attitude towards the person who caused the event, and, if the attitudes towards a person and an event are similar, the event is easily ascribed to the person". The same applies to attitudes towards people and other objects. Heider's theory predicts the emergence of a *balanced* state where positive and negative relations among individuals are in harmony with their attitudes to people, things, ideas, situations etc. If actor A likes actor B in the balanced state, then A shares with B a positive or negative appraisal of any other individual C; vice versa, if A dislikes B, then A disagrees with any B's appraisal of C.

In other words, if a social network corresponds to a complete signed graphs where each actor has a positive or negative appraisal of all other actors, then it tends to a balanced state satisfying four simple axioms (Fig. 3):

1. a friend of my friend is my friend;
2. a friend of my enemy is my enemy;
3. an enemy of my friend is my enemy;
4. an enemy of my enemy is my friend.

The violation of these rules leads to tensions and cognitive dissonances that have to be somehow resolved; the description of such tension resolving mechanisms is beyond Heider's theory and their mathematical modeling is a topic of ongoing research.

If the signed complete graph is balanced, then its node can be divided into two opposing factions, where members of the same factions are "friends" and every

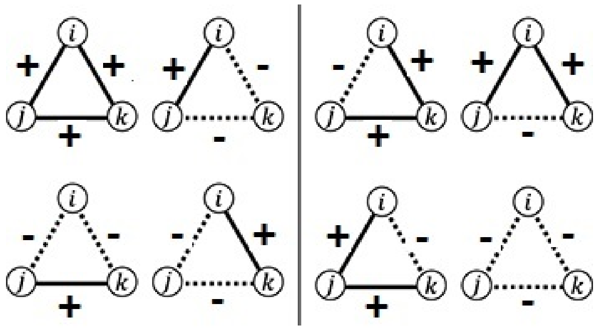

Fig. 3. Balanced (left) vs. imbalanced (right) triads

two actors from different factions are "enemies" (if all relations are positive, then one of the factions is empty). This property can be used as a definition of the structural balance in a more general situation where the graph is not necessarily complete [8]; a strongly connected graph is decomposed into two opposing factions if and only if the product of signs along any path connecting two nodes i and j is the same and depends only on the pair (i, j) (equivalently, the product of all signs over a semicycle is positive). Experimental studies reveal that many large-scale networks are close to the balanced state [17].

Dynamical processes over social networks

Spread of information (including fake news and rumours), evolution of attitudes, beliefs and actions related to them, strengthening and weakening of social ties are examples of dynamic processes unfolding over social networks. In this section, we give a brief overview of the most "mature" results on social dynamics obtained in the systems and control literature, and the related mathematical concepts.

Opinions and models of opinion formation

As discussed in [22], an individual's opinion is his/her cognitive orientation towards some object, e.g. particular issue, event, action or another person. Opinions can stand for displayed attitudes or subjective certainties of belief. Mathematically, opinions are usually modeled as either elements of a discrete set (e.g. the decision to support some action to withstand it, the name of a presidential election candidate to vote for) or scalar and vector quantities, belonging to a subset of the Euclidean space.

One of the simplest model with discrete opinions is known as the *voter model* [10]. Each node of a network is associated to an actor ("voter") whose opinion is a binary value (0 or 1). At each step, a random voter is selected who replaces his/her opinion by the opinion of a randomly chosen neighbor in the graph. Other examples include, but are not limited to, Granovetter's threshold model [25] (an individual supports some action if some proportion of his/her neighbors support it), Schelling model or spatial segregation [42] (the graph stands for a network of geographic locations, and an opinion is the preferred node to live), Axelrod's model of culture dissemination [3] (an opinion stands for a set of cultural traits) and the Ising model of phase transition adapted to social behaviors [44]. The aforementioned models are usually examined by tools of advanced probability theory and statistical physics [9].

Control-theoretic studies on opinion dynamics have primarily focused on models with real-valued ("continuous") opinions, which can attain a continuum of values. We consider only microscopic or agent-based models of opinion formation, portraying evolution of individual opinions and described by systems of differential or recurrent equations, whose dimension is proportional to the size of the network (number of social actors). As the number of actors tends to infinity, the dynamics transforms into a macroscopic (statistical, Eulerian, fluid-based, mean-field) model that portrays the evolution of opinion *distribution* (a density function or a probability measure) and is usually described by a partial differential or integral equation [7].

Models of rational consensus. Social power

A simple model of opinion formation was proposed by French [21], examined by Harari [26] and generalized by DeGroot [14] as an iterative procedure to reach a consensus. Each social actor in a network keeps a scalar or vector opinion $x_i \in \mathbb{R}^d$ (e.g. a vector of subjective probabilities [14; 47]). At each stage of the opinion evolution, the opinions are simultaneously recalculated based on a simple rule of iterative averaging

$$x_i(t+1) = \sum_{j=1}^{n} w_{ij} x_j(t) \quad \forall i = 1, \dots, n, \;\; t = 0, 1, \dots \tag{2}$$

Here n stands for the number of social actors, and $W = (w_{ij})$ is a stochastic[2] matrix of *influence weights*. The matrix W should be compatible with the graph of a social network, that is, an actor allocates influence weight to the adjacent actors (including him/herself if the node has a self-arc). The influence can be thought of as a finite resource each individual distributes to self and the others [22]. In the original French's model, this distribution is uniform: if a node has out-degree m, each neighboring node gets the weight $1/m$, e.g. the graph structure in Fig. 4 induces the following rule of opinion update (for simplicity, we assume that opinions are scalar)

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \end{bmatrix} = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 \end{pmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.$$

A continuous-time counterpart of model (2) was proposed by Abelson [1]

$$\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij}(x_j(t) - x_i(t)), \quad i = 1, \dots, n, \quad t \geq 0. \tag{3}$$

[2] A $n \times n$ matrix W is (row-)stochastic if all its entries are nonnegative $w_{ij} \geq 0$ and each row sums up to 1, i.e. $\sum_{j=1}^{n} w_{ij} = 1$ for each $i = 1, \dots, n$.

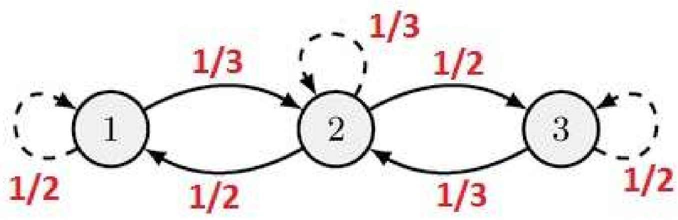

Fig. 4. An example of the French model with $n = 3$ agents

The coefficients $a_{ij} \geq 0$ play the role of infitesimal influence weights, the matrix $A = (a_{ij})$ is not necessarily stochastic yet should be compatible with the graph of the social network: $a_{ij} > 0$ if and only if $(i, j) \in \mathcal{E}$.

The most typical behavior of the French-DeGroot and the Abelson model is *consensus* of the opinions, that is, their convergence to a common value $x_i(t) \xrightarrow[t\to\infty]{} x_*$ that depends on the initial condition. For this reason, these dynamical systems are often referred to as the *consensus protocols* and have been thoroughly studied in the literature on multi-agent systems (see the survery in [39; 40]); first consensus criteria have been found by Harari [26] and Abelson [1]. Consensus is established, for instance, whenever the graph of the social network contains a globally reachable node and, in the discrete-time case, this node has a self-arc.

It is remarkable, however, that the original goal of French's work was not concerned with consensus but rather aimed at finding numerical characteristics of *social power*, that is, an individual's capability to influence the group's ultimate decisions. The consensus is established in the French-DeGroot model if and only if the matrix W has a unique left eigenvector p corresponding to eigenvalue 1 such that[3]

$$p^\top W = p^\top, \quad \sum_{i=1}^{n} p_i = 1.$$

[3] Stochastic matrices satisfying this property are known as fully regular or SIA (stochastic indecomposable aperiodic) matrices; such a matrix serves as a transition matrix of some regular (ergodic) Markov chains [39]. The vector p is nonnegative in view of the Perron-Frobenius theory and stands for the unique stationary distribution of the corresponding Markov chain.

Using (2), one shows that

$$\sum_i p_i x_i(k+1) = \sum_j (\sum_i p_i w_{ij}) x_j(k) = \sum_j p_j x_j(k) = \sum_i p_i x_i(k) = \ldots = \sum_i p_i x_i(0),$$

and passing to the limit as $k \to \infty$, one finds the consensus value x_* as follows

$$x_* = (\sum_i p_i) x_* = \sum_i p_i x_i(0).$$

The component p_i can be thus considered as the influence of actor i's opinion on the final opinion of the group. This measure of influence, or *social power*, of an individual is similar to the eigenvector centrality (computed for the normalized adjacency matrix).

Most of the properties of the French-DeGroot models retain their validity in the case of asynchronous *gossip-based* communication [18] where at each stage of the opinion iteration two randomly chosen actors interact, whereas the remaining actors' opinions remain unchanged. A gossip-based model portrays spontaneous interactions between people in real life and can be considered as a model of opinion formation over a random temporal (time-varying) graph.

From consensus of opinions to community cleavage

Whereas the French-DeGroot and the Abelson model predict consensus, real social groups often exhibit disagreement and clustering of opinions, even when the graph is strongly connected. Thus a realistic dynamic model of opinion formation should be able to explain both consensus and disagreement and help to disclose conditions that prevent the individuals from reaching a consensus. To find such models is a long-standing challenging problem in mathematical sociology referred to as Abelson's "diversity puzzle" or the problem of *community cleavage* [22; 39].

Most of the opinion cleavage models studied in control theory stem from the French-DeGroot and the Abelson models. The three main classes of such models are

1. models with stubborn individuals;

2. homophily-based (bounded confidence) models;
3. models with antagonistic interactions.

Models with stubborn individuals

The **first** class of models explains disagreement by stubborness or "zealotry" of some individuals who are resistant to opinion assimilation. In the French-DeGroot model (2), such an actor assigns the maximal weight $w_{ii} = 1$ to him/herself being unaffected by the others' opinions ($w_{ij} = 0$ for $j \neq i$), so that his/her opinion remains unchanged $x_i(t+1) = x_i(t) = \ldots = x_i(0)$. Similarly, in the Abelson's model a stubborn individual assigns zero influence weights $a_{ij} = 0$ to all individuals, so that $\dot{x}_i(t) \equiv 0$. In presence of two or more stubborn actors with different opinions there is no consensus in the group, and the opinions split into several clusters (in the generic situation, all steady opinions are different).

A natural extension of the French-DeGroot model with stubborn individuals is the Friedkin-Johnsen model [22], being an important and indispensable part of the social influence network theory (SINT) [24]. In the Friedkin-Johnsen model, actors are allowed to be "partially" stubborn. Namely, along with the stochastic matrix of influence weights $W = (w_{ij})$ a set of *susceptibility* coefficients $0 \leq \lambda_{11}, \ldots, \lambda_{nn} \leq 1$ (or, equivalenltly, a diagonal matrix $0 \leq \Lambda \leq I_n$) is introduced that measure how strong is the influence of the actor's initial opinion on all subsequent opinions:

$$x_i(t+1) = \lambda_{ii} \sum_j w_{ij} x_j(k) + (1 - \lambda_{ii}) x_i(0), \quad \forall i = 1, \ldots, n \; \forall t = 0, 1, \ldots$$

The French-DeGroot model appears as a special case of the Friedkin-Johnsen system where none of the actors is anchored at his/her initial opinion ($\lambda_{ii} = 1$ for all i). If $\lambda_{ii} < 1$ for some i, then the matrix ΛW is typically Schur stable (has the spectral radius < 1). Assuming for simplicity that the opinions $x_i(t)$ are scalars, the vector $x(t) = (x_1(t), \ldots, x_n(t))^\top$ they constitute converges to

$$x(\infty) = \lim_{t\to\infty} x(t) = Vx(0), \quad V = (I - \Lambda W)^{-1}(I - \Lambda).$$

The matrix V ("control matrix" [22]) is stochastic, but its rows are different. The average steady opinion of the group is thus found as

$$\frac{1}{n}\sum_{j=1}^{n} x_j(\infty) = \sum_j c_j x_j(0), \quad c_j = \frac{1}{n}\sum_{i=1}^{n} v_{ij}.$$

The vector $c = (c_1, \ldots, c_n)$ is a natural generalization of French's social power and measures the influence of the actors' initial opinions on the average opinion of the group; it can thus also be thought of as a centrality measure. As discussed in [39], in the case where $\Lambda = \alpha I$, $\alpha \in (0, 1)$ this centrality measure (first introduced in [23]) is nothing else than the conventional PageRank.

Homophily-based models

The principle of homophily can be formulated as "birds of a feather fly together" [35]. Social actors tend to interact with like-minded individuals and assimilate their opinions easier than an opinions of a dissimilar person. This principle is prominently illustrated by models of bounded confidence, similar to the French-DeGroot and the Abelson model yet allowing the matrix (w_{ij}) (respectively, (a_{ij})) to depend on the system's state. The Hegselmann-Krause model [27] is an extension of (2), stipulating that $w_{ij}(x_i, x_j) = 0$ if $|x_i - x_j| \geq \varepsilon$, where $\varepsilon > 0$ is an actor's *confidence bound.* A gossip-based version of this model is due to Deffuant and Weisbuch [13]. A detailed discussion of bounded confidence models (which e.g. may contain partially stubborn individuals) and their convergence properties is available at the survey [40].

Antagonistic interactions

Abelson [1] suggested that one of the reasons for disagreement can be "boomerang" (reactance, anticonformity) effect: an attempt to convince other people can make them to adopt an opposing position. In other words, social actors do not always bring their opinions closer, and the convex-combination mechanisms of the models (2),(3) has to be generalized to allow repulsion between their opinions. This idea is in harmony with Heider's theory of structural balance, predicting that individuals disliking each other should have opposite positions on every issue. Although presence of negative ties in opinion formation models has not been secured experimentally, the mathematical theory of networks with positive and negative ties is important since such networks arise in abundance in biology and economics.

A natural extension of the French-DeGroot model allowing negative ties is known as the "discrete-time Altafini model" [32] (historically, its continuous-time counterpart was first proposed by Altafini, see the survey in [43]) and deals with a system (2), where the weights w_{ij} can be positive and negative, however, their absolute values constitute a stochastic matrix $(|w_{ij}|)$. It appears that a structurally balanced graph leads to bimodal polarization (or bipartite consensus) of the opinions: the actors in the two opposing factions converge on the two opposite opinions x_* and $(-x_*)$, where x_* depends on the initial opinion distribution. An imbalanced strongly connected graph induces, however, a degenerate behavior where all opinions converge to 0. In both situations, the opinions remain bounded. More sophisticated dynamic models have been proposed recently [43] that are able to explain clustering of opinions over graphs without structural balance property.

Inference of networks' structures from opinion dynamics

Data-driven inference of a network's characteristics using observations of some dynamical processes over the network is a topic of an active research in statistics, physics and signal processing. In social networks, a natural dynamical process is opinion formation, which, as has been already mentioned, is closely related to centrality measures and other structural properties of a social graph.

The existing works on identification of opinion formation processes are primarily focused on linear models such as the French-DeGroot model [47] or the Friedkin-Johnsen model and its extensions [41]. Two different methods to infer the network's structure are the *finite-horizon* and the *infinite horizon* identification procedures. In the finite-horizon approach the opinions are observed for T subsequent rounds of conversation. Then, if enough observations are available, the parameters of the model (e.g. the matrix of influence weights $W = (w_{ij})$ can be estimated as the matrix best fitting the dynamics for $0 \leq k \leq T$, by using classical identification techniques.

The infinite-horizon approach, instead, performs the estimation based on the observations of the initial and final opinions' profiles. One of the first works on the inference of the network topology is [47], in which the authors consider French-DeGroot models with stubborn agents. This identification procedure has been adopted also in [41] to estimate the influence matrix of the Friedkin-Johnsen model, under the assumption that the matrix is sparse (i.e. agents are influenced by few friends). In these works the estimation problem is non-convex and deterministic or stochastic relaxation techniques are used to approximate it by semidefinite programs.

Dynamics of social graphs

Up to now, we have considered dynamics *over* social networks paying no attention to dynamics of networks themselves. Two examples where the network's graph can be thought of as a temporal (time-varying) graphs are the aforementioned models with gossip-based interactions (one may suppose that the graph of a network is random and contains a single arc at each step) and models with bounded confidence (if the opinions of two individuals are distant, the link connecting them may be considered as missing).

Even more sophisticated dynamical models describe the dynamics of *social power* under reflected appraisals [29; 48], which is related to the French-DeGroot and the Friedkin-Johnsen model yet formally does not require to introduce opinions. It is assumed that social actors participate in infinitely many rounds of discussion, and, upon finishing each round, are able to find their social powers p_i (or, more generally, centralities c_i generated by the Friedkin-Johnsen model). The social power (an "appraisal" of an individual by the others) modifies the influence weights the individual assigns to the others in the next round, which, in turn, produces a new vector of social power (or Friedkin-Johnsen centralities).

Another important class of dynamical models describe the evolution of positive and negative ties leading to the structural balance. Whereas the balance is predicted by Heider's theory, it does not provide any explicit description of the mechanisms balancing the network. Several mathematical models have been proposed in the literature [34; 45] to portray the dynamics of structural balance.

Cross-References

Connectivity of Dynamic Graphs; Consensus of Complex Multi-agent Systems; Graphs for Modeling Networked Interactions; Markov Chains and Ranking Problems in Web Search

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