Homogenization of some quasi-linear elliptic equations with gradient constraints

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Abstract We prove a homogenization formula for quasi-linear elliptic equations with gradient constraints on a disperse set, within the framework of monotonic operator theory and compensated compactness methods.

1 Introduction

The aim of this paper is to extend to quasi-linear elliptic equations of monotone type some classical results obtained for the linear case by Cioranescu-SaintJean Paulin (see [6]), with application to the torsion of a cylindrical elastic bar with several cylindrical thin cavities. The mathematical model for a homogeneous isotropic material was first studied in [12]. For example, let Q be a cylindrical bar with identical, periodically distributed cylindrical cavities having generators parallel to those of Q and $\varepsilon > 0$ be a the size of the period. Let Ω be the cross-section of the bar, Ω_ε the cross-section of the domain occupied by the material (i.e. the perforated domain). Denoting by B^i_ε the cross section of a single cavity, of size proportional to $\varepsilon > 0$, we have $\Omega_\varepsilon = \Omega \setminus \bigcup_{i=1}^{N_\varepsilon} B^i_\varepsilon$. According to [12], in the linear homogeneous isotropic case, the study of the elastic torsion of this bar leads to the following problem

$$\begin{aligned} -\Delta u_\varepsilon &= 2\mu\theta && \text{in } \Omega_\varepsilon \\ u_\varepsilon &= \text{const} && \text{on } \partial B^i_\varepsilon \\ u_\varepsilon &= 0 && \text{on } \partial\Omega \end{aligned}$$

where μ represents the rigidity modulus of the material, θ is the twist's angle and u_ε denotes the, so-called, stress function, from which the stress tensor can be recovered. The results of [6], that deal also with more general problems, in particular, characterize the response of the bar under torsion for small ε, proving that u_ε is close to the solution u of a well determined boundary-value problem in the full domain Ω.

The above problem written in a variational form as

$$\int_\Omega \nabla u_\varepsilon \cdot \nabla v \, dx = 2\mu\theta \int_\Omega v \, dx$$

for $u_\varepsilon, v \in H^1_0(\Omega)$ and $\nabla u_\varepsilon, \nabla v = 0$ in B_ε, brings naturally to several generalizations. In this paper, we replace ∇u_ε with a vector of the form $a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right)$ that takes into account space oscillations and non linear dependence on the gradient ∇u_ε (see equation (3)).

In good agreement with what happens in the linear case ([6]), we prove that the solutions u_ε of the nonlinear equation (3) are close in $L^2(\Omega)$ to the solution u of the homogenized problem (19), whose coefficients involve the homogenization formula (15), obtained through the cell-problem (14).

In the proof we combine the extension tools provided by Cioranescu-Saint Jean Paulin in [6] for the linear case, with the compensated compactness method ([15], [16]), that allows to pass to the limit in the present nonlinear setting.

For simpllicity, the problem is studied in $H^1(\Omega)$, but natural generalizations to $W^{1,p}(\Omega)$ are possible, assuming appropriate growth and continuity conditions for the function $a(y, \xi)$, following [8].

In a forthcoming paper, we will discuss the homogenization of variational inequalities arising from minimum problems of the type

$$\min\left\{\int_\Omega \left(f\left(\frac{x}{\varepsilon}, \nabla u\right) - 2gu\right) dx : u \in V^\varepsilon\right\} \tag{1}$$

where $V^\varepsilon \subset H_0^1(\Omega)$ is a more general convex set of constraints, arising in the paper [4].

In addition to the cited model of elastic torsion of a bar, it worths to mention the more known well established electrostatic model with periodical inclusions of conductors for which we refer, e. g., to the pioneering work of J. Rauch and M. Taylor [14], or to the more recent book [11]. Moreover, we point out that several related results for minimum problems are obtained in the framework of Γ-convergence theory. We mention in particular [1] about non-linear elastic materials with stiff and soft inclusions, [3] for the homogenization of media with periodically distributed conductors, and [7], for more general constrained variational problems.

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2 Statement of the problem and main result

Let Ω be a bounded open connected set in $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$ and let $Y = [0,1]^n$ denote the periodicity cell. Let B be the closure of a given Y-periodic open set in $\mathbb{R}^n$ with Lipschitz boundary. We assume that B is disperse, in the sense that $B \cap Y \subset\subset Y$. We also assume that $B \cap Y$ has a finite number of conneted components. Let us define the set of functions

$$K^\varepsilon = \left\{v \in H_0^1(\Omega) : \nabla v(x) = 0 \text{ a.e. in } \varepsilon B \cap \Omega\right\} \tag{2}$$

where ε is a small positive parameter and $\varepsilon B = \{x \in \mathbb{R}^n : \varepsilon^{-1}x \in B\}$. It is known that K^ε is a closed subspace of $H_0^1(\Omega)$.
We consider the following variational equation for $u_\varepsilon \in K^\varepsilon$:

$$\int_\Omega a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right) \nabla\varphi \, dx = \int_\Omega g\varphi \, dx, \quad \forall \varphi \in K^\varepsilon \tag{3}$$

where $g \in L^2(\Omega)$ and $a = a(y, \xi) : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is measurable and Y-periodic in $y \in \mathbb{R}^n$ for every $\xi \in \mathbb{R}^n$ and such that

$$a(y, 0) = 0 \quad \text{for a.e. } y \in \mathbb{R}^n. \tag{4}$$

Moreover, we assume that $a(y, \cdot)$ is strictly monotone with uniform bound and Lipschitz continuous uniformly in y, namely $\exists\, \alpha, L > 0$ such that

$$\alpha|\xi_1 - \xi_2|^2 \leqslant (a(y, \xi_1) - a(y, \xi_2)) \cdot (\xi_1 - \xi_2), \quad \text{for a.e. } y \in \mathbb{R}^n, \forall\, \xi_1, \xi_2 \in \mathbb{R}^n; \tag{5}$$

$$|a(y, \xi_1) - a(y, \xi_2)| \leqslant L|\xi_1 - \xi_2|, \quad \text{for a.e. } y \in \mathbb{R}^n, \forall\, \xi_1, \xi_2 \in \mathbb{R}^n. \tag{6}$$

Our goal is to study the asymptotic behavior of the sequence $\{u_\varepsilon\}$ as ε goes to zero and to prove that the limit of the sequence satisfies, in a suitable sense, a (limit) variational problem, so-called homogenized problem.

Remark 2.1 It is interesting to notice that if g is replaced by

$$g_\varepsilon = \frac{g}{|Y \cap B|} \chi_{\mathbb{R}^n \setminus \varepsilon B} \tag{7}$$

where $\chi_{\mathbb{R}^n \setminus \varepsilon B}$ represents the characteristic function of the set $\mathbb{R}^n \setminus \varepsilon B$, the asymptotic problem does not change. More precisely, let us replace g by $g_\varepsilon = h\chi_{\mathbb{R}^n \setminus \varepsilon B}$ with $h \in L^2(\Omega)$, and compare the behaviour of u_ε and v_ε, the solutions of (3) corresponding to g and g_ε respectively. We observe that by the strict monotonicity of $a(y,\cdot)$ (see (5)) it follows that

$$\alpha \int_\Omega |\nabla u_\varepsilon - \nabla v_\varepsilon|^2\,dx \leqslant \int_\Omega \left[a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right) - a\left(\frac{x}{\varepsilon}, \nabla v_\varepsilon\right)\right]\cdot(\nabla u_\varepsilon - \nabla v_\varepsilon)\,dx$$
$$= \int_\Omega (g - h\chi_{\mathbb{R}^n\cap\varepsilon B})(u_\varepsilon - v_\varepsilon)\,dx$$

then

$$\int_\Omega (g - h\chi_{\mathbb{R}^n\cap\varepsilon B})(u_\varepsilon - v_\varepsilon)\,dx \xrightarrow{\varepsilon\to 0} \int_\Omega (g - h|Y\cap B|)(u - v)\,dx,$$

where $|Y\cap B|$ denotes the Lebesgue measure of $Y\cap B$. If $h = \frac{g}{|Y\cap B|}$ this yields $u = v$, which means that the asymptotic behaviour of u_ε is the same as the one of v_ε. This situation is typical when equation (3) models the behaviour of the electrostatic potential of conductors occupying the region $\Omega\cap B_\varepsilon$: in this case $g_\varepsilon = 0$ outside the conductiors and $g_\varepsilon = -4\pi q$, with q the electric charge of each conducting component (see, e.g., [10]). For a complete treatment of the electrostatic model in the general case see [11].

Proposition 2.2 *For fixed $\varepsilon > 0$ and $g \in L^2(\Omega)$ there exists the unique solution $u_\varepsilon \in K^\varepsilon$ of equation* (3)*. Such solution satisfies the following a priori estimates*

$$\|u_\varepsilon\|_{H_0^1(\Omega)} \leq c, \tag{8}$$

$$\left\|a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right)\right\|_{L^2(\Omega)^n} \leq Lc \tag{9}$$

where $c = \alpha^{-1}c_P\,\|g\|_{L^2(\Omega)}$ is independent of ε, and c_P denotes the constant for the Poincaré inequality in $H_0^1(\Omega)$.

From the a priori estimates (8), (9) and by Rellich's theorem we have, up to a subsequence,

$$u_\varepsilon \rightharpoonup u \quad \text{in } H_0^1(\Omega), \tag{10}$$

$$a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right) \rightharpoonup \hat{a} \quad \text{in } L^2(\Omega)^n, \tag{11}$$

and it is natural to look for a characterization of the limits u and $\hat{a}$ through a suitable boundary-value problem. As it is customary in homogenization, we expect to express the homogenized equation by means of a suitable auxiliary equation in the periodicity cell Y. In order to determine such cell problem, we have taken into account the homogenization of minimum problems of the type

$$\min\left\{\int_\Omega \left(|\nabla u|^2 - 2gu\right)\,dx : u \in V^\varepsilon\right\}, \tag{12}$$

considered in [4] for a quite general convex set $V^\varepsilon \subset H_0^1(\Omega)$. When $V^\varepsilon = K^\varepsilon$, then (3) with $a(y,\xi) = \xi$ is the Euler-Lagrange equation of (12). The results of [4] suggest then to choose the Euler-Lagrange equation of the cell problem corresponding to (12) as a "good candidate" for the cell problem in our case.
From now on, we denote by $H^1_\sharp(Y)$ the subspace of $H^1_{\mathrm{loc}}(\mathbb{R}^n)$ of functions v that are Y-periodic and have mean-value zero in the periodicity cell Y, equipped with the norm $||v||_{H^1_\sharp(Y)} = ||\nabla v||_{L^2(Y)}$. For every given $\xi \in \mathbb{R}^n$, we consider the following closed convex subset of $H^1_\sharp(Y)$

$$K_\xi = \left\{v \in H^1_\sharp(Y) : \xi + \nabla v(y) = 0 \ \textit{a.e. in } B\right\}, \quad \xi \in \mathbb{R}^n. \tag{13}$$

In particular, for $\xi = 0$, K_0 is a closed subspace of $H^1_\sharp(Y)$. In view of the above considerations, we formulate the following cell problem in weak form

$$\begin{cases} \displaystyle\int_Y a(y, \xi + \nabla w_\xi)\cdot\nabla\varphi\,dy = 0, \quad \forall \varphi \in K_0 \\ w_\xi \in K_\xi. \end{cases} \tag{14}$$

Proposition 2.3 *For fixed $\xi \in \mathbb{R}^n$ there exists a unique solution $w_\xi \in K_\xi$ of equation* (14).

In order to formulate the main result concerning the homogenization of equation (3), we define the homogenized operator a_{hom}.

Definition 2.4 *We will call homogenized operator the function $a_{\text{hom}} : \mathbb{R}^n \to \mathbb{R}^n$ defined as*

$$a_{\text{hom}}(\xi) \cdot \eta = \int_{Y \setminus B} a(y, \xi + \nabla w_\xi) \cdot (\eta + \nabla w_\eta)\, dy, \quad \forall\, \xi, \eta \in \mathbb{R}^n, \tag{15}$$

where $w_\xi \in K_\xi$ and $w_\eta \in K_\eta$ are solutions of the cell problem (14).

Such operator has some properties summed up in the next proposition.

Proposition 2.5 *The function $a_{\text{hom}} : \mathbb{R}^n \to \mathbb{R}^n$ defined by* (15) *is strictly monotone, coercive and Lipschitz continuous. In particular it satisfies*

$$a_{\text{hom}}(0) = 0; \tag{16}$$

$$\alpha|\xi_1 - \xi_2|^2 \leqslant (a_{\text{hom}}(\xi_1) - a_{\text{hom}}(\xi_2)) \cdot (\xi_1 - \xi_2), \quad \forall\, \xi_1, \xi_2 \in \mathbb{R}^n; \tag{17}$$

$$|a_{\text{hom}}(\xi_1) - a_{\text{hom}}(\xi_2)| \leqslant L'|\xi_1 - \xi_2|, \quad \forall\, \xi_1, \xi_2 \in \mathbb{R}^n. \tag{18}$$

with Lipschitz constant $L' = L^3\alpha^{-2}\sqrt{2 + \delta^{-1}}$, $\delta = \operatorname{dist}(\partial B \cap Y, Y)$.

At this stage we can state the main result.

Theorem 2.6 *Let u_ε be the unique solution of the equation* (3). *Then $u_\varepsilon \rightharpoonup u$ weakly in $H_0^1(\Omega)$, $u_\varepsilon \to u$ strongly in $L^2(\Omega)$ as $\varepsilon \to 0$, where u is the unique solution of the homogenized equation*

$$\int_\Omega a_{\text{hom}}(\nabla u) \cdot \nabla \varphi \, dx = \int_\Omega g\varphi \, dx, \quad \forall\, \varphi \in H_0^1(\Omega) \tag{19}$$

with a_{hom} defined by (15).

Remark 2.7 The convergence in (10), (11) are not enough to pass to the limit in equation (3), due to the nonlinearity in the equation and the fact that the test functions depend on ε. The proof of (19) is based on the classical energy method (see [15]), comparing the asymptotic behaviour of $F_\varepsilon = a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right)$ and $G_\varepsilon = a\left(\frac{x}{\varepsilon}, \xi + \nabla w_\xi(\frac{x}{\varepsilon})\right)$, for any fixed $\xi \in \mathbb{R}^n$, with the help of compensated compactness argumentsFor the reader's convenience, we recall the useful statement in Proposition 2.8, that is a simple case of a more general result due to L. Tartar ([16]).As in the linear case (see [6]), $F_\varepsilon, G_\varepsilon$ have to be suitably modified (extended) in the sets $\Omega' \cap \varepsilon B$, $\Omega' \subset\subset \Omega$, in order to satisfy the assumptions that permit to pass to the limit by compensated compactness.

Proposition 2.8 *(***Compensated compactness***) Let $u_\varepsilon, u \in H^1(\Omega)$ be such that $u_\varepsilon \rightharpoonup u$ weakly in $H^1(\Omega)$ and $F_\varepsilon, F \in L^2(\Omega; \mathbb{R}^n)$ be such that $F_\varepsilon \rightharpoonup F$ weakly in $L^2(\Omega; \mathbb{R}^n)$, and $\operatorname{div} F_\varepsilon \to \operatorname{div} F$ strongly in $H^{-1}(\Omega)$. Then*

$$\int_\Omega F_\varepsilon \cdot \nabla u_\varepsilon \, \varphi \, dx \to \int_\Omega F \cdot \nabla u \, \varphi \, dx \qquad \forall \varphi \in C_0^\infty(\Omega).$$

3 Proofs

Proof of Proposition 2.2
Let us consider the operator

$$\begin{aligned} A_\varepsilon :\ & K^\varepsilon \to (K^\varepsilon)' \\ & u \mapsto A_\varepsilon u = -\operatorname{div}\left(a\left(\frac{x}{\varepsilon}, \nabla u\right)\right), \end{aligned} \tag{20}$$

for fixed $u \in K^\varepsilon$, defined by the pairing

$$\langle A_\varepsilon u, v\rangle = \int_\Omega a\left(\frac{x}{\varepsilon}, \nabla u\right) \nabla v \, dx.$$

With this notation, equation (3) can be equivalently expressed as

$$A_\varepsilon u_\varepsilon = g\,. \tag{21}$$

For fixed $v \in K^\varepsilon$, by the assumption (5) we have

$$\begin{aligned}\langle A_\varepsilon u - A_\varepsilon v, u - v\rangle &= \int_\Omega \left[a\left(\frac{x}{\varepsilon}, \nabla u\right) - a\left(\frac{x}{\varepsilon}, \nabla v\right)\right] (\nabla u - \nabla v)\, dx \\ &\geqslant \alpha \int_\Omega |\nabla u - \nabla v|^2 \, dx,\end{aligned} \tag{22}$$

then A_ε is strictly monotone. In particular, from this fact, it follows that equation (3) cannot have more than one solution.
On the other hand, A_ε is hemicontinuous, in the sense that, for fixed $u, v, w \in K^\varepsilon$, the function

$$\mathbb{R} \ni t \to \langle A_\varepsilon(u + tv), w\rangle = \int_\Omega a\left(\frac{x}{\varepsilon}, \nabla u + t\nabla v\right) \nabla w \, dx, \tag{23}$$

is continuous in t. In fact, for fixed $t_1, t_2 \in \mathbb{R}$, by the Cauchy-Schwarz inequality and assumption (6) we have

$$\begin{aligned}|\langle A_\varepsilon(u + t_1 v), w\rangle - \langle A_\varepsilon(u + t_2 v), w\rangle| &= \left|\int_\Omega \left[a\left(\frac{x}{\varepsilon}, \nabla u + t_1\nabla v\right) - a\left(\frac{x}{\varepsilon}, \nabla u + t_2\nabla v\right)\right] \nabla w \, dx\right| \\ &\leqslant L|t_1 - t_2| \, \|\nabla v\|_{L^2(\Omega)^n} \, \|\nabla w\|_{L^2(\Omega)^n}\,,\end{aligned} \tag{24}$$

i.e., A_ε is hemicontinuous.
Furthermore, by the assumptions (4) and (5), for any $u \in H_0^1(\Omega)$, we have

$$\langle A_\varepsilon u, u\rangle = \int_\Omega a\left(\frac{x}{\varepsilon}, \nabla u\right) \nabla u \, dx \geqslant \alpha \int_\Omega |\nabla u|^2 \, dx, \tag{25}$$

Then, we can conclude that A_ε is coercive, i.e.,

$$\frac{\langle A_\varepsilon u, u\rangle}{||u||_{H_0^1(\Omega)}} \to +\infty, \tag{26}$$

as $||u||_{H_0^1(\Omega)} \to +\infty$.
In view of the previous steps, by the Hartman-Stampacchia's theorem (see, for example, [9] or [13]), we obtain that A_ε is surjective, hence $\exists\,!\, u_\varepsilon \in K^\varepsilon$ solution of (3).
A priori estimates (8), (9) follow easily from assumptions (4), (5),(6). In fact, since $a(y, \cdot)$ is strictly monotone, by the Cauchy-Schwarz inequality and assumption (4) we have

$$\begin{aligned}\alpha \, \|\nabla u_\varepsilon\|^2_{L^2(\Omega)^n} &= \alpha \int_\Omega |\nabla u_\varepsilon|^2 \, dx \\ &\leqslant \int_\Omega a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right) \nabla u_\varepsilon \, dx \\ &= \int_\Omega g u_\varepsilon \, dx \\ &\leqslant c_P \, \|g\|_{L^2(\Omega)} \, \|\nabla u_\varepsilon\|_{L^2(\Omega)^n}\,,\end{aligned}$$

where c_P denotes the constant for the Poincaré inequality in $H_0^1(\Omega)$, whence

$$\|u_\varepsilon\|_{H_0^1(\Omega)} = \|\nabla u_\varepsilon\|_{L^2(\Omega)^n} \leqslant \frac{c_P \, \|g\|_{L^2(\Omega)}}{\alpha}, \tag{27}$$

that is the estimate (8) with $c = \alpha^{-1} c_P \|g\|_{L^2(\Omega)}$. Moreover, since $a(y, \cdot)$ is Lipschitz-continuous, by the Cauchy-Schwarz inequality and the assumption (4) it follows that

$$\int_\Omega \left| a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right)\right|^2 dx \leqslant L^2 \int_\Omega |\nabla u_\varepsilon|^2 \, dx, \tag{28}$$

then, from (27) and (28) we have (9). □

Proof of Proposition 2.3

In order to prove that (14) has at most one solution, we note that it is possible to choose test functions of the form $\varphi = w_\xi + \psi$, with $\psi \in K_{-\xi}$. Then, if w_1, w_2 are two solutions of (14) for the same value of $\xi \in \mathbb{R}^n$, we have

$$\begin{aligned} 0 &= \int_Y (a(y, \xi + \nabla w_1) - a(y, \xi + \nabla w_2)) ((\nabla w_1 + \nabla \psi) - (\nabla w_2 + \nabla \psi)) \, dy = \\ &= \int_Y (a(y, \xi + \nabla w_1) - a(y, \xi + \nabla w_2)) (\nabla w_1 - \nabla w_2) \, dy = \\ &\geq \alpha \int_Y |\nabla w_1 - \nabla w_2|^2 \, dy, \end{aligned}$$

from which it follows that $w_1 = w_2$. In order to prove the existence result, given $\xi \in \mathbb{R}^n$, we fix an arbitrary test function $\phi_\xi \in K_{-\xi}$, and we introduce the new unknown $z_\xi = w_\xi + \phi_\xi$. Clearly, $z_\xi \in K_0$. Then, w_ξ is a solution of problem (14) of and only if z_ξ solves the following problem:

$$\begin{cases} \displaystyle\int_Y a(y, \xi - \nabla \phi_\xi + \nabla z_\xi) \cdot \nabla \varphi \, dy = 0, \quad \forall \varphi \in K_0 \\ z_\xi \in K_0. \end{cases} \tag{29}$$

It remains to prove the existence of a solution z_ξ. To this end, for the fixed $\phi_\xi \in K_{-\xi}$, let us consider the operator

$$\begin{aligned} \widehat{A}_\xi : \; & K_0 \to (K_0)' \\ & u \mapsto \widehat{A}_\xi u = -\mathrm{div}\left(a\left(y, \xi - \nabla \phi_\xi + \nabla u\right)\right), \end{aligned} \tag{30}$$

defined by the pairing

$$\left\langle \widehat{A}_\xi u, v \right\rangle = \int_Y a\left(y, \xi - \nabla \phi_\xi + \nabla u\right) \nabla v \, dy.$$

Arguing as in the proof of Proposition 2.2, we can show that $\widehat{A}_\xi$ is monotone, hemicontinuous and coercive. Hence, by the Hartman-Stampacchia's theorem, we can show that $\widehat{A}_\xi$ is surjective, which yields the existence of a function $z_\xi \in K_0$ solution of (29), and completes the proof. □

Remark 3.1 From (14), choosing particular test functions $\varphi \in C_0^\infty(Y \backslash B)$, extended by the constant 0 in B, it follows that $div_y\left(a(y, \xi + \nabla w_\xi(y))\right) = 0$ in $Y \setminus B$. Hence, denoting by ν_E the exterior unit normal vector to the boundary of the set E, for a general test function $\varphi \in K_0$, we have

$$\begin{aligned} 0 &= \int_{Y \backslash B} a(y, \xi + \nabla w_\xi) \nabla \varphi \, dy = \\ &= -\int_{Y \backslash B} div_y\left(a(y, \xi + \nabla w_\xi(y))\right) \varphi \, dy - \int_{\partial(Y \backslash B)} a(y, \xi + \nabla w_\xi) \cdot \nu_{Y \backslash B} \, \varphi \, d\sigma = \\ &= \int_{\partial B \cap Y} a(y, \xi + \nabla w_\xi) \cdot \nu_{Y \backslash B} \, \varphi \, d\sigma \end{aligned}$$

provided $a(y, \xi + \nabla w_\xi)$ is smooth enough to perform the integration by parts. Since φ has constant trace on the connected components Γ of the boundary $\partial B \cap Y$, and is Y-periodic, it follows that

$$\int_\Gamma a(y, \xi + \nabla w_\xi) \cdot \nu_B \, d\sigma = 0. \tag{31}$$

More generally, since $F(y) = a(y, \xi + \nabla w_\xi(y)) \in L^2(Y \setminus B)^n$, and $\mathrm{div} F \in L^2(Y \setminus B)^n$, then $F \cdot \nu \in H^{-1/2}(\partial(Y \setminus B))$ and

$$- \int_{Y \setminus B} \mathrm{div} F \varphi,\ dy = \int_{Y \setminus B} F \cdot \nabla \varphi \, dy + \langle F \cdot \nu_{Y \setminus B}, \varphi \rangle \qquad \forall \varphi \in K_0,$$

from which we can say that

$$\langle a(y, \xi + \nabla w_\xi) \cdot \nu_B, \varphi \rangle = 0, \quad \forall \varphi \in K_0, \tag{32}$$

where $\langle\, , \,\rangle$ denotes the duality pairing between $H^{1/2}(\partial(Y \setminus B))$ and $H^{-1/2}(\partial(Y \setminus B))$.

We state now an extension result, that we will use to pass to the limit by compensated compactness in the proof of Theorem 2.6. The proof can be found in [6, Lemma 2] if $n = 2$, [10, Chapter 3, Section 3.2] if $n \geqslant 2$.

Lemma 3.2 *Let $z \in L^2(Y \setminus B)^n$ and $g \in L^2(Y)$ such that*

$$-\mathit{div}\, z = g \qquad\qquad in\ D'(Y \setminus B), \tag{33}$$

$$\int_{Y \setminus B} z \cdot \nabla \varphi \, dy = \int_Y g \varphi \, dy \quad \forall\, \varphi \in C_0^\infty(Y) :\ \nabla \varphi|_B = 0, \tag{34}$$

then there exists $\tilde{z} \in L^2(Y)^n$ such that

$$-\mathit{div}\, \tilde{z} = g \qquad in\ Y\ and\ in\ D'(Y), \tag{35}$$

$$\tilde{z} = z \qquad in\ Y \setminus B, \tag{36}$$

$$z \cdot \nu_B = \tilde{z} \cdot \nu_B \quad in\ Y \cap \partial B, \tag{37}$$

$$\int_{B \cap Y} |\tilde{z}|^2 \, dy \leqslant c \left(\int_Y |g|^2 \, dy + \int_{Y \setminus B} |z|^2 \, dy \right). \tag{38}$$

where ν_B denotes the unit normal vector to the boundary of B, and c is a constant independent of z and g.

Remark 3.3 The result is invariant up to translations of the domain Y in $\mathbb{R}^n$. Moreover, if $g = 0$ the lemma defines a linear and continuous extension operator

$$\begin{aligned} T :\ L^2(Y \setminus B) &\longrightarrow L^2(Y)^n \\ z &\longmapsto Tz = \tilde{z} \end{aligned}$$

such that

$$||Tz||_{L^2(Y)^n} \leq c_T ||Tz||_{L^2(Y \setminus B)^n}, \tag{39}$$

with $c_T > 0$. This operator will be considered on $Y^i = Y + i$, with $i \in \mathbb{Z}^n$.

From here on, we prepare the tools that we will use in the proof of Theorem 2.6 to pass to the limit in the equation (3) by compensated compactness. To this end, we need to modify the flux

$$b_\varepsilon(x) = a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon(x)\right) \tag{40}$$

over the sets εB.
If we set $\Omega_\varepsilon = \Omega \setminus \varepsilon B$ and we take in particular $\varphi \in C_0^\infty(\Omega_\varepsilon)$ extended by 0 in $\Omega \cap \varepsilon B$ in (3) we obtain

$$\int_{\Omega_\varepsilon} b_\varepsilon(x) \nabla \varphi \, dx = \int_{\Omega_\varepsilon} g \varphi \, dx \quad \forall\, \varphi \in C_0^\infty(\Omega_\varepsilon). \tag{41}$$

which means

$$-\mathrm{div}\, b_\varepsilon(x) = g \quad \text{in } D'(\Omega_\varepsilon) \text{ and in } L^2(\Omega_\varepsilon) \tag{42}$$

Proposition 3.4 *Let $z_\varepsilon(y) = b_\varepsilon(\varepsilon y)$, with b_ε defined by (40). Then there exists an extension $\tilde{z}_\varepsilon \in L^2(Y^i)^n$ of $z_\varepsilon \in L^2(Y^i \setminus B)^n$, for $i \in I_\varepsilon(\Omega) = \{k \in \mathbb{Z}^n : \varepsilon Y^k \subset \Omega\}$, such that*

$$-div\, \tilde{z}_\varepsilon(y) = \varepsilon g(\varepsilon y) \quad in \ \ D'(Y^i), \tag{43}$$

$$\tilde{z}_\varepsilon = z_\varepsilon \qquad in\, Y^i \setminus B, \tag{44}$$

$$\int_B |\tilde{z}_\varepsilon|^2\, dy \leqslant c \left(\int_{Y^i} |\varepsilon g(\varepsilon y)|^2\, dy + \int_{Y^i \setminus B} |z_\varepsilon|^2\, dy \right), \tag{45}$$

with c independent of εg and z_ε.

Proof: We observe that, for any $i \in I_\varepsilon(\Omega)$

$$-\mathrm{div}\, z_\varepsilon(y) = \varepsilon g(\varepsilon y) \quad \text{in } Y^i \setminus B. \tag{46}$$

Moreover, setting $Y^i_\varepsilon = \varepsilon Y^i$, from (3) we have

$$\int_{Y^i_\varepsilon \setminus \varepsilon B} b_\varepsilon(x) \nabla \varphi\, dx = \int_{Y^i_\varepsilon} g \varphi\, dx \quad \forall\, \varphi \in C_0^\infty(Y^i_\varepsilon) : \nabla \varphi = 0 \text{ in } \varepsilon B \cap Y^i_\varepsilon. \tag{47}$$

Performing the change of variable $x = \varepsilon y$ in (47) we obtain

$$\int_{Y^i \setminus B} z_\varepsilon(y) \nabla \varphi\, dy = \int_{Y^i} \varepsilon g(\varepsilon y) \varphi\, dy \quad \forall\, \varphi \in C_0^\infty(Y^i) : \nabla \varphi = 0 \text{ in } \varepsilon B \cap Y^i_\varepsilon. \tag{48}$$

Then, by Lemma 3.2 there exists $\tilde{z}_\varepsilon \in L^2(Y^i)^n$ satisfying (43), (44) and (45). □

In order to pass to the limit in (3) it is necessary to obtain equations and estimates in Ω, or at least in any relatively compact open subset Ω' of Ω, using the notation $\Omega' \subset\subset \Omega$. Let us fix $\Omega' \subset\subset \Omega$ and set $J_\varepsilon(\Omega') = \{k \in \mathbb{Z}^n : Y^k_\varepsilon \cap \Omega' \neq \phi\}$. Then, there exists $\varepsilon_0 = \varepsilon_0(\Omega') > 0$ such that $\forall\, \varepsilon < \varepsilon_0$ if $k \in J_\varepsilon(\Omega')$ then $Y^k_\varepsilon \subseteq \Omega$. For $\varepsilon < \varepsilon_0$ the function $\tilde{z}_\varepsilon$ defined by Proposition 3.4 makes sense $\forall\, i \in J_\varepsilon(\Omega')$. More precisely

Proposition 3.5 *Let $\Omega' \subset\subset \Omega$, $\varepsilon < \varepsilon_0(\Omega')$ and $b_\varepsilon(x)$ defined by (40). Then for all $i \in J_\varepsilon(\Omega')$ there exists an extension $\tilde{b}^i_\varepsilon \in L^2(Y^i_\varepsilon)^n$ of $b_\varepsilon \in L^2(Y^i_\varepsilon \setminus \varepsilon B)^n$ such that*

$$-div_x\, \tilde{b}^i_\varepsilon(x) = g(x) \quad in\, Y^i_\varepsilon, \tag{49}$$

$$\tilde{b}^i_\varepsilon = b_\varepsilon \qquad in\, Y^i_\varepsilon \setminus \varepsilon B, \tag{50}$$

$$\int_{\varepsilon B} |\tilde{b}^i_\varepsilon(x)|^2\, dx \leqslant c \left(\int_{Y^i_\varepsilon} |\varepsilon g(x)|^2\, dx + \int_{Y^i_\varepsilon \setminus \varepsilon B} |b_\varepsilon(x)|^2\, dx \right), \tag{51}$$

with c independent of ε, g and b_ε.

Proof: Since $b_\varepsilon \in L^2(Y^i_\varepsilon \setminus \varepsilon B)^n$, $g \in L^2(Y^i_\varepsilon)$, setting $z_\varepsilon(y) = b_\varepsilon(\varepsilon y)$, then the extension $\tilde{z}_\varepsilon(y)$ defined by Proposition 3.4 satisfies (43)-(45). Hence, setting $\tilde{b}^i_\varepsilon = \tilde{z}_\varepsilon(\frac{x}{\varepsilon})$, and performing the change of variable $x = \varepsilon y$ in (43)-(45), conditions (49)-(51) follow.

Corollary 3.6 *For any $\Omega' \subset\subset \Omega$, $\varepsilon < \varepsilon_0(\Omega')$, there exists an extension $\tilde{b}_\varepsilon \in L^2(\Omega')^n$ of $b_\varepsilon|_{\Omega_\varepsilon}$ such that*

$$-div_x \tilde{b}_\varepsilon(x) = g(x) \quad in\, D'(\Omega'), \tag{52}$$

$$\tilde{b}_\varepsilon = b_\varepsilon \qquad in\, \Omega' \setminus \varepsilon B \tag{53}$$

$$\int_{\Omega'} |\tilde{b}_\varepsilon(x)|^2\, dx \leqslant c \left(\int_{\Omega} |\varepsilon g(x)|^2\, dx + \int_{\Omega \setminus \varepsilon B} |b_\varepsilon(x)|^2\, dx \right). \tag{54}$$

Proof: Setting

$$\tilde{b}_\varepsilon(x) = \sum_{i \in J_\varepsilon(\Omega')} \chi_{Y^i_\varepsilon}(x) \tilde{b}^i_\varepsilon(x), \tag{55}$$

statements (52) and (53) are straightforward, whereas estimate (54) follows from (55) and the fact that $\Omega' \subseteq \cup\{Y^i_\varepsilon : i \in J(\Omega')\} \subseteq \Omega$. □

Proposition 3.7 *Let $\{\Omega'_j\}$ be an increasing sequence of open subsets of Ω such that $\Omega'_j \subset\subset \Omega$ and $\cup_j \Omega'_j = \Omega$. Let $\tilde{b}^{(j)}_\varepsilon$ be the function defined in Corollary 3.6 by (55) when $\Omega' = \Omega'_j$. Then there exists $b \in L^2_{\rm loc}(\Omega)^n$ and there exists a subsequence of $\varepsilon \to 0$ (not relabelled), such that for all $j \geq 1$*

$$\tilde{b}^{(j)}_\varepsilon \rightharpoonup b \qquad \textit{weakly in } L^2(\Omega'_j)^n, \tag{56}$$

$$-\mathit{div}_x \tilde{b}^{(j)}_\varepsilon \to -\mathit{div}_x b \quad \textit{strongly in } H^{-1}(\Omega'_j),\ \forall\, j, \tag{57}$$

$$-\mathit{div}_x \tilde{b}^{(j)}_\varepsilon(x) = g(x) = -\mathit{div}_x b \quad \textit{in } D'(\Omega'_j) \textit{ for } \varepsilon < \varepsilon_0(\Omega'_j). \tag{58}$$

Proof: For $j = 1$ we choose a subsequence ε_1 of ε such that the extension $\tilde{b}^{(1)}_{\varepsilon_1}$ of $b_{\varepsilon_1}|_{\Omega_{\varepsilon_1}}$ defined by (55) for $\Omega' = \Omega'_1$ satisfies

$$\tilde{b}^{(1)}_{\varepsilon_1} \rightharpoonup b^{(1)} \text{ weakly in } L^2(\Omega'_1)^n, \tag{59}$$

as $\varepsilon_1 \to 0$. For $j = 2$ we repeat the procedure extracting a subsequence ε_2 of the previous one ε_1, so that the extension $\tilde{b}^{(2)}_{\varepsilon_2}$ (of $b_{\varepsilon_2}|_{\Omega_{\varepsilon_2}}$) satisfies

$$\tilde{b}^{(2)}_{\varepsilon_2} \rightharpoonup b^{(2)} \text{ weakly in } L^2(\Omega'_2)^n, \tag{60}$$

as $\varepsilon_2 \to 0$. Since $\Omega'_1 \subset\subset \Omega'_2$, the limits coincide in the smaller domain, i.e.,

$$b^{(2)} = b^{(1)} \quad \text{ in } \Omega'_1.$$

For any $j \geq 2$ we can proceed from Ω'_{j-1} to Ω'_j in the same way, getting a further subsequence ε_j such that the extension $\tilde{b}^{(j)}_{\varepsilon_j}$ satisfies

$$\tilde{b}^{(j)}_{\varepsilon_j} \rightharpoonup b^{(j)} \text{ weakly in } L^2(\Omega'_j)^n, \tag{61}$$

as $\varepsilon_j \to 0$ and $b^{(j)} = b^{(j-1)}$ in Ω'_{j-1}. For any $j \geq 1$ we now define

$$b(x) = b^{(j)}(x) \quad \forall x \in \Omega'_j. \tag{62}$$

We observe that $b \in L^2_{\rm loc}(\Omega)^n$, since $b^{(j)} \in L^2(\Omega'_j)^n$, $\forall j \geq 1$. Moreover, by construction, if $\varepsilon < \varepsilon_0(\Omega'_j)$ then $-\mathrm{div}_x \tilde{b}^{(j)}_{\varepsilon_j} = g$ in Ω'_j and hence, by (61) and (62) also $-\mathrm{div}_x b = g$ in Ω'_j, for all $j \geq 1$. In particular, this implies that for all $j \geq 1$,

$$-\mathrm{div}_x \tilde{b}^{(j)}_{\varepsilon_j} \to -\mathrm{div}_x b \quad \text{strongly in } H^{-1}(\Omega'_j) \tag{63}$$

as $\varepsilon_j \to 0$. The choice of the diagonal subsequence of ε_j, $j \geq 1$, concludes the proof. □

Now, for any given $\xi \in \mathbb{R}^n$, we consider the solution w_ξ of the cell problem (14). Using its periodic extension to $\mathbb{R}^n$ we define the functions

$$v_\varepsilon(x) = \varepsilon \left[w_\xi\left(\frac{x}{\varepsilon}\right) + \xi \cdot \frac{x}{\varepsilon} \right] = \varepsilon\, w_\xi\left(\frac{x}{\varepsilon}\right) + \xi \cdot x. \tag{64}$$

In virtue of (64) we have

$$v_\varepsilon \to \xi \cdot x \quad \text{strongly in } L^2_{\rm loc}(\mathbb{R}^n), \tag{65}$$

$$\nabla v_\varepsilon = \nabla_y w_\xi + \xi \rightharpoonup \xi \qquad \text{weakly in } L^2_{\rm loc}(\mathbb{R}^n), \tag{66}$$

as $\varepsilon \to 0$.
We are now in the position to introduce an auxiliary operator $a^0 : \mathbb{R}^n \to \mathbb{R}^n$ (see below (78)) that will be the essential tool to prove Theorem 2.6. To this end, we define by $\beta = \beta(y, \xi)$ the function

$$\beta(y, \xi) = a\left(y, \xi + \nabla w_\xi(y)\right). \tag{67}$$

For any $\xi \in \mathbb{R}^n$, the function $\beta(\cdot, \xi) \in [L^2_{\mathrm{loc}}(\mathbb{R}^n)]^n$, it is Y-periodic, and has the following properties:

$$-\mathrm{div}_y\, \beta(y, \xi) = 0 \quad \text{in } \ D'(Y \setminus B), \tag{68}$$

$$\int_{Y\setminus B} \beta(y, \xi) \cdot \nabla\varphi\, dy = 0, \quad \forall\, \varphi \in D'(Y \setminus B) : \ \nabla\varphi|_B = 0. \tag{69}$$

Hence, by Lemma 3.2 (with $g = 0$) there exists an extension

$$\tilde\beta = \tilde\beta(\cdot, \xi) \in L^2(Y)^n \tag{70}$$

such that

$$-\mathrm{div}\, \tilde\beta(y, \xi) = 0 \quad \text{in}\, Y, \text{ in } \ D'(Y), \tag{71}$$

$$\tilde\beta = \beta \quad \text{in}\, Y \setminus B, \tag{72}$$

$$\int_B |\tilde\beta|^2\, dx \leqslant c \int_{Y\setminus B} |\beta|^2\, dx, \tag{73}$$

with c independent of β.
Let us define

$$\tilde\beta_\varepsilon(x) = \tilde\beta\left(\frac{x}{\varepsilon}\right). \tag{74}$$

The εY-periodic function $\tilde\beta_\varepsilon$ has the following properties

$$-\mathrm{div}\tilde\beta_\varepsilon = 0 \text{ in } \mathbb{R}^n, \tag{75}$$

$$\tilde\beta_\varepsilon(x) = \beta\left(\frac{x}{\varepsilon}\right) \text{ in } \mathbb{R}^n \setminus \varepsilon B, \tag{76}$$

and

$$\tilde\beta_\varepsilon \rightharpoonup \frac{1}{|Y|} \int_Y \tilde\beta(y, \xi)\, dy \quad \text{weakly in } L^2_{\mathrm{loc}}(\mathbb{R}^n) \tag{77}$$

Proposition 3.8 *Let $a^0 : \mathbb{R}^n \to \mathbb{R}^n$ be the function defined by*

$$a^0(\xi) = \int_Y \tilde\beta(y, \xi)\, dy \tag{78}$$

where $\beta(y, \xi) = a(y, \xi + \nabla w_\xi)$, $w_\xi \in K_\xi$ solves the cell problem (14), and $\tilde\beta \in L^2(Y)^n$ denotes the extension of β, by means of the operator introduced in Remark 3.3. Then a^0 is strictly monotone, coercive and Lipschitz continuous. More precisely,

$$a^0(0) = 0; \tag{79}$$

$$\alpha|\xi - \eta|^2 \leqslant \left(a^0(\xi) - a^0(\eta)\right) \cdot (\xi - \eta), \quad \forall\, \xi, \eta \in \mathbb{R}^n; \tag{80}$$

$$|a^0(\xi_1) - a^0(\xi_2)| \leqslant L'|\xi_1 - \xi_2|, \quad \forall\, \xi_1, \xi_2 \in \mathbb{R}^n. \tag{81}$$

with Lipschitz constant $L' = c_T L^3 \alpha^{-2}\sqrt{2 + \delta^{-1}}$, $\delta = \mathrm{dist}(\partial B \cap Y, Y)$, and $c_T > 0$ given by (39).

In the proof of Proposition 3.8 we will use the following Lemma.

Lemma 3.9 *Let $G \in [L^2_{\text{per}}(Y)]^n$. If*

$$\int_Y G \cdot \nabla\varphi \, dy = 0, \quad \forall \varphi \in H^1_0(Y) \tag{82}$$

then

$$\int_Y G \cdot \nabla\varphi \, dy = 0, \quad \forall \varphi \in H^1_{\text{per}}(Y). \tag{83}$$

Proof: We split the proof into two steps.
<u>*Step 1*</u> Let $G \in C^\infty_{\text{per}}(\mathbb{R}^n)$. For any $\varphi \in C^\infty_0(Y)$ we have

$$\int_Y G \cdot \nabla\varphi \, dy = 0,$$

so that

$$\operatorname{div} G = 0 \quad \text{in } D'(Y). \tag{84}$$

If $\varphi \in H^1_{\text{per}}(Y)$ then, integrating by parts , by (84) and the periodicity of G we have

$$\int_Y G \cdot \nabla\varphi \, dy = -\int_Y (\operatorname{div} G)\varphi \, dy + \int_{\partial Y} G \cdot n\varphi \, d\sigma = 0, \tag{85}$$

so (83) is proved for $G \in C^\infty_{\text{per}}(\mathbb{R}^n)$.
<u>*Step 2*</u> Let $G \in [L^2_{\text{per}}(Y)]^n$. We will proceed by approximating G by convolution. Let $\rho_h \in C^\infty_0(\mathbb{R}^n)$ be convolution kernels such that $\rho_h \geqslant 0$, $\operatorname{spt}(\rho_h) \subseteq B_{\frac{1}{h}}(0)$ and $\int \rho_h = 1$.
We first show that $G_h = G \star \rho_h$ is Y-periodic, and satisfies (82). Then from step 1, it follows that

$$. \int_Y G_h \cdot \nabla\varphi \, dy = 0, \qquad \text{for all } \varphi \in H^1_{\text{per}}(Y). \tag{86}$$

and then, passing to the limit as $h \to +\infty$, we obtain (83). In order to prove periodicity, let us denote by $(e_i)_{i=1}^n$ the canonical basis of $\mathbb{R}^n$ and consider

$$(G \star \rho_h)(x) = \int_{B_{\frac{1}{h}}(0)} G(y)\rho_h(x-y)\, dy,$$

and

$$(G \star \rho_h)(x + e_i) = \int_{B_{\frac{1}{h}}(0)} G(y)\rho_h(x + e_i - y)\, dy, \tag{87}$$

for any i. Using the periodicity of G and performing the change of variable $y = z + e_i$ in (87) we have

$$\begin{aligned} \int_{B_{\frac{1}{h}}(0)} G(y)\rho_h(x+e_i-y)\, dy &= \int_{B_{\frac{1}{h}}(0)} G(z+e_i)\, \rho_h(x-z)\, dy \\ &= \int_{B_{\frac{1}{h}}(0)} G(z)\, \rho_h(x-z)\, dy = (G \star \rho_h)(x), \end{aligned} \tag{88}$$

which means that $G \star \rho_h$ is Y-periodic. We recall that $G_h = G \star \rho_h \in C^\infty(\mathbb{R}^n)$ and $G_h \to G$ strongly in $L^2_{\text{loc}}(\mathbb{R}^n)$.
Now we prove that G_h satisfies (82) for smooth test functions, i.e.,

$$\int_Y G_h \cdot \nabla\varphi \, dy = 0, \quad \forall\, \varphi \in C^\infty_0(Y). \tag{89}$$

In fact, let $\varphi \in C_0^\infty(Y)$, by Fubini's Theorem and (82) it follows that

$$\begin{aligned}\int_Y G_h \cdot \nabla\varphi \, dy &= \int_Y (G \star \rho_h)(y)\, \nabla\varphi(y)\, dy \\ &= \int_Y \left(\int_{B_{\frac{1}{h}}(0)} G(y-x)\rho_h(x)\, dx \right) \nabla\varphi(y)\, dy \\ &= \int_{B_{\frac{1}{h}}(0)} \left(\int_Y G(y-x)\nabla\varphi(y)\, dy \right) \rho_h(x)\, dx = 0,\end{aligned}$$

where the last equality is due to the fact that

$$\int_Y G(y-x)\nabla\varphi(y)\, dy = \int_{Y-x} G(z)\nabla\varphi(x+z)\, dz = 0,$$

because the support of $\psi(z) = \varphi(x+z)$, which is a subset of $Y - x$, is also contained in Y when $x \in B_{\frac{1}{h}}$ and h is sufficiently large. Finally, (89) implies (82) for G_h, hence by Step 1 we have (86) for G_h, and passing to the limit as $h \to \infty$ we obtain (83) for G. □

Proof of Proposition 3.8 In order to show (79), let us consider the solution of the cell problem (14) for $\xi = 0$, i.e.,

$$\begin{cases} \displaystyle\int_Y a(y, \nabla w_0) \cdot \nabla\varphi \, dy = 0, \quad \forall\varphi \in K_0 \\ w_0 \in K_0. \end{cases} \tag{90}$$

Since $a(y, 0) = 0$ (see assumption (4)) then w_0=const. is solution of the problem (90). Hence, recalling the definition (78) of a^0, from estimate (73) we have

$$\begin{aligned} 0 \leqslant \left| a^0(0) \right| &= \left| \int_Y \tilde{\beta}(y,0)\, dy \right| = \left| \int_Y \tilde{a}(y,0)\, dy \right| \\ &\leqslant \left(c \int_{Y\setminus B} |a(y,0)|^2\, dy \right)^{\frac{1}{2}} = 0, \end{aligned} \tag{91}$$

from which (79).

Now, we prove that a^0 is strictly monotone. Let $\xi, \eta \in \mathbb{R}^n$ be fixed. Considering the identity

$$\begin{aligned} &\left\langle a^0(\xi) - a^0(\eta), \xi - \eta \right\rangle = \\ &= \int_Y [\tilde{a}(y, \xi + \nabla w_\xi) - \tilde{a}(y, \eta + \nabla w_\eta)](\xi + \nabla w_\xi - \eta - \nabla w_\eta)\, dy \\ &+ \int_Y \tilde{a}(y, \xi + \nabla w_\xi)(\nabla w_\eta - \nabla w_\xi)\, dy \\ &+ \int_Y \tilde{a}(y, \eta + \nabla w_\eta)(\nabla w_\xi - \nabla w_\eta)\, dy, \end{aligned} \tag{92}$$

we can first show that the last two terms are zero, i.e.,

$$\int_Y \tilde{a}(y, \xi + \nabla w_\xi)(\nabla w_\eta - \nabla w_\xi)\, dy = 0 = \int_Y \tilde{a}(y, \eta + \nabla w_\eta)(\nabla w_\xi - \nabla w_\eta)\, dy. \tag{93}$$

In fact, since

$$y \mapsto \tilde{a}(y, \xi + \nabla w_\xi(y)) \in [L^2_{\mathrm{per}}(Y)]^n,$$

from (71) it follows that

$$\int_Y \tilde{a}(y, \xi + \nabla w_\xi)\nabla\varphi \, dy = 0, \quad \varphi \in D(Y).$$

Then, in view of Lemma 3.9 it follows that

$$\begin{aligned}&\int_Y \tilde{a}(y,\xi+\nabla w_\xi)\nabla w_\eta\,dy=\int_Y \tilde{a}(y,\xi+\nabla w_\xi)\nabla w_\xi\,dy=\\&=\int_Y \tilde{a}(y,\eta+\nabla w_\eta)\nabla w_\xi\,dy=\int_Y \tilde{a}(y,\eta+\nabla w_\eta)\nabla w_\eta\,dy=0.\end{aligned}\tag{94}$$

From (92), (94), and the fact that $\xi+\nabla w_\xi=0=\eta+\nabla w_\eta$ in B, by the monotoniciy assumption (5) for a we have

$$\begin{aligned}&\langle a^0(\xi)-a^0(\eta),\xi-\eta\rangle=\\&=\int_Y[\tilde{a}(y,\xi+\nabla w_\xi)-\tilde{a}(y,\eta+\nabla w_\eta)](\xi+\nabla w_\xi-\eta-\nabla w_\eta)\,dy\\&=\int_{Y\setminus B}[a(y,\xi+\nabla w_\xi)-a(y,\eta+\nabla w_\eta)](\xi+\nabla w_\xi-\eta-\nabla w_\eta)\,dy\\&\geqslant\alpha\int_Y|\xi+\nabla w_\xi-\eta-\nabla w_\eta|^2\,dy,\end{aligned}\tag{95}$$

which proves that $a^0(\xi)$ is monotone.
Moreover, since w_ξ, w_η are Y-periodic, then

$$\int_Y \nabla w_\xi\,dy=0=\int_Y \nabla w_\eta\,dy,$$

and hence the integral in the last line of (95) can be estimated as

$$\int_Y|\xi+\nabla w_\xi-\eta-\nabla w_\eta|^2\,dy=\int_Y|\xi-\eta|^2\,dy+\int_Y|\nabla w_\xi-\nabla w_\eta|^2\,dy\geq\int_Y|\xi-\eta|^2\,dy,$$

which completes the proof of the strict monotonicity inequality (80).

Let us show that $a^0(\xi)$ is Lipschitz continuous. We split the proof into 3 steps.
Step 1 Let $\xi_1,\xi_2\in\mathbb{R}^n$ be fixed, then

$$\|\xi_1+\nabla w_{\xi_1}-\xi_2-\nabla w_{\xi_2}\|_{L^2(Y)}\leqslant c_1\|\xi_1-\xi_2+\nabla w_{\xi_1-\xi_2}\|_{L^2(Y)}.\tag{96}$$

We choose two test functions M_1 and M_2 defined as

$$M_1=w_{\xi_2}+w_{\xi_1-\xi_2}-w_{\xi_1},\tag{97}$$

$$M_2=w_{\xi_1}-w_{\xi_1-\xi_2}-w_{\xi_2}.\tag{98}$$

where $w_\eta\in K_\eta$ denotes the solution of the cell problem (14), for $\eta=\xi_1,\xi_2$ and $\xi_1-\xi_2$ respectively. Clearly $M_1,M_2\in H_0$, then substituting (97) and (98) into (14) we obtain

$$\int_Y a(y,\xi_1+\nabla w_{\xi_1})\cdot(\nabla w_{\xi_2}+\nabla w_{\xi_1-\xi_2}-\nabla w_{\xi_1})\,dy=0,\tag{99}$$

$$\int_Y a(y,\xi_2+\nabla w_{\xi_2})\cdot(\nabla w_{\xi_1}-\nabla w_{\xi_1-\xi_2}-\nabla w_{\xi_2})\,dy=0.\tag{100}$$

Adding up (99) and (100) we obtain

$$\int_Y[a(y,\xi_1+\nabla w_{\xi_1})-a(y,\xi_2+\nabla w_{\xi_2})]\cdot(\nabla w_{\xi_2}+\nabla w_{\xi_1-\xi_2}-\nabla w_{\xi_1})\,dy=0,$$

that is equivalent to

$$\begin{aligned}A&=\int_Y[a(y,\xi_1+\nabla w_{\xi_1})-a(y,\xi_2+\nabla w_{\xi_2})]\cdot(\xi_1+\nabla w_{\xi_1}-\xi_2-\nabla w_{\xi_2})\,dy\\&=\int_Y[a(y,\xi_1+\nabla w_{\xi_1})-a(y,\xi_2+\nabla w_{\xi_2})]\cdot(\xi_1-\xi_2+\nabla w_{\xi_1-\xi_2})\,dy=B.\end{aligned}\tag{101}$$

Since $a(y,\cdot)$ is strictly monotone we have

$$A \geqslant \alpha \int_Y |\xi_1 + \nabla w_{\xi_1} - \xi_2 - \nabla w_{\xi_2}|^2 \, dy = \alpha \, \|\xi_1 + \nabla w_{\xi_1} - \xi_2 - \nabla w_{\xi_2}\|^2_{L^2(Y)}, \tag{102}$$

on the other hand by the Cauchy-Schwartz inequality and since $a(y,\cdot)$ is Lipschitz continuous we get

$$\begin{aligned} B &\leqslant \left(\int_Y |a(y,\xi_1 + \nabla w_{\xi_1}) - a(y,\xi_2 + \nabla w_{\xi_2})|^2 \, dy \right)^{\frac{1}{2}} \left(\int_Y |\xi_1 - \xi_2 + \nabla w_{\xi_1 - \xi_2}|^2 \, dy \right)^{\frac{1}{2}} \\ &\leqslant L \left(\int_Y |\xi_1 + \nabla w_{\xi_1} - \xi_2 - \nabla w_{\xi_2}|^2 \, dy \right)^{\frac{1}{2}} \left(\int_Y |\xi_1 - \xi_2 + \nabla w_{\xi_1 - \xi_2}|^2 \, dy \right)^{\frac{1}{2}} \\ &= L \, \|\xi_1 + \nabla w_{\xi_1} - \xi_2 - \nabla w_{\xi_2}\|_{L^2(Y)} \, \|\xi_1 - \xi_2 + \nabla w_{\xi_1 - \xi_2}\|_{L^2(Y)}. \end{aligned} \tag{103}$$

Finally, from (101), (102) and (103) we obtain (96) where $c_1 = \dfrac{L}{\alpha}$.

<u>*Step 2*</u>

$$\|\xi + \nabla w_\xi\|_{L^2(Y)} \leqslant c_2 |\xi|, \quad \forall \, \xi \in \mathbb{R}^n. \tag{104}$$

Let $\xi \in \mathbb{R}^n$ be fixed. We consider the following test function

$$z_\xi^\delta = \begin{cases} -\xi \cdot y & \text{if} \quad y \in B, \\ -\left(1 - \frac{\operatorname{dist}(y,B)}{\delta}\right)(\xi \cdot y) + \frac{\operatorname{dist}(y,B)}{\delta} \mu_\xi & \text{if} \quad 0 \leqslant \operatorname{dist}(y,B) \leqslant \delta, \\ \mu_\xi & \operatorname{dist}(y,B) > \delta, \end{cases} \tag{105}$$

where μ_ξ is chosen so that z_ξ^δ has zero mean-value in Y. We observe that $z_\xi^\delta \in K_\xi$. Since $|\nabla z_\xi^\delta| \leqslant |\xi| \left(1 + \frac{1}{\delta}\right)$ we have

$$\left\|\nabla z_\xi^\delta\right\|_{L^2(Y)^n} \leqslant |\xi| \left(1 + \frac{1}{\delta}\right). \tag{106}$$

Then, since $a(y,\cdot)$ is strictly monotone and Lipschitz continuous, by the Cauchy-Schwarz inequality, assumption (4) and taking into account (14) with $\varphi = w_\xi - z_\xi^\delta$ we have

$$\begin{aligned} \alpha \int_Y |\xi + \nabla w_\xi|^2 \, dy &\leqslant \int_Y a(y, \xi + \nabla w_\xi) \cdot (\xi + \nabla w_\xi) \, dy \\ &= \int_Y a(y, \xi + \nabla w_\xi) \cdot (\xi + \nabla z_\xi^\delta) \, dy \\ &\leqslant L \, \|\xi + \nabla w_\xi\|_{L^2(Y)} \left\|\xi + \nabla z_\xi^\delta\right\|_{L^2(Y)}. \end{aligned} \tag{107}$$

Then, by (106) and the fact that $z_\xi^\delta \in K_\xi$ we have

$$\begin{aligned} \left\|\xi + \nabla z_\xi^\delta\right\|^2_{L^2(Y)} &= |\xi|^2 + \int_Y |\nabla z_\xi^\delta|^2 \, dy \\ &\leqslant \left(2 + \frac{1}{\delta}\right) |\xi|^2. \end{aligned} \tag{108}$$

Finally from (107) and (108) we obtain

$$\left(\int_Y |\xi + \nabla w_\xi|^2 \, dy \right)^{\frac{1}{2}} \leqslant \frac{L}{\alpha} \left(2 + \frac{1}{\delta}\right)^{\frac{1}{2}} |\xi|, \tag{109}$$

which is (104) with $c_2 = \dfrac{L}{\alpha} \left(2 + \dfrac{1}{\delta}\right)^{\frac{1}{2}}$.

Step 3 Now, we prove that a^0 is Lipschitz continuous and satisfies (81). Recalling the definition (78) of a^0, using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}|a^0(\xi_1) - a^0(\xi_2)| &= \left| \int_Y [\tilde{a}(y, \xi_1 + \nabla w_{\xi_1}) - \tilde{a}(y, \xi_2 + \nabla w_{\xi_2})]\, dy \right| \\ &\leqslant \left(\int_Y |\tilde{a}(y, \xi_1 + \nabla w_{\xi_1}) - \tilde{a}(y, \xi_2 + \nabla w_{\xi_2})|^2\, dy \right)^{\frac{1}{2}} \\ &\leqslant c_T \left(\int_{Y \setminus B} |a(y, \xi_1 + \nabla w_{\xi_1}) - a(y, \xi_2 + \nabla w_{\xi_2})|^2\, dy \right)^{\frac{1}{2}},\end{aligned} \tag{110}$$

where the last inequality is due to the continuity of the extension operator (see Remark 3.3). Then, by the Lipschitz-continuity (6) of a, estimates (96), and (104) with $\xi = \xi_1 - \xi_2$, we conclude that

$$\begin{aligned}|a^0(\xi_1) - a^0(\xi_2)| &\leq L c_1 c_2 |\xi_1 - \xi_2| \\ &= c_T \frac{L^3}{\alpha^2} \left(2 + \frac{1}{\delta} \right)^{1/2} |\xi_1 - \xi_2|,\end{aligned} \tag{111}$$

that is (81) with $L' = \dfrac{L^3}{\alpha^2} \left(2 + \dfrac{1}{\delta} \right)^{1/2} c_T$. □

In the following proposition we show that the function a^0 introduced in (78) does not depend on the extension operators nor on the particular subsequence and actually coincides with the function a_{hom} defined by (15).

Proposition 3.10 *Let a^0 and a_{hom} be defined by (78) and (15) respectively. Then $a^0 = a_{\text{hom}}$, i.e.*

$$a^0(\xi) \cdot \eta = \int_{Y \setminus B} a(y, \xi + \nabla w_\xi) \cdot (\eta + \nabla w_\eta)\, dy, \quad \forall\, \xi, \eta \in \mathbb{R}^n. \tag{112}$$

Proof: Here, for simplicity of notation, we assume the function $a(y, \xi + \nabla w_\xi)$ regular enough to perform standard integrations by parts (see Remark 3.11 below). We split the proof into three steps.
Step 1 Let us show that

$$\begin{aligned}a^0(\xi) \cdot \eta &= \int_Y \tilde{a}(y, \xi + \nabla w_\xi) \cdot \eta\, dy \\ &= \int_{Y \setminus B} a(y, \xi + \nabla w_\xi) \cdot \eta\, dy - \int_{\partial B} a(y, \xi + \nabla w_\xi) \cdot \nu_B^{\text{est}} (\eta \cdot y)\, d\sigma, \quad \forall\, \xi, \eta \in \mathbb{R}^n\end{aligned} \tag{113}$$

where ν_B^{est} denotes the outward unit normal to ∂B. We observe that

$$\int_Y \tilde{a}(y, \xi + \nabla w_\xi) \cdot \eta\, dy = \int_{Y \setminus B} a(y, \xi + \nabla w_\xi) \cdot \eta\, dy + \int_B \tilde{a}(y, \xi + \nabla w_\xi) \cdot \eta\, dy, \tag{114}$$

furthermore, integrating by parts the second integral of the right hand side with $\eta = \nabla(\eta \cdot y)$ we have

$$\int_B \tilde{a}(y, \xi + \nabla w_\xi) \cdot \eta\, dy = - \int_{\partial B} \tilde{a}(y, \xi + \nabla w_\xi) \cdot \nu_B^{\text{est}} (\eta \cdot y)\, d\sigma + \int_B div\ \tilde{a}(y, \xi + \nabla w_\xi)(\eta \cdot y)\, dy. \tag{115}$$

But $div\ \tilde{a}(y, \xi + \nabla w_\xi) = 0$ by Lemma 3.2 with $z(y) = a(y, \xi + \nabla w_\xi(y))$ and $g = 0$, so that

$$\int_B div\ \tilde{a}(y, \xi + \nabla w_\xi)(\eta \cdot y)\, dy = 0,$$

and then

$$\int_B \tilde{a}(y, \xi + \nabla w_\xi) \cdot \eta \, dy = -\int_{\partial B} a(y, \xi + \nabla w_\xi) \cdot \nu_B^{\text{est}} (\eta \cdot y) \, d\sigma. \tag{116}$$

Hence, by (114), (115) and (116) statement (113) follows.

<u>*Step 2*</u> Let us show that

$$\int_{Y \setminus B} a(y, \xi + \nabla w_\xi) \cdot \nabla w_\eta \, dy = \int_{\partial B \cap Y} a(y, \xi + \nabla w_\xi) \cdot \nu_B^{\text{est}} \, w_\eta \, d\sigma, \quad \forall \xi, \eta \in \mathbb{R}^n, \tag{117}$$

where ν_B^{est} denotes the outward unit normal to ∂B.
Since by (14)

$$-\text{div}\, a(y, \xi + \nabla w_\xi) = 0 \text{ in } Y \setminus B, \tag{118}$$

with $w_\xi \in K_\xi$ we have

$$\int_{Y \setminus B} \text{div}\, a(y, \xi + \nabla w_\xi) w_\eta \, dy = 0, \tag{119}$$

with $w_\eta \in K_\eta$.
Then, integrating by parts (119) we obtain

$$\int_{Y \setminus B} a(y, \xi + \nabla w_\xi) \nabla w_\eta \, dy + \int_{\partial (Y \setminus B)} a(y, \xi + \nabla w_\xi) \cdot \nu_{Y \setminus B}^{\text{est}} \, w_\eta \, d\sigma = 0. \tag{120}$$

Now, taking into account that the integral on ∂Y is zero by the periodicity, replacing $\nu_{Y \setminus B}^{\text{est}} = -\nu_B^{\text{est}}$ we get (117).

<u>*Step 3*</u> By Step 1 and Step 2, we have

$$\begin{aligned} a_{\text{hom}}(\xi) \cdot \eta &= \int_{Y \setminus B} a(y, \xi + \nabla w_\xi) \cdot (\eta + \nabla w_\eta) \, dy \\ &= a^0(\xi) \cdot \eta + \int_{\partial B \cap Y} a(y, \xi + \nabla w_\xi) \cdot \nu_B^{\text{est}} \, \eta \cdot y \, d\sigma \\ &+ \int_{\partial B \cap Y} a(y, \xi + \nabla w_\xi) \cdot \nu_B^{\text{est}} \, w_\eta \, d\sigma \\ &= a^0(\xi) \cdot \eta + \int_{\partial B \cap Y} a(y, \xi + \nabla w_\xi) \cdot \nu_B^{\text{est}} \, (\eta \cdot y + w_\eta) \, d\sigma. \end{aligned} \tag{121}$$

Since the function $(\eta \cdot y + w_\eta)$ has constant trace on (the connected components of) $\partial B \cap Y$, by (31), (32) the last integral of (121) is zero and (112) follows. $\square$

Remark 3.11 In the previous proof, in the general case all boundary integrals can be understood in the sense of the duality between $H^{1/2}$ and $H^{-1/2}$.

Corollary 3.12 *The function a_{hom} has the same properties of a^0.*

Proof of Theorem 2.6

Let $\{\Omega'_j\}$ be an increasing sequence of open subsets of Ω as in Proposition 3.7.Since $a(y, \cdot)$is monotone, it follows that

$$\left(a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon(x)\right) - a\left(\frac{x}{\varepsilon}, \nabla v_\varepsilon(x)\right)\right) \cdot (\nabla u_\varepsilon(x) - \nabla v_\varepsilon(x)) \geqslant 0, \quad \text{for a.e. } x \in \Omega,$$

where u_ε is the solution of (3) and v_ε is defined by (64). Then, for anyfixed $\varphi \in D(\Omega)$, with $\varphi \geqslant 0$ there exists $j \geq 1$ such that $\text{spt}\, \varphi \subset \Omega'_j \subset\subset \Omega$ and we have

$$\int_\Omega \left(a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon(x)\right) - a\left(\frac{x}{\varepsilon}, \nabla v_\varepsilon(x)\right)\right) \cdot (\nabla u_\varepsilon(x) - \nabla v_\varepsilon(x))\varphi(x) \, dx \geqslant 0. \tag{122}$$

Moreover, we observe that $\nabla u_\varepsilon - \nabla v_\varepsilon = -(\xi + \nabla w_\xi(y)) = 0$ in $\varepsilon B \cap \Omega$. Then considering the extensions $\tilde{b}_\varepsilon^{(j)}(x)$ of b_ε defined by (55) for $\Omega' = \Omega'_j$ and the periodic extension of $\beta(\frac{x}{\varepsilon}) = a(\frac{x}{\varepsilon}, \nabla v_\varepsilon(x))$ to $\mathbb{R}^n$ defined by (74),inequality (122) can be cast as

$$\int_{\Omega'_j} \left(\tilde{b}_\varepsilon^{(j)}(x) - \tilde{\beta}_\varepsilon(x)\right) \cdot (\nabla u_\varepsilon(x) - \nabla v_\varepsilon(x))\varphi(x)\, dx \geqslant 0. \tag{123}$$

In view of Remark 2.7, (56), (57), (58), (65), (66), (75), (78) and Proposition 3.10 we have

$$\begin{cases} u_\varepsilon - v_\varepsilon \rightharpoonup u - \xi \cdot x & \text{weakly in } H^1(\Omega), \\ \tilde{b}_\varepsilon^{(j)} - \tilde{\beta}_\varepsilon \rightharpoonup b(x) - a_{\text{hom}}(\xi) & \text{weakly in } L^2(\Omega'_j)^n, \\ -div\, \tilde{b}_\varepsilon^{(j)} - div\, \tilde{\beta}_\varepsilon = g \to g & \text{strongly in } H^{-1}(\Omega'_j). \end{cases}$$

Then, recalling Proposition 2.8, we can pass to the limit in (3) using compensated compactness and we get

$$\int_\Omega (b(x) - a_{\text{hom}}(\xi)) \cdot (\nabla u(x) - \xi)\varphi(x)\, dx \geqslant 0 \quad \forall\, \varphi \in C_0^\infty(\Omega),\ \varphi \geqslant 0, \tag{124}$$

which implies

$$(b(x) - a_{\text{hom}}(\xi)) \cdot (\nabla u(x) - \xi) \geqslant 0 \quad \forall\, \xi \in \mathbb{Q}^n, \forall\, x \in \Omega \setminus N_\xi, \text{ with } |N_\xi| = 0. \tag{125}$$

Now, denoting $N = \bigcup_{\xi \in \mathbb{Q}^n} N_\xi$ it follows that

$$(b(x) - a_{\text{hom}}(\xi)) \cdot (\nabla u(x) - \xi) \geqslant 0 \quad \forall\, \xi \in \mathbb{Q}^n, \forall\, x \in \Omega \setminus N, \text{ with } |N| = 0, \tag{126}$$

which means

$$(b(x) - a_{\text{hom}}(\xi)) \cdot (\nabla u(x) - \xi) \geqslant 0, \quad \text{a.e. in } \Omega,\ \forall\, \xi \in \mathbb{Q}^n. \tag{127}$$

By the continuity of a_{hom} (see Proposition 2.5) it follows that

$$(b(x) - a_{\text{hom}}(\xi), \nabla u(x) - \xi) \geqslant 0, \quad \text{a.e. in } \Omega,\ \forall\, \xi \in \mathbb{R}^n. \tag{128}$$

Choosing $\xi = \nabla u(x) + t\eta$, diving by t, separately for $t > 0$, $t < 0$, and then letting $t \to 0$, by the continuity of a_{hom} we get

$$(b(x) - a_{\text{hom}}(\nabla u(x)), \eta) = 0 \tag{129}$$

and from the arbitrariness of $\eta \in \mathbb{R}^n$ we conclude that

$$b(x) = a_{\text{hom}}(\nabla u(x)). \tag{130}$$

In view of the strict monotonicity of a_{hom} (see Proposition 2.5 and 3.10) we can conclude that the whole sequence u_ε tends to the unique solution u of the homogenized equation (19). □

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