Strichartz estimates for the metaplectic representation

(Article begins on next page)

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Abstract. We provide new estimates for the matrix coefficients of the metaplectic representation, inspired by a formal analogy with the Strichartz estimates which hold for several classes of evolution propagators $U(t)$. The one parameter group of unitary operators $U(t)$ is replaced by a unitary representation of a non-compact Lie group, the group element playing the role of time; the case of the metaplectic or oscillatory representation is of special interest in this connection, because the Schrödinger group is a subgroup of the metaplectic group. We prove uniform weak-type sharp estimates for matrix coefficients and Strichartz-type estimates for that representation. The crucial point is the choice of function spaces able to detect such a decay, which in general will depend on the given group action. The relevant function spaces here turn out to be the so-called modulation spaces from Time-frequency Analysis in Euclidean space, and Lebesgue spaces with respect to Haar measure on the metaplectic group. The proofs make use in an essential way of the covariance of the Wigner distribution with respect to the metaplectic representation.

1. Introduction and statement of the main results

We study some new estimates for matrix coefficients of the metaplectic representations which are inspired by a formal analogy with the dispersive and Strichartz estimates in PDEs (see e.g. [26] and the references therein). Namely, we know that the Schrödinger propagator $U(t) = e^{it\Delta}$ in $\mathbb{R}^n$ satisfies the so-called *dispersive estimate*

$$\|U(t)\psi\|_{L^\infty} \lesssim |t|^{-n/2}\|\psi\|_{L^1}$$

as well as mixed-norm estimates, known as *Strichartz estimates*, which read

$$\|U(t)\psi\|_{L^q(\mathbb{R};L^r(\mathbb{R}^n))} \lesssim \|\psi\|_{L^2(\mathbb{R}^n)}$$

for $\frac{2}{q} + \frac{n}{r} = \frac{n}{2}$, $2 \leq q$, $r \leq \infty$ and $(q,r,n) \neq (2,\infty,2)$.

The propagator $U(t)$ is a strongly continuous unitary representation of the abelian group $\mathbb{R}$. Now, for a non-compact abelian group G, the *irreducible* unitary representations are one-dimensional and their matrix coefficients are just (multiples of) the group characters, with no decay at all. The above decay is in part due to a lack of "coherence" of the irreducible components of $U(t)$: frequency components move in different directions and with different speeds.

Motivated by the importance of decay estimates in representation theory and ergodic theory (see e.g. [19, 21] and the references therein), Strichartz-type estimates seem worth investigating for strongly continuous unitary representations $\mu : G \to \mathcal{U}(H)$ of a non-compact locally compact group G, where H is a Hilbert space. The representation $\mu(g)$ plays now the role of the above propagator $U(t)$. Generally speaking, we are interested in estimates of the type

$$(1.1) \qquad \qquad \|\mu(\cdot)\psi\|_{L^q(G;X_\theta)} \lesssim \|\psi\|_H$$

for some scale of Banach spaces X_θ, valid for a range of pairs (q,θ).

In this note we develop this idea for the metaplectic group $G = Mp(n,\mathbb{R})$, that is the double covering of the symplectic group $Sp(n,\mathbb{R})$, and the corresponding metaplectic, or oscillatory, representation, first constructed by Segal and Shale [24, 25] in the framework of quantum mechanics (see also van Hove [17]) and by Weil [29] in number theory. This is a strongly continuous unitary representation of $Mp(n,\mathbb{R})$ in $L^2(\mathbb{R}^n)$, which turns out to be faithful, so that we can think of $Mp(n,\mathbb{R})$ as a subgroup of $\mathcal{U}(L^2(\mathbb{R}^n))$, and the representation is given just by the inclusion. Following [12] we will therefore denote by $\widehat{S}$ a metaplectic operator and by $S = \pi(\widehat{S}) \in Sp(n,\mathbb{R})$ its projection in the symplectic group.

Now, it turns out that the operator $e^{it\Delta}$ is a particular metaplectic operator, so that a natural candidate for the spaces X_θ in (1.1) would seem to be the Lebesgue spaces. However, the Fourier transform is itself a metaplectic operator, and therefore we should actually look for spaces invariant with respect to the action of the Fourier transform. $U(n)$-invariance (see Section 4) finally suggests, as right function spaces, the modulation spaces M^p, widely used in Time-frequency Analysis; see [12, 15] and also [10, 11] for the original source and a historical perspective.

In short, for a given Schwartz function $\varphi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$, consider the time-frequency shifts $\varphi_z(y) = e^{i\xi \cdot y} \varphi(y - x)$, $z = (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$. Then for $1 \leq p \leq \infty$ we define the M^p norm of $\psi \in \mathcal{S}'(\mathbb{R}^n)$, as

$$\|\psi\|_{M^p} = \Big(\int_{\mathbb{R}^{2n}} |\langle \psi, \varphi_z \rangle|^p \, dz \Big)^{1/p}$$

(with obvious changes when $p = \infty$). Different windows φ give equivalent norms. We have $\mathcal{S}(\mathbb{R}^n) \subset M^p \subset \mathcal{S}'(\mathbb{R}^n)$ for every $1 \leq p \leq \infty$, $M^2 = L^2(\mathbb{R}^n)$, $M^p \subset M^q$ if and only if $p \leq q$, $(M^p)' = M^{p'}$ if $p < \infty$. Modulation space norms measure the phase space concentration of a function; roughly speaking we can think of a function in M^p as a function having L^p decay at infinity and FL^p local regularity. Let us also observe that modulation spaces have been recently applied in PDEs by

several authors, see e.g. [3, 4, 5, 23, 28] and the references therein (some of their properties are collected in Section 2).

We begin with a uniform pointwise decay estimate for matrix coefficients.

Theorem 1 (Uniform pointwise estimate). *The following estimate holds:*

$$(1.2) \qquad |\langle \widehat{S}\psi, \varphi\rangle| \lesssim (\lambda_1(S)\ldots\lambda_n(S))^{-1/2}\|\psi\|_{M^1}\|\varphi\|_{M^1}$$

for $\widehat{S} \in Mp(n, \mathbb{R})$, $\psi, \varphi \in \mathcal{S}(\mathbb{R}^n)$, where $\lambda_1(S), \ldots, \lambda_n(S)$ are the singular values ≥ 1 of $S = \pi(\widehat{S}) \in Sp(n, \mathbb{R})$.

The result is sharp as far as the decay is concerned (see Section 4).

As a consequence we can obtain the following weak-type estimates.

Corollary 2 (Uniform weak-type estimate). *Let $G = Mp(n, \mathbb{R})$, endowed with its Haar measure. The following estimate holds:*

$$(1.3) \qquad \|\langle \widehat{S}\psi, \varphi\rangle\|_{L^{4n,\infty}(G)} \lesssim \|\psi\|_{M^1}\|\varphi\|_{M^1},$$

for $\psi, \varphi \in \mathcal{S}(\mathbb{R}^n)$.

Here $L^{4n,\infty}$ is the weak-type L^{4n} space on $G = Mp(n, \mathbb{R})$.

Corollary 2 refines a result by Howe [18], who proved that for fixed $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^n)$ the matrix coefficient in (1.3) is in $L^{4n+\epsilon}$ for every $\epsilon > 0$ but in general not in L^{4n}. In fact, in the proof of this result we use the KAK decomposition and, in the subsequent Proposition 5, bi-invariant functions as in [18].

Estimates for matrix coefficients have a long tradition in representation theory, see for example [7, 9, 18, 19, 22] and the references therein. Usually, dealing with a unitary representation of a group G in a Hilbert space H, one takes φ_1, φ_2 in K-invariant finite dimensional subspaces of H, $K \subset G$ being a maximal compact subgroup, and the constants in the estimates will depend on the dimension of such subspaces. Sometimes this finiteness condition is replaced by taking φ_1, φ_2 in higher order Sobolev-type spaces, and often an ϵ-loss in the decay appears, as above (see e.g. [21]). On the contrary, in (1.3) we have the low regularity space M^1, and functions in M^1 do not need to have any differentiability, even in a fractional sense.

Weak-type estimates for matrix coefficients such as (1.3) seem of great interest in their own right; for example, they could play a key role in extending Cowling's strengthened version of the Kunze-Stein phenomenon [8] to groups of rank higher than 1.

As a consequence of the dispersive estimates we therefore obtain the following Strichartz-type estimates.

Theorem 3 (Strichartz estimates). *Let $G = Mp(n, \mathbb{R})$, endowed with its Haar measure. The following estimates hold:*

$$\|\widehat{S}\psi\|_{L^q(G;M^r)} \lesssim \|\psi\|_{L^2},$$

for

$$\frac{4n}{q} + \frac{1}{r} \leq \frac{1}{2}, \quad 2 \leq q, r \leq \infty.$$

The range of admissible pairs (q, r) in Theorem 3 is represented in Figure 1, which also shows a comparison with the case of the Schrödinger group (as already observed, the one-parameter group $e^{it\Delta}$ is a subgroup of $Mp(n, \mathbb{R})$). Notice however that the exponent r refers to different function spaces; in fact we have $L^r \subset M^r$ for $2 \le r \le \infty$, with strict inclusion when $r > 2$. As one can see, the admissibility condition implies $q \ge 8n$. Also, we have a whole region of admissible pairs, and not just a segment, because the modulation spaces M^r are nested, unlike the Lebesgue spaces. Let us observe that, compared with the trivial estimate for $q = \infty, r = 2$, the other admissible pairs (q, r) represent a gain (loss) of time (space) decay at infinity, as in the case of the propagator $e^{it\Delta}$, but we do no longer have any smoothing effect, as expected: among the metaplectic operators we also meet linear changes of variables, which do not produce smoothing in any reasonable space. This is in turn related to the fact that $M^1 \subset M^\infty$ in (1.2).
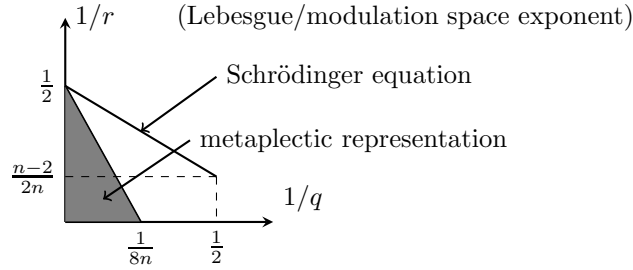

Figure 1: Admissible pairs for Strichartz estimates

Let us observe that similar estimates seem worth investigating for other unitary representations, e.g. the oscillatory representation restricted to subgroups of $Mp(n, \mathbb{R})$ (cf. [1, 2, 6]), unitary representations of linear Lie groups such as $SL(n, \mathbb{R})$ or more general semisimple Lie groups. Part of the problem is to identify low regularity spaces strictly tailored to the given representation, playing the role of the modulation spaces used here. We plan to carry on this investigation in future work.

The paper is organized as follows. In Section 2 we recall some preliminary results on time-frequency methods used in the proofs of the main results. That material is mainly extracted from [12]; see also [14]. Section 3 is devoted to the proof of the above results. Finally in Section 4 we collected some concluding remarks.

2. Preliminaries

We recall here a number of definitions and results that we will use in the following. We refer to [12, 16, 20] for details.

2.1. Notation

We denote by $\langle \cdot, \cdot \rangle$ the inner product in $L^2(\mathbb{R}^n)$, linear in the first argument. The notation $A \lesssim B$, for expressions $A, B \geq 0$, means $A \leq CB$ for a constant C depending only on the dimension n and parameters which are fixed in the context. We also write $A \asymp B$ for $A \lesssim B$ and $B \lesssim A$.

The symplectic group is denoted by $Sp(n, \mathbb{R})$, and we set $U(2n, \mathbb{R}) := Sp(n, \mathbb{R}) \cap O(2n, \mathbb{R}) \simeq U(n)$. We also set

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

We will need the Young inequality for weak type spaces, which reads as follows.

On a measure space X, for $0 < p < \infty$ the weak-type Lebesgue space $L^{p,\infty}(X)$ is defined as the space of measurable functions $f : X \to \mathbb{C}$ such that

$$\|f\|_{L^{p,\infty}} := \sup_{\lambda > 0} \{\lambda \cdot (\mathrm{meas}\{x : |f(x)| \geq \lambda\})^{1/p}\} < \infty.$$

Let now G be a unimodular locally compact Hausdorff group. Let

$$1 < p, q, r < \infty, \quad \frac{1}{p} + \frac{1}{r} = \frac{1}{q} + 1.$$

Then there exists a constant $C_{p,q,r} > 0$ such that for all f in $L^p(G)$ and g in $L^{r,\infty}(G)$ we have

$$(2.1) \qquad \|f * g\|_{L^q(G)} \leq C_{p,q,r} \|g\|_{L^{r,\infty}(G)} \|f\|_{L^p(G)}.$$

2.2. Modulation spaces

Fix a window function $\varphi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$. The short-time Fourier transform (STFT) of a function/temperate distribution $\psi \in \mathcal{S}'(\mathbb{R}^n)$ with respect to φ is defined by

$$V_\varphi \psi(x, \xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i\xi \cdot y} \psi(y) \overline{\varphi(y - x)} dy, \quad x, \xi \in \mathbb{R}^n.$$

For $1 \leq p, q \leq \infty$ and a Schwartz function $\varphi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$, the modulation space $M^{p,q}(\mathbb{R}^n)$ is defined as the space of $\psi \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|\psi\|_{M^{p,q}} := \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |V_\varphi \psi(x, \xi)|^p dx \right)^{q/p} d\xi \right)^{1/q} < \infty,$$

with obvious changes if $p = \infty$ or $q = \infty$.

If $p = q$, then we write M^p instead of $M^{p,p}$.

We will also need a variant, sometimes called Wiener amalgam norm in the literature, defined by

$$\|\psi\|_{W(FL^p, L^q)} := \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |V_\varphi \psi(x, \xi)|^p d\xi \right)^{q/p} dx \right)^{1/q},$$

where the Lebesgue norms appear in the inverse order. Both these norms provide a measure of the time-frequency concentration of a function and are widely used in Time-frequency Analysis [12, 15].

We have $M^{p_1,q_1} \subseteq M^{p_2,q_2}$ if and only if $p_1 \le p_2$ and $q_1 \le q_2$. Similarly $W(FL^{p_1}, L^{q_1}) \subseteq W(FL^{p_2}, L^{q_2})$ if and only if $p_1 \le p_2$ and $q_1 \le q_2$.

The duality goes as expected:

$$(2.2) \qquad (M^{p,q})' = M^{p',q'}, \quad 1 \le p, q < \infty,$$

and in particular

$$|\langle f, g \rangle| \lesssim \|f\|_{M^p} \|g\|_{M^{p'}}.$$

Particularly important is the case of the Gelfand triple

$$M^1 \subset L^2(\mathbb{R}^n) \subset M^\infty.$$

We observe that

$$\mathcal{S}(\mathbb{R}^n) \subset M^1 \subset L^2(\mathbb{R}^n)$$

with dense and strict inclusions. For atomic characterizations of the space M^1 we refer to [12, 15].

We will also use the complex interpolation theory for modulation spaces, which reads as follows: for $1 \le p, q, p_i, q_i \le \infty$, $i = 0, 1$, $0 \le \theta \le 1$,

$$\frac{1}{p} = \frac{1-\vartheta}{p_0} + \frac{\vartheta}{p_1}, \quad \frac{1}{q} = \frac{1-\vartheta}{q_0} + \frac{\vartheta}{q_1},$$

we have

$$(M^{p_0,q_0}, M^{p_1,q_1})_\vartheta = M^{p,q}.$$

2.3. The Wigner distribution

We now introduce a quadratic time-frequency distribution which will play a key role in the following. Again it represents a basic tool in the analysis of signals [15] and in phase space Quantum Mechanics [12, 13]. We refer to [12, 13] for details.

The cross-Wigner distribution $W(\psi, \varphi)$ of functions $\psi, \varphi \in L^2(\mathbb{R}^n)$ is defined to be

$$W(\psi, \varphi)(x, \xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i\xi \cdot y} \psi\left(x + \frac{y}{2}\right) \overline{\varphi\left(x - \frac{y}{2}\right)} dy.$$

We also set $W\psi = W(\psi, \psi)$.

We recall the important *Moyal identity* (see e.g. [12, Theorem 182]):

$$(2.3) \qquad \langle W\psi, W\varphi \rangle_{L^2(\mathbb{R}^{2n})} = (2\pi)^{-n} |\langle \psi, \varphi \rangle|^2.$$

We will also need the following estimates.

Proposition 4. *We have*

$$(2.4) \qquad \|W(\psi, \varphi)\|_{L^1(\mathbb{R}^{2n})} \lesssim \|\varphi\|_{M^1} \|\psi\|_{M^1},$$

$$(2.5) \qquad \int_{\mathbb{R}^n} \sup_{x \in \mathbb{R}^n} |W(\psi, \varphi)(x, \xi)| d\xi \lesssim \|\varphi\|_{M^1} \|\psi\|_{M^{\infty,1}}$$

and

$$(2.6) \qquad \int_{\mathbb{R}^n} \sup_{\xi \in \mathbb{R}^n} |W(\psi, \varphi)(x, \xi)| dx \lesssim \|\varphi\|_{M^1} \|\psi\|_{W(FL^\infty, L^1)}.$$

Proof. Formula (2.4) is proved in [12, Proposition 3.6.5].

Let us prove (2.5) and (2.6). It is easy to see that

$$(2.7) \qquad |W(\psi, \varphi)(x, \xi)| = 2^n |V_\varphi \psi(2x, 2\xi)|$$

so that it is sufficient to prove similar estimates with $W(\psi, \varphi)(x, \xi)$ replaced by $V_\varphi \psi(x, \xi)$. To this end we recall from [15, Lemma 11.3.3] that, for $\varphi, \varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ such that $\|\varphi_0\| \neq 0$ and $\psi \in \mathcal{S}'(\mathbb{R}^n)$ we have

$$(2.8) \qquad |V_\varphi \psi(x, \xi)| \lesssim \frac{1}{\|\varphi_0\|^2} (|V_{\varphi_0}\psi| * |V_\varphi \varphi_0|)(x, \xi),$$

for all $(x, \xi) \in \mathbb{R}^{2n}$.

Now, we apply this inequality with a fixed Schwartz window φ_0 and we also observe that

$$|V_\varphi \varphi_0(x, \xi)| = |V_{\varphi_0}\varphi(-x, -p)|.$$

The desired estimates for $V_\varphi \psi(x, \xi)$ then follow by applying the Young inequality for mixed-norm Lebesgue spaces in (2.8). $\qquad\square$

We have already recalled in the Introduction the existence of the metaplectic representation $\mu : Mp(n, \mathbb{R}) \to \mathcal{U}(L^2(\mathbb{R}^n))$. One of the most important property of the cross–Wigner distribution (essentially, the defining property of the metaplectic representation) is its covariance with respect to the action of metaplectic operators (see e.g. [12, Corollary 2.17]). Namely

$$(2.9) \qquad W(\widehat{S}\psi, \widehat{S}\varphi)(z) = W(\psi, \varphi)(S^{-1}z), \quad z \in \mathbb{R}^n \times \mathbb{R}^n.$$

for every $\widehat{S} \in Mp(n, \mathbb{R})$, with projection $S \in Sp(n, \mathbb{R})$.

3. Proof of the main results

In this section we prove Theorem 1, Corollary 2 and Theorem 3.

Proof of Theorem 1. By the Moyal identity (2.3) and the covariance property (2.9), we have

$$\begin{aligned} |\langle \widehat{S}\psi, \varphi \rangle|^2 &= (2\pi)^n \langle W(\widehat{S}\psi), W\varphi \rangle_{L^2(\mathbb{R}^{2n})} \\ &= (2\pi)^n \langle W\psi(S^{-1}\cdot), W\varphi \rangle_{L^2(\mathbb{R}^{2n})}. \end{aligned}$$

We now can write $S^{-1} = S_1 U_1$ with $S_1 \in Sp(n, \mathbb{R})$ positive definite and $U_1 \in U(2n, \mathbb{R})$. Hence, by an orthogonal change of variable we obtain

$$|\langle \widehat{S}\psi, \varphi \rangle|^2 = (2\pi)^n \langle W\psi(S_1 \cdot), W\varphi(U_1^T \cdot) \rangle_{L^2(\mathbb{R}^{2n})}.$$

We now diagonalize S_1, $S_1 = U_2^T D U_2$ where

$$D = \mathrm{diag}(\lambda_1, \dots, \lambda_n, \lambda_1^{-1}, \dots, \lambda_n^{-1})$$

with $\lambda_1 \geq \dots \geq \lambda_n \geq \lambda_n^{-1} \geq \dots \geq \lambda_1^{-1} > 0$ and $U_2 \in U(2n, \mathbb{R})$. With a further change of variable we obtain

$$|\langle \widehat{S}\psi, \varphi \rangle|^2 = (2\pi)^n \langle W\psi(U_2^T D \cdot), W\varphi(U_1^T U_2^T \cdot) \rangle_{L^2(\mathbb{R}^{2n})}.$$

Let

$$F_1 = W\psi(U_2^T \cdot) = W(\widehat{U}_2 \psi), \qquad F_2 = W\varphi(U_1^T U_2^T \cdot) = W(\widehat{U}_2 \widehat{U}_1 \varphi).$$

We estimate

$$\langle W\psi(U_2^T D \cdot), W\varphi(U_1^T U_2^T \cdot) \rangle_{L^2(\mathbb{R}^{2n})}$$

$$= \int_{\mathbb{R}^{2n}} F_1(\lambda_1 x_1, \dots, \lambda_n x_n, \lambda_1^{-1}\xi_1, \dots, \lambda_n^{-1}\xi_n) \overline{F_2(x, \xi)} \, dx \, d\xi$$

$$\leq \int_{\mathbb{R}^{2n}} \sup_{\xi \in \mathbb{R}^n} |F_1(\lambda_1 x_1, \dots, \lambda_n x_n, \xi_1, \dots, \xi_n)| \sup_{x \in \mathbb{R}^n} |F_2(x, \xi)| \, dx \, d\xi$$

$$= \lambda_1^{-1} \dots \lambda_n^{-1} \int_{\mathbb{R}^n} \sup_{\xi \in \mathbb{R}^n} |F_1(x, \xi)| dx \int_{\mathbb{R}^n} \sup_{x \in \mathbb{R}^n} |F_2(x, \xi)| d\xi$$

$$\lesssim \lambda_1^{-1} \dots \lambda_n^{-1} \|\widehat{U}_2 \psi\|_{M^1} \|\widehat{U}_2 \psi\|_{W(\mathcal{F}L^\infty, L^1)} \|\widehat{U}_2 \widehat{U}_1 \varphi\|_{M^1} \|\widehat{U}_2 \widehat{U}_1 \varphi\|_{M^{\infty,1}},$$

where we used, in the last line, Proposition 4.

Using the inclusions

$$M^1 = M^{1,1} \hookrightarrow M^{\infty,1}, \quad M^1 = W(\mathcal{F}L^1, L^1) \hookrightarrow W(\mathcal{F}L^\infty, L^1)$$

we continue the above estimate as

$$\langle W\psi(U_2^T D \cdot), W\varphi(U_1^T U_2^T \cdot) \rangle_{L^2(\mathbb{R}^{2n})} \lesssim \lambda_1^{-1} \dots \lambda_n^{-1} \|\widehat{U}_2 \psi\|_{M^1}^2 \|\widehat{U}_2 \widehat{U}_1 \varphi\|_{M^1}^2.$$

It is then sufficient to show that

(3.1) $$\|\widehat{U}_2 \psi\|_{M^1} \leq C\|\psi\|_{M^1}$$

and

(3.2) $$\|\widehat{U}_2 \widehat{U}_1 \varphi\|_{M^1} \leq C\|\varphi\|_{M^1}$$

for a constant $C > 0$ independent of $\widehat{U}_1, \widehat{U}_2$.

Let us verify (3.1), which implies (3.2) too.

This follows by observing that, for a suitable choice of the window, $\widehat{U}$ is an isometry of M^p, for $1 \leq p \leq \infty$, $U \in U(2n, \mathbb{R})$. In fact, if φ_0 denotes a (conveniently normalized) Gaussian, we have $\widehat{U}\varphi_0 = c\varphi_0$, $|c| = 1$ ([12, Proposition 252]; see also [14]), so that

$$|V_{\varphi_0}\widehat{U}\psi(z)| = |V_{\widehat{U}\varphi_0}\widehat{U}\psi(z)| = |V_{\varphi_0}\psi(U^{-1}z)|, \quad z \in \mathbb{R}^{2n},$$

whence $\|\widehat{U}\psi\|_{M^p} = \|\psi\|_{M^p}$.   □

In order to prove Corollary 2 we need the following preliminary result.

Proposition 5. *Let* $\alpha > 0$, $\beta > 0$. *Consider the function*

$$h(S) = (\lambda_1(S) \ldots \lambda_n(S))^{-\alpha}$$

on $Sp(n, \mathbb{R})$, *where* $\lambda_1(S) \ldots \lambda_n(S)$ *are the singular values* ≥ 1 *of the symplectic matrix* S.

We have $h \in L^{\beta,\infty}$ *on* $Sp(n, \mathbb{R})$, *with respect to the Haar measure, if* $\alpha\beta \geq 2n$.

Proof. We have to estimate the measure of the set

$$\mathcal{D}_\lambda = \{S \in Sp(n, \mathbb{R}) : h(S) \geq \lambda\}, \quad \lambda > 0$$

or equivalently

$$\int_{Sp(n,\mathbb{R})} \chi_{\mathcal{D}_\lambda} dS,$$

where $\chi_{\mathcal{D}_\lambda}$ is the indicator function of $\mathcal{D}_\lambda$. Observe that $\mathcal{D}_\lambda = \emptyset$ if $\lambda > 1$ so that we can suppose $0 < \lambda \leq 1$.

Recall that if $f : Sp(n, \mathbb{R}) \to \mathbb{C}$ is $U(2n, \mathbb{R})$-bi-invariant, its integral with respect to the Haar measure is given by

$$(3.3) \quad \int_{Sp(n,\mathbb{R})} f(S)dS$$

$$= C \int_{t_1 \geq \ldots \geq t_n \geq 0} f(a_t) \prod_{i<j} \sinh \frac{t_i - t_j}{2} \prod_{i \leq j} \sinh \frac{t_i + t_j}{2} \, dt_1 \ldots dt_n$$

for some constant $C > 0$, where

$$a_t = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}, \quad t = \mathrm{diag}(t_1, \ldots, t_n), \ (t_1, \ldots, t_n) \in \mathbb{R}^n.$$

We use formula (3.3) with $f = \chi_{\mathcal{D}_\lambda}$ since h, and therefore f, is $U(2n, \mathbb{R})$-bi-invariant. We have

$$h(a_t) = e^{-\alpha(t_1 + \ldots + t_n)/2}.$$

Hence $h(a_t) \geq \lambda$ if and only if

$$t_1 + t_2 + \ldots + t_n \leq A_\lambda := -2\log\lambda/\alpha.$$

By (3.3),

$$\text{meas}\,\mathcal{D}_\lambda = C \int_{\substack{t_1 \geq t_2 \geq \dots \geq t_n \geq 0 \\ t_1 + \dots + t_n \leq A_\lambda}} \prod_{i<j} \sinh \frac{t_i - t_j}{2} \prod_{i \leq j} \sinh \frac{t_i + t_j}{2}\, dt_n \dots dt_1.$$

Now we have

$$\prod_{i<j} \sinh \frac{t_i - t_j}{2} \prod_{i \leq j} \sinh \frac{t_i + t_j}{2}$$

$$\leq \exp\Big(\sum_{i<j} \Big(\frac{t_i - t_j}{2} + \frac{t_i + t_j}{2}\Big) + t_1 + \dots + t_n\Big)$$

$$= \exp\big((n-1)t_1 + (n-2)t_2 + \dots + t_{n-1} + t_1 + \dots + t_n\big)$$

$$= e^{t_n} e^{2t_{n-1}} e^{3t_{n-2}} \dots e^{nt_1}.$$

By first integrating with respect to the variable t_n from $t_n = 0$ to $t_n = A_\lambda - t_{n-1} - \dots - t_1$, we obtain

$$\text{meas}\,\mathcal{D}_\lambda \leq C \int_{\substack{t_1 \geq t_2 \geq \dots \geq t_{n-1} \geq 0 \\ t_{n-1} \leq A_\lambda - t_{n-2} - \dots - t_1}} e^{A_\lambda} e^{t_{n-1}} \dots e^{(n-1)t_1}\, dt_{n-1} \dots dt_1.$$

Now we can repeat the same argument for t_{n-1} and so on. We obtain

$$\text{meas}\,\mathcal{D}_\lambda \leq C e^{nA_\lambda} = C \lambda^{-2n/\alpha}, \quad 0 < \lambda \leq 1.$$

Hence $\text{meas}\,\mathcal{D}_\lambda \leq C' \lambda^{-\beta}$ if $2n/\alpha \leq \beta$, which is the desired result. $\qquad \square$

Proof of Corollary 2. Using (1.2) it is sufficient to prove that the function

$$\widehat{S} \longmapsto (\lambda_1(S) \dots \lambda_n(S))^{-1/2}$$

is in $L^{4n,\infty}$ on $Mp(n,\mathbb{R})$ with respect to the Haar measure. Since this function factorizes through $Sp(n,\mathbb{R})$, it is enough to prove that the function

$$h(S) := (\lambda_1(S) \dots \lambda_n(S))^{-1/2}$$

is in $L^{4n,\infty}$ on $Sp(n,\mathbb{R})$. This follows from Proposition 5 with $\alpha = 1/2$ and $\beta = 4n$. $\qquad \square$

We are now ready to prove the Strichartz estimates for the metaplectic representation.

Proof of Theorem 3. We know that

$$(3.4) \qquad\qquad\qquad \|\widehat{S}\psi\|_{L^2} = \|\psi\|_{L^2}$$

for $\psi \in L^2(\mathbb{R}^n)$, which gives the desired Strichartz estimate for $q = \infty$, $r = 2$, because $M^2 = L^2$, and also for $q = \infty$, $2 \leq r \leq \infty$, because $L^2 \hookrightarrow M^r$ for $r \geq 2$. Hence from now on we can suppose $q < \infty$.

By Theorem 1 and (2.2) we have

$$\|\widehat{S}\psi\|_{M^\infty} \lesssim (\lambda_1(S)\ldots\lambda_n(S))^{-1/2}\|\psi\|_{M^1}.$$

By interpolation with (3.4) we obtain, for every $2 \le r \le \infty$,

$$(3.5) \qquad \|\widehat{S}\psi\|_{M^r} \lesssim (\lambda_1(S)\ldots\lambda_n(S))^{-(\frac{1}{2}-\frac{1}{r})}\|\psi\|_{M^{r'}}.$$

Let $G = Mp(n,\mathbb{R})$, as in the statement. We apply the usual TT^* method (see [26, page 75]) to the operator $T\psi = \widehat{S}\psi$. To prove that $T : L^2 \to L^q(G; M^r)$ continuously, we will verify that $TT^* : L^{q'}(G; M^{r'}) \to L^q(G; M^r)$ continuously.

We have

$$T^*F(\cdot) = \int_G \widehat{S}^{-1}F(\widehat{S}, \cdot)d\widehat{S}$$

if $F(\widehat{S}, x)$ is, say, a continuous function on $G \times \mathbb{R}^n$ with compact support.

Hence

$$[TT^*F](\widehat{S}, \cdot) = \int_G \widehat{S}\widehat{S'}^{-1}F(\widehat{S'}, \cdot)d\widehat{S'}.$$

Using (3.5) we can estimate this expression, for every $2 \le r \le \infty$, $1 \le q \le \infty$, as

$$\|TT^*F\|_{L^q(G;M^r)} \le \| \int_G \|\widehat{S}\widehat{S'}^{-1}F(\widehat{S'}, \cdot)\|_{M^r} d\widehat{S'}\|_{L^q(G)}$$

$$(3.6) \qquad \le \| \int_G (h \circ \pi)(\widehat{S}\widehat{S'}^{-1})\|F(\widehat{S'}, \cdot)\|_{M^{r'}} d\widehat{S'}\|_{L^q(G)},$$

where we set $h(S) = (\lambda_1(S)\ldots\lambda_n(S))^{-(\frac{1}{2}-\frac{1}{r})}$ as a function on $Sp(n,\mathbb{R})$ and $\pi : G = Mp(n,\mathbb{R}) \to Sp(n,\mathbb{R})$ is the projection.

Finally suppose that the pair (q, r) satisfies $2 \le q, r \le \infty$ and $\frac{4n}{q} + \frac{1}{r} \le \frac{1}{2}$; see Figure 1. Observe that this implies $q > 2$ and we are also supposing $q < \infty$, which implies $r > 2$. Choose

$$\alpha = \frac{1}{2} - \frac{1}{r}, \quad \frac{1}{\beta} = \frac{2}{q}.$$

We see that $\alpha, \beta > 0$ and $\alpha\beta \ge 2n$ so that, by Proposition 5, we have $h \in L^{\beta,\infty}$ in $Sp(n,\mathbb{R})$ and $h \circ \pi \in L^{\beta,\infty}$ on G. Moreover we have $\frac{1}{q}+1 = \frac{1}{q'}+\frac{1}{\beta}$, $1 < q, q', \beta < \infty$. Hence we can apply the weak-type Young inequality (2.1) on G to the last expression in (3.6), and we see that it is therefore dominated by $\|F\|_{L^{q'}(G;M^{r'})}$.

This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

4. Concluding remarks

4.1. The motivation for modulation spaces

Let us point out the main elements which led us to consider the modulation space M^1 and its dual M^∞ as natural candidates for the dispersive estimate (1.2).

Estimate (1.2) clearly does not hold with M^1 and M^∞ replaced by L^1 and L^∞, respectively, because, for example, the pointwise multiplication by $e^{it|x|^2}$ is a metaplectic operator but Lebesgue norms do not detect any decay as $|t| \to +\infty$. Hence we focused on a space which controls L^1 decay in space *and* L^1 decay in momentum, as M^1 indeed does.

But in the course of the proof of Theorem 1 we also used in an essential way another property of M^1, namely that the set of operators $\widehat{U}$ are uniformly bounded on M^1 when $U = \pi(\widehat{U})$ varies in $U(2n, \mathbb{R})$, as proved in (3.1).

Motivated by these issues, it would be very interesting to get characterizations of function spaces, in particular modulation spaces, in terms of symplectic invariance.

4.2. Sharpness of the results

It is easy to see that the exponent $-1/2$ in (1.2) is sharp. In fact, one can apply that estimate to a Gaussian function ψ and the metaplectic operator $\widehat{S}\psi(x) = c\sqrt{\det L}\,\psi(Lx)$ (for suitable $c \in \mathbb{C}$, $|c| = 1$), with $L = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, $\lambda_1, \ldots, \lambda_n \geq 1$. We have $S = (\lambda_1^{-1}, \ldots, \lambda_n^{-1}, \lambda_1, \ldots, \lambda_n)$ (cf. [12, Proposition 116]) and

$$\|\widehat{S}\psi\|_{M^\infty} \asymp (\lambda_1 \ldots \lambda_n)^{-1/2},$$

as proved in [5, Lemma 3.2] (and in [27, Lemma 1.8] in the case $\lambda_1 = \ldots = \lambda_n$).

Let us observe that the exponent $4n$ in (1.3) is sharp as well; in fact Howe [18] proved that for fixed $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^n)$ the matrix coefficients in general do not belong to L^{4n}.

Acknowledgments

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