NEW RESULTS ON CYCLE–SLIPPING IN PENDULUM–LIKE SYSTEMS

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Abstract

In this paper, we examine dynamics of multidimensional control systems obtained as feedback interconnections of stable linear blocks and periodic nonlinearities. The simplest of such systems is the model of mathematical pendulum (with viscous friction), so we call such systems *pendulum-like*. Other examples include, but are not limited to, coupled vibrating units, networks of oscillators, Josephson junction arrays and numerous synchronization circuits used in radio and telecommunication engineering. Typically, a pendulum-like system has infinite sequence of equilibria, and one of the central problems addressed in the theory of such systems is to find the conditions of "global stability", or *gradient-like* behavior ensuring that every solution converges to one of the equilibria points. If a system is gradient-like, another problem arises, being the main concern of this paper: can we find the terminal equilibrium, given the initial condition of the system? It is well known that solutions do not converge, in general, to the nearest equilibrium; this phenomenon is known as *cycle-slipping*. For a pendulum, cycle-slipping corresponds to multiple rotations of the pendulum about its suspension point. In this paper, we estimate the number of slipped cycles for general pendulum-like systems by means of periodic Lyapunov functions and the Kalman-Yakubovich-Popov lemma.

Key words

Periodic nonlinearity, stability, Lyapunov function

1 Introduction

A number of systems, arising in natural sciences and engineering, can be represented by a *feedback interconnection* of an asymptotically stable linear stationary system and *periodic nonlinearity*. The simplest example of such a system is a pendulum with viscous damping, inspiring thus the term "pendulum-like". Sometimes, systems with periodic nonlinearities are also called *synchronization* (synchronous control) systems [Leonov, 2006; Hoppensteadt, 1983] in view of numerous applications to synchronization circuits such as phase,frequency and delay-locked loops (PLL/FLL/DLL) [Margaris, 2004; Best, 2003; Leonov and Kuznetsov, 2014]. Other examples include, but are not limited to, electric motors, power generators, single and coupled Josephson junctions, networks of coupled oscillators and neurons [Baker and Blackburn, 2005; Stoker, 1950; Blekhman, 2000; Monteiro et al., 2003; Hoppensteadt and Izhikevich, 2000; Imry and Schulman, 1978; Qin and Chen, 2004].

Pendulum-like systems are featured by complex multistable dynamics with infinite sequences of stable and unstable equilibria points (in fact, their natural phase space is a cylindric or toric manifold [Kudrewicz and Wasowicz, 2007; Leonov et al., 1996]). Many effects in such systems, e.g. oscillations, hidden attractors and "cycle slipping" [Chicone and Heitzman, 2013; Leonov et al., 2015b; Leonov et al., 2015a; Best et al., 2016; Dudkowski et al., 2016] cannot be examined by tools of classical nonlinear control and require special techniques. One of the central problems, concerned with dynamics of systems, is the convergence of all solutions to equilibria points. This counterpart of global asymptotic stability in pendulum–like systems with unique equilibrium is referred to as the *gradient-like* behavior [Leonov, 2006; Duan et al., 2007]; the gradient-like behavior excludes, in particular, limit cycles and other hidden attractors.

Another important problem concerned with pendulum–like systems is to find the equilibrium to which the solution is attracted. In general, the solution does not converge to the nearest equilibrium, as exemplified by the mathematical pendulum with friction or the stepper motor [Stoker, 1950]. The pendulum can make several turns around the suspension point before calming down at the lower equilibrium. Similarly, a stepper motor can skip steps. This phenomenon is known as *cycle slipping*. Cycle slipping in synchronization circuits is usually considered to be undesired behavior as the continuous changing of the phase error leads to demodulation and decoding errors. This motivated extensive research on the cycle slipping phenomena in the engineering community. For more than 50 years (since the seminal paper [Viterbi, 1963]) these studies have mainly focused on *stochastic* cycle slipping, caused by random noise [Ascheid and Meyr, 1982; Sancho et al., 2014]. However, solutions of a deterministic system with periodic nonlinearities can also slip several cycles under some initial conditions; obtaining non-conservative estimates for the phase error increment is a non-trivial problem even for a low-order dynamics. Using the absolute-stability techniques and the Kalman-Yakubovich-Popov (KYP) lemma, some frequency-domain estimates were obtained in [Ershova and Leonov, 1983] and later reformulated in LMI form in [Yang and Huang, 2007; Lu et al., 2008]. Subsequently estimates from [Ershova and Leonov, 1983] have been significantly improved and extended to discrete-time and distributed parameter systems in [Leonov et al., 1992; Smirnova et al., 2006; Perkin et al., 2013; Perkin et al., 2014].

In this paper, the frequency-domain estimates for the number of slipped cycles are extended to the systems with external disturbances. Obviously, if such a disturbance persistently excites the solution (being e.g. harmonic or other periodic oscillatory signal), the solution no longer converges to an equilibrium point but rather oscillates. In synchronization systems, such disturbances are typically modeled as combinations of stationary random signals and polyharmonic signals [Hill and Cantoni, 2000; Cataliotti et al., 2007; Schilling et al., 2010] to be rejected or damped. In this paper, we deal with other type of disturbances that have finite limit at infinity (being thus combinations of constant and decaying signals), enabling thus the disturbed system to have equilibria. Frequency-algebraic criteria for gradient-like behavior of synchronization systems with disturbances have been proved in [Smirnova et al., 2018a; Smirnova et al., 2018b]. In this paper, we obtain several frequency-algebraic estimates for the number of cycles slipped under the influence of an external force.

2 The problem setup

Consider a phase synchronization system with external disturbances described by the equations

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + b\xi(t) \in \mathbb{R}^m, \\ \frac{d}{dt}\sigma(t) &= c^*x(t) + \rho\xi(t) \in \mathbb{R}, \\ \xi(t) &= \psi(\sigma(t)) + f(t) \in \mathbb{R}. \end{aligned} \tag{1}$$

Here $A \in \mathbb{R}^{m\times m}$, $b, c \in \mathbb{R}^m$, $\rho \in \mathbb{R}$ are constant, the symbol * stands for Hermitian conjugation.

Henceforth the following assumptions are adopted.

A1. The pair (A, b) is controllable, the pair (A, c) is observable, the matrix A is a Hurvitz matrix.

A2. There exist a constant $\lambda \in \mathbb{R}$ and a function $q(\cdot) \in \mathbb{C}[0, +\infty) \cap L^1[0, +\infty)$ such that

$$f(t) = q(t) + \lambda \tag{2}$$

$$\lim_{t\to+\infty} q(t) = 0. \tag{3}$$

A3. The function $\psi(\cdot)$ is non-constant, $\mathbb{C}^1$–smooth and Δ–periodic ($\psi(\sigma + \Delta) = \psi(\sigma)$). Hence

$$\mu_1 \triangleq \inf_{\zeta\in[0,\Delta)} \psi'(\zeta), \quad \mu_2 \triangleq \sup_{\zeta\in[0,\Delta)} \psi'(\zeta) \tag{4}$$

are finite and satisfy the inequality $\mu_1\mu_2 < 0$.

A4. The function $\varphi(\zeta) \triangleq \psi(\zeta) + \lambda$ has simple isolated roots. It is obvious that

$$\xi(t) = \varphi(\sigma(t)) + q(t). \tag{5}$$

The main problem for any synchronization system is the convergence of all solutions (gradient–like behavior). In papers [Smirnova et al., 2018a; Smirnova et al., 2018b] the frequency–algebraic conditions which guaranteed that any solution of forced synchronization system converged as $t \to +\infty$ have been established.

In this paper we go on with stability investigation of forced synchronization systems, addressing the problem of cycle-slipping. We extend the results from [Perkin et al., 2013] to forced synchronization systems. The estimates are formulated in terms of the transfer function of the linear part:

$$K(p) = -\rho + c^*(A - pE_m)^{-1}b \quad (p \in \mathbb{C}), \tag{6}$$

where E_l is a unit $l \times l$–matrix.

In case $f(t) \not\equiv 0$ the estimates become more complicated. They depend on parameters of external disturbance, namely, the constants

$$d_0 \triangleq \sup_{t\geq 0} |q(t)|,\ d_1 \triangleq \int_0^\infty |q(t)|dt,\ d_2 \triangleq \int_0^\infty q^2(t)dt.$$

3 Preliminaries

Introducing the new variable

$$z(t) \triangleq x(t) - \int_0^t e^{A(t-\tau)} bq(\tau)\, d\tau, \tag{7}$$

the equations (1) are rewritten as

$$\begin{aligned} \dot{z}(t) &= Az(t) + b\varphi(\sigma(t)), \\ \dot{\sigma}(t) &= c^* z(t) + \rho(\varphi(\sigma(t)) + g(t)), \\ g(t) &\triangleq \frac{1}{\rho} \int_0^t c^* e^{A(t-\tau)} bq(\tau)\, d\tau + q(t). \end{aligned} \tag{8}$$

Henceforth we primarily deal with (8). A2 entails [Gelig, 1966; Smirnova et al., 2018b] that

$$\begin{aligned} & g(t) \to 0 \text{ as } t \to +\infty, \\ & g(t) \in L_1[0, +\infty) \cap L_2[0, +\infty). \end{aligned} \tag{9}$$

In particular, constants g_0, K_1, K_2 exist such that

$$\sup_{t \geq 0} |g(t)| \leq g_0, \int_0^\infty |g(t)| dt \leq K_1, \int_0^\infty g^2(t) dt \leq K_2. \tag{10}$$

These constants depend on d_0, d_1, d_2 and A, b, c, ρ. Using (8), constants $C_1, C_2 > 0$ exist such that

$$|\dot{\sigma}(t)| \leq C_1 |e^{At}||z(0)| + \rho |g(t)| + C_2. \tag{11}$$

We also introduce one auxiliary construction, based on the Kalman-Yakubovich-Popov (KYP) lemma. Transform (8) into the system

$$\begin{aligned} &\frac{d}{dt} y(t) = Qy(t) + L\eta(t) \in \mathbb{R}^{m+1}, \\ &\frac{d}{dt} \sigma(t) = D^* y(t) + \rho g(t) \in \mathbb{R}, \end{aligned} \tag{12}$$

$$y(t) \triangleq (z(t)^\top, \varphi(\sigma(t)))^\top,\ \eta(t) \triangleq \frac{d}{dt} \varphi(\sigma(t)),$$

$$Q \triangleq \begin{bmatrix} A & b \\ O & 0 \end{bmatrix},\ L \triangleq \begin{bmatrix} O \\ 1 \end{bmatrix},\ D \triangleq \begin{bmatrix} c \\ \rho \end{bmatrix}.$$

Determine the quadratic form of $y \in \mathbb{R}^{m+1}, \eta \in \mathbb{R}$

$$\begin{aligned} G(y, \eta) &= 2y^* H(Qy + L\eta) + \varepsilon (y^* D)^2 + \delta (y^* L)^2 + \\ &+ \varkappa (y^* L)(D^* y) - \tau (D^* y - \mu_1^{-1} \eta)(\mu_2^{-1} \eta - D^* y) \end{aligned}$$

where $H = H^*$ is a real $m \times m$–matrix, ε, δ, $\tau > 0$ and $\varkappa$ are parameters. The KYP lemma [Gelig et al., 2004] states that the frequency–domain inequality

$$\begin{aligned} & Re\{\varkappa K(\imath\omega) - \tau (K(\imath\omega) + \mu_1^{-1} \imath\omega)^* (K(\imath\omega) + \\ & + \mu_2^{-1} \imath\omega)\} - \varepsilon |K(\imath\omega)|^2 - \delta \geq 0, \quad \forall \omega \geq 0 \end{aligned} \tag{13}$$

implies the existence of the matrix $H = H^*$ such that

$$G(y, \eta) \leq 0,\ \ \forall y \in \mathbb{R}^{m+1}, \eta \in \mathbb{R}. \tag{14}$$

The matrix H will be used to design Lyapunov–type functions. We shall also use the functions

$$\Phi(\zeta) \triangleq \sqrt{(1 - \mu_1^{-1} \psi'(\zeta))(1 - \mu_2^{-1} \psi'(\zeta))} \tag{15}$$

$$r_j(k, \varkappa, \alpha) = \frac{\int_0^\Delta (\psi(\zeta) + \lambda)\, d\zeta + (-1)^j \frac{\alpha}{\varkappa k}}{\int_0^\Delta |\psi(\zeta) + \lambda|\, d\zeta} \quad (j = 1, 2), \tag{16}$$

$$r_{0j}(k, \varkappa, \alpha) = \frac{\int_0^\Delta (\psi(\zeta) + \lambda)\, d\zeta + (-1)^j \frac{\alpha}{\varkappa k}}{\int_0^\Delta \Phi(\zeta) |\psi(\zeta) + \lambda|\, d\zeta} \quad (j = 1, 2). \tag{17}$$

The estimates of slipped cycles exploit the functionals

$$I_T \triangleq \int_0^T U(\varkappa, \varepsilon, \tau; t)\, dt,$$

$$\begin{aligned} U(\varkappa, \varepsilon, \tau; t) &\triangleq \varkappa \rho \varphi(\sigma(t)) g(t) + 2(\tau + \varepsilon) \rho \dot{\sigma}(t) g(t) + \\ &+ \tau \rho (|\mu_1^{-1}| - \mu_2^{-1}) \dot{\varphi}(\sigma(t)) g(t) - (\tau + \varepsilon) \rho^2 g^2(t). \end{aligned} \tag{18}$$

Estimates (10), (11) allow to get an estimate on I_T:

$$|I_T| \leq M_0(\varkappa, \tau, \varepsilon), \quad \forall T > 0. \tag{19}$$

4 Estimates for the number of slipped cycles

Introduce the symmetric matrices

$$T_j(k, \varkappa; \alpha) \triangleq$$

$$\triangleq \begin{bmatrix} \varepsilon & \frac{a \varkappa r_j(k, \varkappa; \alpha)}{2} & 0 \\ \frac{a \varkappa r_j(k, \varkappa; \alpha)}{2} & \delta & \frac{a_0 \varkappa r_{0j}(k, \varkappa; \alpha)}{2} \\ 0 & \frac{a_0 \varkappa r_{0j}(k, \varkappa; \alpha)}{2} & \tau \end{bmatrix}, \tag{20}$$

where $j = 1, 2$, $\varepsilon, \delta, \tau > 0$, $a \in [0, 1]$ and $a_0 = 1 - a$.

Theorem 1. Let $\psi(\sigma(0)) = -\lambda$. Suppose there exist $k \in \mathbb{N}$, $\varkappa \neq 0$, $\varepsilon, \delta, \tau > 0$, $a \in [0, 1]$ such that the following conditions are fulfilled:

1) the frequency–domain inequality (13) holds;
2) there exists a matrix $H = H^*$, satisfying (14) such that the matrices $T_j(k, \varkappa; y^*(0) H y(0) + M_0(\varkappa, \tau, \varepsilon))$ $(j = 1, 2)$, where $M_0(\varkappa, \tau, \varepsilon)$ is from (19), are positive definite.

Then the solution with the initial condition $(x(0), \sigma(0))$ *slips less than k cycles* [Ershova and Leonov, 1983], that is,

$$|\sigma(t) - \sigma(0)| < k\Delta \quad \forall t \geq 0. \tag{21}$$

Proof. For the solution at hand, denote

$$S \triangleq y^*(0)Hy(0) + M_0(\varkappa, \tau, \varepsilon). \quad (22)$$

Let ε_0 be so small that matrices $T_j(k, \varkappa; S + \varepsilon_0)$ are positive definite. Consider the functions

$$F_j(\zeta) \triangleq \varphi(\zeta) - r_j|\varphi(\zeta)|, \quad (23)$$

$$\Psi_j(\zeta) \triangleq \varphi(\zeta) - r_{0j}|\varphi(\zeta)|\Phi(\zeta), \quad (j = 1, 2). \quad (24)$$

Here

$$r_j \triangleq r_j(k, \varkappa, S + \varepsilon_0),\ r_{0j} \triangleq r_{0j}(k, \varkappa, S + \varepsilon_0). \quad (25)$$

We now introduce the functions

$$W(t) \triangleq y^*(t)Hy(t), \quad (26)$$

$$V_j(t) \triangleq W(t) + \varkappa \int\limits_{\sigma(0)}^{\sigma(t)} (aF_j(\zeta) + a_0\Psi_j(\zeta))\, d\zeta \quad (j = 1, 2). \quad (27)$$

Then in virtue of system (12) we have

$$\begin{aligned} \tfrac{dV_j(t)}{dt} &= 2y^*(t)H(Qy(t) + L\eta(t)) + \\ &+ \varkappa\,(aF_j(\sigma(t)) + a_0\Psi_j(\sigma(t)))\,\dot\sigma(t). \end{aligned} \quad (28)$$

Condition 1) of Theorem 1 implies (14) is true. Thus

$$\begin{gathered} \dot V_j(t) \le -\varepsilon(\dot\sigma(t) - \rho g(t))^2 - \\ -\varkappa(\dot\sigma(t) - \rho g(t))\varphi(\sigma(t)) - \delta(\varphi(\sigma(t)))^2 - \\ -\tau(\dot\sigma(t) - \rho g(t) - \mu_1^{-1}\dot\varphi(\sigma(t))) \times \\ \times(\dot\sigma(t) - \rho g(t) - \mu_2^{-1}\dot\varphi(\sigma(t))) + \\ +\varkappa(\varphi(\sigma(t)) - ar_j|\varphi(\sigma(t))| - \\ -a_0 r_{0j}|\varphi(\sigma(t))|\Phi(\sigma(t)))\dot\sigma(t), \end{gathered} \quad (29)$$

whence

$$\frac{dV_j(t)}{dt} \le -Z_j(\dot\sigma(t), |\varphi(\sigma(t))|, \dot\sigma(t)\Phi(\sigma(t))) + U(t), \quad (30)$$

where $U(t)$ is defined by (18) and $Z_j(\alpha, \beta, \gamma)$ is a quadratic form with matrix $T_j(k, \varkappa; S + \varepsilon_0)$. Due to condition 2),

$$\frac{dV_j(t)}{dt} \le U(t) \quad (j = 1, 2), \quad (31)$$

and thus

$$V_j(t) \le W(0) + M_0(\varkappa, \tau, \varepsilon) \quad (j = 1, 2). \quad (32)$$

Suppose that (21) is violated and for some $\bar t > 0$

$$\sigma(\bar t) = \sigma(0) + k\Delta. \quad (33)$$

By definition of V_1, we have

$$\begin{aligned} &V_1(\bar t) = W(\bar t) + \\ &+\varkappa k \left(a\int\limits_0^\Delta F_1(\zeta)\,d\zeta + a_0\int\limits_0^\Delta \Psi_1(\zeta)\,d\zeta \right). \end{aligned} \quad (34)$$

Note that $\varphi(\sigma(\bar t)) = 0$, $y(\bar t) = (z(\bar t), 0)^T$ and

$$W(\bar t) = z^*(\bar t)H_0 z(\bar t),\ H_0 \in \mathbb{R}^{m\times m},\ H_0^* = H_0 \quad (35)$$

(here H_0 is a submatrix of H).

Substituting $\bar y = (z, 0)^T$ and $\eta = 0$ into (14), one has

$$G(\bar y, 0) = 2z^*H_0Az + (\varepsilon + \tau)(c^*z)^2 \le 0,\ \forall z \in \mathbb{R}^m. \quad (36)$$

Thus $H_0A + A^TH_0$ is negative definite and, due to A1, H_0 is positive definite. On the other hand,

$$\int\limits_0^\Delta F_1(\zeta)\,d\zeta = \int\limits_0^\Delta \Psi_1(\zeta)\,d\zeta = \frac{S + \varepsilon_0}{\varkappa k}. \quad (37)$$

The latter equation implies that

$$V_1(\bar t) \ge W(0) + M_0(\varkappa, \tau, \varepsilon) + \varepsilon_0, \quad (38)$$

which contradicts (32). We have proved that

$$\sigma(t) < \sigma(0) + k\Delta, \quad \forall t > 0. \quad (39)$$

Using the function $V_2(t)$, one proves that

$$\sigma(t) > \sigma(0) - k\Delta, \quad \forall t > 0, \quad (40)$$

which finishes the proof of Theorem 1. □

Now we discard the constraint $\psi(\sigma(0)) + \lambda = 0$. Let

$$\sigma_0 - \Delta < \sigma(0) < \sigma_0, \quad \psi(\sigma_0) = -\lambda. \quad (41)$$

Introduce the functions ($\varphi(\zeta) \triangleq \psi(\zeta) + \lambda$),

$$\nu_1(k, \varkappa, \alpha) \triangleq \frac{\int\limits_{\sigma(0)}^{\sigma_0} \varphi(\zeta)\, d\zeta + k \int\limits_0^{\Delta} \varphi(\zeta)\, d\zeta - \frac{\alpha}{\varkappa}}{\int\limits_{\sigma(0)}^{\sigma_0} |\varphi(\zeta)|\, d\zeta + k \int\limits_0^{\Delta} |\varphi(\zeta)|\, d\zeta},$$

$$\nu_2(k, \varkappa, \alpha) \triangleq \frac{\int\limits_{\sigma_0-\Delta}^{\sigma(0)} \varphi(\zeta)\, d\zeta + k \int\limits_0^{\Delta} \varphi(\zeta)\, d\zeta + \frac{\alpha}{\varkappa}}{\int\limits_{\sigma_0-\Delta}^{\sigma(0)} |\varphi(\zeta)|\, d\zeta + k \int\limits_0^{\Delta} |\varphi(\zeta)|\, d\zeta},$$

$$\nu_{01}(k, \varkappa, \alpha) \triangleq \frac{\int\limits_{\sigma(0)}^{\sigma_0} \varphi(\zeta)\, d\zeta + k \int\limits_0^{\Delta} \varphi(\zeta)\, d\zeta - \frac{\alpha}{\varkappa}}{\int\limits_{\sigma(0)}^{\sigma_0} |\hat{\varphi}(\zeta)|\, d\zeta + k \int\limits_0^{\Delta} |\hat{\varphi}(\zeta)|\, d\zeta},$$

$$\nu_{02}(k, \varkappa, \alpha) \triangleq \frac{\int\limits_{\sigma_0-\Delta}^{\sigma(0)} \varphi(\zeta)\, d\zeta + k \int\limits_0^{\Delta} \varphi(\zeta)\, d\zeta + \frac{\alpha}{\varkappa}}{\int\limits_{\sigma_0-\Delta}^{\sigma(0)} |\hat{\varphi}(\zeta)|\, d\zeta + k \int\limits_0^{\Delta} |\hat{\varphi}(\zeta)|\, d\zeta},$$

where $\hat{\varphi}(\zeta) = \Phi(\zeta)\varphi(\zeta)$, and the symmetric matrices

$$P_j(k, \varkappa; \alpha) \triangleq$$
$$\triangleq \begin{bmatrix} \varepsilon & \frac{a\varkappa\nu_j(k,\varkappa;\alpha)}{2} & 0 \\ \frac{a\varkappa\nu_j(k,\varkappa;\alpha)}{2} & \delta & \frac{a_0\varkappa\nu_{0j}(k,\varkappa;\alpha)}{2} \\ 0 & \frac{a_0\varkappa\nu_{0j}(k,\varkappa;\alpha)}{2} & \tau \end{bmatrix}, \qquad (42)$$

with $a_0 = 1 - a$ and $j = 1, 2$.

Theorem 2. Suppose $\sigma(0) \in (\sigma_0 - \Delta, \sigma_0)$. Suppose there exist $k \in \mathbb{N}$, $\varkappa \neq 0$, $\varepsilon, \delta, \tau > 0$, $a \in [0, 1]$ such that the following conditions are fulfilled:

1) the frequency–domain inequality (13) holds.
2) there exists a real matrix $H = H^*$, satisfying (14) such that the matrices $P_j(k, \varkappa; y^*(0)Hy(0) + M_0(\varkappa, \tau, \varepsilon))$ $(j = 1, 2)$ are positive definite.

Then the solution with the initial data $(x(0), \sigma(0))$ slips less than $k + 1$ cycles:

$$|\sigma(t) - \sigma(0)| < (k + 1)\Delta \quad \forall t \geq 0. \qquad (43)$$

Proof. Similar to the proof of Theorem 1, let ε_0 be so small that matrices $P_j(k, \varkappa; S + \varepsilon_0) > 0$, where S is defined by (22). Consider the functions

$$\bar{F}_j(\zeta) \triangleq \varphi(\zeta) - \nu_j|\varphi(\zeta)|,$$
$$\bar{\Psi}_j(\zeta) \triangleq \varphi(\zeta) - \nu_{0j}|\varphi(\zeta)|\Phi(\zeta),$$

where $j = 1, 2$ and the constants are as follows

$$\nu_j \triangleq \nu_j(k, \varkappa, S + \varepsilon_0),\ \nu_{0j} \triangleq \nu_{0j}(k, \varkappa, S + \varepsilon_0) \qquad (44)$$

Define Lyapunov–type functions

$$\bar{V}_j(t) \triangleq W(t) + \varkappa \int\limits_{\sigma(0)}^{\sigma(t)} (a\bar{F}_j(\zeta) + a_0\bar{\Psi}_j(\zeta))\, d\zeta \qquad (45)$$

where $W(t)$ is defined by (26).

Compute the derivative of $\bar{V}_j$ in virtue of system (12):

$$\begin{array}{l} \frac{d\bar{V}_j(t)}{dt} = 2y^*(t)H(Qy(t) + L\eta(t)) + \\ +\varkappa\left(a\bar{F}_j(\sigma(t)) + a_0\bar{\Psi}_j(\sigma(t))\right)\dot{\sigma}(t). \end{array} \qquad (46)$$

Condition 1) of the theorem guarantees that the inequality (14) is valid. Then repeating the estimate (29) with ν_j instead of r_j and ν_{0j} instead of r_{0j} we get

$$\frac{d\bar{V}_j(t)}{dt} \leq -\bar{Z}_j(\dot{\sigma}(t), |\varphi(\sigma(t))|, \dot{\sigma}(t)\Phi(\sigma(t)) + U(t), \qquad (47)$$

where $U(t)$ is defined by (18) and $\bar{Z}(\alpha, \beta, \gamma)$ is a quadratic form with matrix $P_j(k, \varkappa; S + \varepsilon_0)$.

Condition 2) of the theorem implies that

$$\frac{d\bar{V}_j(t)}{dt} \leq U(t) \quad (j = 1, 2), \qquad (48)$$

and therefore

$$\bar{V}_j(t) \leq \bar{V}_j(0) + \int\limits_0^t U(t)\, dt \qquad (49)$$

entailing that

$$\bar{V}_j(t) \leq W(0) + M_0(\varkappa, \tau, \varepsilon) \quad (j = 1, 2) \qquad (50)$$

where M_0 is defined in (19). Suppose now that

$$\sigma(\bar{t}) = \sigma_0 + k\Delta \qquad (51)$$

for some $\bar{t} > 0$. Consider the Lyapunov function $\bar{V}_1$:

$$\bar{V}_1(\bar{t}) = W(\bar{t}) + \varkappa \int\limits_{\sigma(0)}^{\sigma_0+k\Delta} (a\bar{F}_1(\zeta) + a_0\bar{\Psi}_1(\zeta))\, d\zeta =$$
$$= W(\bar{t}) + \varkappa \int_{\sigma(0)}^{\sigma_0} (a\bar{F}_1(\zeta) + a_0\bar{\Psi}_1(\zeta))\, d\zeta +$$
$$+\varkappa k \int_0^{\Delta} (a\bar{F}_1(\zeta) + a_0\bar{\Psi}_1(\zeta))\, d\zeta.$$

A straightforward computation shows that

$$\begin{array}{l} \int\limits_{\sigma(0)}^{\sigma_0} \bar{F}_1(\zeta)\, d\zeta + k \int\limits_0^{\Delta} \bar{F}_1(\zeta)\, d\zeta = \\ = \int\limits_{\sigma(0)}^{\sigma_0} \varphi(\zeta)\, d\zeta + k \int\limits_0^{\Delta} \varphi(\zeta)\, d\zeta - \\ -\nu_1(\int\limits_{\sigma(0)}^{\sigma_0} |\varphi(\zeta)|\, d\zeta + k \int\limits_0^{\Delta} |\varphi(\zeta)|\, d\zeta) = \\ = \varkappa^{-1}(W(0) + M_0(\varkappa, \tau, \varepsilon) + \varepsilon_0). \end{array} \qquad (52)$$

Similarly

$$\int_{\sigma(0)}^{\sigma_0} \bar{\Psi}_1(\zeta)\, d\zeta + k \int_{0}^{\Delta} \bar{\Psi}_1(\zeta)\, d\zeta = \\ = \varkappa^{-1}(W(0) + M_0(\varkappa,\tau,\varepsilon) + \varepsilon_0). \tag{53}$$

It follows from (52), (53) that

$$\bar{V}_1(t) = W(\bar{t}) + W(0) + M_0(\varkappa,\tau,\varepsilon) + \varepsilon_0. \tag{54}$$

By noticing that $\varphi(\sigma(\bar{t})) = \varphi(\sigma_0 + k\Delta) = 0$, the equations (35),(36) imply that

$$W(\bar{t}) \geq 0, \tag{55}$$

and therefore

$$\bar{V}_1(t) \geq W(0) + M_0(\varkappa,\tau,\varepsilon) + \varepsilon_0, \tag{56}$$

which contradicts (50). We have proved that

$$\sigma(t) < \sigma_0 + k\Delta, \quad \forall t > 0. \tag{57}$$

Suppose now that

$$\sigma(\bar{t}) = \sigma_0 - (k+1)\Delta \tag{58}$$

with $\bar{t} > 0$. Consider the Lyapunov function $\bar{V}_2$:

$$\bar{V}_2(\bar{t}) = W(\bar{t}) - \\ -\varkappa a \left(\int_{\sigma_0-(k+1)\Delta}^{\sigma_0-\Delta} \bar{F}_2(\zeta)\, d\zeta + \int_{\sigma_0-\Delta}^{\sigma(0)} \bar{F}_2(\zeta)\, d\zeta \right) - \\ -\varkappa a_0 \left(\int_{\sigma_0-(k+1)\Delta}^{\sigma_0-\Delta} \bar{\Psi}_2(\zeta)\, d\zeta + \int_{\sigma_0-\Delta}^{\sigma(0)} \bar{\Psi}_2(\zeta)\, d\zeta \right). \tag{59}$$

It is easy to see that

$$\int_{\sigma_0-(k+1)\Delta}^{\sigma_0-\Delta} \bar{F}_2(\zeta)\, d\zeta + \int_{\sigma_0-\Delta}^{\sigma(0)} \bar{F}_2(\zeta)\, d\zeta = \\ = \int_{\sigma_0-(k+1)\Delta}^{\sigma_0-\Delta} \bar{\Psi}_2(\zeta)\, d\zeta + \int_{\sigma_0-\Delta}^{\sigma(0)} \bar{\Psi}_2(\zeta)\, d\zeta = \\ = -\varkappa^{-1}(W(0) + M_0(\varkappa,\tau,\varepsilon) + \varepsilon_0). \tag{60}$$

As a result

$$\bar{V}_2(t) = W(\bar{t}) + W(0) + M_0(\varkappa,\tau,\varepsilon) + \varepsilon_0 > \\ > W(0) + M_0(\varkappa,\tau,\varepsilon), \tag{61}$$

which contradicts (50). We have established that

$$\sigma(t) > \sigma_0 - (k+1)\Delta, \quad \forall t > 0, \tag{62}$$

which finishes the proof of (43) and Theorem 2. □

5 The case of non-differentiable nonlinearity

In this section we consider the case when there is no information about the derivative $\psi'(\sigma)$. So we have to waive the C^1-smoothness of $\psi(\sigma)$ and confine ourselves to the assumption of $\psi(\sigma) \in C(\mathbb{R})$. Then we cannot use the auxiliary construction (12) and the quadratic form $G(y,\eta)$ with $\tau > 0$. We have to admit that $\tau = 0$. Consequently the frequency inequality (13) takes the form

$$\varkappa Re\{K(\imath\omega)\} - \varepsilon|K(\imath\omega)|^2 - \delta \geq 0 \quad \forall \omega \geq 0. \tag{63}$$

The inequality (63) guarantees [Gelig et al., 2004] the existence of matrix $H_0 = H_0^*$ such that

$$G_0(z,\xi) \triangleq 2z^*H_0(Az + b\xi) + \varepsilon(c^*z + \rho\xi)^2 + \\ +\varkappa(c^*z + \rho\xi)\xi + \delta\xi^2 < 0 \quad \forall z \in \mathbb{R}^m,\ \xi \in \mathbb{R}. \tag{64}$$

Since (64) implies that

$$2z^*H_0Az \leq -\varepsilon(c^*z)^2 \quad \forall z \in \mathbb{R}^m$$

we conclude that H_0 is positive definite.

Theorem 3. Suppose there exist $k \in \mathbb{N}$, $\varkappa \neq 0$ and $\varepsilon, \delta > 0$ such that the following requirements are true:

1) the frequency–domain inequality (63) holds;
2) there exists a matrix $H_0 = H_0^*$, satisfying (64) such that inequalities

$$4\varepsilon\delta > \varkappa^2 r_j^2(k,\varkappa, z^*(0)H_0z(0) + M_0(\varkappa,0,\varepsilon)) \\ (j = 1,2), \tag{65}$$

where r_j are defined by (16) and M_0 is taken from (19), are valid.

Then the solution with the initial condition $(x(0), \sigma(0))$ slips less than k cycles, i.e. the estimate (21) is true.

Proof. The proof of this theorem is alike that of Theorem 1. Let

$$S_0 \triangleq z^*(0)H_0z(0) + M_0(\varkappa,0,\varepsilon) \tag{66}$$

and ε_0 be so small that the inequalities

$$4\varepsilon\delta > \varkappa^2 r_j^2(k,\varkappa,S_0 + \varepsilon_0) \quad (j = 1,2) \tag{67}$$

are valid. We introduce the Lyapunov–type functions

$$V_j(t) \triangleq W(t) + \varkappa \int_{\sigma(0)}^{\sigma(t)} F_j(\zeta)\, d\zeta \\ (j = 1,2), \tag{68}$$

where

$$W(t) \triangleq z^*(t)H_0z(t) \quad z \in \mathbb{R}^m \tag{69}$$

and F_j are defined by (23). Then we compute the derivatives of Lyapunov–type functions in virtue of system (8):

$$\begin{aligned}&\frac{dV_j(t)}{dt}=2z^*(t)H_0(Az(t)+b\varphi(\sigma(t)))+\\&+\varkappa F_j(\sigma(t))\dot\sigma(t).\end{aligned}\tag{70}$$

It follows from (64) and (8) that

$$\begin{aligned}&\dot V_j(t)\le -\varepsilon\dot\sigma^2(t)-\delta\varphi^2(\sigma(t))-\\&-r_j\varkappa|\varphi(\sigma(t))|\dot\sigma(t)+U(\varkappa,\varepsilon,0;t),\end{aligned}\tag{71}$$

where $U(\varkappa,\varepsilon,\tau;t)$ is defined by (18). From (71) by virtue of (67) we get the estimate

$$V_j(t)\le W(0)+M_0(\varkappa,0,\varepsilon)\quad (j=1,2).\tag{72}$$

Suppose that

$$\sigma(\bar t)=\sigma(0)-k\Delta.\tag{73}$$

Then

$$V_2(\bar t)=W(\bar t)-k\varkappa\int_0^\Delta F_2(\zeta)\,d\zeta.\tag{74}$$

Notice that

$$\int_0^\Delta F_2(\zeta)\,d\zeta=-\frac{S_0+\varepsilon_0}{\varkappa k}.\tag{75}$$

Since H_0 is positive definite one has

$$V_2(\bar t)\ge S+\varepsilon_0>W(0)+M_0(\varkappa,0,\varepsilon),\tag{76}$$

which contradicts (72). So

$$\sigma(t)>\sigma(0)-k\Delta.\tag{77}$$

Using the function $V_1(t)$ we can easily prove that

$$\sigma(t)<\sigma(0)+k\Delta.\tag{78}$$

By (77) and (78) Theorem 3 is proved. □

Example. Consider the phase–locked loop (PLL) with the proportional–integrating low–pass filter and the sine–shaped characteristic of phase detector:

$$\begin{aligned}&\dot x(t)=-\tfrac{1}{T}x(t)-(1-s)(\varphi(\sigma(t))+q(t))\\&\dot\sigma(t)=x(t)-sT(\varphi(\sigma(t))+q(t))\\&\varphi(\sigma)=\sin\sigma-\beta\\&(s,\beta\in(0,1),T>0),\end{aligned}\tag{79}$$

with the transfer function

$$K(p)=T\frac{sTp+1}{Tp+1}\quad (p\in\mathbb{C}).\tag{80}$$

In the case of $q(t)\equiv 0$ the system has been investigated in [Ershova and Leonov, 1983; Leonov et al., 1992].

Let us apply Theorem 3. The inequality (63) with $\varkappa=1$ is equal to the inequality

$$\omega^2(T^3s-\varepsilon T^4s^2-\delta T^2)+(T-\varepsilon T^2-\delta)\ge 0,\quad \forall\omega.\tag{81}$$

The optimal values for the varying parameters δ and ε are as follows from [Gelig et al., 2004]:

$$\varepsilon=T^{-1}(1+s)^{-1},\delta=T(1-\varepsilon T),\tag{82}$$

whence

$$\varepsilon\delta=\frac{s}{(1+s)^2}.\tag{83}$$

The matrix H_0 is determined by special algorithm [Ershova and Leonov, 1983]:

$$H_0=\frac{1}{2}T\varepsilon=\frac{1}{2(1+s)}.\tag{84}$$

So the condition 2) of Theorem 3 is equal to the inequality

$$\frac{2\sqrt{s}}{1+s}>\frac{2\pi\beta+\frac{1}{k}(\frac{z^2(0)}{2(1+s)}+M_0(0;1;\varepsilon))}{4(\beta\arcsin\beta+\sqrt{1-\beta^2})}.\tag{85}$$

(Notice that if

$$\frac{2\sqrt{s}}{1+s}\le\frac{2\pi\beta}{4(\beta\arcsin\beta+\sqrt{1-\beta^2})}.$$

the condition 2) of Theorem 3 is violated for any k.)

It follows from (85) that the upper estimate for the number of slipped cycles is the number

$$\begin{aligned}&m=\\&=\left[\frac{\frac{1}{2}z^2(0)+(1+s)M_0(0,1,\varepsilon)}{8\sqrt{s}(\beta\arcsin\beta+\sqrt{1-\beta^2})-2\pi\beta(1+s)}\right]+1,\end{aligned}\tag{86}$$

where $[\cdot]$ is used for the integer floor. In Fig. 1 the case of $s=0.4, z(0)=T\beta$ is illustrated. The value of $M_0(0,1,\varepsilon)$ is supposed small enough. Namely,

$$\begin{aligned}&\left[\frac{\frac{1}{2}z^2(0)+(1+s)M_0(0,1,\varepsilon)}{8\sqrt{s}(\beta\arcsin\beta+\sqrt{1-\beta^2})-2\pi\beta(1+s)}\right]=\\&=\frac{1}{2}\left[\frac{z^2(0)}{8\sqrt{s}(\beta\arcsin\beta+\sqrt{1-\beta^2})-2\pi\beta(1+s)}\right]\end{aligned}$$

The domains with the number of slipped cycles less than m are situated under the corresponding curves.

The estimate (86) remains valid for any continuous nonlinear 2π–periodic function

$$\varphi(\sigma)=\sin\sigma-\beta+\varphi_0(\sigma)$$

where $(\sin\sigma-\beta)\varphi_0(\sigma)\ge 0$, $\varphi_0(\arcsin\beta)=$ $=\varphi_0(\pi-\arcsin\beta)=0$, and

$$0\le\int_0^{2\pi}\varphi_0(\sigma)d\sigma\le 2\pi\beta.$$

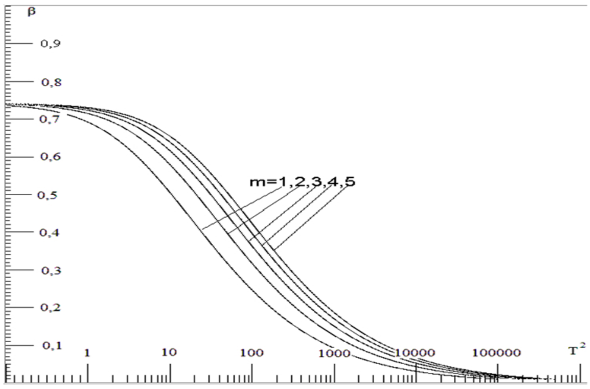

Figure 1. Domains with the number of slipped cycles less than m

6 Conclusion

The problem of cycle-slipping for multidimensional control systems with periodic nonlinearities and external disturbances has been considered. By means of periodic Lyapunov functions and Kalman-Yakubovich-Popov lemma, new estimates for the number of slipped cycles has been established. These results allow to estimate, which equilibrium (of the infinite sequences) attracts the solution for a given initial condition. In our future works, we are going to apply the developed theory to analysis of synchronization in mechanical and physical systems, such as e.g. coupled pendulums [Czolczynksi et al., 2012], vibration units (rotos) [Blekhman, 1988] and Josephson junction arrays [Qin and Chen, 2004].

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