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Hybrid metamaterials combining pentamode lattices and phononic plates

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We propose a design strategy for hybrid metamaterials with alternating phononic plates and pentamode units that produce complete bandgaps for elastic waves. The wave control relies on the simultaneous activation of two scattering mechanisms in the constituent elements. The approach is illustrated by numerical results for a configuration comprising phononic plates with cross-like cavities. We report complete bandgaps of tunable width due to variations of geometric parameters. We show that the wave attenuation performance of the hybrid metamaterials can be further enhanced through implementation of lightweight multiphase material compositions. These give rise to efficient wave attenuation in challenging low-frequency regions. The proposed design strategy is not limited to the analyzed cases alone and can be applied to various designs of phononic plates with cavities, inclusions or slender elements. *Published by AIP Publishing.*

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Phononic and acoustic metamaterials demonstrate unusual mechanical properties^{1,2} and the ability to control elastic waves by producing bandgaps^{3–5} or negative group velocity.^{6,7} They draw these remarkable functionalities from their engineered architectures, giving rise to unconventional dynamic characteristics in various frequency ranges. Numerous two-dimensional (2D) configurations with periodic patterns have been designed to activate wave manipulation mechanisms, resulting in omnidirectional, complete bandgaps for plane-polarized elastic waves (2D bandgaps).⁸ Applications of such designs to three-dimensional (3D) geometries are usually characterized by poor attenuation of oblique or normally incident waves.^{8,9} Common examples are phononic plates with voids¹⁰ or internal resonators^{11–13} that can manipulate waves in the plane of a plate, while waves with out-of-plane wavevector components can propagate freely.^{9,13,14} This issue substantially limits the potential of 2D metamaterials for engineering applications, including seismic wave shielding,^{15,16} vibration mitigation,^{3,6,13} or wave focusing and splitting.^{17,18}

Here, we propose a design strategy specially aimed at extending 2D bandgaps in phononic plates to a full 3D setting. We show that hybrid metamaterials, consisting of phononic plates interlayered by pentamode lattice units, exhibit complete 3D bandgaps due to the simultaneous activation of wave scattering in the plates and the hybrid structure.

Pentamode lattices belong to a class of “extremal materials” as introduced by Milton and Cherkaev.^{19–21} These essentially 3D structures consist of periodic repetitions of four tapered bars meeting at point-like joints in a diamond-like lattice. Ideal pentamodes have zero shear modulus, and

thus exhibit fluid-like dynamics, inhibiting the propagation of shear waves at any frequency.^{19,22,23} Realistic structures are characterized by a finite, non-zero effective shear modulus. Typically, this modulus is much smaller than the effective bulk modulus.^{23,24} Shear and compressional waves are thus weakly coupled. This leads to frequency intervals with a single compressional mode. As we shall show, the hybrid structures formed by a combination of pentamode lattices and phononic plates can be designed to produce 3D bandgaps. Such metastructures enable bandgap tuning by adjusting the geometrical parameters and maintain structural integrity due to incorporated spheres at the joints.

A typical phononic plate has an essentially 2D configuration if its cross-section is invariant along the thickness. This simplifies theoretical analysis of the plate dynamics, possible optimization procedures, and manufacturing processes. A 2D formulation of the related elastodynamic problem for the cross-sectional geometry (assuming an infinite thickness of the plate) enables the decoupling of motions into in-plane modes with displacements $\{u_x, u_y\}$ and out-of-plane (or transverse) modes with displacements u_z .^{8,9} Scattering mechanisms for these mode families are governed by a 2D elasticity tensor and a shear modulus, respectively. This results in 2D bandgaps at different frequencies for different mode types.^{9,11} In a 3D plate of finite thickness, the separation of modes is, in general, not possible. For waves in the cross-sectional plane, the band structures of in-plane and out-of-plane modes are superimposed, while for oblique incident waves, the two mode types are coupled, leading to the closing of bandgaps (see Figs. S1–S3 in the [supplementary material](#)). In order to induce complete 3D bandgaps, one needs to introduce a wave attenuation mechanism in the out-of-plane direction, suppressing the coupled modes. This is

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typically done by developing a new 3D configuration from scratch, neglecting the extensive knowledge and data available for 2D designs. Our approach, in contrast, relies on preserving and using the wave attenuation abilities of 2D phononic plates in 3D hybrid metamaterials.

An example of the proposed hybrid metamaterials is a combination of phononic plates with cross-like cavities (known for their ability to induce wide 2D bandgaps¹⁰) interlayered by pentamode units. The metamaterial unit cell [Fig. 1(a), on the left] can be periodically repeated along the z axis to form a 1D meta-chain or populated along the three axial directions into a 3D structure. An extended face-centered-cubic lattice, typical for the pentamode, is thus replaced by a tetragonal lattice with the Brillouin zone shown in Fig. 1(b).²⁵ To maintain the structural stability, we introduce elastic spheres connecting the lattice bars to each other and to the plates. The center and end diameters of a bar are denoted by D and d , respectively; the radius of the connection spheres is R . The bar length is $\sqrt{3}a/4$, where a denotes the height of the pentamode element [Fig. 1(a), on the right]. The cross-like cavity is defined by length b , width c , and depth h . The unit cell dimensions are $a_1 \times a_1 \times a_3$ with $a_1 = a + 2R$ and $a_3 = a + 2h$. The examples considered here are for structures made of the isotropic titanium alloy Ti6Al4V²⁶ with Young's modulus $E = 120$ GPa, Poisson's ratio $\nu = 0.33$, and mass density $\rho = 4450$ kg/m³.

Wave dispersion is evaluated numerically in Comsol Multiphysics 5.2 by applying Bloch-Floquet boundary conditions at the three pairs of the plate faces and solving the related eigenfrequency problem for wavenumbers along the borders of the irreducible Brillouin zone [Fig. 1(b)]. Figure 2 shows the dispersion relation for the hybrid metamaterial with $a = 16$ mm, $d = 0.2$ mm, $D = 1.2$ mm, $R = 0.1a$, $h = 0.2a$, $b = 0.9a_1$, and $c = 0.25a_1$. These values are chosen to provide the widest bandgaps for the transverse and in-plane modes in the pentamode lattice and the phononic plate, respectively.^{10,22} The color of the bands designates the mode polarization $p = \int_V |u_z|^2 dV / \int_V (|u_x|^2 + |u_y|^2 + |u_z|^2) dV$, where V is the material volume in the unit cell. Specifically, blue indicates in-plane modes, and red indicates out-of-plane modes.

The band structure diagram in Fig. 2 exhibits a complete 3D bandgap highlighted in dark gray. For waves propagating parallel to the Oxy plane, it originates from Bragg scattering in a phononic plate of thickness $2h$ with stress-free bottom and upper faces. Such a plate has a 2D bandgap between 55.7 kHz and 96.4 kHz, indicated in gray.

To understand the band-gap formation process for waves along $\Gamma - Z$, $M - A$ and $X - R$ directions, we note that the

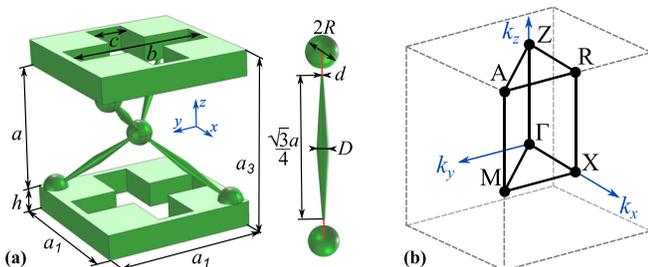


FIG. 1. (a) The unit cell of a hybrid metamaterial with cross voids and (b) irreducible Brillouin zone for a tetragonal lattice.

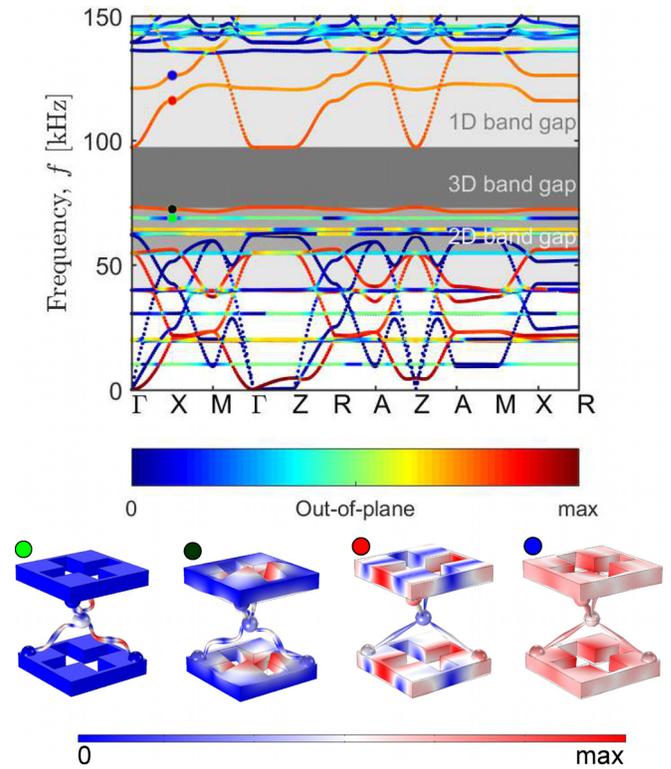


FIG. 2. Band structure for a hybrid monomaterial metastructure with cross-like voids. The color of the dispersion bands indicates the mode polarization. The colored circles refer to the vibration patterns at the selected frequencies given at the bottom.

structure of the hybrid metamaterial resembles the lattice of a zincblende crystal with tetrahedral coordination and alternating masses at lattice sites.²⁷ Thus, the wave propagation along the z axis can be approximated by a dispersion relation of a 1D diatomic chain (Fig. S4),²⁸ which is formed by two masses (a plate with half-spheres and a central sphere) connected by springs (inclined bars). The corresponding dispersion relation (see Sec. II in the [supplementary material](#) for details) is characterized by an extremely wide 1D bandgap highlighted by light gray in Fig. 2. As the real hybrid metastructure is formed by elastic plates with distributed (not lumped) masses, the vibration modes of the plates give rise to additional bands in the band structure (e.g., the mode indicated by the blue circle in Fig. 2). The Bloch-Floquet conditions at the plate boundaries also generate an additional set of modes, represented, for instance, by the localized mode marked by the green circle. As a result, the 3D bandgap has a narrow width, as compared to that of the diatomic chain, limited to the frequencies of the 2D bandgap for the plate modes. Vibration patterns at the bounds of the 3D bandgap (red and black circles) reveal strong interactions between the bars and the plates.

The introduced analogy with a diatomic chain suggests the universality of the proposed design strategy. In other words, hybrid metamaterials can be constructed for any phononic plate exhibiting 2D bandgaps (see, e.g., Fig. S6). This analogy also indicates an important role of the central sphere in the wave attenuation mechanism for the hybrid designs. On the one hand, the decrease in its mass results in the shift of the upper bandgap bound to higher frequencies. On the

other hand, the smaller the radius of the sphere, the smaller the effective axial stiffness of the bars due to the vanishing contact areas between the bars. Our simulations show that there is no bandgap for $R < 0.06a$ [Fig. S7(a)]. For larger R values, a 3D bandgap of almost constant width is induced, as the variations of the central sphere remain small compared to the mass of the plate.²⁹ The revealed dependence of the band-gap width on R is opposite to that for pure pentamodes, in which wide bandgaps for shear waves are obtained for vanishing contacts between the bars.²²

Figures 3(a)–3(d) show the shapes of iso-frequency contours for the lowest out-of-plane mode (a) and (b), originating from zero frequency, and the first mode above the bandgap (c) and (d) for the two planes of the Brillouin zone [see Fig. 1(b)]. The symmetries of these contours reflect the rotation and reflection symmetries of the unit cell. For waves in the plane of the phononic plate, the hybrid metamaterial is

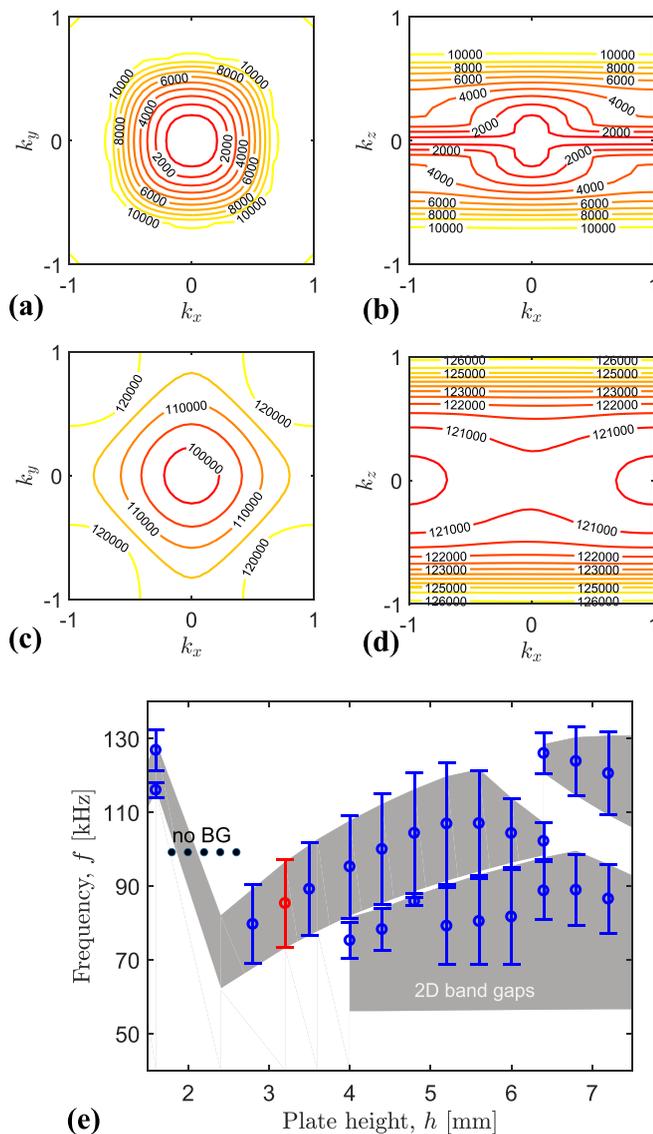


FIG. 3. (a)–(d) Directionality of plane waves propagating in Γ - X - M and Γ - X - R - Z planes of the Brillouin zone at low (a,b) and high (c,d) frequencies. The frequencies (in Hz) associated with the contours are labeled. (e) The bars indicate the width of 3D bandgaps for hybrid metamaterials with varying thicknesses h of the plate. The red bar corresponds to the bandgap in Fig. 2. The shaded region shows the 2D bandgaps for phononic plates of the corresponding thickness.

isotropic, whereas for waves with non-zero components k_z , it is strongly anisotropic at any frequency. Similar behavior is observed for other modes (see Figs. S8 and S9).

Next, we demonstrate that the 3D bandgap exists in a wide range of the geometric parameters, i.e., the wave-attenuation mechanisms are not limited to a particular geometric configuration of the hybrid metamaterial. Figure 3(e) presents the bandgap width versus the plate height h , with the other parameters fixed. The gray shading indicates the frequencies of the 2D bandgaps for the plates of thickness $2h$ with cross-like cavities. In most cases, the 3D bandgap occurs within the frequencies of the 2D bandgaps. However, for other parameters of the hybrid meta-structure, one can extend the 3D bandgap to slightly wider ranges (see Fig. S7). Note that by varying the plate thickness, one can tune the gap frequencies or even close the bandgap, as e.g., for $1.6 < h < 2.8$ mm. For $4 \leq h \leq 6$ mm, the mode marked with the black circle in Fig. 2, is shifted towards higher frequencies and separates the bandgap into two parts. As the thickness increases ($h > 6$ mm), other plate modes enter the band-gap range and split it further. Similar tunability can be achieved by varying the center diameter D of the bars [Fig. S8(b)].

We further analyze bi-material configurations with different material phases for the plate and lattice units. The key ideas here are to improve the structural integrity by decreasing the weight of the plates and to obtain more *light-weight* configurations. As an example, we consider a unit cell in Fig. 1(a) with the plates made of Nylon³⁰ (Young's modulus $E^{(p)} = 2$ GPa, Poisson's ratio $\nu^{(p)} = 0.41$, and mass density $\rho^{(p)} = 1200$ kg/m³) and the pentamode bars made of titanium alloy [Fig. 4(a)]. Our simulations reveal a 3D bandgap of 22% gap width for $h = 3.5$ mm. The mid-gap frequency 22.9 kHz is about 4 times lower than that of the corresponding mono-material (titanium) configuration [Fig. 2(a)], and the effective material density $\rho^{eff} = 270$ kg/m³ (evaluated as the sum of a material phase density multiplied by its volume fraction) is 3.3 times smaller than $\rho^{eff} = 892$ kg/m³ for the mono-structure. Hence, apart from the improved integrity, the bi-material hybrid configuration enables the generation of 3D bandgaps in the challenging low-frequency range. This is a distinguishing feature of these designs as compared to other mass-lattice meta-structures in the literature, where low-frequency wave attenuation is achieved through the introduction of heavy masses.^{5,26,30,31}

Finally, we estimate the efficiency of wave attenuation in the bi-material configurations by performing transmission analysis. The related frequency-domain finite-element simulations are performed for 5 unit-cell samples with periodic boundary conditions at the lateral faces, excited by time-harmonic normal displacements of amplitude $u_{z0} = 1$ μ m at one end, while the other end is attached to a perfectly matched layer (of 5 unit-cell size). For waves propagating in the $\Gamma - Z$ and $\Gamma - X$ directions [Fig. 4(b)], the curves in Figs. 4(c) and 4(d) represent the magnitude of normalized transmitted displacements $\sqrt{u_x^2 + u_y^2 + u_z^2}/u_{z0}$ averaged upon one unit cell. Elastic and damped material responses are denoted by the black and red curves, respectively. The transmission drops agree well with the bandgaps (shaded

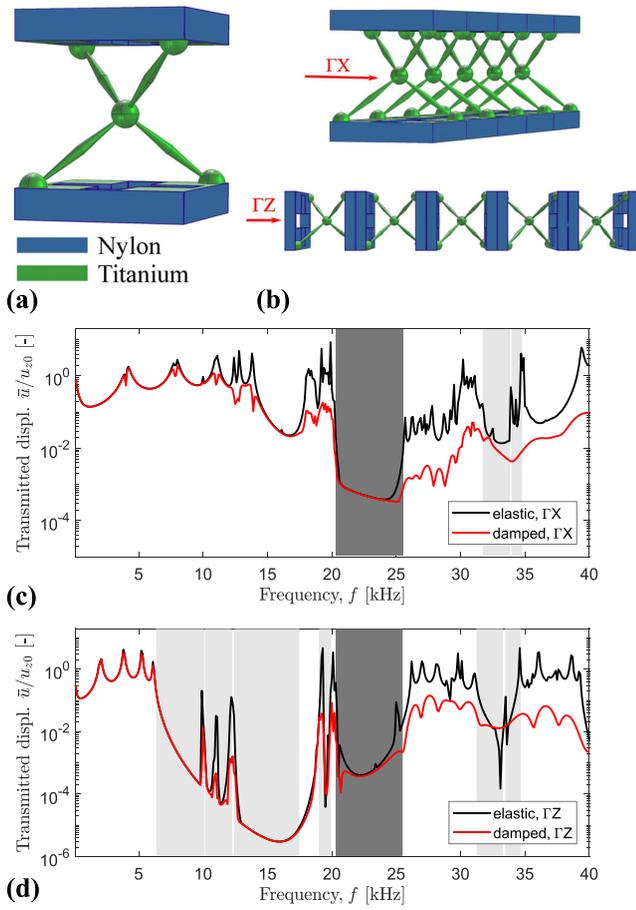


FIG. 4. (a) Unit cell of a two-phase hybrid metamaterial; (b) meta-structures composed of five unit cells. (c) and (d) Normalized transmitted displacements $\sqrt{u_x^2 + u_y^2 + u_z^2}/u_{z0}$ vs. frequency. Shaded regions indicate directional (light gray) and complete 3D (dark gray) bandgaps.

regions) or occur at frequencies of modes not excited by the applied loading [Fig. S5(b) in the [supplementary material](#)]. The small discrepancies can be attributed to the finite sizes of the samples. Note that at the frequencies of the 3D bandgap (dark gray shading), the transmission drops by three orders of magnitude, indicating excellent wave attenuation performance of the hybrid designs. The material dissipation in Nylon is implemented using a Rayleigh model with coefficients $\alpha = 1 \text{ s}^{-1}$ and $\beta = 4e - 7 \text{ s}$, in agreement with the reported experimental data.³⁰ For titanium, we introduce the loss factor η in the stress-strain relation $\sigma = D(1 + i\eta)\epsilon$ and assign $\eta = 0.001 \text{ Pa s}$ corresponding to minimum experimentally measured losses.³² The transmission magnitude in the damped case is lower compared to the elastic case, in agreement with the predictions for lossy composites.^{33,34} The amplified damping behavior at $f > 25 \text{ kHz}$ can be explained by the inapplicability of the Rayleigh model at these frequencies.³⁰

In summary, we have proposed a design strategy for hybrid metamaterials producing 3D bandgaps for elastic waves by combining phononic plates with pentamode units. We have illustrated the idea considering an example of hybrid metamaterials considering a specific phononic plate and demonstrated the universality of the strategy for plates with various wave attenuation mechanisms. This paves the way for the development of numerous 3D metamaterials with

target wave attenuation characteristics by fully exploiting the advantages of 2D configurations. For instance, one can apply powerful topology optimization techniques to design 2D geometries with required dynamic characteristics at much lower computational costs as compared to 3D cases, and then introduce them into hybrid designs with pentamode lattices by ensuring the presence of 2D bandgaps.

The proposed hybrid designs guarantee structural integrity through reinforcement of the critical joints. This becomes of importance when considering finite deformations.³⁵ Wave attenuation occurs for a wide range of configurations and is shown to be highly tunable by varying geometric parameters. Hence, it relies on the intrinsic structure of the proposed designs, rather than on a specific choice of geometric properties. This feature opens the way to the development of meta-structures for broadband wave attenuation by employing rainbow-type designs.³¹ Moreover, we have shown that *multiphase* designs of the hybrid metamaterials can further produce low-frequency attenuation characteristics in lightweight structures. The illustrative example of polymeric plates and stiff pentamodes demonstrates the potential for a broad range of engineering applications aimed at wave and vibration attenuation.

See [supplementary material](#) for 3-D band structures of phononic plates (I), details of the equivalent mass-spring model for hybrid metamaterials (II), examples for bi-material architectures (III), and geometrical variations of the design (IV).

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