A Binary Operator Generated by Homographic Function

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Abstract: In this work we are discussing the binary operator that we can generated by homographic function. By means of this operator, that we can see as a generalized sum, we can create a group.

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In recent works we have shown some generalized sums and related groups, which are based on transcendental functions. Let us note that the generalized sums are binary operators which widespread the addition of real numbers. Besides the investigation of groups generated by transcendental functions, we have also considered groups involving generalized integers [1-5]. Here we consider the group having a binary operator generated by homographic function.

Let us consider two elements x and y of reals **R,** and a related binary operation. We indicate this composition law by the notation $x \oplus y$, a generalized sum so that (**R**,⊕) is giving a group.

Let us remember that a group is a set A having an operation ⊕ which is combining the elements of A. That is, the operation combines any two elements a,b to form another element of the group denoted $a \oplus b$. To qualify $(A,\oplus)$ as a group, the set and operation must satisfy the following requirements. *Closure***:** For all a,b in A, the result of the operation $a \oplus b$ is also in A. *Associativity:* For all a,b and c in A, it holds $(a \oplus b) \oplus c = a \oplus (b \oplus c)$. *Neutral (or identity) element*: An element e exists in A, such that for all elements a in A, it is $e \oplus a = a \oplus e = a$. *Opposite (or inverse) element:* For each a in A, there exists an element b in A such that $a \oplus b = b \oplus a = e$, where e is the identity. If a group is Abelian, a further requirement is the c*ommutativity:* For all a,b in A, $a \oplus b = b \oplus a$.

To have a given generalized sum, we follow an approach based on a "generation" [1,6-8].

Let us have a function $G(x)$, which is invertible $G^{-1}(G(x))=x$. A deformation *generator* can define the *group law* $\Phi(x,y)$ [6,7]:

$$\Phi(x,y)=G(G^{-1}(x)+G^{-1}(y)) \quad \text{or} \quad x\oplus y=G(G^{-1}(x)+G^{-1}(y)) \ .$$

In this manner the *group law* is giving the *generalized sum,* as we can call the binary operator of the group.

Let us consider the homographic function.

$$G(x)=\frac{x+1}{x-1} \ ; \ G^{-1}(x)=\frac{x+1}{x-1} \ ; \ G^{-1}(G(x))=\frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1}=\frac{x+1+x-1}{x+1-x+1}=x$$

$$x\oplus y=G(G^{-1}(x)+G^{-1}(y))=G(\frac{x+1}{x-1}+\frac{y+1}{y-1})=G(\frac{2xy-2}{(x-1)(y-1)})=\frac{\frac{2xy-2}{(x-1)(y-1)}+1}{\frac{2xy-2}{(x-1)(y-1)}-1}$$

So that: $x\oplus y=\dfrac{3xy-x-y-1}{xy+x+y-3}$ (1).

(1) is the generalized sum based on the homographic function..

To have a finite value of (1), we need $xy+x+y-3\neq 0$. That is: $y\neq(3-x)/(1+x)$ (*). In this manner we have the closure on finite values.

Let us consider the neutral element *e* of this sum. It is different from zero, as we can easily see if we use 0 in the generalized sum:

$$x\oplus 0=\frac{3x0-x-0-1}{x0+x+0-3}=\frac{-x-1}{x-3}=\frac{x+1}{3-x}$$

Let us note that $x\oplus 0$ is the number that we have in the condition (*). For this reason, let us also avoid 0, from the element of the set (**).

The neutral element e is equal to -1 : $x\oplus(-1)=\frac{-3x-x+1-1}{-x+x-1-3}=\frac{-4x}{-4}=x$

The opposite element of x is $1/x$, so that: $x\oplus(1/x)=\frac{3x/x-x-1/x-1}{x/x+x+1/x-3}=-1$

For the a*ssociativity,* we can show that $x\oplus(y\oplus z)=(x\oplus y)\oplus z$. Actually:

$$x\oplus(y\oplus z)=\frac{3x(y\oplus z)-x-(y\oplus z)-1}{x(y\oplus z)+x+(y\oplus z)-3} \quad ; \quad (x\oplus y)\oplus z=\frac{3(x\oplus y)z-(x\oplus y)-z-1}{(x\oplus y)z+(x\oplus y)+z-3}$$

$$x\oplus(y\oplus z)=\frac{8xyz-4xy-4xz-4yz+4}{4xyz-4x-4y-4z+8}=(x\oplus y)\oplus z$$

Using the binary operator (1), and conditions (*),(**), we can define the neutral element. We have also that the binary operator possesses the associative property. In this manner (1) is a generalized sum of a group.

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