Groupoid of OEIS A003154 Numbers
(star numbers or centered dodecagonal numbers)

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Here we discuss the binary operators of the set made by the OEIS sequence of integers A003154, defined as star numbers or centered dodecagonal numbers. The binary operators can be used to have groupoids.

Written in Torino, 5 September 2019. DOI: 10.5281/zenodo.3387054

In [1] we can find discussed the Star Numbers. These numbers are representing the cells in a generalized Chinese checkers board (or "centered" hexagram). To the star numbers are linked some sequences of integers [1,2]. In [3], these numbers are also defined as centered dodecagonal numbers. As illustrated by Omar E. Pol, these number shave a classic representation in the form of stars, but they can also be represented by n-1 concentric hexagons around a central element. In general, centered polygonal numbers are those numbers represented by a central dot, surrounded by polygonal layers with a constant number of sides. Here we consider the Oeis A003154 numbers as a groupoid.
A groupoid is an algebraic structure made by a set with a binary operator [4]. The only restriction on the operator is closure. This properties means that, applying the binary operator to two elements of a given set S, we obtain a value which is itself a member of S. If this operation is associative and we have a neutral element and opposite elements into the set, then the groupoid becomes a group. So let us consider OEIS A003154 numbers.
The numbers have the following form [3]:

$$S_n=6n(n-1)+1=6n^2-6n+1$$

As we did in some previous discussions (see for instance [5]), we can find a binary operator, which satisfy the closure. Let us follow the same approach as in [6-8].

We have: $S_n=6n^2-6n+1=6(n-1)^2+6(n-1)+1$

Let us use numbers A_n:, so that: $A_n=(n-1)$. We have that:

$$A_{n+m}=(n-1)+(m-1)+1=(n+m-1)$$

So we can define a binary operation such as: $A_{n+m}=A_n \oplus A_m=A_n+A_m+1$.

We have that: $S_n = 6A_n^2 + 6A_n + 1$; $A_n = -\frac{1}{2} \pm \frac{1}{12}(12+24S_n)^{1/2} = -\frac{1}{2} \pm \frac{1}{6}(3+6S_n)^{1/2}$ (1)

Let us consider in (1) the positive sign:

$$A_{n+m} = A_n + A_m + 1 = -\frac{1}{2} + \frac{1}{6}(3+6S_n)^{1/2} - \frac{1}{2} + \frac{1}{6}(3+6S_m)^{1/2} + 1$$

Then it must be:

$$A_{n+m} = -\frac{1}{2} + \frac{1}{6}(3+6S_{n+m})^{1/2} \quad = \quad \frac{1}{6}(3+6S_n)^{1/2} + \frac{1}{6}(3+6S_m)^{1/2}$$

$$\frac{1}{6}(3+6S_{n+m})^{1/2} = \frac{1}{6}(3+6S_n)^{1/2} + \frac{1}{6}(3+6S_m)^{1/2} + \frac{1}{2}$$

So we have:

$$(3+6S_{n+m}) = ((3+6S_n)^{1/2} + (3+6S_m)^{1/2} + 3)^2 \quad =$$

$$(3+6S_n) + (3+6S_m) + 9 + 6(3+6S_n)^{1/2} + 6(3+6S_m)^{1/2} + 2(3+6S_n)^{1/2}(3+6S_m)^{1/2}$$

Then:

$$S_{n+m} = S_n + S_m + 2 + (3+6S_n)^{1/2} + (3+6S_m)^{1/2} + \frac{1}{3}(3+6S_n)^{1/2}(3+6S_m)^{1/2}$$

The generalized sum for the star numbers is given as:

$$S_n \oplus S_m = S_n + S_m + 2 + (3+6S_n)^{1/2} + (3+6S_m)^{1/2} + \frac{1}{3}(3+6S_n)^{1/2}(3+6S_m)^{1/2} \quad (2)$$

From (1), we have the recursive relation: $S_{n+1} = S_n \oplus S_1$. Starting from number $S_1 = 1$, we have: 13, 37, 73, 121, 181, 253, 337, 433, 541, 661, 793, 937, 1093, 1261, 1441, 1633, 1837, 2053, 2281, 2521, and so on. The same as http://oeis.org/A003154 .

The recursive relation is:

$$S_{n+1} = S_n + 1 + 2 + (3+6S_n)^{1/2} + 3 + (3+6S_n)^{1/2}$$

$$S_{n+1} = S_n + 6 + 2(3+6S_n)^{1/2}$$

The square root:

$$(3+6S_n)^{1/2}$$

gives the sequence: 3, 9, 15, 21, 27, 33, 39, 45, etc.

Let us consider in (1) the negative sign:

$$A_{n+m}=A_n+A_m+1=-\frac{1}{2}-\frac{1}{6}(3+6S_n)^{1/2}-\frac{1}{2}-\frac{1}{6}(3+6S_m)^{1/2}+1$$

We have:

$$S_n\oplus S_m=S_n+S_m+2-(3+6S_n)^{1/2}-(3+6S_m)^{1/2}+\frac{1}{3}(3+6S_n)^{1/2}(3+6S_m)^{1/2} \quad (3)$$

From (3), with the number $S_1=1$ we have the relation: $S_n=S_n\oplus S_1$. Therefore, S_1 is a neutral element, as we can easily see:

$$S_n+1+2-(3+6S_n)^{1/2}-(3+6)^{1/2}+\frac{1}{3}(3+6S_n)^{1/2}(3+6)^{1/2} =$$

$$S_n-(3+6S_n)^{1/2}+(3+6S_n)^{1/2}=S_n$$

Using (3) and starting from number $S_2=13$, we have: 37, 73, 121, 181, 253, 337, 433, 541, 661, 793, 937, 1093, 1261, 1441, 1633, 1837, 2053, 2281, 2521, and so on. Again, it is same as http://oeis.org/A003154 .

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