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Fusing incomplete preference rankings in design for manufacturing applications through the ZM_{II}-technique

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ABSTRACT

The authors recently presented a technique (denominated "*ZM*") to fuse multiple (*subjective*) preference rankings of some objects of interest - in manufacturing applications - into a common unidimensional *ratio* scale (Franceschini, Maisano 2019). Although this technique can be applied to a variety of decision-making problems in the Manufacturing field, it is limited by a response mode requiring the formulation of *complete* preference rankings, i.e. rankings that include all objects. Unfortunately, this model is unsuitable for some practical contexts – such as decision-making problems characterized by a relatively large number of objects, field surveys, etc. – where respondents can barely identify the more/less preferred objects, without realistically being able to construct complete preference rankings.

The purpose of this paper is to develop a new technique (denominated "ZM_{II}") which also "tolerates" *incomplete* preference rankings, e.g., rankings with the more/less preferred objects only. This technique borrows the underlying postulates from the Thurstone's *Law of Comparative Judgment* and uses the *Generalized Least Squares* method to obtain a ratio scaling of the objects of interest, with a relevant uncertainty estimation.

Preliminary results show the effectiveness of the new technique even for relatively incomplete preference rankings. Description is supported by an application example concerning the design of a coach-bus seat.

Keywords: Design for Manufacturing, Group decision making, Incomplete preference ranking, Thurstone's Law of Comparative Judgment, Generalized least squares, ZM_{II}-technique, Incomparability.

INTRODUCTION

A rather common decision-making problem is articulated as follows (Keeney and Raiffa, 1993; Franceschini et al., 2007; Coaley, 2014):

- A set of *objects* (o_1, o_2, …) should be compared based on the degree of a specific *attribute*;
- A set of *judges* (j_1, j_2, …) individually express their *subjective* judgments on (at least a portion of) these objects. In fact, judges may refrain from judging some objects, e.g., those for which they are unable to express their own judgement because of practical impediment or lack of adequate knowledge;
- The above judgments should be fused into a single *collective judgment*.

Considering the Manufacturing and Quality Engineering/Management fields, possible examples concern: (i) the fusion of customer expectations on a set of product requirements (Nahm, et al. 2013), (ii) the fusion of judgments by reliability and maintenance engineers on the design of appropriate preventive maintenance interventions, or (iii) the fusion of the opinions of designers and marketing experts on the brand image of several competing products (Lin et al. 2017; Zheng et al. 2019).

The scientific literature encompasses a plurality of fusion techniques, which may differ from each other for at least three features: (i) the *response mode* for collecting subjective judgments (e.g., ratings, rankings, paired-comparison relationships, etc.); (ii) the type of fusion model (e.g., heuristic, mathematical, statistical, fuzzy models, etc.) (Hosseini and Al Khaled, 2016; Wang et al. 2017; Çakır, 2018), and (iii) the type of *collective judgment* (e.g., object rankings, ordinal/interval/ratio scale values, etc.). For an exhaustive discussion of the existing techniques, we refer the reader to the vast literature and reviews (Coaley, 2014; De Vellis, 2016).

Regardless of the peculiarities of the individual fusion techniques, a key element for their success is the simplicity of the response mode (Harzing et al., 2009; Franceschini et al., 2019). For example, various studies show that comparative judgments of objects (e.g., "o_i is more/less preferred than o_j") are simpler and more reliable than judgments in absolute terms (e.g., "the degree of the attribute of o_i is low/intermediate/high") (Edwards, 1957; Harzing et al., 2009; Vanacore et al., 2019).

The authors have recently presented a fusion technique, denominated "*ZM*-technique", which combines the Thurstone's *Law of Comparative Judgment* (LCJ) (Thurstone, 1927; Edwards, 1957) with a response mode based on preference rankings (Franceschini and Maisano, 2019). The resulting collective judgment is expressed in the form of a *ratio* scaling (Roberts, 1979; Franceschini et al., 2019). An important requirement of the *ZM*-technique is that, apart from "regular" objects (i.e., o_1, o_2, …), preference rankings also include two "dummy" or "anchor" objects: i.e., o_Z, which corresponds to the *absence* of the attribute of interest, and o_M, which corresponds to the *maximum-imaginable* degree of the attribute (Franceschini and Maisano, 2019).

Borrowing the language from Mathematics' Order Theory, the original version of the *ZM*-technique requires judges to formulate *linear* preference rankings, i.e., *complete* rankings that include all (regular and dummy) objects, according to a hierarchical sequence with relationships of *strict preference* (e.g., "$o_i > o_j$") and/or *indifference* (e.g., "$o_i \sim o_j$") (Nederpelt and Kamareddine, 2004). For the sake of simplicity, such decision-making problem will be hereafter referred to as "complete ranking problem" or, even more simply, as "complete problem".

This is certainly a limitation, as it makes the response mode unsuitable for some practical contexts where ranking many objects can be problematic. It has also been observed that when formulating preference rankings judges tend to focus on the more/less preferred objects, providing more reliable judgments about them, to the detriment of the remaining objects (Lagerspetz, 2016; Harzing et al., 2009). Another limitation of the *ZM*-technique – and the traditional LCJ too (Montag, 2006) – is the impossibility to estimate the uncertainty related to the resulting scaling of objects.

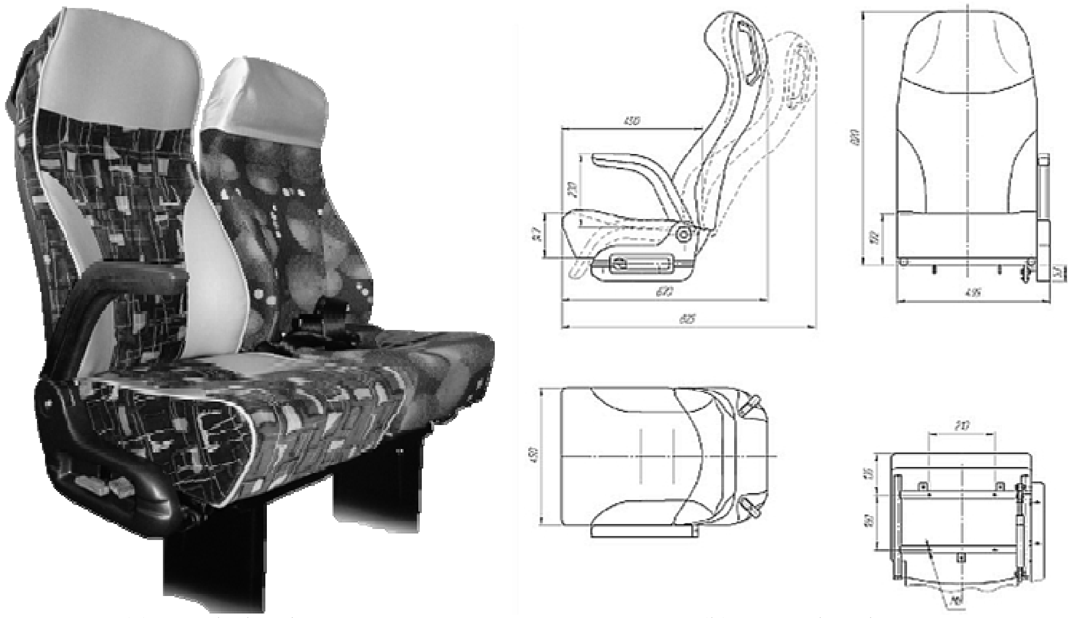
The above considerations raise the following research question: "How could the *ZM*-technique be modified/improved to (1) make the response more user-friendly and reliable and (2) determine a (statistically sound) estimate of the uncertainty related to the solution?".

The aim of this paper is to address the previous research question, proposing a new version of the *ZM*-technique – denominated "ZM_{II}" – that overcomes the limitations of the original version, while preserving the basic principles. The ZM_{II}-technique replaces *complete* preference rankings with *incomplete* ones, such as rankings which are focussed exclusively on the more/less preferred objects. Borrowing the language from Order Theory, these other rankings can be classified as *partial*, i.e., apart from strict preference and indifference relationships, they may also contain *incomparability* relationships (e.g., "$o_i \parallel o_j$") among (some of) the objects (Nederpelt and Kamareddine, 2004). For the sake of simplicity, a decision-making problem characterised by this type of rankings will be hereafter referred to as "incomplete ranking problem" or, even more simply, as "incomplete problem". Although the problem of fusing incomplete preference rankings is well present in the scientific literature, it seems that there is no work which explicitly deals with their fusion into a ratio scaling (Gulliksen and Tucker, 1961; Kendall, 1963; Vincke, 1982; Alvo and Cabilio, 1991; González-Pachón and Romero, 2001).

The rest of the paper is organized into five sections. Section "Case study outline" introduces a case study concerning the design of a civilian coach-bus seat, which will accompany the theoretical description of the new fusion technique. Section "Background information" briefly recalls the LCJ and *ZM*-technique. Section "Methodology" illustrates the new version of the fusion technique, which includes the construction of an overdetermined system of equations and its solution through the *Generalized Least Squares* (GLS) method (Kariya and Kurata, 2004); this section also shows that the new technique allows to estimate the uncertainty related to the solution, "propagating" the uncertainty of input data. Section "Application example" applies the new technique to the aforesaid case study. Section "Conclusions" summarizes the original contributions of this paper and its practical implications, limitations and suggestions for future research. Further details and technicalities on the new fusion technique are contained in the Appendix section.

CASE STUDY OUTLINE

The conceptual description of the ZM_{II}-technique will be accompanied by a practical application to the following real-life case study. A company designing and manufacturing passenger seats for different civilian means of transport (buses, trains, planes, ferries, etc.) should innovate the design of its range of coach-bus seats (see Figure 1). A team of designers decide to accomplish this objective through the *Quality Function Deployment* (QFD) tool (Kowalska et al., 2018; Zheng et al., 2019).



(a) Rendering image (b) CAD drawing

Figure 1. Example of a passenger coach-bus seat.

One of the initial phases of the QFD is the identification and prioritization of *customer requirements* (CRs) (Franceschini et al., 2015; Franceschini and Maisano, 2015). Assuming that the marketing function of the company has identified the twelve major CRs (o_1 to o_{12} in Table 1), the next step is to prioritize them according to their importance. The new fusion technique can be used to merge the individual (subjective) CR judgments of a panel of twenty regular coach-bus passengers (j_1 to j_{20}). The case study will be developed after the description of the new fusion technique.

Table 1. List of the major CRs related to a civilian coach-bus seat, from the perspective of passengers.

Abbr.	**Description**
o_1	Arm rests not too narrow
o_2	Does not make you sweat
o_3	Does not soak up a spilt drink
o_4	Hole in tray for coffee cup
o_5	Arm rest folds right away
o_6	Does not hit person behind when you recline
o_7	Comfortable when you recline
o_8	Comfortable seat belt
o_9	Enough leg room
o_{10}	Magazines can be easily removed from rack
o_{11}	Seat belt feels safe
o_{12}	Comfortable (does not give you back ache)

BACKGROUND INFORMATION

This section is organized in two subsections, which respectively recall the Thurstone's LCJ and the *ZM*-technique.

Thurstone's Law of Comparative Judgment

Thurstone (1927) postulated the existence of a *psychological continuum*, i.e., an abstract and unknown unidimensional scale, in which objects are positioned depending on the degree of a certain *attribute* – i.e., a specific feature of the objects, which evokes a subjective response in each judge. The position of a generic *i*-th object (o_i) is postulated to be distributed normally, in order to reflect the intrinsic judge-to-judge variability: $o_i \sim N(x_i, \sigma_i^2)$, where x_i and σ_i^2 are the unknown mean value and variance related to the degree of the attribute of that object. E.g., considering the case study in the section "Case study outline", twelve CRs are supposed to be positioned in the psychological continuum according to their degree of importance for a panel of twenty coach-bus passengers.

Considering two generic objects, o_i and o_j, it can be asserted that:

$$o_i - o_j \sim N(x_i - x_j,\ \sigma_i^2 + \sigma_j^2 - 2 \cdot \rho_{ij} \cdot \sigma_i \cdot \sigma_j), \quad (1)$$

where ρ_{ij} is the Pearson coefficient denoting the correlation between the positions of objects o_i and o_j. The probability that the position of o_i in the psychological continuum is higher than that of o_j can be expressed as:

$$p_{ij} = P(o_i - o_j > 0) = 1 - \Phi\left[\frac{0-(x_i - x_j)}{\sqrt{\sigma_i^2 + \sigma_j^2 - 2 \cdot \rho_{ij} \cdot \sigma_i \cdot \sigma_j}}\right], \quad (2)$$

Φ being the cumulative distribution function of the standard normal distribution $z \sim N(0, 1)$.

The LCJ (*case V*) includes the following additional simplifying assumptions (Thurstone, 1927; Edwards, 1957): $\sigma_i^2 = \sigma^2\ \forall i$, $\rho_{ij} = \rho, \forall i, j$, and $2 \cdot \sigma^2 \cdot (1-\rho) = 1$. Eq. 2 can therefore be expressed as:

$$p_{ij} = P(o_i - o_j > 0) = 1 - \Phi[-(x_i - x_j)]. \quad (3)$$

Although p_{ij} is unknown, it can be estimated using the information contained in a set of (subjective) judgments by a number (m) of judges (Thurstone, 1927). Precisely, each judge expresses his/her judgment for each paired comparison (i.e., $\forall i, j$) through relationships of *strict preference* (e.g., "$o_i > o_j$" or "$o_i < o_j$") or *indifference* (e.g., "$o_1 \sim o_2$"). Then, for each judge who prefers o_i to o_j, a frequency indicator f_{ij} is incremented by one unit. In the case the two objects are considered indifferent, f_{ij} is conventionally incremented by 0.5, so that:

$$f_{ij} = m_{ij} - f_{ji}, \quad (4)$$

m_{ij} being the total number of judges who express their judgment for the i,j-th paired comparison. In general, $m_{ij} \leq m$ since judges may sometimes refrain from expressing their judgments on some of the possible paired comparisons.

The observed proportion of judges that prefer o_i to o_j can be used to estimate the unknown probability p_{ij}:

$$\hat{p}_{ij} = \frac{f_{ij}}{m_{ij}}. \tag{5}$$

Of course, the relationship of complementarity $\hat{p}_{ij} = 1 - \hat{p}_{ji}$ holds.

Returning to Eq. 3, it can be expressed as:

$$\hat{p}_{ij} = 1 - \Phi[-(x_i - x_j)], \tag{6}$$

from which:

$$x_i - x_j = -\Phi^{-1}(1 - \hat{p}_{ij}). \tag{7}$$

It can be noticed that, if all judges express the same judgment, the model is no more viable ($\hat{p}_{ij}$ values of 1.00 and 0.00 would correspond to $-\Phi^{-1}(1 - \hat{p}_{ij})$ values of $\pm\infty$). A simplified approach for tackling this problem is associating values of $\hat{p}_{ij} \geq 0.977$ with $-\Phi^{-1}(1 - 0.977) = 1.995$ and values of $\hat{p}_{ij} \leq 0.023$ with $-\Phi^{-1}(1 - 0.023) = -1.995$. More sophisticated solutions to deal with this issue have been proposed (Gulliksen, 1956; Edwards, 1957).

Extending the reasoning to all possible paired comparisons for which $m_{ij} \geq 1$ (i.e., for at least one judge there is a paired-comparison relationship of strict preference or indifference), the relevant $\hat{p}_{ij}$ values can be determined and the following system of equations can be constructed:

$$\begin{cases} \vdots \\ x_i - x_j + \Phi^{-1}(1 - \hat{p}_{ij}) = 0 \qquad \forall i, j : m_{ij} \geq 1. \\ \vdots \end{cases} \tag{8}$$

Since the rank of the system is lower than the number (n) of unknowns of the problem (i.e., x_i, $\forall i$) – and the system itself would be indeterminate – the following conventional condition was introduced by Thurstone (1927):

$$\sum_{\forall i} x_i = 0. \tag{9}$$

Eqs. 8 and 9 are then aggregated into a new system, which is *over-determined* (i.e., it has rank n while the total number of equations – q – is higher than n) and *linear* with respect to the unknowns:

$$\begin{cases} \begin{bmatrix} \vdots \\ x_i - x_j + \Phi^{-1}\left(1-\hat{p}_{ij}\right) = 0 \qquad \forall i, j : m_{ij} \geq 1 \\ \vdots \end{bmatrix} \\ \sum_{\forall i} x_i = 0 \end{cases} . \tag{10}$$

This system can be expressed in matrix form as:

$$\begin{cases} \vdots \\ \sum_{k=1}^{n} \left(a_{hk} \cdot x_k\right) - b_h = 0 \quad \forall h \in [0, q] \\ \vdots \end{cases} \Rightarrow \quad \boldsymbol{A} \cdot \boldsymbol{X} - \boldsymbol{B} = \boldsymbol{0}, \tag{11}$$

$\boldsymbol{X} = [\ldots, x_i, \ldots]^T \in R^{n \times 1}$ being the column vector containing the unknowns of the problem, a_{hk} being a generic element of matrix $\boldsymbol{A} \in R^{q \times n}$, and b_h being a generic element of vector $\boldsymbol{B} \in R^{n \times 1}$. For details on the construction of $\boldsymbol{A}$ and $\boldsymbol{B}$, see (Gulliksen, 1956).

In the case each judge expresses his/her judgment on the totality of the $C_2^n = n \cdot (n-1)/2$ paired comparisons, the system in Eq. 10 is *complete* – i.e., with $q = C_2^n + 1$ equations – and can be solved in a closed form as (Thurstone, 1927):

$$\hat{x}_j = \sum_{i=1}^{m} \Phi^{-1}\left(1-\hat{p}_{ij}\right) \qquad \forall j . \tag{12}$$

The LCJ unfortunately has some limitations, including the following ones:

1. the response mode is relatively tedious for judges;
2. the LCJ results into an *interval* scaling, i.e., objects are defined on a scale with meaningful distance but arbitrary zero point (Thurstone, 1927; Roberts, 1979);
3. the solution can be determined only when the system of equations is "complete";
4. no uncertainty estimation is provided.

ZM-technique

The *ZM*-technique was proposed to overcome some of the above limitations of the LCJ (Franceschini and Maisano, 2019). A significant drawback of the LCJ response mode is that paired comparisons can be tedious and complex to manage, since much repetitious information is required from judges. This problem can be overcome asking each judge to formulate a *preference ranking*, i.e., a sequence of objects in order of preference (more preferred ones in the top positions and less preferred ones in the bottom ones).

Apart from regular objects (o_1, o_2, …), judges should include two special *dummy* objects in their rankings: one (o_Z) corresponding to the *absence* of the attribute of interest and one (o_M) corresponding to the *maximum-imaginable* degree of the attribute (Franceschini and Maisano, 2019). E.g., considering the case study in the section "Case study outline", o_Z corresponds to a fictitious CR of no importance at all, while o_M corresponds to a fictitious CR of the maximum-imaginable importance. When dealing with these special objects, two important requirements should be considered by judges:

1. o_Z should be positioned at the bottom of a preference ranking, i.e., there should not be any other object with preference lower than o_Z. In the case the attribute of another object is judged to be absent, that object will be considered indifferent to o_Z and positioned at the same hierarchical level.
2. o_M should be positioned at the top of a preference rankings, i.e., there should not be any other object with preference higher than o_M. In the case the attribute of another object is judged to be the maximum-imaginable, that object will be considered indifferent to o_M and positioned at the same hierarchical level.

Next, the preference rankings of judges can be turned into paired-comparison data (e.g., the four-object ranking $(o_3 \sim o_1) > o_2 > o_4$ is turned into the $C_2^4 = 6$ paired-comparison relationships: "$o_1>o_2$", "$o_1\sim o_3$", "$o_1>o_4$", "$o_2<o_3$", "$o_2>o_4$", and "$o_3>o_4$"; it can be noticed that this response mode forces judges to be *transitive* (e.g., if "$o_1>o_2$" and "$o_2>o_4$", then "$o_1>o_4$").

Next, the traditional LCJ can be applied to the resulting paired-comparison data and a scaling (x) of the objects can be determined (Eq. 12). Through the following transformation, the resulting scaling (x) is transformed into a new one (y), which is defined in the conventional range [0, 100]:

$$\hat{y}_i = \hat{y}_i(\hat{\boldsymbol{X}}) = 100 \cdot \frac{\hat{x}_i - \hat{x}_Z}{\hat{x}_M - \hat{x}_Z} \quad \forall i, \tag{13}$$

where: $\hat{x}_Z$ and $\hat{x}_M$ are the scale values of o_Z and o_M, resulting from the LCJ; $\hat{x}_i$ is the scale value of a generic i-th object, resulting from the LCJ; $\hat{y}_i$ is the scale value of a generic i-th object in the new scale y. This transformation can also be expressed in vector form as:

$$\hat{\boldsymbol{Y}} = \hat{\boldsymbol{Y}}(\hat{\boldsymbol{X}}) = [\ldots, \hat{y}_i(\hat{\boldsymbol{X}}), \ldots]^T = \left[\ldots, 100 \cdot \frac{\hat{x}_i - \hat{x}_Z}{\hat{x}_M - \hat{x}_Z}, \ldots\right]^T, \tag{14}$$

being $\hat{\boldsymbol{Y}}$ a column vector whose components result from a system of n decoupled equations. Since scale y "inherits" the *interval* property from scale x and has a conventional zero point that corresponds to the absence of the attribute (i.e., $\hat{y}_Z$), it can be reasonably considered as a *ratio* scale (Roberts, 1979; Franceschini et al., 2019). We note that the two dummy objects, o_Z and o_M, are used to "anchor" the x scaling to the y scaling (Paruolo et al., 2013).

Although the *ZM*-technique simplifies the response mode and allows to obtain a ratio scaling, it still does not solve other relevant limitations of the traditional LCJ:

- The procedure is not applicable to the system in Eq. 10 when it is not complete (i.e., there is at least one (i, j) paired comparison for which $m_{ij} = 0$).
- It does not contemplate neither the variability of $\hat{p}_{ij}$ values, which are actually treated as deterministic parameters (not probabilistic ones), nor the propagation of this variability on the $\hat{\boldsymbol{X}}$ solution (and therefore on the transformed solution, $\hat{\boldsymbol{Y}}$).

  In fact, since f_{ij} is determined considering a sample of m_{ij} paired comparisons (as illustrated in subsection "Thurstone's LCJ"), it will be distributed binomially; $\hat{p}_{ij}$ is the best estimator of p_{ij}, according to the information available. In formal terms:

$$f_{ij} \sim B[\mu_{f_{ij}} \approx m_{ij} \cdot \hat{p}_{ij}, \sigma^2_{f_{ij}} \approx m_{ij} \cdot \hat{p}_{ij} \cdot (1 - \hat{p}_{ij})], \tag{15}$$

  In the hypothesis that $m_{ij} \cdot \hat{p}_{ij} \geq 5$, when $0 \leq \hat{p}_{ij} \leq 0.5$, or $m_{ij} \cdot \left(1 - \hat{p}_{ij}\right) \geq 5$, when $0.5 < \hat{p}_{ij} \leq 1$, the following approximations can be reasonably introduced (Ross, 2014):

$$\begin{aligned} f_{ij} &\sim N[\mu_{f_{ij}} \approx m_{ij} \cdot \hat{p}_{ij}, \sigma^2_{f_{ij}} \approx m_{ij} \cdot \hat{p}_{ij} \cdot (1 - \hat{p}_{ij})] \\ p_{ij} &\sim N\left[\mu_{p_{ij}} \approx \hat{p}_{ij}, \sigma^2_{p_{ij}} \approx \frac{\hat{p}_{ij} \cdot (1 - \hat{p}_{ij})}{m_{ij}}\right] \end{aligned} . \tag{16}$$

  It is worth remarking that, even when all judges express their judgments for all the possible paired comparisons (i.e., $m_{ij} = m \ \forall i, j$), the variance of p_{ij} may change from one paired comparison to one other, as it also depends on the relevant $\hat{p}_{ij}$ value.

METHODOLOGY

Response-mode simplification

Although the formulation of preference rankings is less tedious and complex to manage than the direct formulation of paired-comparison relationships, it still may be problematic for some practical situations, e.g., asking judges to rank more than a handful of objects during a field survey or a telephone/street interview may put a very high demand on their cognitive abilities (Harzing et al., 2009; Lenartowicz and Roth, 2001). To overcome this obstacle, a more flexible response mode that tolerates *incomplete* preference rankings can be adopted. Below is a list of some possible types of incomplete preference rankings.

- Preference rankings including only the more preferred objects (or "*t*-objects", where "*t*" stands for "top") and the less preferred ones (or "*b*-objects", where "*b*" stands for "bottom"); from now on, these rankings will be denominated "Type-*t&b*" rankings. The *t* parameter is conventionally defined as the number of regular objects (i.e. excluding the two dummy objects) within the *t*-

objects, while the b parameter is conventionally defined as the number of regular objects within b-objects. In the example in Figure 2(a), t=b=2.

- Preference rankings including only the more preferred objects (i.e., t-objects) among those available; see the example in Figure 2(b). From now on, these rankings will be denominated "Type-t" rankings.
- Preference rankings excluding the two dummy objects (o_Z and o_M). A preference ranking of this type, when also including all regular objects, will hereafter be referred to as "quasi-complete" (see the example in Figure 2(c)).
- Combining the previous types of incomplete preference rankings, one can obtain Type-$t\&b$ or Type-t preference rankings that exclude the dummy objects.

Figure 2 also shows that a generic incomplete ranking can be transformed into a "reconstructed" ranking, which includes all the (dummy and regular) objects. E.g., considering Type-$t\&b$ rankings, the objects that are not considered by judges can be allocated at an intermediate hierarchical level with respect to the t- and b-objects, with mutual *incomparability* relationships. As for Type-t rankings, the objects that are not considered by judges can be allocated at a lower hierarchical level with respect to the t-objects. As for the rankings that do not include o_Z and o_M, they can be reconstructed in compliance with the following constraints:

1. o_M will – by definition – be at a higher hierarchical level with respect to the other objects, with the exception of those positioned at the highest hierarchical level (o_i). A relationship of incomparability with the latter objects ("$o_M \parallel o_i$") indicates hesitation between a possible relationship of indifference ("$o_M \sim o_i$") or strict preference ("$o_M > o_i$");
2. o_Z will – by definition – be at a lower hierarchical level with respect to the other objects, with the exception of those positioned at the lowest hierarchical level (o_i). A relationship of incomparability with the latter objects ("$o_Z \parallel o_i$") indicates hesitation between a possible relationship of indifference ("$o_Z \sim o_i$") or strict preference ("$o_Z < o_i$").

For the sake of simplicity, both the incomplete and reconstructed rankings will hereafter be referred to as "incomplete", without distinction.

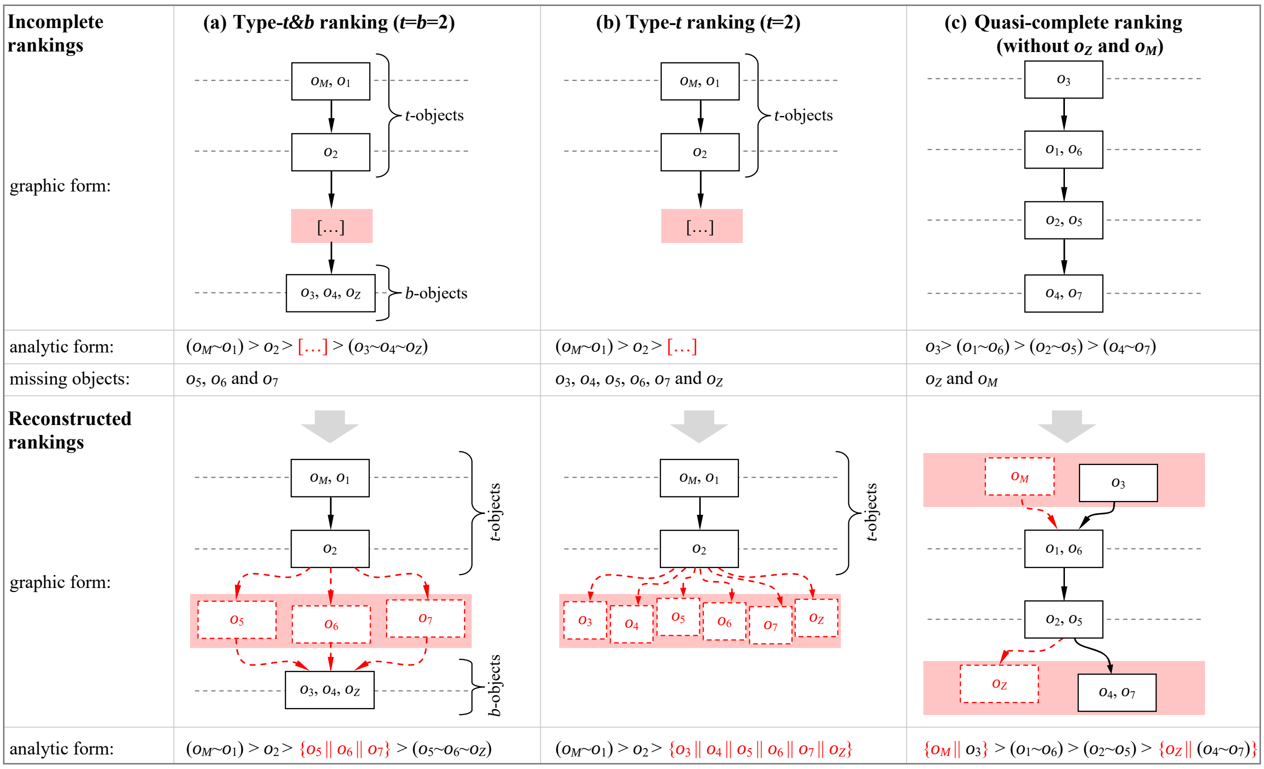


Figure 2. Example of three different types of incomplete rankings formulated by judges. These rankings can be turned into reconstructed rankings, which include all the (regular and dummy) objects; for ease of understanding, the reconstructed parts are marked in red.

Artificial generation of incomplete rankings

This subsection illustrates the artificial "deterioration" of a complete preference ranking to generate a set of corresponding incomplete rankings. Simulating practical circumstances where the formulation of complete rankings can be problematic, this mechanism can be used to generate incomplete rankings. Let us focus on the example in Figure 3, in which an (initial) complete preference ranking (a) is decomposed exclusively into paired-comparison relationships of strict preference and indifference. Then, this complete ranking is artificially deteriorated into several incomplete preference rankings that are compatible with it, e.g., quasi-complete, Type-*t&b* or Type-*t* rankings, as respectively exemplified in Figure 3(b), (c) and (d). For the incomplete rankings, new paired-comparison relationships of incomparability gradually replace those of strict preference and indifference, which characterize the complete ranking. The compatibility between the incomplete rankings and the "source" complete ranking is given by the fact that – excluding the paired-comparison relationships of incomparability – the remaining ones are identical (Fahandar et al., 2017).

The *degree of completeness* of a generic *k*-th preference ranking can be quantitatively described by the synthetic indicator:

$$c_k = \frac{\text{No. of "usable" paired comparison relationships in the } k^{\text{th}} \text{ preference ordering}}{C_2^n}, \quad (17)$$

Figure 3. Example of deterioration of a complete preference ranking (a), generating several incomplete/reconstructed rankings (b, c, d, e and f); for ease of understanding, the reconstructed parts of the latter rankings are marked in red.

which expresses the fraction of "usable"[1] paired-comparison relationships (i.e. these of strict preference or indifference), with respect to the total ones: $C_2^n = n \cdot (n-1)/2$, where n is the total number of (regular and dummy) objects of the problem. E.g., Figure 3 reports the c_k values related to the rankings (see below the tables with the paired-comparison relationships). For complete preference rankings, $c_k = 1$.

[1] The adjective "usable" indicates that the strict-preference and indifference relationships are the ones contributing to the solution of the decision-making problem of interest (cf. Sect. A.1 in the Appendix).

Interestingly, even very incomplete rankings may contain a relevant portion of usable paired-comparison relationships. E.g., consider the Type-*t* ranking in Figure 3(f), in which the two more preferred regular objects were merely identified, without being ordered. Despite the apparently high degree of incompleteness, half of the total paired-comparison relationships are usable (c_k = 50%).

The indicator c_k can be extended from a single (*k*-th) preference ranking to sets of (*m*) preference rankings – such as those characterizing a whole (complete or incomplete) decision-making problem. We thus define a new aggregated indicator (c):

$$c = \frac{\sum_{k=1}^{m} \text{No. of "usable" paired comparison relationships in the } k^{\text{th}} \text{ preference ordering}}{\sum_{k=1}^{m} \text{Total no. of "usable" paired comparison relationships in the } k^{\text{th}} \text{ preference ordering}} = \\ = \frac{\sum_{k=1}^{m} c_k \cdot C_2^n}{m \cdot C_2^n} = \frac{\sum_{k=1}^{m} c_k}{m} \quad . (18)$$

Eq. 18 shows that c can be interpreted as the arithmetic average of the c_k values of the preference rankings under consideration.

GLS solution

In general, the system in Eq. 11 will not necessarily be complete, as the number of equations (q) could be lower than $C_2^n + 1$ (i.e., for any paired comparison with $m_{ij} = 0$, no equation can be formulated) and therefore cannot be solved through the LCJ.

The literature dealt with the problem of solving such incomplete systems through the *Ordinary Least Squares* (OLS) method. For example, Gulliksen (1956) discusses some approximate numerical methods for the OLS solution formula to Eq. 11 (Kariya and Kurata, 2004; Ross, 2014):

$$\hat{\boldsymbol{X}} = \left(\boldsymbol{A}^T \cdot \boldsymbol{A}\right)^{-1} \cdot \boldsymbol{A}^T \cdot \boldsymbol{B} \,. \tag{19}$$

Additionally, it can be demonstrated that, in the case in which the system of equations is complete, the LCJ solution coincides with the OLS one – i.e., that one minimizing the sum of the squared residuals related to the equations in Eq. 11 (Gulliksen, 1956):

$$\sum_{h=1}^{q} \left[\sum_{k=1}^{n} \left(a_{hk} \cdot x_k \right) - b_h \right]^2 , \tag{20}$$

n being the number of elements in $\hat{\boldsymbol{X}}$, and q being the total number of equations available. In general, the OLS solution is possible even for incomplete systems, as long as $q \geq n$; this condition is easily met in practice (Gulliksen, 1956).

Even though the OLS method provides an effective solution to the problem of interest, it does not provide any practical estimate of the uncertainty associated with the elements of $\hat{\boldsymbol{X}}$. In fact, although it is possible to calculate the covariance matrix of $\hat{\boldsymbol{X}}$ as:

$$\Sigma_X = \left(\boldsymbol{A}^T \cdot \boldsymbol{A}\right)^{-1}, \tag{21}$$

it is of no practical use for this specific problem, as the uncertainties of the $\hat{\boldsymbol{X}}$ elements are identical and not affected by the real uncertainty of input data (i.e., $\hat{p}_{ij}$ values, see subsection "*ZM-technique*") (Gulliksen, 1956). This limitation can be overcome using the *Generalized Least Squares* (GLS) method, which is more articulated than the OLS method as it includes several additional steps (see the qualitative representation in Figure 4).

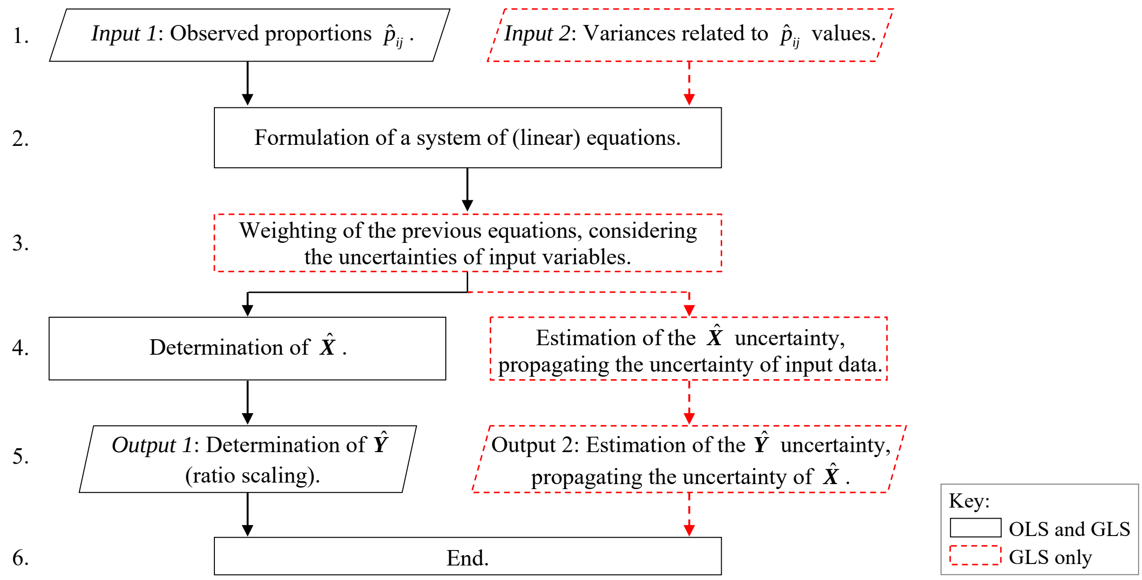


Figure 4 – Flow chart representing the main steps of the OLS and GLS solution to the problem of interest; the GLS solution includes several additional steps (see red dashed blocks) with respect to the OLS one.

The idea of applying the GLS to the problem of interest, when it is characterized by incomplete rankings, had already been advanced several decades ago by Arbuckle and Nugent (1973), who contemplated this and other goodness-of-fit criteria, such as *maximum likelihood*. These techniques, however, have not been applied extensively, probably due to some computational constraints that are nowadays overcome. Additionally, the GLS solution proposed by Arbuckle and Nugent (1973) was combined with a classic response mode, based on the direct formulation of paired-comparison relationships.

From a technical point of view, the GLS method allows obtaining a solution that minimizes the weighted sum of the squared residuals related to the equations in Eq. 11, i.e.:

$$\sum_{h=1}^{q} w_h \cdot \left[\sum_{k=1}^{n} \left(a_{hk} \cdot x_k\right) - b_h\right]^2, \tag{22}$$

in which weights (w_h) take into account the uncertainty in the $\hat{p}_{ij}$ values. It can be demonstrated that, for a generic equation related to a generic *i,j*-th paired comparison: $w_h = \left[\frac{\partial\Phi^{-1}\left(1-\hat{p}_{ij}\right)}{\partial\hat{p}_{ij}}\right]^2 \Big/ \sigma^2_{p_{ij}}$ (Arbuckle and Nugent, 1973).

Next, weights are aggregated into a (squared) matrix $\boldsymbol{W} \in R^{(q-1)\times(q-1)}$, whose construction is illustrated in the "Detailed description of the GLS" subsection (in the Appendix), and $\boldsymbol{X}$ can be estimated as:

$$\hat{\boldsymbol{X}} = \left(\boldsymbol{A}^T \cdot \boldsymbol{W} \cdot \boldsymbol{A}\right)^{-1} \cdot \boldsymbol{A}^T \cdot \boldsymbol{W} \cdot \boldsymbol{B}. \tag{23}$$

Combining Eqs. 23 and 14, the final (ratio) scaling $\hat{\boldsymbol{Y}}$ can be obtained as:

$$\hat{\boldsymbol{Y}} = \hat{\boldsymbol{Y}}[\hat{\boldsymbol{X}}] = \hat{\boldsymbol{Y}}[(\boldsymbol{A}^T \cdot \boldsymbol{W} \cdot \boldsymbol{A})^{-1} \cdot \boldsymbol{A}^T \cdot \boldsymbol{W} \cdot \boldsymbol{B}]. \tag{24}$$

Next, the uncertainty related to the elements in $\hat{\boldsymbol{Y}} = [..., \hat{y}_i, ...]^T \in R^{n\times 1}$ can be determined by applying the relationship:

$$\Sigma_{\boldsymbol{Y}} = \boldsymbol{J}_{\boldsymbol{Y}(\hat{X})} \cdot [(\boldsymbol{A}^T \cdot \boldsymbol{W} \cdot \boldsymbol{A})^{-1}] \cdot \boldsymbol{J}^T_{\boldsymbol{Y}(\hat{X})}. \tag{25}$$

where $\boldsymbol{J}_{\boldsymbol{Y}(\hat{X})} \in R^{n\times n}$ is a Jacobian matrix containing the partial derivatives related to the equations of the system in Eq. 14, with respect to the elements of $\hat{\boldsymbol{X}}$. For details, see the "Detailed description of the GLS" subsection (in the Appendix).

Assuming that the p_{ij} and $\hat{y}_i$ values are approximately normally distributed, a 95% confidence interval related to each $\hat{y}_i$ value can be computed as:

$$\hat{y}_i \pm U_i = \hat{y}_i \pm 2 \cdot \sigma_i \quad \forall i, \tag{26}$$

U_i being the so-called *expanded uncertainty* of $\hat{y}_i$ with a coverage factor $k = 2$ and $\sigma_i = \sqrt{\Sigma_{\mathbf{Y},(i,i)}}$ (JCGM 100:2008, 2008).

APPLICATION EXAMPLE

Returning to the case study in section "Case study outline", twenty regular bus-coach passengers (j_1 to j_{20}, i.e., *judges*) should prioritize the twelve CRs (o_1 to o_{12}, i.e., *objects*) in Table 1, according to their *importance* (i.e., *attribute*).

Let us initially assume that the passengers interviewed have adequate knowledge of the CRs and are in the practical conditions to formulate their complete rankings, which represent a complete problem (see Table 2(a)). These rankings are then translated into a number of paired-comparison relationships (i.e., $C_2^{12} = 66$ for each preference ranking, resulting in total $66 \cdot 20 = 1320$ paired-comparison relationships) and the LCJ is applied, producing the scaling in Table 3(a) (see also the graphical representation in Figure 5). These results are already referred to the conventional scale (y), which is included in the range [0, 100]. Consistently with the considerations in subsection "GLS solution", the results of the LCJ are identical to those of the OLS. On the other hand, the application of the GLS method produces a very close – although non-identical – result. The difference stems from the fact that – unlike LCJ and OLS – the GLS takes into account the uncertainties related to $\hat{p}_{ij}$ values. The GLS solution is therefore superior from both a conceptual and practical point of view.

To study the effectiveness of the GLS in the presence of incomplete rankings, the complete rankings in Table 2(a) were intentionally deteriorated, replacing some of the relationships of strict preference (">" and "<") and indifference ("~"), with incomparability relationships ("||"), according to the logic described in the subsection "Artificial generation of incomplete rankings". Rankings with different degrees of completeness were generated, under the assumption that they were produced by a panel of relatively "heterogeneous" passengers, e.g., in terms of knowledge of the CRs, ability to concentrate, education level, etc.. The deterioration of complete rankings was performed combining the following "deterioration parameters" (cf. subsection "Artificial generation of incomplete rankings"):

1. *Type of ranking* (hereafter abbreviated as "Ranking type"), which can be "Complete", "Quasi-complete", "Type-t&b" and "Type-t";
2. *Ability of the judge to manage* o_Z *and* o_M (hereafter abbreviated as "Manage o_Z/o_M?"), which can be "Yes" or "No";
3. *Value of t and/or b* (hereafter abbreviated as "t/b value"), which can be "1", "2" and "3";
4. *Ability of the judge to order the t- and/or b-objects* (hereafter abbreviated as "Order t/b-objects?"), which can be "Yes" or "No".

Table 2(b) contains the resulting incomplete or – to be more precise – non-necessarily-complete rankings[2]. These rankings constitute a new incomplete problem, characterized by a relatively low degree of completeness ($c \approx 62.2\%$).

[2] In fact, one of the initial twenty complete rankings – i.e., that by j_4 – was not intentionally deteriorated, under the assumption that j_4 is a passenger with a high capacity to provide precise and detailed responses.

Table 2. Incomplete preference rankings (in the semi-last column) that are obtained by deteriorating some complete "source" rankings (in the second column). It can be noticed that the deterioration mechanism may differ from ranking to ranking, as it is obtained combining the four deterioration parameters: "Ranking type", "Manage o_Z/o_M?", "t/b value" and "Order t/b-objects?".

Judge	(a) Complete problem	Ranking type	Manage o_Z/o_M?	t/b value	Order t/b-block(s)?	(b) Incomplete problem	c_k
j_1	$(o_M\sim o_1\sim o_8\sim o_{12})>o_9>o_7>o_6>o_{11}>o_2>o_4>(o_3\sim o_{10}\sim o_Z\sim o_5)$	Type-$t\&b$	Yes	3	Yes	$(o_M\sim o_1\sim o_8\sim o_{12})>\{o_2\Vert o_4\Vert o_6\Vert o_7\Vert o_9\Vert o_{11}\}>(o_3\sim o_{10}\sim o_Z\sim o_5)$	83.5%
j_2	$(o_M\sim o_8)>o_{11}>o_9>(o_7\sim o_{12})>o_1>(o_4\sim o_2)>(o_{10}\sim o_6\sim o_3)>(o_5\sim o_Z)$	Type-$t\&b$	N/A	3	No	$\{o_M\Vert o_8\Vert o_9\Vert o_{11}\}>\{o_1\Vert o_2\Vert o_4\Vert o_7\Vert o_{12}\}>\{o_Z\Vert o_3\Vert o_5\Vert o_6\Vert o_{10}\}$	71.4%
j_3	$(o_M\sim o_9)>o_1>o_{11}>o_{12}>o_8>o_5>(o_3\sim o_2)>o_7>(o_6\sim o_{10}\sim o_4\sim o_Z)$	Type-t	No	3	Yes	$\{o_M\Vert o_9\}>o_1>o_{11}>\{o_Z\Vert o_2\Vert o_3\Vert o_4\Vert o_5\Vert o_6\Vert o_7\Vert o_8\Vert o_{10}\Vert o_{12}\}$	49.5%
j_4	$(o_M\sim o_9\sim o_{12})>o_8>o_1>(o_4\sim o_2)>o_7>o_{11}>(o_6\sim o_3)>(o_{10}\sim o_Z\sim o_5)$	Complete	Yes	N/A	N/A	$(o_M\sim o_9\sim o_{12})>o_8>o_1>(o_4\sim o_2)>o_7>o_{11}>(o_6\sim o_3)>(o_{10}\sim o_Z\sim o_5)$	100.0%
j_5	$o_M>o_1>o_9>(o_8\sim o_4)>(o_6\sim o_{11}\sim o_{12})>(o_2\sim o_7)>o_3>(o_{10}\sim o_Z\sim o_5)$	Type-$t\&b$	No	1	Yes	$\{o_M\Vert o_1\}>\{o_2\Vert o_3\Vert o_4\Vert o_6\Vert o_7\Vert o_8\Vert o_9\Vert o_{11}\Vert o_{12}\}>\{o_Z\Vert (o_{10}\sim o_5)\}$	57.1%
j_6	$(o_M\sim o_{12})>(o_9\sim o_8)>o_1>(o_{11}\sim o_2)>(o_{10}\sim o_3\sim o_6)>o_7>(o_4\sim o_5\sim o_Z)$	Type-t	Yes	3	Yes	$(o_M\sim o_{12})>(o_9\sim o_8)>\{o_Z\Vert o_1\Vert o_2\Vert o_3\Vert o_4\Vert o_5\Vert o_6\Vert o_7\Vert o_{10}\Vert o_{11}\}$	50.5%
j_7	$o_M>(o_1\sim o_8)>o_9>o_{12}>o_2>(o_7\sim o_4)>o_{11}>o_5>(o_6\sim o_{10}\sim o_Z\sim o_3)$	Type-$t\&b$	N/A	1	No	$\{o_M\Vert o_1\Vert o_8\}>\{o_2\Vert o_4\Vert o_5\Vert o_7\Vert o_9\Vert o_{11}\Vert o_{12}\}>\{o_Z\Vert o_3\Vert o_6\Vert o_{10}\}$	67.0%
j_8	$(o_M\sim o_9)>(o_2\sim o_7\sim o_1)>o_5>(o_8\sim o_{12})>o_{11}>(o_6\sim o_4)>(o_{10}\sim o_Z\sim o_3)$	Type-t	Yes	3	Yes	$(o_M\sim o_9)>(o_2\sim o_7\sim o_1)>\{o_Z\Vert o_3\Vert o_4\Vert o_5\Vert o_6\Vert o_8\Vert o_{10}\Vert o_{11}\Vert o_{12}\}$	60.4%
j_9	$(o_M\sim o_{11}\sim o_{12})>o_9>(o_1\sim o_6\sim o_7)>o_5>(o_2\sim o_3)>o_4>o_8>(o_Z\sim o_{10})$	Type-t	Yes	3	Yes	$(o_M\sim o_{11}\sim o_{12})>o_9>\{o_Z\Vert o_1\Vert o_2\Vert o_3\Vert o_4\Vert o_5\Vert o_6\Vert o_7\Vert o_8\Vert o_{10}\}$	50.5%
j_{10}	$(o_M\sim o_{12})>(o_{11}\sim o_5)>o_9>o_7>(o_4\sim o_2)>(o_8\sim o_6)>o_1>(o_{10}\sim o_Z\sim o_3)$	Type-$t\&b$	N/A	1	No	$\{o_M\Vert o_{12}\}>\{o_1\Vert o_2\Vert o_4\Vert o_5\Vert o_6\Vert o_7\Vert o_8\Vert o_9\Vert o_{11}\}>\{o_Z\Vert o_3\Vert o_{10}\}$	56.0%
j_{11}	$(o_M\sim o_{12})>(o_7\sim o_8\sim o_1)>o_{11}>o_9>o_2>o_3>(o_{10}\sim o_6\sim o_4)>(o_5\sim o_Z)$	Type-t	Yes	3	Yes	$(o_M\sim o_{12})>(o_7\sim o_8\sim o_1)>\{o_Z\Vert o_2\Vert o_3\Vert o_4\Vert o_5\Vert o_6\Vert o_9\Vert o_{10}\Vert o_{11}\}$	60.4%
j_{12}	$(o_M\sim o_8\sim o_9\sim o_{12})>(o_{11}\sim o_2)>(o_1\sim o_5)>o_6>o_7>(o_3\sim o_{10})>(o_Z\sim o_4)$	Type-t	Yes	3	Yes	$(o_M\sim o_8\sim o_9\sim o_{12})>\{o_Z\Vert o_1\Vert o_2\Vert o_3\Vert o_4\Vert o_5\Vert o_6\Vert o_7\Vert o_{10}\Vert o_{11}\}$	50.5%
j_{13}	$o_M>o_8>(o_1\sim o_{12})>o_9>o_2>(o_{11}\sim o_5)>o_7>o_{10}>(o_Z\sim o_6\sim o_3\sim o_4)$	Type-$t\&b$	N/A	3	No	$\{o_M\Vert o_1\Vert o_8\Vert o_{12}\}>\{o_2\Vert o_5\Vert o_7\Vert o_9\Vert o_{10}\Vert o_{11}\}>\{o_Z\Vert o_3\Vert o_4\Vert o_6\}$	70.3%
j_{14}	$(o_M\sim o_1)>o_8>o_{12}>(o_2\sim o_5)>(o_9\sim o_6)>(o_{11}\sim o_4)>o_3>o_7>(o_Z\sim o_{10})$	Type-t	N/A	3	No	$\{o_M\Vert o_1\Vert o_8\Vert o_{12}\}>\{o_Z\Vert o_2\Vert o_3\Vert o_4\Vert o_5\Vert o_6\Vert o_7\Vert o_9\Vert o_{10}\Vert o_{11}\}$	44.0%
j_{15}	$o_M>o_{12}>(o_9\sim o_2)>o_1>(o_8\sim o_6\sim o_7)>(o_{11}\sim o_{10})>o_5>(o_4\sim o_Z\sim o_3)$	Type-$t\&b$	N/A	3	No	$\{o_M\Vert o_2\Vert o_9\Vert o_{12}\}>\{o_1\Vert o_6\Vert o_7\Vert o_8\Vert o_{10}\Vert o_{11}\}>\{o_Z\Vert o_3\Vert o_4\Vert o_5\}$	70.3%
j_{16}	$(o_M\sim o_8\sim o_9)>o_{12}>o_7>(o_4\sim o_5)>o_6>o_1>o_{11}>o_3>o_{10}>o_2>o_Z$	Type-t	No	3	Yes	$\{o_M\Vert (o_8\sim o_9)\}>o_{12}>\{o_Z\Vert o_1\Vert o_2\Vert o_3\Vert o_4\Vert o_5\Vert o_6\Vert o_7\Vert o_{10}\Vert o_{11}\}$	48.4%
j_{17}	$o_M>o_{11}>o_2>(o_4\sim o_8\sim o_1)>(o_5\sim o_{12})>(o_3\sim o_9)>o_7>o_{10}>(o_Z\sim o_6)$	Type-t	No	2	Yes	$\{o_M\Vert o_{11}\}>o_2>\{o_Z\Vert o_1\Vert o_3\Vert o_4\Vert o_5\Vert o_6\Vert o_7\Vert o_8\Vert o_9\Vert o_{10}\Vert o_{12}\}$	38.5%
j_{18}	$(o_M\sim o_9\sim o_{12})>o_1>o_2>(o_6\sim o_8)>o_7>(o_{11}\sim o_5)>o_4>(o_{10}\sim o_Z\sim o_3)$	Quasi-complete	No	N/A	N/A	$\{o_M\Vert (o_9\sim o_{12})\}>o_1>o_2>(o_6\sim o_8)>o_7>(o_{11}\sim o_5)>o_4>\{o_Z\Vert (o_{10}\sim o_3)\}$	95.6%
j_{19}	$(o_M\sim o_8\sim o_{12})>o_2>o_7>(o_{11}\sim o_1)>o_9>(o_6\sim o_4)>o_3>(o_{10}\sim o_Z\sim o_5)$	Type-t	No	3	Yes	$\{o_M\Vert (o_8\sim o_{12})\}>o_2>\{o_Z\Vert o_1\Vert o_3\Vert o_4\Vert o_5\Vert o_6\Vert o_7\Vert o_9\Vert o_{10}\Vert o_{11}\}$	48.4%
j_{20}	$(o_M\sim o_9\sim o_{11})>o_{12}>o_8>o_1>o_5>(o_2\sim o_7)>o_4>(o_6\sim o_{10}\sim o_Z\sim o_3)$	Type-$t\&b$	N/A	4	No	$\{o_M\Vert o_8\Vert o_9\Vert o_{11}\Vert o_{12}\}>\{o_1\Vert o_2\Vert o_5\Vert o_7\}>\{o_Z\Vert o_3\Vert o_4\Vert o_6\Vert o_{10}\}$	71.4%

o_1 to o_{12} are the regular objects, while o_Z to o_M are the dummy objects;
">", "~" and "||" respectively depict the *strict preference*, *indifference* and *incomparability* relationships;
$\{o_i \Vert o_j \Vert \ldots\}$ is a generic block containing incomparable objects;
$(o_i \sim o_j \sim \ldots)$ is a generic block containing indifferent objects;
the degree of completeness of a generic k-th preference ranking is depicted by c_k (see Eq. 17);
the degree of completeness of the whole incomplete problem is depicted by $c = (\Sigma c_k)/m \approx 62.2\%$ (see Eq. 18).

Table 3(b) (see also the graphical representation in Figure 5(b)) contain the results of the application of the GLS to the paired-comparison relationships resulting from the incomplete problem in Table 2(b). The GLS results obtained for the complete problem (in Table 3(a)) can be then used as a "gold standard" to evaluate the goodness of the GLS results obtained for the corresponding incomplete problem. Quite surprisingly, these two typologies of results are very close, both in terms of accuracy and dispersion. It can be noticed that results tend to worsen especially for the less preferred objects (see for example the relatively wider deviations and uncertainty bands) related to o_3, o_4, o_5, o_6 and o_{10}; this is probably due to the relatively lower amount of usable paired-comparison relationships that include these objects, which are often omitted from Type-*t* incomplete rankings.

Table 3. Solutions (i.e., *y* scalings) resulting from the application of different fusion techniques (i.e., LCJ, OLS and/or GLS) to the complete problem in Table 2(a) and the incomplete problem in Table 2(b). GLS solutions are associated with relevant uncertainties (cf. Eq. 26).

Objects	(a) Complete problem		(b) Incomplete problem
	LCJ=OLS	GLS	GLS
o_Z	0.0	0.0 ±6.8	0.0 ±11.2
o_M	100.0	100.0 ±7.3	100.0 ±8.5
o_1	68.5	67.4 ±8.0	70.0 ±10.7
o_2	53.3	53.1 ±7.8	56.6 ±11.2
o_3	15.6	18.6 ±8.3	6.7 ±15.6
o_4	31.2	31.8 ±7.6	24.1 ±14.0
o_5	33.3	33.7 ±7.4	17.5 ±13.7
o_6	29.7	31.9 ±7.7	23.0 ±14.1
o_7	47.1	46.9 ±7.7	45.5 ±12.1
o_8	70.9	70.3 ±7.9	75.9 ±10.5
o_9	77.4	74.8 ±8.2	77.4 ±10.6
o_{10}	8.3	10.6 ±8.7	7.2 ±14.8
o_{11}	58.5	55.7 ±7.7	55.9 ±10.8
o_{12}	81.0	79.1 ±8.3	83.9 ±10.5

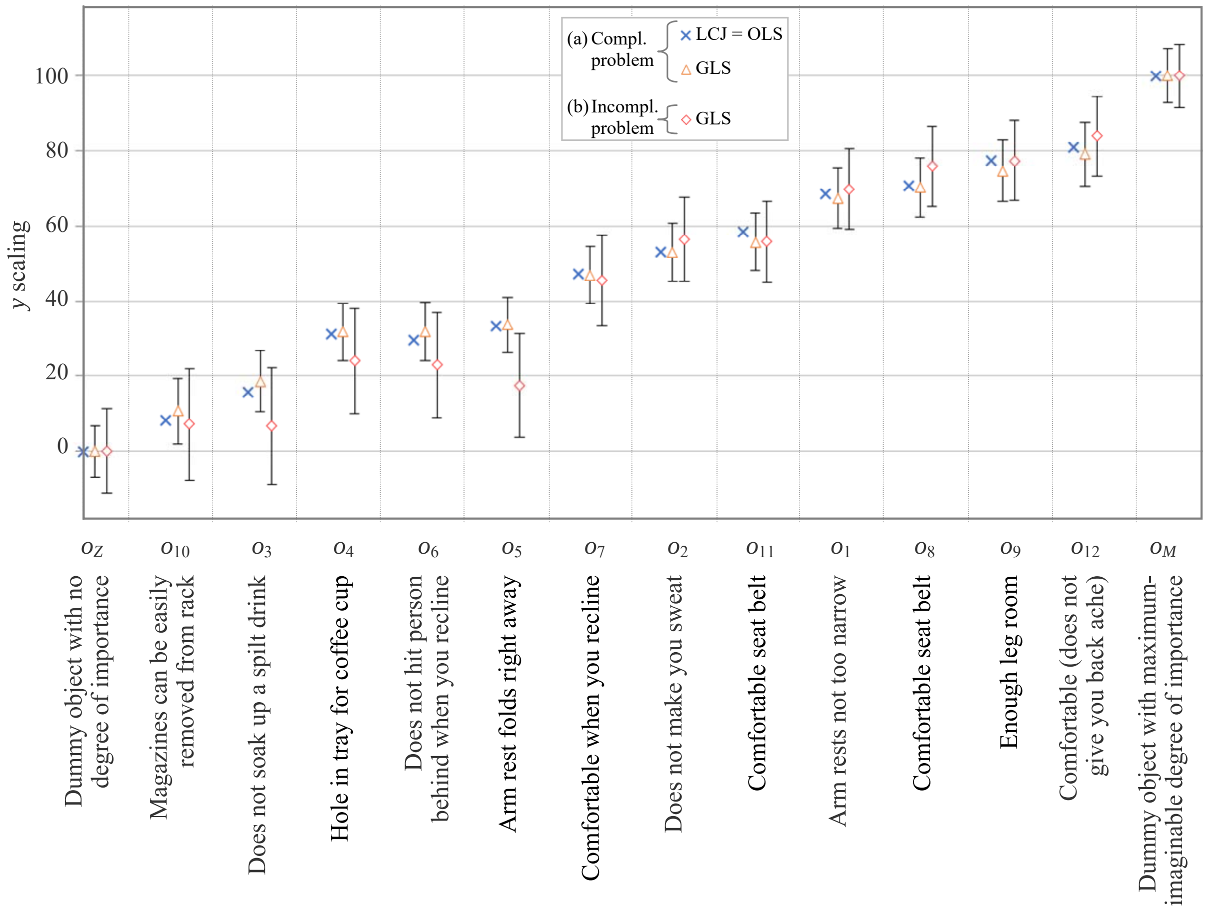


Figure 5. Graphical representation of the solutions (i.e., *y* scalings) in Table 5. Objects (o_i, i.e., the CRs related to a civilian coach-bus seat, as illustrated in section "Case study outline") are sorted in ascending order with respect to the corresponding GLS results obtained for the complete problem (represented by triangles).

The above results prove that the proposed technique seems effective, even for relatively incomplete rankings. The authors plan to study the effect of the afore-mentioned deterioration parameters on the solution accuracy, through a conspicuous amount of factorial experiments (Box, Hunter, Hunter 1978). Preliminary results show that, unsurprisingly, the solution accuracy tends to decrease dramatically for problems characterized by many type-*t* rankings. On the other hand, results are much better for problems characterized by many type-*t&b* rankings. For more information, please refer to the "Additional study of the solution accuracy" subsection, in the Appendix.

CONCLUSIONS

The ZM_{II}-technique allows to fuse multiple *incomplete* preference rankings into a *ratio* scaling, with a relevant uncertainty estimation. This technique represents an important improvement over the *ZM*-technique, proposed in (Franceschini and Maisano, 2019), whose application is limited to *complete* problems.

From a technical point of view, the ZM_{II}-technique is based on the formulation of a system of equations – borrowing the underlying postulates/assumptions of the LCJ – and its solution through the GLS method. From a practical point of view, the new response mode makes the technique more flexible and adaptable to a variety of contexts in which the concentration effort of judges cannot realistically be too high (e.g., decision-making problems with a relatively large number of objects, telephone or street interviews, etc.). This flexibility encourages the reliability of input data, as it prevents judges from providing forced and unreliable responses. Furthermore, the fusion technique can also be applied to problems characterized by "heterogeneous" preference rankings (e.g., partly complete and partly incomplete, with different forms of incompleteness).

Based on the above considerations, the ZM_{II}-technique reasonably represents an appropriate response to the previously formulated research question: "How could the *ZM*-technique be modified/improved to (1) make the response more user-friendly and reliable and (2) determine a (statistically sound) estimate of the uncertainty related to the solution?".

Preliminary results show that the ZM_{II}-technique is largely automatable, computationally efficient and provides relatively accurate results, even when preference rankings are characterized by a relatively low degree of completeness. Additionally, it seems that much better results can be obtained for incomplete rankings containing both the more and the less preferred elements (i.e., Type-*t&b* rankings).

Regarding the future, we will test the new technique in a more organic way. Precisely, we plan to investigate the accuracy of the solution depending on various factors, such as (i) level of completeness of the problem, (ii) number of judges, (iii) number of objects, (iv) degree of agreement between judges, etc..

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APPENDIX

A.1 Detailed description of the GLS

This subsection provides a detailed illustration of the GLS-method application to the problem of interest. From an operational point of view, the GLS requires the definition of a (squared) weight matrix (***W***), which encapsulates the uncertainty related to the equations of the system. A practical way to define ***W*** is to apply the *Multivariate Law of Propagation of Uncertainty* (MLPU) to the system in Eq. 10, referring to the input variables affected by uncertainty (i.e., $\hat{p}_{ij}$ values) (Kariya and Kurata, 2004); these variables can be collected in the column vector $\boldsymbol{\xi}$. Precisely, $\boldsymbol{W}$ can be determined propagating the uncertainty of the elements in $\boldsymbol{\xi}$ to the equations of the system:

$$\boldsymbol{W} = \left[\boldsymbol{J}_{\xi} \cdot \Sigma_{\xi} \cdot \boldsymbol{J}_{\xi}^{T}\right]^{-1}, \tag{A1}$$

where $\boldsymbol{J}_{\xi}$ is the Jacobian matrix containing the partial derivatives of the first members of Eq. 10, with respect to the elements in $\boldsymbol{\xi}$, and $\boldsymbol{\Sigma}_{\xi}$ is the covariance matrix of $\boldsymbol{\xi}$.

By applying the GLS method to the system in Eq. 11, a final estimate of $\boldsymbol{X}$ can be obtained as (Kariya and Kurata, 2004):

$$\hat{\boldsymbol{X}} = \left(\boldsymbol{A}^{T} \cdot \boldsymbol{W} \cdot \boldsymbol{A}\right)^{-1} \cdot \boldsymbol{A}^{T} \cdot \boldsymbol{W} \cdot \boldsymbol{B}. \tag{A2}$$

The uncertainty of the solution can be estimated through a covariance matrix $\Sigma_{\boldsymbol{X}}$, which can be obtained by applying the following relationship:

$$\Sigma_{\boldsymbol{X}} = \left(\boldsymbol{A}^{T} \cdot \boldsymbol{W} \cdot \boldsymbol{A}\right)^{-1}. \tag{A3}$$

$\Sigma_{\boldsymbol{X}}$ – unlike the homologous matrix resulting from the OLS method (see Eq. 21, in the subsection “GLS solution”) – is of considerable practical use, since it is obtained by propagating the real uncertainty of input data.

Focussing on the problem of interest, the vector containing the input variables affected by uncertainty is $\boldsymbol{\xi} = [\ldots, p_{ij}, \ldots]^{T} \in R^{(q-1)\times 1}$. On the other hand, the partial derivatives in the Jacobian matrix $\boldsymbol{J}_{\xi} \in R^{(q-1)\times(q-1)}$ can be determined in a closed form, by approximating terms $\Phi^{-1}(1-\hat{p}_{ij})$ (see Eq. 10) through the following formula (Aludaat and Alodat, 2008):

$$\Phi^{-1}(1-\hat{p}_{i,j}) \approx k\sqrt{\frac{-\ln[1-(1-2\cdot\hat{p}_{i,j})^{2}]}{\sqrt{\pi/8}}} \qquad \begin{cases} 0 \le \hat{p}_{i,j} \le 0.5 \to k = 1 \\ 0.5 < \hat{p}_{i,j} \le 1 \to k = -1 \end{cases}, \tag{A4}$$

from which:

$$\begin{array}{ll} \dfrac{\partial\left[\Phi^{-1}(1-\hat{p}_{i,j})\right]}{\partial\hat{p}_{i,j}} \approx \left|\dfrac{\sqrt{2}\cdot(2\cdot\hat{p}_{i,j}-1)}{\sqrt{-2\cdot\sqrt{2\cdot\pi}\cdot\ln(-4\cdot\hat{p}_{i,j}^{2}+4\cdot\hat{p}_{i,j})\cdot\hat{p}_{i,j}\cdot\left(1-\hat{p}_{i,j}\right)}}\right| & \text{for } \hat{p}_{i,j} \ne 0.5 \\ \dfrac{\partial\left[\Phi^{-1}(1-\hat{p}_{i,j})\right]}{\partial\hat{p}_{i,j}} \approx 2.506628 & \text{for } \hat{p}_{i,j} = 0.5 \end{array}. \tag{A5}$$

The matrix $\boldsymbol{\Sigma}_{\xi} \in R^{(q-1)\times(q-1)}$ diagonally contains the variances related to the input variables, i.e., $\hat{p}_{ij}$ terms:

$$\sigma_{p_{ij}}^2 = \frac{\hat{p}_{ij} \cdot (1 - \hat{p}_{ij})}{m_{ij}}. \tag{A6}$$

The relevant covariances can be neglected, upon the reasonable assumption that the estimates of different p_{ij} values are (statistically) independent from each other.

Next, it is possible to determine the matrix $\boldsymbol{W}$ (Eq. A1) and, subsequently, $\hat{\boldsymbol{X}}$ (Eq. A2) with the relevant uncertainty (Eq. A3); this solution is defined on an interval scale (x), as illustrated in the "Thurstone's LCJ" subsection.

Through Eq. 14, the x scaling can be transformed into a new one (y scaling), which is included in the conventional range [0, 100]. The uncertainty related to the elements in $\hat{\boldsymbol{Y}} = [\ldots, \hat{y}_i, \ldots]^T \in R^{n\times1}$ can be determined by applying the relationship:

$$\Sigma_{\boldsymbol{Y}} = \boldsymbol{J}_{\boldsymbol{Y}(\hat{x})} \cdot \Sigma_{\boldsymbol{X}} \cdot \boldsymbol{J}_{\boldsymbol{Y}(\hat{x})}^T, \tag{A7}$$

where $\boldsymbol{J}_{\boldsymbol{Y}(\hat{x})} \in R^{n\times n}$ is a Jacobian matrix containing the partial derivatives related to the equations in Eq. 24, with respect to the elements of $\boldsymbol{X}$. In the hypothesis that the n (regular and dummy) objects are ordered as (o_Z, o_M, o_1, o_2, o_3, …) and therefore $\hat{\boldsymbol{X}} = [\hat{x}_Z, \hat{x}_M, \hat{x}_1, \hat{x}_2, \hat{x}_3, \ldots]^T$ and $\hat{\boldsymbol{Y}} = [\hat{y}_Z, \hat{y}_M, \hat{y}_1, \hat{y}_2, \hat{y}_3, \ldots]^T$, $\boldsymbol{J}_{\boldsymbol{Y}(\hat{x})}$ would be:

$$\boldsymbol{J}_{\boldsymbol{Y}(\hat{x})} = \begin{bmatrix} \frac{\partial y_Z}{\partial x_Z} & \frac{\partial y_Z}{\partial x_M} & \frac{\partial y_Z}{\partial x_1} & \cdots & \frac{\partial y_Z}{\partial x_i} & \cdots \\ \frac{\partial y_M}{\partial x_Z} & \frac{\partial y_M}{\partial x_M} & \frac{\partial y_M}{\partial x_1} & \cdots & \frac{\partial y_M}{\partial x_i} & \cdots \\ \frac{\partial y_1}{\partial x_Z} & \frac{\partial y_1}{\partial x_M} & \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_i} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial y_i}{\partial x_Z} & \frac{\partial y_i}{\partial x_M} & \frac{\partial y_i}{\partial x_1} & \cdots & \frac{\partial y_i}{\partial x_i} & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} \frac{100}{\hat{x}_M - \hat{x}_Z} & 0 & 0 & \cdots & 0 & \cdots \\ 0 & \frac{-100}{\hat{x}_M - \hat{x}_Z} & 0 & \cdots & 0 & \cdots \\ -100 \cdot \frac{(\hat{x}_M - \hat{x}_1)}{(\hat{x}_Z - \hat{x}_M)^2} & 100 \cdot \frac{\hat{x}_Z - \hat{x}_1}{(\hat{x}_M - \hat{x}_Z)^2} & \frac{-100}{\hat{x}_M - \hat{x}_Z} & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -100 \cdot \frac{(\hat{x}_M - \hat{x}_i)}{(\hat{x}_Z - \hat{x}_M)^2} & 100 \cdot \frac{\hat{x}_Z - \hat{x}_i}{(\hat{x}_M - \hat{x}_Z)^2} & 0 & \cdots & \frac{-100}{\hat{x}_M - \hat{x}_Z} & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots \end{bmatrix}. \tag{A8}$$

Combining Eqs. A7 and A3, $\boldsymbol{\Sigma_Y}$ can be expressed as:

$$\Sigma_{\boldsymbol{Y}} = \boldsymbol{J}_{\boldsymbol{Y}(\hat{x})} \cdot [(\boldsymbol{A}^T \cdot \boldsymbol{W} \cdot \boldsymbol{A})^{-1}] \cdot \boldsymbol{J}_{\boldsymbol{Y}(\hat{x})}^T. \tag{A9}$$

A.2 Additional study of the solution accuracy

This subsection contains an additional study of the solution accuracy, depending on the degree of completeness of a generic (incomplete) problem. The methodological approach is articulated into three points:

- A number of complete problems (with twelve regular objects and twenty judges) are randomly generated, determining the relevant solutions.

- These complete problems are then artificially deteriorated into incomplete problems, determining the new corresponding solutions. This deterioration process is consistent with that one exemplified in the section "Artificial generation of incomplete rankings".
- The solution of each incomplete problem is compared with the solution of the corresponding "source" complete problem, which can be interpreted as a "gold standard" or "true value". This comparison is performed using the response indicator $\bar{\varepsilon}$, which expresses the accuracy of the solution of a certain incomplete problem, with respect to the solution of the "source" complete problem:

$$\bar{\varepsilon} = \sqrt{\frac{\sum_{i=1}^{n}\left(\hat{y}_i - \hat{y}_{i_{\text{(complete)}}}\right)^2}{n}}, \tag{A10}$$

  $\hat{y}_i$ being the scale value of the i-th object, resulting from the solution of the incomplete problem (i.e., $\hat{\boldsymbol{Y}} = [\ldots, \hat{y}_i, \ldots]^T$ see section "*ZM*-technique");

  $\hat{y}_{i_{\text{(complete)}}}$ being the scale value of the i-th object, resulting from the solution of the complete problem (i.e., $\hat{\boldsymbol{Y}}_{\text{(complete)}} = [\ldots, \hat{y}_{i\text{(complete)}}, \ldots]^T$);

  n being the number of (dummy and regular) objects.

  The closer the $\hat{y}_{i_{\text{(complete)}}}$ values get to the $\hat{y}_i$ values, the more $\bar{\varepsilon}$ will tend to decrease. For complete rankings, the calculation of this response indicator will obviously "degenerate" into $\bar{\varepsilon} = 0$.

Unlike the incomplete problem exemplified in the section "Application example", the new incomplete problems consist of "homogeneous" preference rankings, i.e., all rankings are characterized by the same form of incompleteness (e.g., all Type-*t* rankings with *t*=2, unordered *t* objects and without o_Z/o_M). Table A.1 shows the form of incompleteness of the (incomplete) rankings related to each incomplete problem, depending on the four deterioration parameters: "Ranking type", "Manage o_Z/o_M?", "*t/b* value" and "Order *t/b* objects?" (cf. section "Application example").

Table A.1. General characteristics of the simulated problems and corresponding $\bar{\varepsilon}$ values. The preference rankings related to each problem are characterized by the same combination of the deterioration parameters: "Ranking type", "Manage o_Z/o_M?", "*t/b* value", and "Order *t/b* objects?".

Problem no.	Ranking type	Manage o_Z/o_M?	*t/b* value	Order *t/b*-objects?	*c*	$\bar{\varepsilon}$
1	Complete	Yes	N/A	N/A	100.0%	0.0
2	Complete	Yes	N/A	N/A	100.0%	0.0
3	Quasi-complete	No	N/A	N/A	96.6%	0.9
4	Quasi-complete	No	N/A	N/A	96.0%	2.1
5	Type-*t&b*	Yes	1	Yes	65.1%	5.2
6	Type-*t&b*	Yes	1	Yes	60.9%	4.9
7	Type-*t&b*	Yes	2	Yes	77.0%	5.1
8	Type-*t&b*	Yes	2	Yes	74.8%	2.4
9	Type-*t&b*	Yes	3	Yes	89.7%	3.3
10	Type-*t&b*	Yes	3	Yes	87.7%	1.9
11	Type-*t&b*	No	1	Yes	61.2%	6.5
12	Type-*t&b*	No	1	Yes	57.5%	5.0
13	Type-*t&b*	No	2	Yes	73.1%	6.3
14	Type-*t&b*	No	2	Yes	71.4%	2.7
15	Type-*t&b*	No	3	Yes	85.8%	5.3
16	Type-*t&b*	No	3	Yes	84.3%	2.4
17	Type-*t&b*	N/A	1	No	58.6%	6.8
18	Type-*t&b*	N/A	1	No	56.1%	5.1
19	Type-*t&b*	N/A	2	No	66.2%	8.1
20	Type-*t&b*	N/A	2	No	65.5%	3.9
21	Type-*t&b*	N/A	3	No	70.7%	6.9
22	Type-*t&b*	N/A	3	No	70.4%	8.3
23	Type-*t*	Yes	1	Yes	32.3%	25.9
24	Type-*t*	Yes	1	Yes	31.6%	32.8
25	Type-*t*	Yes	2	Yes	44.7%	19.8
26	Type-*t*	Yes	2	Yes	43.8%	21.0
27	Type-*t*	Yes	3	Yes	55.4%	18.7
28	Type-*t*	Yes	3	Yes	54.8%	18.0
29	Type-*t*	No	1	Yes	30.7%	25.9
30	Type-*t*	No	1	Yes	30.2%	32.5
31	Type-*t*	No	2	Yes	43.2%	19.7
32	Type-*t*	No	2	Yes	42.4%	20.7
33	Type-*t*	No	3	Yes	53.9%	18.4
34	Type-*t*	No	3	Yes	53.2%	17.9
35	Type-*t*	N/A	1	No	30.2%	26.1
36	Type-*t*	N/A	1	No	29.7%	32.6
37	Type-*t*	N/A	2	No	39.5%	19.8
38	Type-*t*	N/A	2	No	39.1%	20.0
39	Type-*t*	N/A	3	No	46.5%	17.4
40	Type-*t*	N/A	3	No	46.0%	18.4

The graph in Figure A.1 shows a very strong correlation between the solution accuracy (depicted by $\bar{\varepsilon}$) and the degree of completeness of the incomplete problems (depicted by *c*, cf. Eq. 18). This trend is fairly well approximated by a second order polynomial model ($R^2 \approx 90.2\%$). Also, it is not surprising that the solution accuracy deteriorates significantly when switching from problems with Type-*t&b* rankings (red-square series) to problems with Type-*t* rankings (green-triangle series).

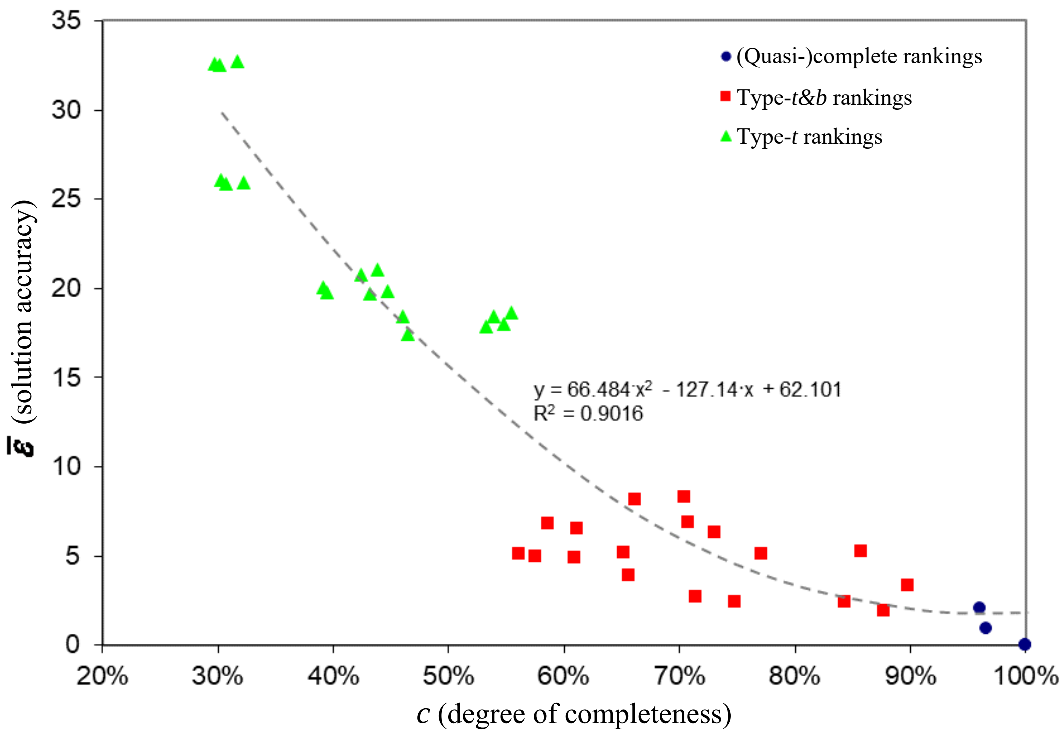

Figure A.1. Graphical representation of the results in Table 5.