# Groupoids of OEIS A002378 and A016754 Numbers
# (oblong and odd square numbers)

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Here we discuss the binary operators of the sets made by the OEIS sequences of integers A002378 and A016754. A002378 are defined as oblong numbers.

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Let us use the definition of the first type of groupoid given in [1]: it is an algebraic structure on a set with a binary operator. The only restriction on the operator is closure. This properties means that, applying the binary operator to two elements of a given set S, we obtain a value which is itself a member of S. Here, we consider the groupoids of the sets of the numbers given by OEIS sequences A002378 and A016754 [2,3], which are told as oblong and odd squares (centered octagonal) numbers.

An A002378 number is also known as a promic, pronic, or heteromecic number (formerly M1581 N0616). It is an integer having the following form [2]:

$$O_n = n(n+1)$$

OEIS gives: 2, 6, 12, 20, 30, 42, 56, 72, 90, 110, 132, 156, 182, 210, 240, 272, 306, 342, 380, 420, 462, 506, 552, 600, 650, 702, 756, 812, 870, 930, 992, 1056, and so on.
Ref. 2 tells that $4O_n+1$ are the odd squares A016754 numbers.
An A016754 number (odd square number) is defined as [3]:

$$o_n = (2n+1)^2$$

So we have [3]: 1, 9, 25, 49, 81, 121, 169, 225, 289, 361, 441, 529, 625, 729, 841, 961, 1089, 1225, 1369, 1521, and so on.
As we did in some previous discussions (see for instance [4,5]), we can find a binary operator, which satisfy the closure, of given sets of numbers. In [4], we considered the groupoids of Mersenne, Fermat, Cullen and Woodall numbers. Here, we follow the same approach as in [6], for Carol and Kynea numbers.

Here how to find the operator for A002378 numbers. Let us use:

$$(n(n+1)+1/4)^{1/2} = (n^2+n+1/4)^{1/2} = ((n+1/2)^2)^{1/2}$$

So we define:

$$(O_m+1/4)^{1/2}=(m+1/2)=A_m \quad ; \quad (O_n+1/4)^{1/2}=(n+1/2)=A_n \quad ;$$

$$(O_{m+n}+1/4)^{1/2}=(m+n+1/2)=A_{m+n}$$

We use numbers $A_m$ to help us in the calculation. We have for them the binary operator:

$$A_{m+n}=A_m\oplus A_n=A_m+A_n-1/2=(m+1/2)+(n+1/2)-1/2=m+n+1/2$$

Therefore: $(O_{m+n}+1/4)^{1/2}=A_{m+n}$ = $A_m\oplus A_n=A_m+A_n-1/2$

$$(O_{m+n}+1/4)^{1/2}=(O_m+1/4)^{1/2}+(O_n+1/4)^{1/2}-1/2$$

We can find the binary operator for the Oblong numbers as:

$$O_{m+n}+1/4=O_m+O_n+3/4+2(O_m+1/4)^{1/2}(O_n+1/4)^{1/2}-(O_m+1/4)^{1/2}-(O_n+1/4)^{1/2}$$

$$O_{m+n}=O_m+O_n+1/2+2(O_m+1/4)^{1/2}(O_n+1/4)^{1/2}-(O_m+1/4)^{1/2}-(O_n+1/4)^{1/2}$$

So we have the binary operator defined as:

$$O_m\oplus O_n=O_m+O_n+1/2+2(O_m+1/4)^{1/2}(O_n+1/4)^{1/2}-(O_m+1/4)^{1/2}-(O_n+1/4)^{1/2}$$

Associativity:

$$O_m\oplus(O_n\oplus O_p)=O_m\oplus O_{n+p}=O_{m+n+p} \quad ; \quad (O_m\oplus O_n)\oplus O_p=O_{m+n}\oplus O_p=O_{m+n+p}$$

From this binary operation, we can have the recursive relation: $O_{n+1}=O_n\oplus O_1$ .

From $O_1=2$ , we have: 6, 12, 20, 30, 42, 56, 72, 90, 110, 132, 156, 182, 210, 240, 272, 306, 342, 380, 420, 462, and so on.

Let us consider the odd square numbers.
Here how to find the operator for A016754 numbers: $o_n=(2n+1)^2$ . Let us use:

$$o_m^{1/2}=(2m+1)=A_m \quad ; \quad o_n^{1/2}=(2n+1)=A_n \quad ; \quad o_{m+n}^{1/2}=(2(m+n)+1)=A_{m+n}$$

We use numbers $A_m$ to help us in the calculation. So we have the binary operator:

$$A_{m+n}=A_m\oplus A_n=A_m+A_n-1=(2m+1)+(2n+1)-1=2(m+n)+1$$

Therefore: $o_{m+n}^{1/2}=A_{m+n}$ = $A_m\oplus A_n=A_m+A_n-1$

We can find the binary operator for the odd square numbers as:

$$o_{m+n}=o_m+o_n+1+2o_m^{1/2}o_n^{1/2}-2o_m^{1/2}-2o_n^{1/2}$$

So we have the binary operator  defined as:

$$o_m \oplus o_n = o_m + o_n + 1 + 2o_m^{1/2}o_n^{1/2} - 2o_m^{1/2} - 2o_n^{1/2}$$

Associativity:

$$o_m \oplus (o_n \oplus o_p) = o_m \oplus o_{n+p} = o_{m+n+p} \quad ; \quad (o_m \oplus o_n) \oplus o_p = o_{m+n} \oplus o_p = o_{m+n+p}$$

From this binary operation, we can have the recursive relation: $o_{n+1}=o_n \oplus o_1$ .

From $o_1=9$ , we have: 25, 49, 81, 121, 169, 225, 289, 361, 441, 529, 625, 729, 841, 961, 1089, 1225, 1369, 1521, 1681, 1849, and so on.

**References**

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