Groupoids of OEIS A002378 and A016754 Numbers (oblong and odd square numbers)

Groupoids of OEIS A002378 and A016754 Numbers (oblong and odd square numbers)

Amelia Carolina Sparavigna
Department of Applied Science and Technology, Politecnico di Torino, Italy.

Here we discuss the binary operators of the sets made by the OEIS sequences of integers A002378 and A016754. A002378 are defined as oblong numbers.

Written in Torino, 16 June 2019. DOI: 10.5281/zenodo.3247003

Let us use the definition of the first type of groupoid given in [1]: it is an algebraic structure on a set with a binary operator. The only restriction on the operator is closure. This properties means that, applying the binary operator to two elements of a given set S, we obtain a value which is itself a member of S. Here, we consider the groupoids of the sets of the numbers given by OEIS sequences A002378 and A016754 [2,3], which are told as oblong and odd squares (centered octagonal) numbers.

An A002378 number is also known as a promic, pronic, or heteromecic number (formerly M1581 N0616). It is an integer having the following form [2]:

$$O_n = n(n+1)$$

OEIS gives: 2, 6, 12, 20, 30, 42, 56, 72, 90, 110, 132, 156, 182, 210, 240, 272, 306, 342, 380, 420, 462, 506, 552, 600, 650, 702, 756, 812, 870, 930, 992, 1056, and so on.
Ref. 2 tells that $4O_n+1$ are the odd squares A016754 numbers.
An A016754 number (odd square number) is defined as [3]:

$$o_n = (2n+1)^2$$

So we have [3]: 1, 9, 25, 49, 81, 121, 169, 225, 289, 361, 441, 529, 625, 729, 841, 961, 1089, 1225, 1369, 1521, and so on.
As we did in some previous discussions (see for instance [4,5]), we can find a binary operator, which satisfy the closure, of given sets of numbers. In [4], we considered the groupoids of Mersenne, Fermat, Cullen and Woodall numbers. Here, we follow the same approach as in [6], for Carol and Kynea numbers.

Here how to find the operator for A002378 numbers. Let us use:

$$(n(n+1)+1/4)^{1/2} = (n^2+n+1/4)^{1/2} = ((n+1/2)^2)^{1/2}$$

So we define:

$$(O_m+1/4)^{1/2}=(m+1/2)=A_m \quad ; \quad (O_n+1/4)^{1/2}=(n+1/2)=A_n \quad ;$$

$$(O_{m+n}+1/4)^{1/2}=(m+n+1/2)=A_{m+n}$$

We use numbers A_m to help us in the calculation. We have for them the binary operator:

$$A_{m+n}=A_m \oplus A_n=A_m+A_n-1/2=(m+1/2)+(n+1/2)-1/2=m+n+1/2$$

Therefore: $(O_{m+n}+1/4)^{1/2}=A_{m+n} \;=\; A_m \oplus A_n=A_m+A_n-1/2$

$$(O_{m+n}+1/4)^{1/2}=(O_m+1/4)^{1/2}+(O_n+1/4)^{1/2}-1/2$$

We can find the binary operator for the Oblong numbers as:

$$O_{m+n}+1/4=O_m+O_n+3/4+2(O_m+1/4)^{1/2}(O_n+1/4)^{1/2}-(O_m+1/4)^{1/2}-(O_n+1/4)^{1/2}$$

$$O_{m+n}=O_m+O_n+1/2+2(O_m+1/4)^{1/2}(O_n+1/4)^{1/2}-(O_m+1/4)^{1/2}-(O_n+1/4)^{1/2}$$

So we have the binary operator defined as:

$$O_m \oplus O_n=O_m+O_n+1/2+2(O_m+1/4)^{1/2}(O_n+1/4)^{1/2}-(O_m+1/4)^{1/2}-(O_n+1/4)^{1/2}$$

Associativity:

$$O_m \oplus (O_n \oplus O_p)=O_m \oplus O_{n+p}=O_{m+n+p} \quad ; \quad (O_m \oplus O_n) \oplus O_p=O_{m+n} \oplus O_p=O_{m+n+p}$$

From this binary operation, we can have the recursive relation: $O_{n+1}=O_n \oplus O_1$.

From $O_1=2$, we have: 6, 12, 20, 30, 42, 56, 72, 90, 110, 132, 156, 182, 210, 240, 272, 306, 342, 380, 420, 462, and so on.

Let us consider the odd square numbers.
Here how to find the operator for A016754 numbers: $o_n=(2n+1)^2$. Let us use:

$$o_m^{1/2}=(2m+1)=A_m \quad ; \quad o_n^{1/2}=(2n+1)=A_n \quad ; \quad o_{m+n}^{1/2}=(2(m+n)+1)=A_{m+n}$$

We use numbers A_m to help us in the calculation. So we have the binary operator:

$$A_{m+n}=A_m \oplus A_n=A_m+A_n-1=(2m+1)+(2n+1)-1=2(m+n)+1$$

Therefore: $o_{m+n}^{1/2}=A_{m+n} \;=\; A_m \oplus A_n=A_m+A_n-1$

We can find the binary operator for the odd square numbers as:

$$o_{m+n}=o_m+o_n+1+2o_m^{1/2}o_n^{1/2}-2o_m^{1/2}-2o_n^{1/2}$$

So we have the binary operator defined as:

$$o_m\oplus o_n=o_m+o_n+1+2o_m^{1/2}o_n^{1/2}-2o_m^{1/2}-2o_n^{1/2}$$

Associativity:

$$o_m\oplus(o_n\oplus o_p)=o_m\oplus o_{n+p}=o_{m+n+p} \quad ; \quad (o_m\oplus o_n)\oplus o_p=o_{m+n}\oplus o_p=o_{m+n+p}$$

From this binary operation, we can have the recursive relation: $o_{n+1}=o_n\oplus o_1$.

From $o_1=9$, we have: 25, 49, 81, 121, 169, 225, 289, 361, 441, 529, 625, 729, 841, 961, 1089, 1225, 1369, 1521, 1681, 1849, and so on.

References

[1] Stover, Christopher and Weisstein, Eric W. "Groupoid." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/Groupoid.html
[2] oeis.org/A002378
[3] oeis.org/A016754
[4] Sparavigna, Amelia Carolina. (2019, April 9). On the generalized sums of Mersenne, Fermat, Cullen and Woodall Numbers. Zenodo. http://doi.org/10.5281/zenodo.2634312
[5] Sparavigna, Amelia Carolina (2019). Composition Operations of Generalized Entropies Applied to the Study of Numbers, International Journal of Sciences 8(04): 87-92. DOI: 10.18483/ijSci.2044
[6] Sparavigna, Amelia Carolina. (2019, June 6). Binary Operators of the Groupoids of OEIS A093112 and A093069 Numbers (Carol and Kynea Numbers). Zenodo. http://doi.org/10.5281/zenodo.3240465