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Zig-zag theories differently accounting for layerwise effects of multilayered composites

Andrea Urraci[1], Ugo Icardi[2]

Dipartimento di Ingegneria Meccanica e Aerospaziale, Politecnico di Torino, Italy

Abstract**—***This paper essays the effects of the choice of through-thickness representation of variables and of zig-zag functions within a general theory by the authors from which the theories considered are particularized. Characteristic feature, coefficients are calculated using symbolic calculus, so to enable an arbitrary choice of the representation. Such choice and that of zig-zag functions is shown to be always immaterial whenever coefficients are recalculated across the thickness by enforcing the fulfillment of elasticity theory constraints. Assigning a specific role to each coefficient is shown immaterial. Moreover, the order of representation of displacements can be freely exchanged with one another and, most important, zig-zag functions can be omitted if part of coefficients are calculated enforcing the interfacial stress field compatibility. Vice versa, accuracy of theories that only partially satisfy constraints, is shown to be strongly dependent upon the assumptions made. Applications to laminated and soft-core sandwich plates and beams having different length-to-thickness ratios, different material properties and thickness of constituent layers, various boundary conditions and distributed or localized loading are presented. Solutions are found in analytic form assuming the same trial functions and expansion order for all theories. Numerical results show which simplifications are yet accurate and therefore admissible.*

Keywords**—Composite and sandwich plates, zig-zag theories, interlaminar transverse shear/normal stress continuity, localized and distributed loadings, FEA 3-D elastostatic solutions.**

I. INTRODUCTION

Laminated and sandwich composites, which find continuously increasing applications because of their superior specific strength and stiffness, better energy absorption, fatigue properties and corrosion resistance than traditional materials, need to be analyzed with specific structural models.

Indeed, differently from non-layered materials their displacement field can no longer be C1-continuous, but instead has to be C°-continuous, i.e. slope discontinuities must occurs at the interfaces of layers with different properties as only in this way local equilibrium equations can be satisfied, which means that out-of-plane shear and normal stresses and the transverse normal stress gradient must be continuous at interfaces (zig-zag effect).

As a result of the strong differences between in-plane and transversal properties, 3-D stress fields arise whose out-of-plane components can assume the same importance as those in the plane and which play a fundamental role in the formation and growth mechanisms of damage and for failure. Considered that multilayer composites are used for the construction of primary structures, this being the only way to fully exploit their advantages, so far many multilayered theories of various order and degree have been developed, wherein sandwiches are described as multilayered structures whenever cell scale effect of honeycomb core aren't the object of the analysis. Sandwiches are often described as three-layer laminates where the core is assumed as the intermediate layer being shear resistant in the transverse direction, free of in-plane normal and shear stresses and deformable in the thickness direction (see, e.g. Frostig and Thomsen [1]). But often higher-order sandwich theories are considered wherein in-plane and transverse displacements vary nonlinearly across the thickness, taking different forms in the faces and the core (see, Rao and Desai [2] and Yang et al. [3]), or a separate representation is used for each of them (see, Cho et al. [4]).

A broad discussion of this matter is found, among many others, in the papers by Carrera and co-workers [5-9], Demasi [10], Vasilive and Lur'e [11], Reddy and Robbins [12], Lur'e, and Shumova [13], Noor et al. [14], Altenbach [15], Khandan et al. [16] and Kapuria and Nath [17] and the book by Reddy [18]. As shown by the quoted contributions, theories can be categorized into equivalent single-layer (ESL) formulations borrowed from those for isotropic materials, which completely disregard layerwise effects and therefore they are only suitable for predicting overall response quantities (but not even for all loading, material properties and stack-up and certainly not for sandwiches as shown e.g. [19] to [25]) and layerwise formulations which differently account for layerwise and zig-zag effects, presenting a different degree of accuracy in predicting through-thickness displacement and stress fields and a different computational burden. Layerwise theories further

subdivide into discrete-layer (DL) and zig-zag (ZZ) theories (acronyms used throughout the paper are defined in Table 1), the former assuming a representation apart for each layer having its own d.o.f., which could overwhelm the computational capacity when structures of industrial interest are analyzed but that is still always accurate irrespective for lay-up, layer properties, loading and boundary conditions considered, the latter incorporating layewrise functions into a global representation.

TABLE 1
Acronyms; in bold the new teories; (n) degree of displacements

Acronym	Description(section)	Acronym	Description(section)
FEA-3D	Mixed solid finite elements [31].	MHWZZA4	Mixed HW theory [25].
HRZZ	Mixed HR theory, $u^{(3)}_{\alpha,\beta}$, $u^{(0)}_{\varsigma}$, [25].	ZZA	$u^{(3)}_{\alpha,\beta}$, $u^{(4)}_{\varsigma}$ [23] (2.3).
HRZZ4	Mixed HR theory, $u^{(3)}_{\alpha,\beta}$, $u^{(4)}_{\varsigma}$, [25]	ZZA*	$u^{(3)}_{\alpha,\beta}$, $u^{(4)}_{\varsigma}$ [24] (2.4).
HSDT_34	Enriched adaptive versions of HSDT $u^{(3)}_{\alpha,\beta}$, $u^{(4)}_{\varsigma}$ (2.6).	ZZA*_43	$u^{(4)}_{\alpha,\beta}$, $u^{(3)}_{\varsigma}$ (2.6).
HWZZ	HW mixed version of ZZA [25] (2.5).	**ZZA*_43PRM**	$u^{(4)}_{\alpha,\beta}$ $u^{(3)}_{\varsigma}$ (2.6).
HWZZ_RDF	Modified HWZZ theory (2.6).	ZZA_RDF	Modified ZZA theory (2.3).
HWZZM	HWZZ with different zig-zag functions [24] (2.7).	ZZA-XX	Zig-zag general theory with exponential representation (3).
HWZZM*	HWZZ with different zig-zag functions [24] (2.7).	ZZA-XX'	Zig-zag general theory with power representation (3).
MHR	Mixed HR theory with Murakami's zig-zag function for $u^{(3)}_{\alpha,\beta}$, ($u^{(4)}_{\varsigma}$ lacking), [25]	ZZA_X1 ZZA_X2 ZZA_X3 ZZA_X4	$u^{(3)}_{\alpha,\beta}$, $u^{(4)}_{\varsigma}$ (3).
MHR4	Mixed HR theory, displacements with Murakami's zig-zag function [25].	**ZZA_X1*** **ZZA_X2*** **ZZA_X3*** **ZZA_X4***	$u^{(3)}_{\alpha,\beta}$, $u^{(4)}_{\varsigma}$ (3).
MHWZZA	Mixed HW theory [25].		

ZZ subdivide in turn into displacement-based and mixed theories, as strains and stresses could be chosen separately from one another using mixed variational theorems, and into physically-based (DZZ) and kinematic-based (MZZ) theories, since layerwise contributions are expressed differently. Regardless of this, ZZ have anyway intermediate features between ESL and DL that allow them to strike the right balance between accuracy and cost saving and so to meet designers' demand of theories in a simple already accurate form. Layerwise contributions are embodied in DZZ as the product of linear [26] or nonlinear [27] zig-zag functions and unknown zig-zag amplitudes, while they are a priori assumed to feature a periodic change of the slope of displacements at interfaces in MZZ, as it happens for periodic lay-ups but applying it in any case, consequently they are referred as kinematic-based theories. Stresses being assumed apart from kinematics, MZZ constitute mixed theories that because their layerwise functions are insensitive to the physical characteristics of the lamination and their kinematics is simplified in general accurately describe through-thickness stress fields but not always as accurately displacements [23-25]. Generally but still not always, because mixed formulations are also known as discussed in [25], stresses of DZZ derive from kinematics which, having to adequately represent them through strain-displacement relations, has to be rather complex so to a priori satisfy interfacial stress compatibility conditions (hence DZZ generally constitute physically-based theories), provides through-thickness displacement fields that are adequately reproduced even at interfaces where the slope doesn't reverse and so MZZ fail.

Carrera Unified Formulation (CUF) [6], which to date is extensively used to carry out analyses of multi-layered structures, as it allows displacements to take arbitrary forms that can be chosen by the user as an input and therefore allows to study general loading and boundary conditions, can be ascribed as a first approximation in the field of MZZ theories since no

physical constraints are enforced to define layerwise functions and it gets existing ESL and MZZ as particularizations. However also refined DZZ [23-25] have shown a comparable degree of generality and flexibility of use compared to CUF, resulting even more efficient because they allow the same accuracy with fewer variables. According to this, the last generation of DZZ with coefficients redefined across each physical or computational lamina must be tested for many other challenging cases similar to those already studied by researchers who used CUF and it is also necessary to check whether further generalizations can be produced as these DZZ have a computational burden still comparable to that of ESL and therefore less than that of the CUF and the DL, at least for the cases so far explored. As shown in [25], DZZ with five fixed d.o.f. can be developed and successfully applied to the analysis of challenging benchmarks with a through-thickness variable kinematic representation able to satisfy all constraints by the elasticity theory, which can be arbitrarily chosen by the user so that accurate theories can be derived with feature similar to those of CUF and HT theories with a hierarchical set of locally defined polynomials (see, Catapano et al. [28] and de Miguel et al. [29]), which neither incorporate zig-zag contributions nor require post-processing steps, but instead require a larger number of degrees of freedom and a larger expansion order of analytical solutions. As also shown in [25], advanced forms of DZZ can be developed considering forms of representation different for each displacement with zig-zag functions completely different from those usually considered until now, or even omitting them without any accuracy loss, provided that a sufficient number of coefficients is enforced which can be redefined across the thickness through the enforcement of physical constraints, therefore in a way to some extent similar to global-local superposition theories by Zhen and Wanji, e.g. [30].

Further research is required to get even whether these findings hold in general for arbitrary lay-ups, loading and boundary conditions and material properties of constituent layers. To contribute to this matter, in this paper theories developed by the authors in [23-25] are retaken and new ones obtained assuming differently the representation of displacement components are considered. They are applied to a number of challenging cases whose lay-up, material properties of layers, dimensions, boundary conditions, normalizations, trial functions and expansion order used to find analytical solutions are reported in Tables 2a and 2b, while mechanical properties of are reported in Table 2c. The results of theories for the challenging elastostatic cases with strong layerwise effects considered, which comprise localized loading and clamped edges, are compared each other and to exact solutions and to 3-D FEA [31] results, in order to show on a broader series of theories and benchmarks than in [25] that whenever the expressions of coefficients of displacements are determined a priori by enforcing the fulfillment of the full set of interfacial stress compatibility conditions, of stress boundary conditions, as well as local equilibrium equations at a number of selected point sufficient to determine all coefficients, the choice of the representation and of zig-zag functions can be arbitrary without the results changing. Under these conditions it will be proven that those zig-zag functions can even be omitted, with self-evident advantages from the computational standpoint. The numerical results also aim to demonstrate that when only a partial set of conditions is satisfied, vice versa the accuracy of theories returns to depend heavily on the type of representation and on zig-zag functions chosen.

TABLE 2a
Geometry, loading and boundary conditions

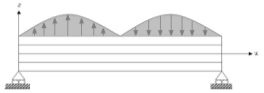
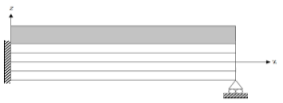
Case	Layer thickness	Material	Sketch	Loading	Lx/h
a (§)	[0.1h/0.4h]s	[Gr-Ep / Foam / Gr-Ep]		$p^0(x) = p^0{}_u \sin(2\pi x / L_x)$ if $0 \le x \le L_x$	10
b (*§)	[0.1h/0.4h]s	[Gr-Ep / Foam / Gr-Ep]			10
c (*§)	[0.1h/0.4h]s	[Gr-Ep / Foam / Gr-Ep]		$p^0(x) = \begin{cases} p^0{}_u & \text{if } 0 \le x \le L_x/2 \\ -p^0{}_u & \text{if } L_x/2 \le x \le L_x \end{cases}$	10
d (*§)	[(2h/7) / (4h/7) / (h/7)]	[n/n/n]		$p^0(x) = p^0{}_u$ if $0 \le x \le L_x$	14.286
e (§)	[(2h/7) / (4h/7) / (h/7)]	[n/n/n]			5.714
Murakami's function assumption not satisfied by u_α (*), u_β (□) or u_ς (§)					

TABLE 2b
Trial functions, expansion order and normalizations.

Case	Trial functions	Expansion	Normalization
a	$u^0(x,y)=\sum_{m=1}^{M} A_m \cos\left(\frac{2m\pi x}{L_x}\right);$ $w^0(x,y)=\sum_{m=1}^{M} C_m \sin\left(\frac{2m\pi x}{L_x}\right);$ $\gamma_x^0(x,y)=\sum_{m=1}^{M} D_m \cos\left(\frac{2m\pi x}{L_x}\right);$	1	$\bar{u}=\frac{u(0,z)}{hp^0}$ $\bar{w}=\frac{w\left(\frac{L_x}{4},z\right)}{hp^0}$ $\bar{\sigma}_{xx}=\frac{\sigma_{xx}\left(\frac{L_x}{4},z\right)}{p^0(L_x/h)^2}$ $\bar{\sigma}_{xz}=\frac{\sigma_{xz}(0,z)}{p^0}$ $\bar{\sigma}_{zz}=\frac{\sigma_{zz}\left(\frac{L_x}{4},z\right)}{p^0}$
b		1	
c		1	
d	$u^0(x,y)=\sum_{i=1}^{I} A_i\left(\frac{x}{L}\right)^i;$ $w^0(x,y)=\sum_{i=1}^{I} C_i\left(\frac{x}{L}\right)^i;$ $\gamma_x^0(x,y)=\sum_{i=1}^{I} D_i\left(\frac{x}{L}\right)^i$	9	$\bar{u}=\frac{u(L_x,z)}{hp^0}$ $\bar{w}=\frac{w(L_x,z)}{hp^0}$ $\bar{\sigma}_{xx}=\frac{\sigma_{xx}(L_x,z)}{p^0(L_x/h)^2}$ $\bar{\sigma}_{xz}=\frac{A\sigma_{xz}(L_x,z)}{P^0L_x}$ $\bar{\sigma}_{zz}=\frac{\sigma_{zz}(L_x,z)}{p^0}$
e		9	$\bar{u}=\frac{u(L_x,z)}{hp^0}$ $\bar{w}=\frac{w(L_x,z)}{hp^0}$ $\bar{\sigma}_{xx}=\frac{\sigma_{xx}(L_x,z)}{p^0(L_x/h)^2}$ $\bar{\sigma}_{xz}=\frac{\sigma_{xz}(L_x,z)}{P^0}$ $\bar{\sigma}_{zz}=\frac{\sigma_{zz}(L_x,z)}{p^0}$

TABLE 2c
Mechanical properties of constituent layers.

Material name	E1[GPa]	E2[GPa]	E3[GPa]	G12 [GPa]	G13 [GPa]	G23 [GPa]	υ12	υ13	υ23
Foam	0.035	0.035	0.035	0.0123	0.0123	0.0123	0.4	0.4	0.4
Gr-Ep	132.38	10.76	10.76	5.65	5.65	3.61	0.24	0.24	0.49
n [iso]	-	-	M1	-	-	-	0.33	0.33	0.33

M1 E_u/E_l=1.6, E_u/E_c=166.6·10^5; [iso]=isotropic $E_1=E_2=E_3$ $G_1=G_2=G_3$

II. THEORETICAL FRAMEWORK

The feature of zig-zag theories retaken from [23] to [25] and those of new ones proposed in this paper are discussed below, but just displacement, and only for mixed formations also strain and stress fields, will be discussed since governing equations as well as any intermediate step can be obtained in a straightforward way using standard techniques. As in [23] to [25] a symbolic calculus tool is used, which allows to get closed form expressions of coefficients as a result of the enforcement of physical constraints once and for all and which allows the user to arbitrarily choose the form of representation, the rest being automatic. For clarity, first the notations and the basic assumptions used (common for all the theories) are defined, then theories are examined.

2.1 Notations and basic assumptions

Layers are assumed to be linear elastic, with a uniform thickness h^k and to be perfectly bonded to each other (bonding resin interlayer disregarded). According, sandwiches are described in homogenized form as multi-layered structures with a thick soft intermediate layer as the core. A Cartesian coordinate reference system (x, y, z) is assumed as the reference frame, (x, y) being on the middle reference plane Ω, so z being the thickness coordinate, and the overall thickness of the laminate is indicated by *h*. Lx and Ly symbolize the plate side-length in the x- and y-directions. Symbols $^{(k)}z^+$ and $^{(k)}z^-$ are assumed to represent the thickness coordinates just passed or just below the interface *k*, respectively. Subscripts $_k$ and superscripts ^{k} chosen as appropriate indicate that quantities belong to the layer k, while ^{u} and $_l$ indicate upper and lower faces of the laminate. In-plane and transverse components of elastic displacements are indicated as u_α and u_ς. Strains, assumed to be infinitesimal, and stresses are symbolized by ε_{ij} and σ_{ij}, respectively. A comma is used to indicate spatial derivatives (e.g., $(.)_{,x}=\partial/\partial x$, $(.)_{,z}=\partial/\partial z$). The middle-plane displacements u^0, v^0, w^0 and the rotations of the normal θ_α (summing shear rotation Γ_α^0 and flexural notation $-w^0(\alpha,\beta)_{,\alpha}$) are assumed as the only degrees of freedom for each theory. To be concise, symbols x, y, z can be replaced through the paper by Greek letters (e.g. $\alpha=1,2\equiv x,y$; $\varsigma=3\equiv z$).

Note that throughout this paper the appellation of higher-order theories is reserved to ZZA [23], ZZA*, HWZZM, HWZZM* retaken from [24], HWZZ [25], HSDT_34, ZZA*_43, ZZA-XX, ZZA-XX', ZZA_RDF, HWZZ_RDF and ZZA_X1 to _X4 retaken from an article completed just before the present one and submitted for publication to another journal and ZZA*_43PRM, ZZA_X1* to X4* introduced in this paper. The appellation lower-order is attributed to theories HRZZ, HRZZ4, MHWZZA, MHWZZA4, MHR and MHR4 from [25]. It is reminded that acronyms and basic features of all theories considered in this paper are explained in Table 1.

2.2 Methodology of solution

Solutions are searched in analytical form as a truncated series expansion of unknown amplitudes A_{Δ}^{i} and trial functions $\Re^{i}(x,y)$ that individually satisfy the prescribed boundary conditions, which is symbolized as:

$$\Delta = \sum_{i=1}^{m_{\Delta}} A_{\Delta}^{i} \ \Re^{i}(x,y) \tag{1}$$

Here Δ symbolises in turns the d.o.f. u^0, v^0, w^0, θ_α. The candidate solution (1) is substituted into the Principle of Virtual Work, or within mixed variational functional and the solution is found using Raileigh-Ritz's like and Lagrange multipliers methods. Namely, $\Re^{i}(x,y)$ are chosen to satisfy geometric boundary conditions, while mechanical boundary conditions are satisfied using Lagrange multipliers method. A number of unknown amplitudes are determined in this latter way, according to the type and number of conditions for each specific case, while the remaining ones are determined deriving the above mentioned functionals with respect to each amplitude A_{Δ}^{i} and equating to zero. Table 2b reports the expressions of $\Re^{i}(x,y)$ for each specific case, along with the expansion order used. The same representation and the same order of expansion are shared by all theories for each examined problem, to carry out comparisons under the same conditions.

Discontinuous loading distributions considered in the numerical applications are studied without using a series expansion with a very large number of components because the symbolic calculus tool used to construct theories allows to represent loading as a general function $\chi(x,y)$ acting on upper and/or lower faces, or just on a part of them, whose relative energy contribution is computed exactly, so that the structural model is simplified and at the same time made more accurate.

The following boundary conditions are enforced on the reference mid-surface at the clamped edge of propped-cantilever beams, herein assumed at $x=0$:

$$u^0(0,0)=0; w^0(0,0)=0; w^0(0,0)_{,x}=0; \Gamma_x^0(0,0)=0 \tag{2}$$

In order to simulate that (2) holds identically across the thickness, the following further boundary conditions are enforced:

$$u_\alpha(0,z)_{,z}=0; u_\varsigma(0,z)_{,z}=0; u_\varsigma(0,z)_{,xz}=0 \tag{3}$$

To ensure that the transverse shear stress resultant force equals the constraint force, it should be also enforced the following mechanical boundary condition

$$\int_{-h/2}^{h/2} \sigma_{xz}(0,z)dz = T \tag{4}$$

The additional support condition $w^0(L,-h/2)=0$ holds at x=L on the lower face $z=-h/2$, while condition (4) is reformulated as:

$$\int_{-h/2}^{h/2} \sigma_{xz}(L,z)dz = T_L \tag{4'}$$

As mentioned above, the latter mechanical boundary conditions are enforced using Lagrange multipliers method. At simply-supported edges, the following boundary conditions are enforced on the reference mid-plane:

$$\begin{aligned} &w^0(0,y)=0; w^0(L_x,y)=0; w^0(0,y)_{,xx}=0; w^0(L_x,y)_{,xx}=0 \\ &w^0(x,0)=0; w^0(x,L_y)=0; w^0(x,0)_{,yy}=0; w^0(x,L_y)_{,yy}=0 \end{aligned} \tag{5}$$

The constraints being assumed to act at $x=0$, $x=L_x$ and $y=0$, $y=L_y$ (L_x, L_y being the length of sides parallel to x, y axes). In the applications, plates in cylindrical bending are considered, square or rectangular plates having been already extensively studied in [24],[25]. The boundary conditions for cylindrical bending follows in a straightforward way from (5) assuming that no variations occur in the y, so that (x, z) is the plane where bending take place.

2.3 ZZA displacement-based theory

This zig-zag theory, developed in [23], applied in [24, 25] and from which other theories of this paper are particularized or constitute a generalization, postulates the following displacement field across the thickness:

$$\begin{aligned} u_\alpha(x,y,z) &= [\ u^0(x,y)+z(\Gamma^0_\alpha(x,y)-w^0(x,y)_{,\alpha})\ ]_0+[\ F^u_\alpha(z)\ ]_i+[\ \sum_{k=1}^{n_i}\Phi^k_\alpha(x,y)(z-z_k)H_k(z)+\sum_{k=1}^{n_3}{}_\alpha C^k_u(x,y)H_k(z)\ ]_c \\ u_\varsigma(x,y,z) &= [\ w^0(x,y)\ ]_0+[\ F^\varsigma(z)\ ]_i+[\ \sum_{k=1}^{n_i}\Psi^k(x,y)(z-z_k)H_k(z)+\sum_{k=1}^{n_i}\Omega^k(x,y)(z-z_k)^2H_k(z)+\sum_{k=1}^{n_3}C^k_\varsigma(x,y)H_k(z)\ ]_c \end{aligned} \tag{6}$$

Summing up, the linear contribution $[...]_0$ incorporates and introduces only the d.o.f. of the theory and has coefficients already all defined, while higher- $[...]_i$ and layerwise $[...]_c$ contributions contain coefficients whose expressions have to be determined in terms of the d.o.f. , of their derivatives, of geometric and of material properties by enforcing the fulfillment of stress boundary conditions

$$\sigma_{\alpha\varsigma}=\sigma_{\varsigma\varsigma,\varsigma}=0;\sigma_{\varsigma\varsigma}=p^0(\pm) \tag{7}$$

at upper (+) and lower (-) laminate bounding faces, $p^0(\pm)$ being the transverse distributed loading acting on these faces, as well as the enforcement of local equilibrium equations:

$$\sigma_{\alpha\beta,\beta}+\sigma_{\alpha\varsigma,\varsigma}=b_\alpha \quad ; \quad \sigma_{\alpha\varsigma,\alpha}+\sigma_{\varsigma\varsigma,\varsigma}=b_\varsigma \tag{8}$$

at selected points across the thickness and of stress-compatibility equations at material layer interfaces

$$\begin{aligned} &\sigma_{\alpha\varsigma}({}^{(k)}z^+)=\sigma_{\alpha\varsigma}({}^{(k)}z^-);\sigma_{\varsigma\varsigma}({}^{(k)}z^+)=\sigma_{\varsigma\varsigma}({}^{(k)}z^-); \\ &\sigma_{\varsigma\varsigma,\varsigma}({}^{(k)}z^+)=\sigma_{\varsigma\varsigma,\varsigma}({}^{(k)}z^-) \end{aligned} \tag{9}$$

In details, any combination of independent functions $[\ F^u_\alpha(z)\ ]_i$ and $[\ F^\varsigma(z)\ ]_i$ of any degree can be chosen

$$\begin{aligned} [\ F^u_\alpha(z)\ ]_i &= [\ C^i_\alpha(x,y)z^2+D^i_\alpha(x,y)z^3+(Oz^4...)\ ]_i=[\ {}_3(.)\tilde{}_\alpha\ ]_i+[\ (Oz^4...)\ ]_i \\ [\ F^\varsigma(z)\ ]_i &= [\ b^i(x,y)z+c^i(x,y)z^2+d^i(x,y)z^3+e^i(x,y)z^4+(Oz^5...)\ ]_i=[\ {}_4(.)\tilde{}_\varsigma\ ]_i+[\ (Oz^5...)\ ]_i \end{aligned} \tag{10}$$

Since coefficients of contributions $Oz^4...$ and $Oz^5...$ can be always determined by enforcing (8) in a suited number of points selected across the thickness. In this way, a variable kinematics representation is allowed that enables ZZA to adapt to changes in solutions across the thickness, which is worth the name of "adaptive" theory given to this theory. Instead, the expressions of coefficients C^i_α, D^i_α, b^i to e^i are determined by enforcing the fulfilment of stress boundary conditions (7), while those of Φ^k_α, Ψ^k and Ω^k are determined so that the continuity (9) of out-of-plane stresses and of the transverse normal stress gradient $\sigma_{\varsigma\varsigma,\varsigma}$ at layer interfaces are met. The remaining layerwise contributions C^k_u and C^k_ς are determined restoring the continuity of displacements

$$u_\alpha({}^{(k)}z^+)=u_\alpha({}^{(k)}z^-);u_\varsigma({}^{(k)}z^+)=u_\varsigma({}^{(k)}z^-) \tag{11}$$

at mathematical layer interfaces whenever the expressions of $[\ F^u_\alpha(z)\ ]_i$ and $[\ F^\varsigma(z)\ ]_i$ change. The readers can find all the details herein omitted for brevity in [23-25]. It is worth mentioning that SEUPT technique [23] can be used in order to obtain a C° formulation of the ZZA theory, as well as of all the other theories of this paper which follow.

A variant of ZZA, called ZZA_RDF and retaken from a paper recently submitted for publication is here reported, whose coefficients assumes a different role than ZZA. Particularly, Ω^k, Ψ^k, Φ_α^k impose the fulfillment of local equilibrium equations at different points across the thickness (for i>1, where I is the number of computational layer) while C_α^i, d^i and e^i impose the continuity of out-of-plane stresses at the interfaces between layers. Because of some laminations, stresses could be erroneously predicted to vanish for z=0, a different reference plane with a distance far $h_d > h/2$ from the bottom face is assumed:

$$
\begin{aligned}
&u_\alpha(x,y,z) = [\ u^0(x,y) + (z - h_d + h/2)(\Gamma_\alpha^0(x,y) - w^0(x,y)_{,\alpha})\ ]_0 + [\ C_\alpha^i(x,y)z^2 + D_\alpha^i(x,y)z^3\ ]_i + [\ \sum_{k=1}^{n_i}\Phi_\alpha^k(x,y)(z-z_k)H_k(z) + \sum_{k=1}^{n_3}{}_\alpha C_u^k(x,y)H_k(z)\ ]_c \\
&u_\varsigma(x,y,z) = [\ w^0(x,y)\ ]_0 + [\ b^i(x,y)z + c^i(x,y)z^2 + d^i(x,y)z^3 + e^i(x,y)z^4\ ]_i + [\ \sum_{k=1}^{n_i}\Psi^k(x,y)(z-z_k)H_k(z) + \sum_{k=1}^{n_i}\Omega^k(x,y)(z-z_k)^2H_k(z) + \sum_{k=1}^{n_3}C_\varsigma^k(x,y)H_k(z)\ ]_c \\
&(h_d \le z \le h_d + h)
\end{aligned}
\tag{11'}
$$

Results of this theory are indistinguishable from those of ZZA in the present numerical applications. As a consequence, it will be demonstrated that is not necessary to assign a specific role to coefficients if the full set of physical constraints is imposed and coefficients are redefined layer-by-layer across the thickness.

2.4 ZZA* displacement-based theory

This theory, which is the first developed by the authors [24] to prove that constraints (9) can be enforced without explicitly incorporating zig-zag functions and relative amplitudes Φ_α^k, Ψ^k and Ω^k inside $[...]_c$ and from which other theories follow as generalizations in this paper, has the following displacement field:

$$
\begin{aligned}
&u_\alpha(x,y,z) = [\ u^0(x,y) + z(\Gamma_\alpha^0(x,y) - w^0(x,y)_{,\alpha})\ ]_0 + \{\ \sum_{k=1}^{n_3}{}_k\tilde{B}_\alpha^i(x,y)z + [C_\alpha^i(x,y)z^2] + [D_\alpha^i(x,y)z^3 + D_\alpha^i(x,y)z_j^3] + \sum_{k=1}^{n_i}{}_k\tilde{C}_\alpha^i(x,y)\ \}_{i+c} \\
&u_\varsigma(x,y,z) = [\ w^0(x,y)\ ]_0 + \{\ [b^i(x,y)z + \sum_{k=1}^{n_3}{}_k\tilde{b}^i(x,y)z] + [c^i(x,y)z^2 + \sum_{k=1}^{n_i}{}_k\tilde{c}^i(x,y)z^2] + [d^i(x,y)z^3] + e^i(x,y)z^4 + \sum_{k=1}^{n_i}{}_k\tilde{d}^i(x,y)\ \}_{i+c}
\end{aligned}
\tag{12}
$$

Contributions $[...]_0$ are the same as ZZA, while ${}_k\tilde{B}_\alpha^i$, C_α^i, ${}_k\tilde{C}_\alpha^i$ assume the same purpose as Φ_α^k, C_α^i and ${}_\alpha C_u^k$ inside ZZA, ${}_k\tilde{b}^i$ and ${}_k\tilde{c}^i$ perform the same function as Ω^k and Ψ^k while ${}_k\tilde{d}^i$ the function of C_ς^k because once redefined at material interfaces they allow to satisfy the continuity of interlaminar stresses and of displacements. Again C_α^i, D_α^i, b^i, c^i, d^i and e^i allow stress boundary conditions (7) and local equilibrium equations (8) to be met. Therefore similar to ZZA the redefinition of coefficients allows the representation to vary according to the conditions imposed. In details, b^i and c^i enable the fulfillment of stress boundary conditions relating to $\sigma_{\varsigma\varsigma}$ and $\sigma_{\varsigma\varsigma,\varsigma}$ at the lower bounding face, then they are assumed to vanish in subsequent layers. Coefficients C_α^i, D_α^i, d^i and e^i allow to satisfy (8) at two points per layer, excluding the upper layer, and also enable stress-free boundary conditions on $\sigma_{\alpha\varsigma}$ and three equilibrium equations to be satisfied at a single point for the bottom layer. The remaining variables allow to meet three equilibrium equations at a single point across the upper layer and the stress boundary conditions on the upper surface. The present ZZA* theory and other subsequent theories that also do not explicitly encompass zigzag functions can be still considered as physically-based zig-zag theories, because constraints are enforced in order to determine the expressions of coefficients.

In order to improve accuracy, the intermediate layers and the last one can be subdivided each into two or more computational layers, so that more equilibrium points can be enforced and, consequently, the representation order can be increased (but not necessarily, because coefficients can be evaluated at different positions using the same power of the thickness coordinate). To enable decomposition into computational layers, a sufficient number of contributions must be contained in the displacements field so an appropriate expansion order of summations in (12) has to be chosen. However, it will be shown by the present numerical results that even for the challenging benchmarks considered, just a third/fourth order overall representation is required to obtain very accurate results.

The new theories developed forward in section 3 of this paper will be aimed to prove that there is no need to assign a specific role to each coefficient, as just done for ZZA_RDF (explained in section 2.3), that the order and type of representation can arbitrarily vary and independently for the single displacement and in a more general way that zig-zag layerwise contributions don't have to be incorporated explicitly (like for ZZA*).

Numerical illustrations will show that ZZA* and the new theories with the features just mentioned achieve the same accuracy of ZZA with a lower processing time, but their most important advantage is that the computational effort of the preparation phase, which is performed once and for all via symbolic calculus, decreases strongly.

2.5 HWZZ mixed theory

This theory was developed in [25] as a mixed Hu-Washizu version of ZZA wherein displacement strain and stress fields are assumed separately from each other in order to limit the computational cost while maintaining accuracy. For this purpose, only the contributions of each field deemed essential are preserved. Displacements derive from those of ZZA neglecting zig-zag contributions by Ω^k, along with higher-order and adaptive contributions $Oz^4...$ and $Oz^5...$ (10) because numerical test have shown their secondary importance for displacements, while they remain of primary importance for stress fields. Consistent with these assumptions, no decomposition into computational layers is allowed for displacements, so also contributions C_u^k, C_v^k, C_w^k are omitted, then the displacement field simplifies to

$$u_\alpha(x,y,z)=\left[\ u^0(x,y)+z(\Gamma_x^0(x,y)-w^0(x,y)_{,\alpha})\ \right]_0+\left[\ C_\alpha^i(x,y)z^2+D_\alpha^i(x,y)z^3\ \right]_i+\left[\ \sum_{k=1}^{n_i}\Phi_\alpha^k(x,y)(z-z_k)H_k(z)\ \right]_c \tag{13}$$

$$u_\varsigma(x,y,z)=\left[\ w^0(x,y)\ \right]_0+\left[\ b^i(x,y)z+c^i(x,y)z^2+d^i(x,y)z^3+e^i(x,y)z^4\ \right]_i+\left[\ \sum_{k=1}^{n_i}\Psi^k(x,y)(z-z_k)H_k(z)\ \right]_c$$

Out-of-plane strains ε_{zz}, γ_{xz}, γ_{yz}, are obtained from those of ZZA neglecting zig-zag contributions by Ω^k, but preserving those by C_u^k, C_v^k, C_w^k that enable decomposition into computational layers, while in-plane strains ε_x, ε_y, ε_{xy} directly follow from (13) and from strain-displacement relations. Once defined strains in this way, membrane stresses σ_{xx}, σ_{yy}, σ_{xy} are derived in a straightforward way from stress-strain relations, while out-of-plane stresses σ_{xz}, σ_{yz}, σ_{zz} are obtained by integrating local equilibrium equations, in order to recovery small stress jumps resulting from omission of contributions by Ω^k in (13).

2.6 Theories HWZZ_RDF, ZZA*_43 and HSDT_34

A mutation of HWZZ, named HWZZ_RDF is assessed, whose master displacement, strain and stress fields are the same of HWZZ, except for terms c^i which are calculated for i>1 by imposing the continuity of the transverse normal stress gradient at the interfaces, instead of imposing the fulfillment of local equilibrium equations. HWZZ_RDF is considered together with other variants discussed forward in order to prove that it is unnecessary to assign a specific role to every single coefficient, instead the role can be freely exchanged without any accuracy loss, i.e. regardless of which coefficient is defined by imposing any of (7) to (9) and (11).

The results of theories ZZA*_43 and HSDT_34 developed by the authors in a paper that is nearing completion are presented in the numerical applications in order to verify together with the new theories developed in section 3 that order and form of representation of displacements across the thickness can be changed freely, provided that a sufficient number of coefficients is incorporated so as to allow the fulfillment of (7) to (9) and (11). The features of these theories are summarized forward, since they provide rather accurate results and therefore can be used in practical applications.

ZZA*_43 is a reconstruction of ZZA* which is obtained assuming each in-plane displacements as a fourth-order piecewise polynomial across the thickness, while the transverse displacement is assumed as a piecewise cubic polynomial:

$$u_\alpha(x,y,z)=\left[\ u^0(x,y)+z(\Gamma_\alpha^0(x,y)-w^0(x,y)_{,\alpha})\ \right]_0+\left\{\ \sum_{k=1}^{n_\Delta}{}_k\tilde{B}_\alpha^i(x,y)z\ +[C_\alpha^i(x,y)z^2]+[D_\alpha^i(x,y)z^3+E_\alpha^i(x,y)z^4]+\sum_{k=1}^{n_i}{}_k\tilde{C}_\alpha^i(x,y)\ \right\}_{i+c} \tag{14}$$

$$u_\varsigma(x,y,z)=\left[\ w^0(x,y)\ \right]_0+\left\{\ [b^i(x,y)z+\sum_{k=1}^{n_\Delta}{}_k\tilde{b}^i(x,y)z]+[c^i(x,y)z^2+\sum_{k=1}^{n_i}{}_k\tilde{c}^i(x,y)z^2]+[d^i(x,y)z^3]+\sum_{k=1}^{n_i}{}_k\tilde{d}^i(x,y)\ \right\}_{i+c}$$

Namely, the expansion order of displacements is reversed with respect to ZZA*, while in accordance with it zig-zag functions aren't explicitly incorporated since their role is played by coefficients to be redefined across the thickness. As in ZZA*, terms ${}_k\tilde{B}^i_\alpha$, C^i_α, ${}_k\tilde{C}^i_\alpha$ play the role that in ZZA is played by Φ^k_α, C^i_α and ${}_\alpha C^k_u$, while ${}_k\tilde{b}^i$ and ${}_k\tilde{c}^i$ play the same role as Ω^k, Ψ^k and ${}_k\tilde{d}^i$ play the same role of C^k_ς. Terms C^i_α, D^i_α, E^i_α, b^i, c^i and d^i allow to satisfy and therefore are defined by local stress boundary conditions (7) and local equilibrium equations (8). However, numerical tests have shown that even permuting the role of coefficients, the result does not change, so what has been said previously about the role of the single coefficients can be varied arbitrarily, therefore it is not even necessary to explicitly define the functions of each one. As an example, the results of a variant ZZA*_43PRM, which is obtained assuming ${}_k\tilde{B}^i_\alpha$ to play as C^i_α of ZZA, ${}_k\tilde{C}^i_\alpha$ to play as Φ^k_α, C^i_α as ${}_\alpha C^k_u$, ${}_k\tilde{b}^i$ as C^k_ς and ${}_k\tilde{c}^i$, ${}_k\tilde{d}^i$ to play as Ω^k, Ψ^k, respectively, are reported in the numerical applications, which appear undistinguishable from those of ZZA*_43.

HSDT_34 is a simplified theory derived as a particularization of ZZA starting from cubic in-plane displacements and a fourth-order transverse displacement, which get a piecewise variation in absence of zig-zag functions by virtue of the redefinition of coefficients, which for this reason have a superscript i:

$$\begin{aligned} u_\alpha(x,y,z) &= [\ u^0(x,y) + z(\Gamma^0_\alpha(x,y) - w^0(x,y)_{,\alpha})\ ]_0 + B^i_\alpha(x,y)z + C^i_\alpha(x,y)z^2 + D^i_\alpha(x,y)z^3 \\ u_\varsigma(x,y,z) &= [\ w^0(x,y)\ ]_0 + b^i(x,y)z + c^i(x,y)z^2 + d^i(x,y)z^3 + e^i(x,y)z^4 \end{aligned} \tag{15}$$

This theory is considered for the twofold purpose of assessing whether accurate results can get by reducing the order of representation with respect to previous theories and of demonstrating through numerical applications that redefinition of coefficients is the key aspect that enables to improve accuracy, as it transforms a poor theory that without redefining the coefficients is a ESL into a layerwise one, and which also allows the elimination of zig-zag functions and still respecting (9), so ultimately confirming the results of other theories of this paper.

2.7 HWZZM and HWZZM* mixed theories

This theory was constructed in [25] starting from the same assumptions of HWZZ but using different zig-zag functions, in order to show that the choice of these functions is immaterial since the same results of ZZA and ZZA* are achieved whenever constraints (7) to (9) and (11) are simultaneously enforced. HWZZM is characterized by the following displacement field

$$\begin{aligned} {}^{\mathfrak{I}}u_\alpha(x,y,z) &= [\ u^0(x,y) + z(\Gamma^0_\alpha(x,y) - w^0(x,y)_{,\alpha})\ ]_0 + [\ F^u_\alpha(z)\ ]_i + [\ A^{u_\alpha}_k(z)[\frac{2z}{z_{k+1}-z_k} - \frac{z_{k+1}+z_k}{z_{k+1}-z_k}\ ] + C^k_\alpha(x,y)]_c \\ {}^{\mathfrak{I}}u_\varsigma(x,y,z) &= [\ w^0(x,y)\ ]_0 + [\ F^\varsigma(z)\ ]_i + [A^{u_\varsigma}_k(z)[\frac{2z}{z_{k+1}-z_k} - \frac{z_{k+1}+z_k}{z_{k+1}-z_k}] + B^{u_\varsigma}_k(z)[\frac{(2z)^2}{z_{k+1}-z_k}] + C^k_\varsigma(x,y)]_c \end{aligned} \tag{16}$$

from which expressions out-of-plane strains ε_{zz}, γ_{xz}, γ_{yz} are derived and from which its simplified counterpart used for obtaining membrane strains ε_x, ε_y, ε_{xy} follows in a straightforward way by neglecting adaptive contributions as in (13) (here ${}^{\mathfrak{I}}(.)$ states that displacements refer to the computational layer $\mathfrak{I}$). It could be noticed that layerwise functions in (16) derive from Murakami's zig-zag functions $M^k(z) = (-1)^k\zeta^k$ assuming

$$\zeta^k = a^k z - b^k, a^k = \frac{2}{z_{k+1}-z_k}, b^k = \frac{z_{k+1}+z_k}{z_{k+1}-z_k} \tag{16'}$$

HWZZM is still a physically-based zig-zag theory because expression of amplitudes $A^{u_\varsigma}_k$ and $B^{u_\varsigma}_k$ are not assumed a priori, but instead they are derived at each interface by enforcing the fulfilment of stress compatibility conditions (9), therefore all remains completely similar to ZZA.

As for HWZZ, membrane stresses σ_{xx}, σ_{yy}, σ_{xy} derive from stress-strain relations and the strains obtained as outlined above, while out-of-plane master stresses come from integration of local equilibrium equations.

In [25], various theories have been derived from HWZZM by assuming a priori and differently from each other the expressions of amplitudes $A_k^{u_\alpha}$, $A_k^{u_\varsigma}$, $B_k^{u_\varsigma}$ that characterize the displacement components (16). Because these theories have proven to be much less accurate than theories where zig-zag amplitudes are defined on a physical basis, they are not considered in this paper.

Another mixed theory, called HWZZM*, which is retaken from [25], is considered for sake of comparison. It is obtained starting from the following displacement field which, similarly to that of ZZA*, does not explicitly contain any zig-zag function:

$$u_\alpha(x,y,z) = \left[\ u^0(x,y) + z(\Gamma_\alpha^0(x,y) - w^0(x,y)_{,\alpha})\ \right]_0 + \left\{\ \sum_{k=1}^{n_\Im} {}_k\tilde{B}_\alpha^i(x,y)z\ \right\}_c + \left\{\ [C_\alpha^i(x,y)z^2] + [D_\alpha^i(x,y)z^3]\ \right\}_i$$
$$u_\varsigma(x,y,z) = \left[\ w^0(x,y)\ \right]_0 + \left\{\ [b^i(x,y)z] + [c^i(x,y)z^2] + [d^i(x,y)z^3] + e^i(x,y)z^4\ \right\}_i + \left\{\ \sum_{k=1}^{n_\Im} {}_k\tilde{b}^i(x,y)z\ \right\}_c \quad (16'')$$

Similarly to HWZZ and HWZZM, no decomposition into mathematical layers is allowed for the displacement field of HWZZM* and ${}_k\tilde{c}^i$, ${}_k\tilde{C}_\alpha^i$, ${}_k\tilde{d}^i$ are omitted. In-plane master strains are still obtained by (16') while ${}_k\tilde{C}_\alpha^i$ and ${}_k\tilde{d}^i$ are restored for master out-of-plane ones, since subdivision into computational layers is allowed for these strains. Again, in-plane stresses are obtained from stress-strain relations, while σ_{xz} , σ_{yz} , σ_{zz} are computed by integration of (8). Numerical discussion will show that also HWZZM* obtains results indistinguishable from those of ZZA and other higher-order theories, so corroborating that zig-zag functions can be omitted without any loss of accuracy if the full set of physical constrains (7-9, 11) is enforced and coefficients are redefined layer-by-layer.

III. NEW THEORIES OF THIS PAPER

Theories are now introduced which are aimed at contributing to demonstrate that the representation form of the displacement field can be assumed arbitrarily whenever the full set of constraints (7) to (9) and (11) is enforced and the coefficients of the representation are redefined across the thickness. Under these conditions, it is aimed to show that the explicit incorporation of zig-zag contributions is unnecessary.

Firstly, two quite general theories retaken from a previous paper recently submitted and called ZZA-XX and ZZA-XX' are here reported which are considered in the numerical applications. The displacement field of ZZA-XX is defined as an infinite series of products of initially unknown amplitudes and exponential functions of the thickness coordinate z_i :

$$u_\alpha(x,y,z) = \left[\sum_{k=1}^{\infty} C_{k_\alpha}^i(x,y)e^{(k\ z_i/h_i)}\right]_\Im + \sum_{j=1}^{\Im} D_{j_\alpha}^i(x,y)$$
$$u_\varsigma(x,y,z) = \left[\sum_{k=1}^{\infty} C_{k_\varsigma}^i(x,y)e^{(k\ z_i/h_i)}\right]_\Im + \sum_{j=1}^{\Im} D_{j_\varsigma}^i(x,y) \quad (17)$$

Coefficients are redefines for each computational layer $\Im$ of thickness h_i and calculated by imposing the full set of physical constrains (7-9, 11), where the expansion order can be chosen arbitrary by user, even if at least (7-9, 11) have to be imposed to obtain maximal accuracy. The displacement field of ZZA-XX' is:

$$u_\alpha(x,y,z) = \left[\sum_{k=1}^{\infty} C_{k_\alpha}^i(x,y)z^{(k)}\right]_\Im + \sum_{j=1}^{\Im} D_{j_\alpha}^i(x,y)$$
$$u_\varsigma(x,y,z) = \left[\sum_{k=1}^{\infty} C_{k_\varsigma}^i(x,y)z^{(k)}\right]_\Im + \sum_{j=1}^{\Im} D_{j_\varsigma}^i(x,y) \quad (17')$$

and the meaning of symbols if the same of (17). Starting from ZZA-XX and ZZA-XX' more general theories can be obtained, whose displacement field is assumed differently from one another and differently for even and odd layers. Such theories can be viewed as particularizations of the general representation

$$u_\alpha(x,y,z) = \left[\ u_\alpha{}^0(x,y) + z(\Gamma_\alpha^0(x,y) - w^0(x,y)_{,\alpha})\ \right]_0 + \left[\ \sum_{k=1}^{3} C_{k_\alpha}^i(x,y)\,{}^iF_\alpha^k(z) + C_\alpha^i\ \right]_{i+c}$$
$$u_\varsigma(x,y,z) = \left[\ w^0(x,y)\ \right]_0 + \left[\ \sum_{k=1}^{4} D_k^i(x,y)\,{}^iG^k(z) + C_\varsigma^i\ \right]_{i+c} \quad (17'')$$

obtained through a different specification of ${}^{i}F^{\alpha}_{k}$ and ${}^{i}G_{k}$, respectively for the in-plane and the transverse displacement component. Four different particularizations ZZA-X1 to ZZA-X4 have been so far already considered by the authors, to which another 4 new ones will be added here. Those already considered previously are particularized from (17) as follows:

ZZA_X1

	$u_\alpha \rhd {}^{i}F^{k}_{\alpha} = \left\{ \begin{matrix} z^{(k+1)/2} if\ k=1,3 \\ e^{(kz/(2h))}\ if\ k=2 \end{matrix} (k_{\max}=3) \right\}$	odd layers	$u_\alpha \rhd {}^{i}F^{k}_{\alpha} = \left\{ z^k \ \ (k=1,2,3) \right\}$
even layers	$u_\beta \rhd {}^{i}F^{k}_{\beta} = \left\{ z^k \ \ (k=1,2,3) \right\}$	(i=1,3,…)	$u_\beta \rhd {}^{i}F^{k}_{\beta} = \left\{ \begin{matrix} z^{(k+1)/2} if\ k=1,3 \\ e^{(kz/(2h))}\ if\ k=2 \end{matrix} (k_{\max}=3) \right\}$ (18)
(i=2,4,…)	$u_\varsigma \rhd {}^{i}G^{k} = \left\{ \begin{matrix} z^{(k+1)/2} if\ k=1,3 \\ e^{(kz/(2h))}\ if\ k=2,4 \end{matrix} (k_{\max}=4) \right\}$		$u_\varsigma \rhd {}^{i}G^{k} = \left\{ z^k \ \ (k=1,2,3,4) \right\}$

ZZA_X2

	$u_\alpha \rhd {}^{i}F^{k}_{\alpha} = \left\{ z^k \ \ (k=1,2,3) \right\}$	odd layers	$u_\alpha \rhd {}^{i}F^{k}_{\alpha} = \left\{ \begin{matrix} z^{(k+1)/2} if\ k=1,3 \\ e^{(kz/(2h))}\ if\ k=2 \end{matrix} (k_{\max}=3) \right\}$
even layers	$u_\beta \rhd {}^{i}F^{k}_{\beta} = \left\{ \begin{matrix} z^{(k+1)/2} if\ k=1,3 \\ e^{(kz/(2h))}\ if\ k=2 \end{matrix} (k_{\max}=3) \right\}$	(i=1,3,…)	$u_\beta \rhd {}^{i}F^{k}_{\beta} = \left\{ \begin{matrix} z^{(k+1)/2} if\ k=1,3 \\ e^{(kz/(2h))}\ if\ k=2 \end{matrix} (k_{\max}=3) \right\}$ (19)
(i=2,4,…)	$u_\varsigma \rhd {}^{i}G^{k} = \left\{ \begin{matrix} z^{(k+1)/2} if\ k=1,3 \\ e^{(kz/(2h))}\ if\ k=2,4 \end{matrix} (k_{\max}=4) \right\}$		$u_\varsigma \rhd {}^{i}G^{k} = \left\{ z^k \ \ (k=1,2,3,4) \right\}$

ZZA_X3

$$u_\alpha \rhd {}^{i}F^{k}_{\alpha} = \left\{ \begin{matrix} z^{(k+1)/2} if\ k=1,3 \\ e^{(kz/(2h))}\ if\ k=2 \end{matrix} (k_{\max}=3) \right\}$$

all layers

$$u_\beta \rhd {}^{i}F^{k}_{\beta} = \left\{ \begin{matrix} z^{(k+1)/2} if\ k=1,3 \\ e^{(kz/(2h))}\ if\ k=2 \end{matrix} (k_{\max}=3) \right\} \quad (20)$$

(i=1,2,3,4,…)

$$u_\varsigma \rhd {}^{i}G^{k} = \left\{ \begin{matrix} z^{(k+1)/2} if\ k=1,3 \\ e^{(kz/(2h))}\ if\ k=2,4 \end{matrix} (k_{\max}=4) \right\}$$

ZZA_X4

	$u_\alpha \rhd {}^{i}F^{k}_{\alpha} = \left\{ z^k \ \ (k=1,2,3) \right\}$	odd layers	$u_\alpha \rhd {}^{i}F^{k}_{\alpha} = \left\{ \begin{matrix} z^k if\ k=1,3 \\ e^{(z/h)}\ if\ k=2 \end{matrix} (k_{\max}=3) \right\}$
even layers	$u_\beta \rhd {}^{i}F^{k}_{\beta} = \left\{ \begin{matrix} z/h \\ \sin(z/h)\ if\ k=2 \\ e^{(kz/(2h))}\ if\ k=3 \end{matrix} (k_{\max}=3) \right\}$	(i=1,3,…)	$u_\beta \rhd {}^{i}F^{k}_{\beta} = \left\{ \begin{matrix} z/h \\ \sin(z/h)\ if\ k=2 \\ e^{(kz/(2h))}\ if\ k=3 \end{matrix} (k_{\max}=3) \right\}$ (21)
(i=2,4,…)	$u_\varsigma \rhd {}^{i}G^{k} = \left\{ \begin{matrix} z^k if\ k=1,3 \\ e^{(z/h)}\ if\ k=2,4 \end{matrix} (k_{\max}=4) \right\}$		$u_\varsigma \rhd {}^{i}G^{k} = \left\{ z^k \ \ (k=1,2,3,4) \right\}$

Hereafter, four new theories ZZA_X1* to ZZA_X4* are considered. They are new variants of (17'') that don't contain any zig-zag functions. Like (18) to (21), their purpose is to demonstrate what claimed from theoretical standpoint, i.e. that through-the-thickness representations can be chosen arbitrary for each displacements and from point to point, without any loss of accuracy. It will be shown that such theories will provide results indistinguishable from others obtained by other higher-order theories, if coefficients are redefined layer-by-layer across the thickness and are calculated by imposing the fulfillment of (7) to (9) and (11):

ZZA_X1*

	$u_\alpha \rhd {}^{i}F^{k}_{\alpha} = \left\{ z^k \ \ (k=1,2,3) \right\}$	odd layers	$u_\alpha \rhd {}^{i}F^{k}_{\alpha} = \left\{ \begin{matrix} z^{(k+1)/2} if\ k=1,3 \\ e^{(kz/(2h))}\ if\ k=2 \end{matrix} (k_{\max}=3) \right\}$
even layers	$u_\beta \rhd {}^{i}F^{k}_{\beta} = \left\{ \begin{matrix} z^{(k+1)/2} if\ k=1,3 \\ e^{(kz/(2h))}\ if\ k=2 \end{matrix} (k_{\max}=3) \right\}$	(i=1,3,…)	$u_\beta \rhd {}^{i}F^{k}_{\beta} = \left\{ z^k \ \ (k=1,2,3) \right\}$ (22)
(i=2,4,…)	$u_\varsigma \rhd {}^{i}G^{k} = \left\{ z^k \ \ (k=1,2,3,4) \right\}$		$u_\varsigma \rhd {}^{i}G^{k} = \left\{ \begin{matrix} z^{(k+1)/2} if\ k=1,3 \\ e^{(kz/(2h))}\ if\ k=2,4 \end{matrix} (k_{\max}=4) \right\}$

ZZA_X2*

	$u_\alpha \rhd {}^{i}F^{k}_{\alpha} = \left\{ \begin{matrix} z^{(k+1)/2} if\ k=1,3 \\ e^{(kz/(2h))}\ if\ k=2 \end{matrix} (k_{\max}=3) \right\}$	odd layers	$u_\alpha \rhd {}^{i}F^{k}_{\alpha} = \left\{ z^k \ \ (k=1,2,3) \right\}$
even layers	$u_\beta \rhd {}^{i}F^{k}_{\beta} = \left\{ \begin{matrix} z^{(k+1)/2} if\ k=1,3 \\ e^{(kz/(2h))}\ if\ k=2 \end{matrix} (k_{\max}=3) \right\}$	(i=1,3,…)	$u_\beta \rhd {}^{i}F^{k}_{\beta} = \left\{ \begin{matrix} z^{(k+1)/2} if\ k=1,3 \\ e^{(kz/(2h))}\ if\ k=2 \end{matrix} (k_{\max}=3) \right\}$ (23)
(i=2,4,…)	$u_\varsigma \rhd {}^{i}G^{k} = \left\{ z^k \ \ (k=1,2,3,4) \right\}$		$u_\varsigma \rhd {}^{i}G^{k} = \left\{ \begin{matrix} z^{(k+1)/2} if\ k=1,3 \\ e^{(kz/(2h))}\ if\ k=2,4 \end{matrix} (k_{\max}=4) \right\}$

ZZA_X3*

$$u_\alpha \triangleright {}^iF_\alpha^k = \left\{ {}^{z^{(k+1)/2}\ if\ k=1,3}_{e^{(kz/(2h))}\ if\ k=2} (k_{\max}=3) \right\}$$

all layers

$$u_\beta \triangleright {}^iF_\beta^k = \left\{ {}^{z/h}_{\sin(z/h)\ if\ k=2}{}^{}_{e^{(kz/(2h))}\ if\ k=3} (k_{\max}=3) \right\} \quad (24)$$

(i=1,2,3,4,…)

$$u_\varsigma \triangleright {}^iG^k = \left\{ {}^{z^{(k+1)/2} if\ k=1,3}_{e^{(kz/(2h))}\ if\ k=2,4} (k_{\max}=4) \right\}$$

ZZA_X4*

even layers

$$u_\alpha \triangleright {}^iF_\alpha^k = \left\{ {}^{z^k if\ k=1,3}_{e^{(z/h)}\ if\ k=2} (k_{\max}=3) \right\}$$

$$u_\beta \triangleright {}^iF_\beta^k = \left\{ {}^{z/h}_{\sin(z/h)\ if\ k=2}{}^{}_{e^{(kz/(2h))}\ if\ k=3} (k_{\max}=3) \right\}$$

(i=2,4,…)

$$u_\varsigma \triangleright {}^iG^k = \left\{ z^k \quad (k=1,2,3,4) \right\}$$

odd layers

$$u_\alpha \triangleright {}^iF_\alpha^k = \left\{ z^k \quad (k=1,2,3) \right\}$$

(i=1,3,…)

$$u_\beta \triangleright {}^iF_\beta^k = \left\{ {}^{z/h}_{\sin(z/h)\ if\ k=2}{}^{}_{e^{(kz/(2h))}\ if\ k=3} (k_{\max}=3) \right\} \quad (25)$$

$$u_\varsigma \triangleright {}^iG^k = \left\{ {}^{z^k if\ k=1,3}_{e^{(z/h)}\ if\ k=2,4} (k_{\max}=4) \right\}$$

IV. NUMERICAL ASSESSMENTS AND DISCUSSION

Henceforth the accuracy of theories will be assessed considering a number of elasto-static challenging benchmarks having pronounced layerwise effects deriving from geometric and material properties, lay-ups and loading that are often typical of cases of industrial interest. These benchmarks are selected as they require simultaneously a very accurate modeling of transverse normal deformation effects. Of interests, here are transverse anisotropy effects, accordingly strong different elastic properties of layers are considered rather than ones from just reoriented layers, which is simply omitted. Cases are retaken from [24] and [25], wherein it is indicated who studied them previously.

The prefixed purpose of numerical illustrations is to show that refined physically-based zig-zag theories of this paper very accurately describe layerwise effects, with a smaller number of d.o.f. than widespread theories described in [5-8], [11-17], whenever the full set of physical constraints (7) to (9) and (11) is enforced, which lets the present theories to assume the appellation of physically-based, the following holds.

(i) The functions expressing the variation of displacements across the thickness can be arbitrarily chosen without having any difference in the results; (ii) also zig-zag functions can be arbitrarily chosen and (iii) even be omitted, if a sufficient number of coefficients is still considered, whose expressions are re-determined across the thickness, with the advantage of containing computational of phases carried out in symbolic form; (iv) a different representation varying from region to region across the thickness can be freely chosen as desired and different zig-zag functions can be assumed for each single displacement (variable-kinematics form) without any accuracy loss, (v) a specific role need not be assigned to individual coefficients whenever their expressions are re-defined across the thickness by imposing (7) to (9) and (11), as the role can be freely exchanged if overall the same conditions are imposed.

Because industrial designers are interested to know whether lower order theories of low cost can be used to carry out analysis, the present study consider these theories in order to confirm that, as shown in literature, they can be accurate for certain cases but not in general; consequently, (vi) a partial fulfilment of (7) to (9) and (11) even with a fairly high degree of representation implies a loss of validity of (i) to (v), so that accuracy back to being strongly dependent upon the representation adopted.

4.1 CASE A

It concerns a simply supported sandwich beam under two half-waves sinusoidal loading applied at the upper face, with graphite-epoxy faces and foam core and a side length-to-thickness ratio of 10. This case is a modified version of that formerly studied in [25], in order to prove that because of the presence of an inflection point in the loading distribution the effects played by the transverse normal deformability on equilibrium of transverse shear and normal stresses are still equally enhanced. This is confirmed by the numerical results of Fig. 1 and Table 4a for this case because, indeed, inaccurate results are obtained by lower-order theories like MHWZZA, MHWZZA4, HRZZ, HRZZ4 but in particular, by MHR and MHR4 in mixed form that assume a polynomial or piecewise representation of the transverse displacement across the thickness, respectively and kinematic-based zig-zag functions. The reason is that such lower-order theories cannot accurately reproduce the trend of transverse normal and shear stresses across the thickness, not even when out-of-plane stresses are derived from the membrane components through the integration of local equilibrium equations. Errors cannot be recovered even when strain and stress fields are assumed apart and with an already rather high order, like in the case of mixed theory MHWZZA

that incorporates strains and stress fields of HWZZ, as well as also even when the transverse displacement comes from ZZA, as in the case of MHWZZA4, a sign that a description capable of representing the effects of normal transverse deformation and the relative stress is required without any simplification being used. The results of the theories considered in this paper reported in Fig. 1 and Table 4a leave out theories that had already demonstrated their inadequacy in [24,25]. These omitted theories are all those having simplifying assumptions that result in a poor description of displacement and stress fields, such as all kinematic based theories with immutable zigzag contributions because defined once and for all. Since errors characterize mixed theories HRZZ and HRZZ4 are lower than MHR and MHR4 that include Murakami's zig-zag functions, it is demonstrated the superiority of physically-based theories over kinematic based ones. Results show that only higher-order adaptive theories (ZZA, ZZA*, HWZZM , HWZZM*, HWZZ, HSDT_34, ZZA*_43, ZZA-XX, ZZA-XX', ZZA_RDF, HWZZ_RDF, ZZA_X1 to _X4, ZZA*_43PRM and ZZA_X1* to X4*), whose coefficients are redefined layer-by-layer across the thickness and that impose the fulfillment of full set of physical constraints (7) to (9) and (11) are always accurate and that their results are indistinguishable from one another. This therefore shows that zig-zag functions can be freely changed, so, even they can be assumed in Murakami's form, provided that amplitudes are recalculated at each interface. Moreover, it is shown that they can be omitted if coefficients are recomputed across the thickness, as shown by the results of ZZA-X1 to ZZA-X4 and ZZA-X1* to ZZA-X4* having a different representation form for each displacement, which is differently assumed form point to point across the thickness.

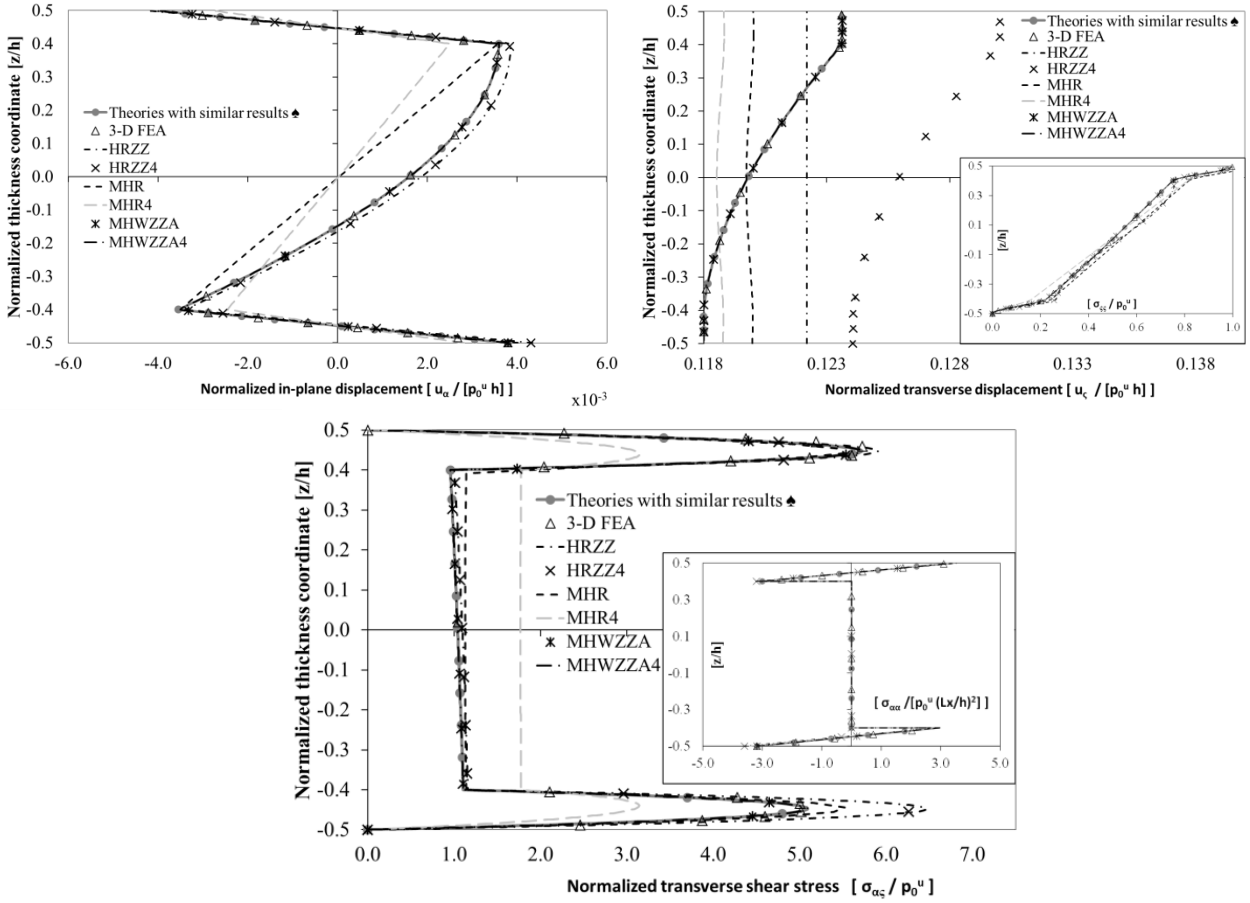


FIGURE 1: Normalized in-plane displacement, transverse displacement, transverse shear stress, in-plane stress (in the inset) and transverse normal stress (in the inset) for case a. Symbol ♠ indicates that theories ZZA, ZZA*, HWZZM, HWZZM*, HWZZ, HSDT_34, ZZA*_43, ZZA-XX, ZZA-XX', ZZA_RDF, HWZZ_RDF, ZZA_X1 to _X4, ZZA*_43PRM and ZZA_X1* to X4*obtain results which differ for less than 1%.

Table 3, shows that processing time of higher-order theories is still always comparable to those of ESL and that lower-order ones, which however cannot obtain a similar or acceptable degree of accuracy, so their advantage in terms of computational burden is only apparent. It could be seen from Table 3 that anyway theories wherein zig-zag functions are omitted are the most efficient theories considered in this paper being able to achieve the same accuracy of theories incorporating zig-zag functions with a lower processing time.

TABLE 3
Processing time [s]; errors: ♥ > 3%; ♣ > 10%.

Type	Theory	Features	Cases	a	b	c	d	e
	FSDT	Reference theory		2.3014	2.3045	2.4710	5.9211	6.1245
New theories	ZZA_X1*	Arbitrary representation		3.7673	3.7015	4.2532	11.4310	12.0417
	ZZA_X2*			3.8680	3.8254	4.3774	11.7316	12.3256
	ZZA_X3*			3.9032	3.8444	4.3979	11.9151	12.5217
	ZZA_X4*			3.8969	3.8535	4.4288	11.8392	12.4289
Mixed HR	HRZZ	Uniform w^0		♥5.3990♣	♥5.5423♣	♥7.4582♣	♥18.4274♣	♥18.2287♣
	HRZZ4	Polynomial w^4		♥5.4094♣	♥5.2737♣	♥11.0258♣	♥18.9174♣	♥18.4489♣
	MHR	Murakami's zig zag u^3,v^3		4.3663	♥4.5619♣	♥4.8186♣	♥6.7118♣	♥6.9651♣
	MHR4	Murakami's zig zag u^3,v^3,w^4		♥4.3310♣	♥4.3291♣	♥4.9692♣	♥6.1582♣	♥6.4367♣
Mixed HW	HWZZ			4.4949	4.5726	5.2894	12.0471	12.3960
	HWZZ_RDF			4.3621	4.2988	4.9147	13.2034	13.9751
	HWZZM*	No zig-zag functions		3.6891	3.6223	4.1572	11.1507	11.7533
	HWZZM	Murakami's zig zag u^3,v^3,w^4		4.0107	3.9760	4.5548	12.1959	12.8123
	MHWZZA			4.4726	♥4.6131♣	♥5.1828♣	♥7.2626♣	♥7.9157♣
	MHWZZA4			4.6211	♥4.7301♣	♥5.2798♣	♥7.1210♣	♥7.8126♣
	ZZA	Adaptive u^3,v^3,w^4		4.9770	4.9120	5.6127	15.0929	15.8988
	ZZA_RDF			4.7548	4.6900	5.3309	14.4480	15.1919
	ZZA*	No zig-zag functions u^3,v^3,w^4		3.7181	3.6988	4.2332	11.3110	11.9627
	HSDT_34			3.7371	3.6694	4.2077	11.3441	11.9652
	ZZA*_43	No zig-zag functions u^4,v^4,w^3		3.7393	3.7005	4.2142	11.3391	11.8843
	ZZA*_43PRM			3.7438	3.6955	4.2329	11.3713	11.9072
	ZZA_X1	Arbitrary representation		3.7705	3.7064	4.2545	11.4102	12.0588
	ZZA_X2			3.8508	3.7999	4.3686	11.6656	12.3498
	ZZA_X3			3.9127	3.8502	4.3945	11.9047	12.4688
	ZZA_X4			3.9074	3.8509	4.4108	11.9031	12.4591
	ZZA-XX	General representation		9.4659	9.3495	10.6582	28.5583	30.1960
	ZZA-XX'			9.2262	9.1041	10.3746	28.0411	29.4950

On a computer with quad-core CPU@2.60GHz, 64-bit OS and 8.00 GB RAM; FSDT shear correction factor 5/6.

TABLE 4a
Results for case a.

Case a	Position	3-D FEA	Theories with similar results ♠	HRZZ	HRZZ4	MHR	MHR4	MHWZZA	MHWZZA4
u_α x10^{-3}	up/min	-4.1022	-4.1021	-4.2829	-4.2703	-3.9624	-2.7736	-4.1093	-4.1112
	down/max	3.7999	3.7925	4.3434	4.3061	4.0179	2.8125	3.8025	3.8063
u_ς x10^{-1}	up	1.2362	1.2359	1.2220	1.3003	1.2002	1.1882	1.2358	1.2360
	down	1.1799	1.1796	1.2220	1.2406	1.1998	1.1878	1.1795	1.1797
	max	1.2364	1.2361	1.2220	1.3005	1.2004	1.1884	1.2360	1.2362
	min	1.1799	1.1796	1.2220	1.2406	1.1998	1.1878	1.1795	1.1797
$\sigma_{\alpha\alpha}$	up/max	3.4318	3.4329	3.5828	3.5722	3.3149	3.2517	3.4331	3.4328
	down/min	-3.1755	-3.1797	-3.6297	-3.5985	-3.3577	-3.4353	-3.1808	-3.1770
$\sigma_{\alpha\varsigma}$	max	5.7202	5.8346	6.4703	6.3773	5.5537	3.1530	5.7264	5.7349
$\sigma_{\varsigma\varsigma}$	up/max	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Symbol ♠ indicates that theories ZZA, ZZA*, HWZZM , HWZZM*, HWZZ, HSDT_34, ZZA*_43, ZZA-XX, ZZA-XX', ZZA_RDF, HWZZ_RDF, ZZA_X1 to _X4, ZZA*_43PRM and ZZA_X1* to X4*obtain results which differ for less than 1%

4.2 CASE B

In this case the same sandwich beam of case a is considered, except that the lower face is assumed to be damaged, its mechanical properties being reduced of two order of magnitude, according to the ply-discount theory (E_{1111} E_{1122} E_{2222} E_{1212} reduced by $2 \cdot 10^{-2}$) because it happens often in practice, e.g. due to the impact of a foreign object that even if occurring only locally, it still represents an area where the analysis must be as accurate as possible, but also because the strong asymmetry of material properties enhances 3-D elastic effects, becoming a very severe test from the standpoint of theories.

The results for this case given in Fig. 2 and in Table 4b confirm the primary role played by a very accurate modelling of the transverse displacement once a marked difference of elastic properties of faces is considered. It can be seen that the through-thickness variation of the in-plane displacement is quite accurately predicted by all theories, so it does not represent a discriminating quantity, so different representations prove to be adequate. The opposite occurs instead for the transverse displacement, but for this case error on u_ς are not seen to reflect on out-of-plane stresses. Indeed, such stresses are quite accurately predicted also by MHR, MHR4, MHWZZA, MHWZZA4, HRZZ and HRZZ4 since the reduced elastic properties of the lower face, which are somewhat more similar to those of the core rather to those of the upper one, prevent the spreading across the thickness, causing their concentration near to it.

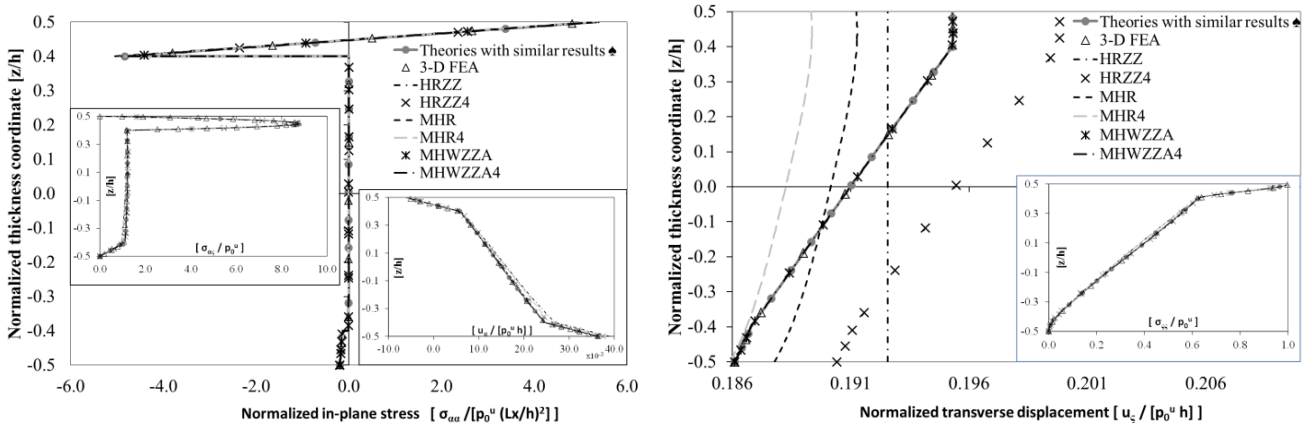


FIGURE 2: Normalized in-plane stress, in-plane displacement (in the inset), transverse shear stress (in the inset), transverse displacement and normalized transverse normal stress (in the inset) for case b.

TABLE 4b
Results for case b.

Case b	Position	3-D FEA	Theories with similar results ♠	HRZZ	HRZZ4	MHR	MHR4	MHWZZA	MHWZZA4
u_α $x10^{-3}$	up/min	-7.3337	-7.3412	-6.4956	-6.4713	-6.2484	-6.5608	-6.3837	-6.3718
	down/max	37.6515	37.7012	38.8976	37.5123	36.1826	37.9917	36.3228	36.3345
u_ς $x10^{-1}$	up	1.9529	1.9529	1.9257	1.9980	1.9124	1.8933	1.9529	1.9529
	down/min	1.8611	1.8609	1.9257	1.9041	1.8779	1.8591	1.8608	1.8609
	max	1.9533	1.9533	1.9257	1.9983	1.9128	1.8937	1.9533	1.9533
$\sigma_{\alpha\alpha}$	up/max	5.3383	5.3330	5.4319	5.4115	5.2253	5.2343	5.3446	5.3465
	down	-0.1926	-0.1925	-0.2062	-0.1989	-0.1918	-0.1919	-0.1966	-0.1921
	min	-4.8293	-4.8261	-5.0463	-4.9921	-4.8346	-4.8520	-4.8078	-4.8196
$\sigma_{\alpha\varsigma}$	max	8.7600	8.8067	8.8180	8.7679	8.5258	8.8528	8.7773	8.7777
$\sigma_{\varsigma\varsigma}$	up	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	max	1.0000	1.0000	1.0061	1.0107	1.0000	1.0000	1.0000	1.0000

Anyway, once again, the only theories shown capable of correctly reproducing stress and displacement through-thickness variations are shown to be ZZA, HWZZ together with all the remaining higher-order theories. The results also show that what clamed in (i) to (vi) holds also for this case since ZZA, ZZA*, HWZZM , HWZZM*, HWZZ, HSDT_34, ZZA*_43, ZZA-XX, ZZA-XX', ZZA_RDF, HWZZ_RDF, ZZA_X1 to _X4, ZZA*_43PRM and ZZA_X1* to X4* achieve the same accuracy degree, their results being indistinguishable from one another. Use of higher-order theories is still confirmed preferable to that of lower-order ones, both because the latter misestimate displacement and stress fields and do not provide advantages in terms of computational costs (see Table 3). As results from this table, higher-order theories that omit layerwise functions are the most efficient theories of this paper, having the same accuracy of other higher-order models and lower processing time.

4.3 CASE C

In this case, a similar simply supported sandwich structure with graphite-epoxy faces and foam core of previous cases a and b is still considered (dimension and material properties are reported in Table 2a). The structure is loaded by a step compressive loading on its upper and lower faces, as indicated in Table 2b. This type of load is considered because it shows stronger layerwise effects than those of sinusoidal loading considered in cases a and b.

Results of Fig. 3 and Table 4c for this case referring to u_α and $\sigma_{\alpha\varsigma}$ are reported at $x/L_x = 0$, while those for u_ς, $\sigma_{\alpha\alpha}$ and $\sigma_{\varsigma\varsigma}$ are reported at $x/L_x = 0.25$, because at these two positions they exhibit peaks. Despite the strong layerwise effects due to the loading considered, theories unexpectedly show results in a better agreement each other and with FEA 3-D than in the previous cases. Anyway, some errors are shown by lower-order theories, that is to say MHR, MHR4, MHWZZA, MHWZZA4, HRZZ, HRZZ4, as regards displacements across the core, as only , ZZA, ZZA*, HWZZM , HWZZM*, HWZZ, HSDT_34, ZZA*_43, ZZA-XX, ZZA-XX', ZZA_RDF, HWZZ_RDF, ZZA_X1 to _X4, ZZA*_43PRM and ZZA_X1* to X4* turn out to be able to provide an adequate information in this case. From the standpoint of accuracy of theories, this is signing a strong case-dependence by loading, as it results from the comparison with previous cases a and b. Even if errors are not very large, theories based on Murakami's zig-zag function whose coefficients are not redefined across the thickness appear inadequate because in the present case the slope of in-plane and transverse displacements don't reverse at interfaces.

In the same way, theories which have a simplified kinematics of u_ς like MHR, MHR4, HRZZ, HRZZ4, MHWZZA and MHWZZA4 are inadequate, not even when using the best theories like ZZA, ZZA*, HWZZM , HWZZM*, HWZZ, HSDT_34, ZZA*_43, ZZA-XX, ZZA-XX', ZZA_RDF, HWZZ_RDF, ZZA_X1 to _X4, ZZA*_43PRM and ZZA_X1* to X4* as post-processors. Errors in u_ς reflect into an overestimation of the transverse shear stress across the upper face, which is more evident for theories MHWZZA and MHWZZA4, and in a underestimation by HRZZ and HRZZ4 across both faces. So this is signing that u_ς assumes a paramount importance under step loading. To highlight the largely case-sensitive behavior of theories based on simplifying hypotheses, and therefore the unpredictability of their predictions, it could be noted that while MHWZZA and MHWZZA4 inadequately predict the variation of the in-plane displacement across the core, their lower-order counterparts MHR, MHR4 give results much closer to FEA-3D due to a mutual compensation of errors.

Only higher-order adaptive theories (ZZA, ZZA*, HWZZM , HWZZM*, HWZZ, HSDT_34, ZZA*_43, ZZA-XX, ZZA-XX', ZZA_RDF, HWZZ_RDF, ZZA_X1 to _X4, ZZA*_43PRM and ZZA_X1* to X4*), whose coefficients are redefined layer-by-layer across the thickness and that impose the fulfillment of full set of physical constraints (7) to (9) and (11) appear always able to get accurate displacements and stresses. Indeed, their results are indistinguishable from one another and from those of 3-D FEA, irrespective of zig-zag functions or representation assumed. So ultimately this confirms that their choice is immaterial and so what clamed in (i) to (vi) also for this case.

The same still holds as regards the computational cost seen for the previous cases, because the little less expensive but inaccurate lower-order theories cannot provide real advantages over higher-order ones, see, Table 3. Higher-order theories that do not contain zig-zag functions are shown again to require a nearly or little more calculation time than their lower-order counterparts (which however are less accurate), proving to be the most efficient theories of this paper.

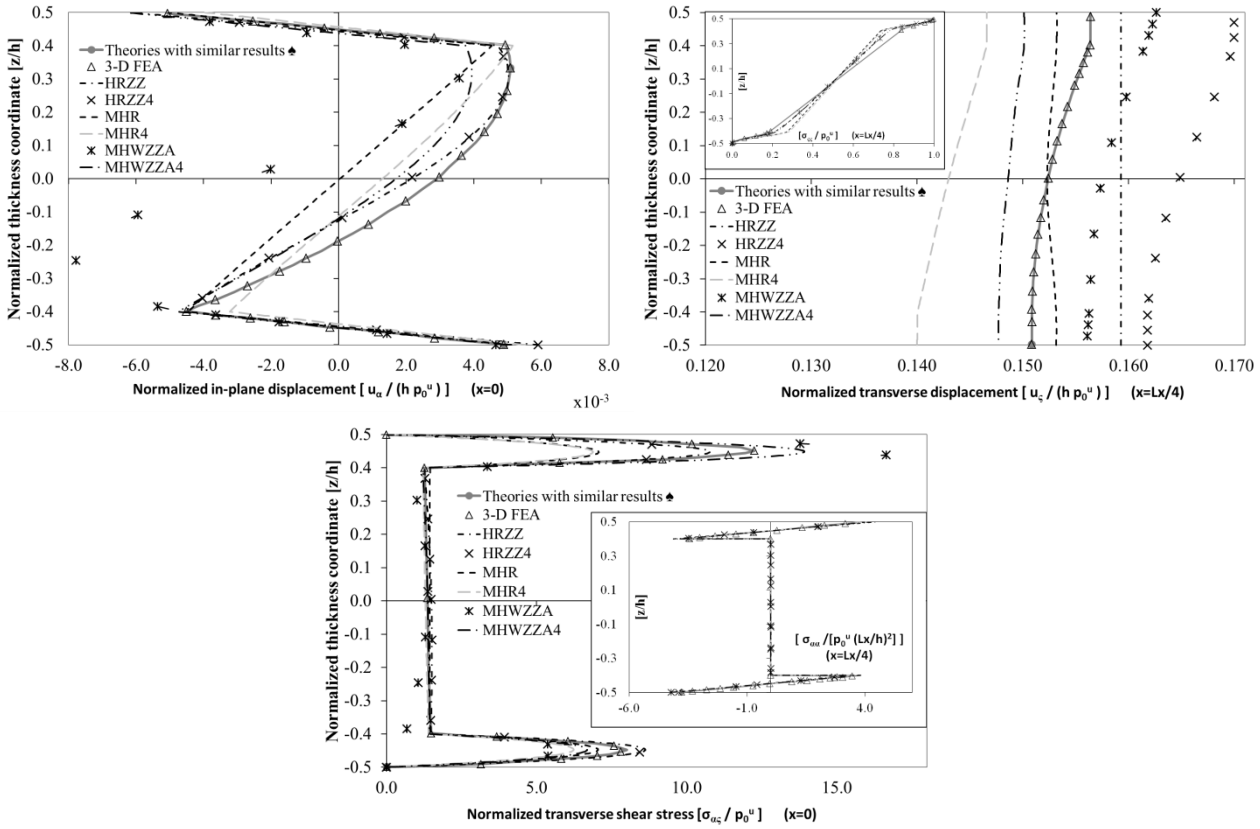


FIGURE 3: Normalized in-plane displacement, transverse displacement, transverse shear stress, transverse normal and in-plane stresses (in the insets) for case c.

TABLE 4c
Results for case c.

Case c	Position	3-D FEA	Theories with similar results ♠	HRZZ	HRZZ4	MHR	MHR4	MHWZZA	MHWZZA4
u_α x10^{-3} (x=0)	up	-5.0694	-5.1284	-6.1288	-6.1299	-5.0448	-3.9548	-6.1457	-6.2844
	down	4.8842	4.8354	5.9176	5.9003	5.1157	5.5149	4.6516	4.9863
	max	5.0889	5.1345	5.9176	5.9003	5.1157	5.5149	4.6516	4.9863
	min	-5.0694	-5.1284	-6.1288	-6.1299	-5.0448	-3.9548	-7.7749	-6.2844
u_ζ x10^{-1} (x=Lx/4)	up	1.5639	1.5640	1.5928	1.6993	1.4660	1.5320	1.6264	1.5013
	down/min	1.5088	1.5089	1.5928	1.6178	1.4001	1.5316	1.5605	1.4769
	max	1.5641	1.5642	1.5928	1.6997	1.4661	1.5323	1.6264	1.5016
$\sigma_{\alpha\alpha}$ (x=Lx/4)	up/max	3.7846	3.7596	4.5093	4.5153	4.1485	4.0898	4.2213	4.2635
	down	-3.7772	-3.7316	-3.9002	-3.9081	-4.0382	-4.0905	-4.2200	-4.2622
	min	-3.7772	-3.7316	-4.1253	-4.1325	-4.0382	-4.0905	-4.2200	-4.2622
$\sigma_{\alpha\zeta}$ (x=0)	max	12.2525	12.2550	10.7842	10.8634	7.0695	6.8761	17.4821	13.9216
$\sigma_{\zeta\zeta}$ (x=Lx/4)	up/max	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

4.4 CASE D

A propped-cantilever sandwich beam under uniform loading and with a length-to-thickness ratio $l_x / h = 14.286$ is considered, whose geometric and material properties are indicated in Table 2a. Fig. 4 and Table 4d show the through-thickness variation of displacements and transverse shear and normal stresses for this case, formerly considered in [25], as predicted by the various theories considered in this paper. Although the structure is rather slim, major discrepancies still exist among the predictions of theories because strong layerwise effects still persist like in the thickest structures.

Because of this, simplified kinematics assumptions of theories MHR, MHR4, MHWZZA, MHWZZA4, HRZZ, HRZZ4 become completely inadequate, although in the literature it is often claimed that even ESL can be used when l_x / h similar to

the one is considered here. Moreover, the stress field is described in an imprecise way by all theories except by ZZA and theories ZZA*, HWZZM, HWZZM*, HWZZ, HSDT_34, ZZA*_43, ZZA-XX, ZZA-XX', ZZA_RDF, HWZZ_RDF, ZZA_X1 to _X4, ZZA*_43PRM and ZZA_X1* to X4*, which indeed provide indistinguishable results from each other. So, it is demonstrated what is claimed in this paper regarding the arbitrary choice of zig-zag functions and of the representation, provided that all physical constraints are contemporaneously satisfied. In particular, the transverse normal stress is misestimate by all lower-order theories. Nevertheless the slope of displacements reverses at both face-core interfaces, theories MHR and MHR4 based upon Murakami's zig-zag function incorrectly predict displacement and stress fields. Instead, theories whose coefficients are redefined layer-by-layer across the thickness by imposing the fulfilment of full set of physical constraints (7) to (9) and (11) are always accurate, irrespective of zig-zag functions or the representation assumed, so confirming what claimed in (i) to (vi).It is confirmed also for this case that higher-order theories are more efficient than lower-order ones, as shown by Table 3, and the most efficient are those that do not explicitly contain zig-zag functions, as thy have the same accuracy but a lower processing time.

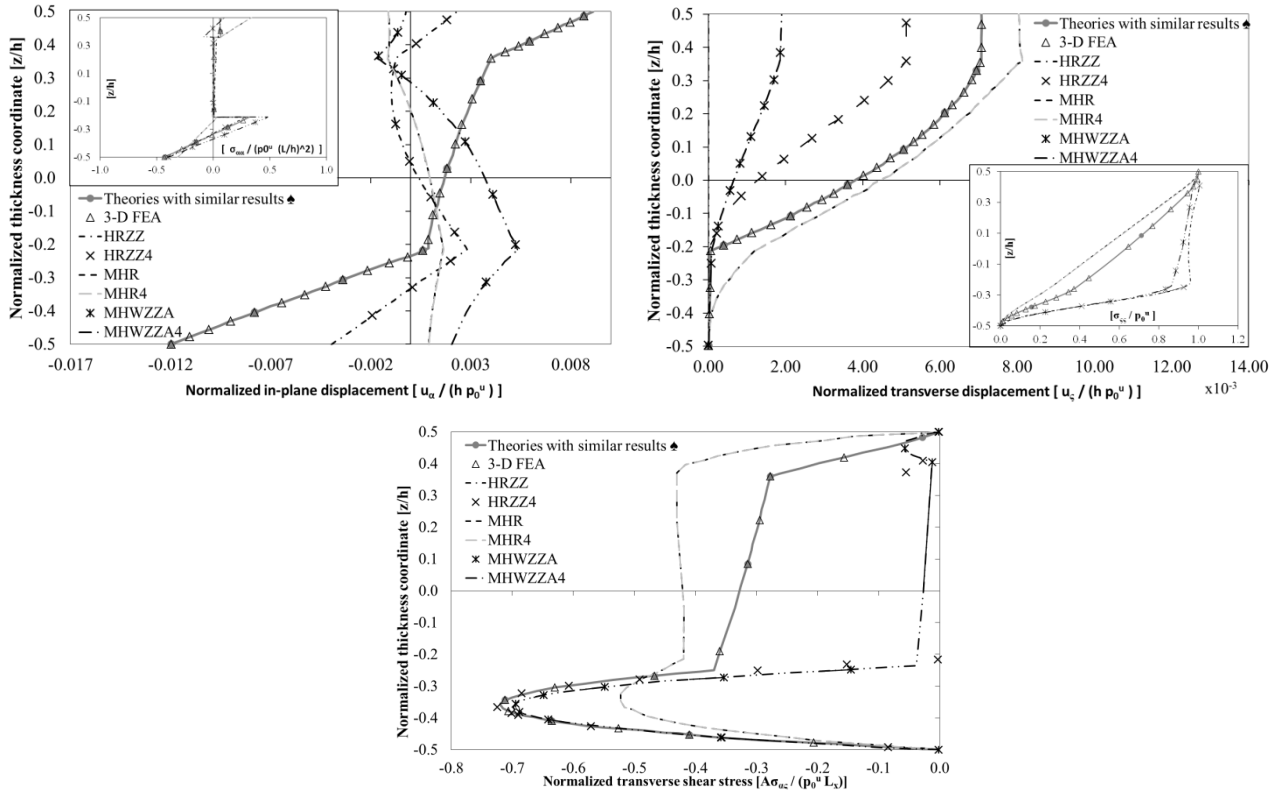


FIGURE 4: Normalized in-plane displacement, in-plane stress (in the inset), transverse displacement, transverse normal stress (in the inset), transverse shear stress for case d.

TABLE 4d
Results for case d.

Case d	Position	3-D FEA	Theories with similar results ♠	HRZZ	HRZZ4	MHR	MHR4	MHWZZA	MHWZZA4
u_α x10^{-3}	up	9.1138	9.1078	2.3007	2.2945	-1.1155	-1.1063	-0.2381	-0.2315
	down	-11.9627	-11.9506	-3.9677	-3.9760	0.9058	0.9143	2.0341	2.0360
	max	9.1138	9.1078	2.8215	2.8208	1.6311	1.6285	5.3391	5.3242
	min	-11.9627	-11.9506	-3.9677	-3.9760	-1.1393	-1.1367	-1.7104	-1.7185
u_ς x10^{-3}	up	7.0780	7.0543	0	5.0978	8.0429	8.0510	1.9002	1.9019
	max	7.0801	7.0815	0	5.1247	8.1313	8.1673	1.9002	1.9002
	down/min	0	0	0	0	0	0	0	0
$\sigma_{\alpha\alpha}$	up	0.0562	0.0554	0.0711	0.0717	0.3440	0.3453	0.1071	0.1061
	down/min	-0.4320	-0.4314	-0.3980	-0.3922	-0.3139	-0.3162	-0.4000	-0.4015
	max	0.3215	0.3252	0.2872	0.2839	0.3440	0.3453	0.4818	0.4808
$\sigma_{\alpha\varsigma}$	min	-0.7192	-0.7189	-0.7200	-0.7245	-0.5228	-0.5168	-0.6989	-0.7043
$\sigma_{\varsigma\varsigma}$	up/max	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

4.5 CASE E

Now it is considered a propped-cantilever sandwich beam under uniform loading like for case d, but actually a thicker structure with $l_x / h = 5.714$ and damaged lower face and core are considered, according to the ply-discount theory (E_{1111} E_{1122} E_{2222} E_{1212} $4 \cdot 10^{-2}$ for Layer 1, E_{1122} E_{2222} E_{1212} E_{1313} E_{2323} $2 \cdot 10^{-2}$ for Layer 2).

Given the strong discrepancies between the indications of the theories shown by the results of Fig. 5 and Table 4e, the present benchmark turns out to be decidedly challenging and therefore particularly selective and suited to highlight the discrepancies among lower- and higher-order theories. Again, only ZZA, ZZA*, HWZZM , HWZZM*, HWZZ, HSDT_34, ZZA*_43, ZZA-XX, ZZA-XX', ZZA_RDF, HWZZ_RDF, ZZA_X1 to _X4, ZZA*_43PRM and ZZA_X1* to X4* are capable to express the variation of displacement and stress fields everywhere by virtue of their better capacity to describe all the variations that occur in the displacement and stress fields, thanks to the possibility they offer of redefining the coefficients of representation in such a way as to satisfy all the requirements of the theory of elasticity.

MHR and MHR4 making use of Murakami's zig-zag functions appear decidedly inaccurate in this case, as they predict a wrong nearly uniform variation of in-plane displacements because of their too simple kinematics. Besides, all lower-order theories (MHR, MHR4, MHWZZA, MHWZZA4, HRZZ and HRZZ4) incorrectly predict the transverse shear stress and in a totally wrong way the transverse normal stress elsewhere across the thickness. Again, HRZZ (that assume a uniform u_ς across the thickness) obtain the worst results as regards the transverse displacement. Table 3 shows again that the processing time of lower-order theories remains similar to that of higher-order adaptive theories, so, the former theories which are less accurate are not useful neither from the point of view of accuracy, nor from computational burden. It is also confirmed that higher-order physically-based zig-zag adaptive theories ZZA, ZZA*, HWZZM , HWZZM*, HWZZ, HSDT_34, ZZA*_43, ZZA-XX, ZZA-XX', ZZA_RDF, HWZZ_RDF, ZZA_X1 to _X4, ZZA*_43PRM and ZZA_X1* to X4* are the only always everywhere accurate and that their results are indistinguishable from each other. Higher-order theories without zig-zag functions are shown again as the most efficient theories of this paper.

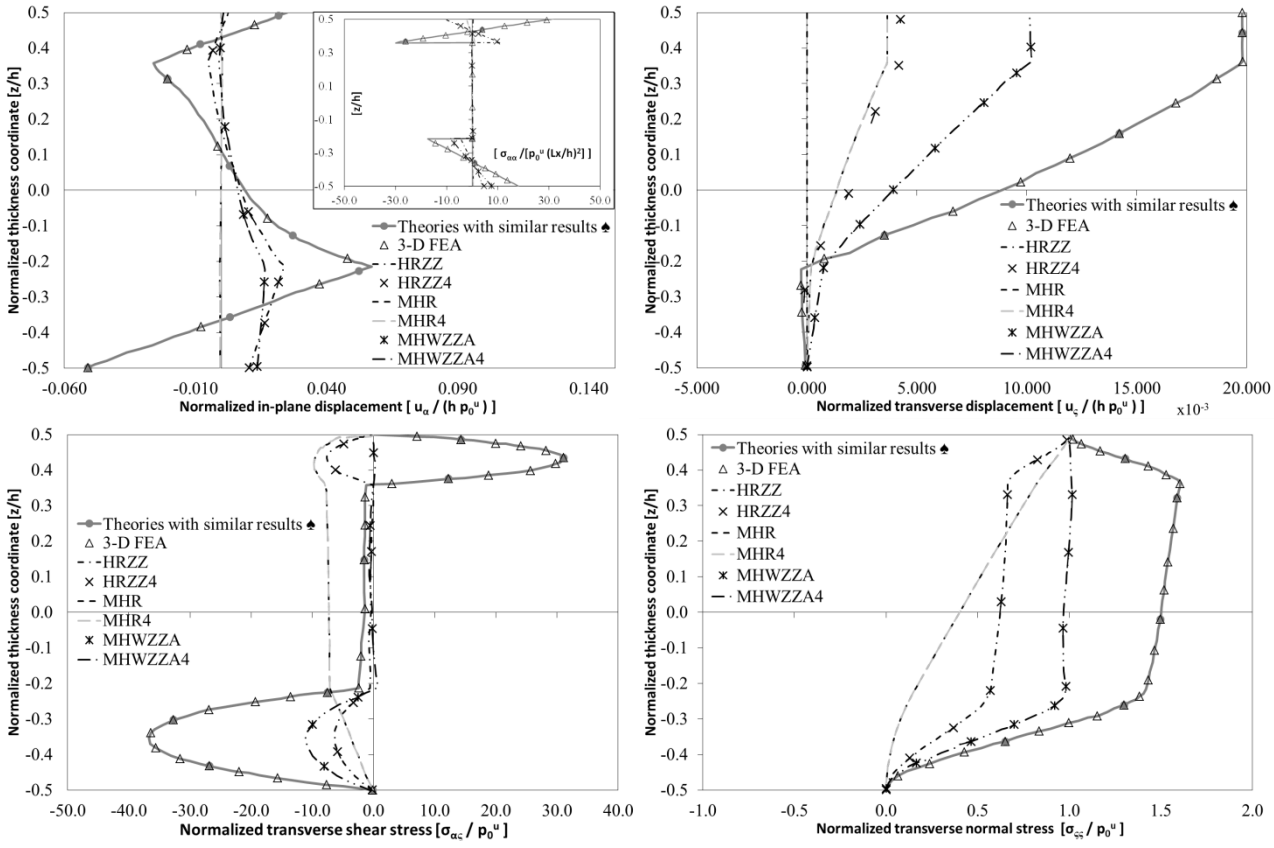


FIGURE 5: Normalized in-plane displacement and stress (in the inset), transverse displacement, transverse shear stress and transverse normal stress for case e.

TABLE 4e
Results for case e.

Case e	Position	3-D FEA	Theories with similar results ♠	HRZZ	HRZZ4	MHR	MHR4	MHWZZA	MHWZZA4
u_α x10^{-2}	up	2.4904	2.4922	0.2563	0.2542	-0.0264	-0.0262	0.0657	0.0656
	down	-5.1031	-5.1067	1.0438	1.0458	-0.0639	-0.0637	1.3600	1.3658
	max	5.7212	5.7306	2.3987	2.3922	0.0191	0.0191	1.6521	1.6626
	min	-5.1031	-5.1067	-0.5210	-0.5220	-0.0705	-0.0703	-0.1118	-0.1115
u_ς x10^{-2}	up	19.8052	19.8180	0	4.2621	3.6543	3.6636	10.1240	10.1369
	down	0	0	0	0	0	0	0	0
	max	19.8219	19.8317	0	4.2701	3.6543	3.6636	10.2232	10.2440
	min	-0.2638	-0.2628	0	-0.1643	0	0	0	0
$\sigma_{\alpha\alpha}$	up	32.0528	32.0557	-10.4441	-10.4670	-2.1000	-2.1188	0.3965	0.3927
	down	18.1345	18.1394	4.5979	4.5857	0.4217	0.4219	7.5403	7.5542
	max	32.0528	32.0557	10.8333	10.8228	1.0373	1.0368	7.5403	7.5542
	min	-29.9257	-29.9093	-10.4441	-10.4670	-2.1000	-2.1188	-8.6119	-8.6280
$\sigma_{\alpha\varsigma}$	max	31.0936	31.0955	0	0	0	0	0.7090	0.7126
	min	-36.6918	-36.6841	-7.6042	-7.6306	-9.7470	-9.7618	-11.1936	-11.1882
$\sigma_{\varsigma\varsigma}$	up	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	max	1.6056	1.5961	1.0000	1.0000	1.0000	1.0000	1.0154	1.0127

V. CONCLUDING REMARKS

Various new zig-zag theories in displacement-based or mixed form, with a different representation of variables across the thickness and differently assumed zig-zag functions, so ultimately differently accounting for layerwise effects, or retaken from previous papers by the authors have been compared. Challenging elastostatic benchmarks with strong layerwise effects, under distributed or localized loading and simply-supported and clamped edges and with distinctly different material properties and thickness of layers, mainly in form of sandwich structures have been considered.

All zig-zag theories have the same five functional degrees of freedom like FSDT and HSDT (that in the cases here examined are very inaccurate), so the number of unknowns is independent from the number of constituent layers. To compare theories under the same conditions, the same trial functions and expansion order are used to obtain solutions in closed form.

The prefixed purpose was to show on a broader series of theories and benchmarks than in the former papers by the authors that whenever the expressions of coefficients of displacements are determined a priori by enforcing the fulfillment of the full set of interfacial stress compatibility conditions, of stress boundary conditions and of local equilibrium equations at a number of selected point sufficient to determine the expressions of all coefficients, the choice of the representation form and of zig-zag functions can be arbitrary without the results changing. When all these conditions are mutually occurring, it has been shown that zig-zag functions can even be omitted, with self-evident advantages from the computational standpoint.

Higher-order zig-zag theories, whose coefficients are redefined layer-by-layer by imposing the fulfillment of interfacial displacement and stress compatibility conditions, stresses boundary conditions at upper and lower bounding faces and local equilibrium equations at different points across the thickness proved to be always those most accurate and efficient, as a computational burden still comparable to that of ESL was required for all benchmarks. In particular, it was demonstrated that zig-zag functions can be freely chosen and variables can be assumed in an arbitrary form, i.e. different form one to another and from region to region across the thickness, without the results changing. According, a specific role does not need to be assigned to individual coefficients of displacements, being sufficient that the total number of coefficients to be determined corresponds to the number of conditions to be imposed. Consequently the expansion order of displacements can be freely chosen if this condition is met and at least it cubic/quartic.

In fact theories ZZA, ZZA*, HWZZ, HWZZ_RDF, ZZA*_43, HSDT_34, HWZZM, ZZA-X1 to ZZA-X4 and ZZA-X1* to ZZA-X4* based on totally different forms of representation but satisfying the conditions mentioned above show indistinguishable results from each other and always prove to be the most accurate and efficient. The most efficient of all are

ZZA*, ZZA*_43, HSDT_34, ZZA-X1 to ZZA-X4 and ZZA-X1* to ZZA-X4* that omit the explicit presence of zig-zag functions, therefore they constitute a convenient option to much expensive 3-D finite element methods and discrete-layer models.

A partial fulfillment of above mentioned constraints implies instead that the accuracy decreases and becomes strongly depending on assumptions made. Lower-order theories HRZZ, HRZZ4, MHWZZA, MHWZZA4 and in particular ones that incorporate Murakami's zig-zag function MHR and MHR4 belong to this category. However in some cases they provide quite accurate results, but in general are rather inaccurate so, it is not possible to deduce any general rule about their usability. The only rule that can be drawn is that the higher-order zigzag theories of this paper with a redefinition of the coefficients obtained through the enforcement of the complete set of physical constraints are always accurate and efficient.

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