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On the design of controllers for hybrid linear systems with impulsive inputs and periodic jumps

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Abstract: In this paper, the problem of designing a controller for a hybrid system with impulsive input and periodic jumps is addressed. In particular, it is shown that any hybrid system with impulsive inputs and periodic jumps can be recast into a discrete-time, linear, time-invariant system, which, in turn, can be used to design a controller by using classical methods. Furthermore, it is shown that, once such a controller has been designed, it can be readily used to control the hybrid system by mean of an interfacing system that is based just on the continuous-time dynamics of the plant to be controlled. Several examples, spanning from aerospace to biomedical applications, are reported in order to corroborate the theoretical results.

1 Introduction

Several physical processes can be modeled through hybrid systems characterized by the interplay of continuous-time dynamics with discrete-time events. For instance, such systems have been proved very useful to model the absorption, distribution, metabolism, and elimination of several drugs that are periodically dosed either through intravenous or oral administration [1–5]. Furthermore, these systems have gained a lot of attention also in aerospace [6], industrial [7], and several other applications. Therefore, it is not surprising that a lot of research effort has been spent to characterize the properties of these systems [8–15]. Two remarkable examples are [16], where reachability and observability of linear impulsive systems are comprehensively characterized through invariant subspaces, and [17], where the structural properties of linear periodic hybrid systems are framed in terms of algebraic and geometric conditions on their data.

The main objective of this paper is to provide a framework that allows to easily design controllers for hybrid systems with periodic jumps that are controlled through impulsive, discrete-time inputs (in the following, referred to as *hybrid system with impulsive inputs*). By allowing nontrivial discrete-time dynamics, such systems constitute a generalization of linear plants that are controlled through periodic, discrete-time inputs (usually referred to as *impulsive systems*), which recently gained a lot of attention [18–21], especially in the context of pharmacokinetics models [2–5, 22, 23].

Differently from [21, 24], where model predictive control (briefly, MPC) techniques are proposed to design the input of an impulsive system, in this paper, it is shown that a controller for a hybrid system with impulsive inputs can be designed by using any tool that allows the design of controllers for discrete-time systems. In particular, it is shown that any hybrid system with impulsive inputs can be recast into a discrete-time, linear, time invariant system, which, in turn, can be used to design a controller for the original system. Once such a discrete-time controller has been designed, it can be readily used to control the hybrid system by using an interfacing plant based on the observability Gramian of the continuous-time dynamics. In such a way, both classical and novel (such as [25, 26]) discrete-time control tools can be used to design a controller for the hybrid system.

The main advantage of the tools given in this paper with respect to the ones given in the literature (as, *e.g.*, the MPC approaches given in [21, 24]) is that they allow the design of a controller for the hybrid system just on the basis of the transfer function of the discrete-time equivalent system, which, in turn, can be obtained directly from the transfer function of the hybrid system. In particular, this allows to directly use frequency-domain methods to design controllers for the hybrid system (see the subsequent Subsections 4.2 and 4.5).

The remainder of the paper is organized as follows: in Section 2, the class of hybrid systems considered in this paper is introduced and some preliminary results are given. In Section 3, it is shown that any hybrid system with impulsive inputs can be recast into a discrete-time plant. Such a system can be used to design a controller, which, in turn, can be readily used to control the hybrid plant by mean of an interfacing system. In Section 4, several examples of application of the proposed technique are given to corroborate the theoretical results. Conclusions are given in Section 5.

2 Notation and preliminary results

In this section, the notation used in this paper is introduced (Subsection 2.1) and some preliminary results about hybrid system with impulsive inputs are given (Subsection 2.2).

2.1 Notation

Let $\mathbb{R}$, $\mathbb{R}_{\geqslant 0}$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{C}$ be the set of real, nonnegative real, integer, natural and complex numbers, respectively. Let $\mathbb{C}_g := \{s \in \mathbb{C} : |s| < 1\}$. The symbol I denotes the identity matrix of suitable dimensions. Letting $A \in \mathbb{R}^{n\times n}$, the symbol $\Lambda(A)$ denotes the spectrum of A. A set $\mathcal{T} \subset \mathbb{R}_{\geqslant 0} \times \mathbb{N}$ is a *hybrid time domain* if, for each $(\tau, \kappa) \in \mathcal{T}$, the set $\mathcal{T} \cap [0, \tau] \times \{0, 1, \dots, \kappa\}$ equals $\bigcup_{k=0}^{\kappa} \mathfrak{I}_k \times \{k\}$, where $\mathfrak{I}_k = [t_k, t_{k+1}]$, $k = 0, \dots, \kappa$, and $t_0 \leqslant t_1 \leqslant \cdots \leqslant t_\kappa \leqslant t_{\kappa+1} = \tau$. A function $\xi : \mathcal{T} \to \mathbb{R}^s$ is a *hybrid arc* if $\mathcal{T}$ is an hybrid time domain and for each fixed $k \in \mathbb{N}$ the map $t \mapsto u(t, k)$ is locally absolutely continuous on $\mathfrak{I}_k := \{t : (t, k) \in \mathcal{T}\}$.

2.2 *Hybrid linear systems with impulsive inputs*

The objective of this subsection consists in defining the class of hybrid systems considered in this paper, namely linear hybrid systems with periodic jumps and discrete-time, impulsive inputs. Towards this end, let $\tau_M \in \mathbb{R}_{>0}$ be given and consider the hybrid time domain

$$\mathcal{T} = \bigcup_{k=0}^{\infty} [k\,\tau_M, (k+1)\,\tau_M] \times \{k\}, \tag{1}$$

together with the linear time invariant (briefly, LTI) hybrid system

$$\dot{x} = A\,x, \tag{2a}$$

$$\overset{+}{x} = E\,x + F\,v, \tag{2b}$$

where $x(t,k) \in \mathbb{R}^n$ denotes the state of the system, $v(k) \in \mathbb{R}^m$ denote the impulsive control inputs, $(t,k) \in \mathcal{T}$, $A \in \mathbb{R}^{n\times n}$, $E \in \mathbb{R}^{n\times n}$ and $F \in \mathbb{R}^{n\times p}$ are matrices of real elements. Letting

$$t_k = k\,\tau_M,\ k \in \mathbb{N},$$

denote the jump times, given an initial condition $x_0 \in \mathbb{R}$ and a function $v : \mathbb{N} \to \mathbb{R}^\mu$, a hybrid arc $x : \mathcal{T} \to \mathbb{R}^n$ is a *solution to system* (2) *with input* v *starting at* x_0 if

1. $x(0,0) = x_0$;
2. for almost all $t \in [t_k, t_{k+1}]$ and all $k \in \mathbb{N}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t,k) = A\,x(t,k);$$

3. $x(t_{k+1}, k+1) = E\,x(t_{k+1}, k) + F\,v(k), \forall k \in \mathbb{N}$.

System (2) is *globally exponentially stable with* $v = 0$ if there exist $c_1, c_2 \in \mathbb{R}_{>0}$ such that $|x(t,k)| < c_1 e^{-c_2 t}|x_0|$, $\forall t \in \mathbb{R}_{>0}$, $\forall k \in \mathbb{N}$. The next statement gives necessary and sufficient condition for exponential stability of system (2) with $v = 0$.

Proposition 1 ([27]). *System* (2) *is globally exponentially stable if and only if* $\Lambda(\tilde{E}) \subset \mathbb{C}_g$, *where* $\tilde{E} = E\exp(A\,\tau_M)$ *is the* monodromy *matrix of system* (2).

Proposition 1 follows by the observation that, if $v = 0$, then the solutions to the hybrid system (2) are given by $x(t,k) = e^{A(t-t_k)}\,\tilde{E}^{k-1}\,x_0$ for all $(t,k) \in \mathcal{T}$, thus implying that $x(t_{k+1}, k+1) = \tilde{E}^k\,x_0$ for all $k \in \mathbb{N}$. Thus, since the eigenvalues of $\tilde{E}$ are continuous with respect to the entries of the matrices A and E and with respect to the period τ_M, global exponential stability of system (2) is robust with respect to small perturbations of these data (an equivalent proof can be obtained by using the robustness analysis given in [11] by using Lyapunov functions).

In [28, 29], the notion of transfer function for LTI hybrid systems has been introduced. Namely, letting the output of system (2) be

$$y(t,k) = C\,x(t,k), \tag{3}$$

with $C \in \mathbb{R}^{m\times n}$, since there is no continuous-time input, the transfer function of system (2) is given by

$$\boldsymbol{W}(\ell,z) = C(\ell\,I - A)^{-1}(z\,I - \tilde{E})^{-1}F. \tag{4}$$

Such a function can be used to compute the output response of system (2), (3). Namely, let the symbols $\boldsymbol{f}(\ell) = \mathcal{L}\left\{f(t)\right\}_{t\to\ell}$ and $\boldsymbol{g}(z) = \mathcal{Z}\left\{g(k)\right\}_{k\to z}$ denote the *Laplace transform* of $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ (with inverse denoted $\mathcal{L}^{-1}\{\boldsymbol{f}(\ell)\}_{\ell\to t}$) and the *Z transform* of $g : \mathbb{N} \to \mathbb{R}$ (with inverse denoted $\mathcal{Z}^{-1}\{\boldsymbol{g}(z)\}_{z\to k}$), respectively [30]. Thus, given an hybrid arc $\xi : \mathcal{T} \to \mathbb{R}^s$, define its hybrid transform

$$\begin{aligned}\boldsymbol{\xi}(\ell,z) &= \mathcal{H}\{\xi(t,k)\}_{t\to\ell,k\to z} \\ &= \mathcal{L}\left\{\mathcal{Z}\left\{\xi(\sigma + k\tau_M, k)\right\}_{k\to z}\right\}_{\sigma\to\ell},\end{aligned}$$

with inverse denoted $\mathcal{H}^{-1}\{\boldsymbol{\xi}(s,z)\}_{\ell\to t,z\to k}$. Hence, by [31, Thm. 1], the output response of system (2) to the input $v : \mathbb{N} \to \mathbb{R}^p$ with initial condition $x_0 = 0$ is given by

$$y(t,k) = \mathcal{H}^{-1}\{\boldsymbol{W}(\ell,z)\boldsymbol{v}(z)\}_{\ell\to t,z\to k},$$

where $\boldsymbol{v}(z) = \mathcal{Z}\left\{v(k)\right\}_{k\to z}$.

In fact, by [17, Thm. 1], the output response of system (2) to the input $v : \mathbb{N} \to \mathbb{R}^p$ with initial condition $x_0 = 0$ is given by

$$y(t,k) = C\,e^{A(t-k\tau_M)}\sum_{j=0}^{k-1}\tilde{E}^{k-1-j}Fv(k),$$

by defining $\boldsymbol{W}(\ell,z)$ as in (4), one has that

$$\begin{aligned}&\mathcal{H}^{-1}\{\boldsymbol{W}(\ell,z)\boldsymbol{v}(z)\}_{\ell\to t,z\to k} \\ &= C\mathcal{L}^{-1}\{(\ell I - A)^{-1}\}_{\ell\to t-k\tau_M}\mathcal{Z}^{-1}\{(zI - \tilde{E})^{-1}F\boldsymbol{v}(z)\}_{z\to k} \\ &= C\,e^{A(t-k\tau_M)}\sum_{j=0}^{k-1}\tilde{E}^{k-1-j}Fv(k).\end{aligned}$$

A complex number $z_0 \in \mathbb{C}$ is a z-pole of $\boldsymbol{W}(\ell,z)$ if

$$\lim_{z\to z_0}\boldsymbol{W}(\ell,z) = \infty,$$

for any $\ell \in \mathbb{C}$.

By [31, Prop. 4], as for classical non-hybrid LTI systems [32], the z-poles of $\boldsymbol{W}(\ell,z)$ correspond to the monodromy modes that are strongly reachable and observable. Therefore, consider the following assumption.

Assumption 1. System (2), (3) is strongly reachable and observable.

Assumption 1 can be easily verified through PBH-like tests. In fact, by [17], system (2), (3) is (strongly) reachable if and only if

$$\operatorname{rank}([\ \tilde{E} - \lambda\,I \quad F\ ]) = n, \quad \forall\lambda \in \Lambda(\tilde{E}), \tag{5a}$$

whereas system (2), (3) is observable if and only if

$$\operatorname{rank}\left(\begin{bmatrix}\tilde{E} - \lambda\,I \\ O(C,A)\end{bmatrix}\right) = n, \quad \forall\lambda \in \Lambda(\tilde{E}), \tag{5b}$$

where $O(C,A)$ is the observability matrix of the pair (C,A),

$$O(C,A) = \begin{bmatrix} C \\ \vdots \\ C\,A^{n-1}\end{bmatrix}.$$

It is worth noticing that (5b) need not imply that the pair (C,A) is observable since $O(C,A)$ need not have full rank.

Under Assumption 1, global exponential stability of the system can be easily verified by inspecting the transfer function of the system, as detailed in the following lemma.

Lemma 1. *Let Assumption* 1 *hold. Then, system* (2), (3) *is globally exponentially stable if and only if the z-poles of* $\boldsymbol{W}(\ell,z)$ *are in* $\mathbb{C}_g$.

Proof: By [31, Prop. 4], the z-poles of $\boldsymbol{W}(\ell, z)$ are the eigenvalues of $\tilde{E}$ that correspond to strongly reachable and observable modes. Therefore, if system (2), (3) is strongly reachable and observable, then the z-poles of $\boldsymbol{W}(\ell, z)$ are the eigenvalues of the matrix $\tilde{E}$. Thus, the statement follows by Proposition 1. $\square$

Note that the transfer function $\boldsymbol{W}(\ell, z)$ can be rewritten as

$$\boldsymbol{W}(\ell, z) = \frac{1}{\det(\ell\, I - A)\, \det(z\, I - \tilde{E})}\, \boldsymbol{U}(\ell, z),$$

where

$$\boldsymbol{U}(\ell, z) = C \operatorname{adj}(\ell\, I - A) \operatorname{adj}(z\, I - \tilde{E})\, F$$

is a polynomial matrix in $\mathbb{R}^{m\times p}[\ell, z]$, *i.e.*, the lowest common denominator $\boldsymbol{d}(\ell, z)$ of the (rational) entries of $\boldsymbol{W}(\ell, z)$ can be factorized as $\boldsymbol{d}(\ell, z) = \boldsymbol{d}_1(\ell)\, \boldsymbol{d}_2(z)$. Therefore, under Assumption 1 and in view of Lemma 1, system (2), (3) is globally exponentially stable if and only if all the roots of $\boldsymbol{d}_2(z) \in \mathbb{R}[z]$ are in $\mathbb{C}_g$. Such a condition can be easily verified by using the Jury criterion [33].

The following lemma shows that if system (2) is globally exponentially stable, then its output response asymptotically tends to the one corresponding to $x_0 = 0$, thus showing that, if system (2) is globally exponentially stable, then the function $\boldsymbol{W}(\ell, z)$ can be used to characterize its steady state response.

Lemma 2. *Assume that system* (2) *is globally exponentially stable. Thus, for each $\delta \in \mathbb{R}_{>0}$ and $x_0 \in \mathbb{R}^n$, there is $\bar{k} \in \mathbb{N}$ such that, for all $(t, k) \in \mathcal{T}$ such that $k \geqslant \bar{k}$,*

$$\left| y(t, k) - \mathcal{H}^{-1}\{\boldsymbol{W}(\ell, z)\boldsymbol{v}(z)\}_{\ell\to t, z\to k} \right| \leqslant \delta.$$

Proof: By [31], the output response of system (2), (3) is given by

$$y(t, k) = C\, e^{A\,(t-t_k)}\, \tilde{E}^k\, x_0 + C\, e^{A\,(t-t_k)} \sum_{j=0}^{k-1} \tilde{E}^{k-1-j}\, F\, v(k).$$

Since $(t - t_k) \in [0, \tau_M]$, there exists $M \in \mathbb{R}_{>0}$ such that $|e^{A\,(t-t_k)}| \leqslant M$ for all $(t, k) \in \mathcal{T}$. Furthermore, if system (2) is globally exponentially stable, then, by Proposition 1, $\Lambda(\tilde{E}) \subset \mathbb{C}_g$ and hence there exists $\varrho \in (0, 1)$ such that $|\tilde{E}| \leqslant \varrho$. Hence, the following inequality hold

$$\begin{aligned} &\left| y(t, k) - C\, e^{A\,(t-t_k)} \textstyle\sum_{j=0}^{k-1} \tilde{E}^{k-1-j}\, F\, v(k) \right| \\ &\quad = \left| y(t, k) - \mathcal{H}^{-1}\{\boldsymbol{W}(\ell, z)\boldsymbol{v}(z)\}_{\ell\to t, z\to k} \right| \\ &\quad\quad = |C\, e^{A\,(t-t_k)}\, \tilde{E}^k\, x_0| \leqslant |C|\, |x_0|\, \varrho^k, \end{aligned}$$

thus implying the statement. $\square$

3 Synthesis of a controller for hybrid LTI system through a discrete-time equivalent system

The main objective of this section is to show that the hybrid system (2) can be recast into a discrete-time LTI system, which, in turn, can be used to design a controller for system (2), (3). Toward this goal, consider the system

$$\overset{+}{\xi} = \tilde{E}\, \xi + F\, v, \tag{6a}$$

$$\psi = O(C, A)\, \xi, \tag{6b}$$

together with its transfer function

$$\boldsymbol{H}(z) = O(C, A)\, (z\, I - \tilde{E})^{-1}\, F, \tag{7}$$

and let $\xi_0 \in \mathbb{R}^n$ be its initial condition.

Remark 1. Note that the discrete-time system (7) differs from the one that can be obtained by considering discrete samples $\tilde{\psi}(k)$ of the output $y(t, k)$ at post-jump times, $\tilde{\psi}(k) = y(t_k, k)$, that is

$$\overset{+}{\xi} = \tilde{E}\, \xi + F\, v, \tag{8a}$$

$$\tilde{\psi} = C\, \xi. \tag{8b}$$

In fact, in view of (5b), system (7) is observable if and only if the hybrid system (2), (3) is observable, whereas, by classical properties of discrete-time linear plants, system (8) is observable if and only if

$$\operatorname{rank}\left(\begin{bmatrix} \tilde{E} - \lambda\, I \\ C \end{bmatrix} \right) = n, \quad \forall \lambda \in \Lambda(\tilde{E}),$$

that is a condition more restrictive than (5b). In particular, there may exist some unstable monodromy modes (*i.e.*, eigenvalues of $\tilde{E}$ not in $\mathbb{C}_g$) that are not observable through the matrix C, but that are observable through $O(C, A)$, as shown in the following example.

Example 1. Consider the hybrid system (2), (3) with data $\tau_M = 1$,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad F = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$E = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & -1 \end{bmatrix}.$$

By letting $\tilde{E} = E\, e^{A\,\tau_M}$, it can be easily verified that system (8) is not detectable and hence it is not possible to design a linear output feedback controller that stabilizes it for all its initial conditions, and hence it is not possible to design a feedback controller that stabilizes the hybrid system (2), (3) by just using post-jump samples of its output. On the other hand, it can be easily verified that system (6) is observable and hence it is possible to design an output feedback controller that stabilizes it.

The main objective of this section is to show that the output response of the discrete-time system (6) comprehensively captures the hybrid dynamics of system (2), (3) and that if one designs a controller for the discrete-time system (6), it can be readily applied to the hybrid system (2), (3) by means of an interfacing system. Toward this end, consider the following proposition, which shows that the output $y(t, k)$ of the hybrid LTI system (2), (3) can be reconstructed form the output of the discrete-time LTI system (6).

Proposition 2. *Assume that $x_0 = \xi_0$. Let $y(t, k)$ and $\psi(k)$ be the output responses of systems* (2), (3) *and* (6) *to the input $v : \mathbb{N} \to \mathbb{R}^p$, respectively. Then, for each $\sigma \in [0, \tau_M]$, there exists $\Gamma : [0, \tau_M] \to \mathbb{R}^{m\times(m\,n)}$ such that, $\forall k \in \mathbb{N}$,*

$$y(t_k + \sigma, k) = \Gamma(\sigma)\, \psi(k).$$

Proof: Note that the output response of system (2), (3) is given by

$$y(t, k) = C\, e^{A\,(t-t_k)}\, x(t_k, k) = \sum_{i=0}^{\infty} \frac{1}{i!} C\, A^i (t - t_k)^i\, x(t_k, k),$$

for all $(t, k) \in \mathcal{T}$. By the by the Cayley-Hamilton theorem [34], there exist $\alpha_0, \ldots, \alpha_{n-1} \in \mathbb{R}$ such that $A^n = \sum_{j=0}^{n-1} \alpha_i\, A^i$. Therefore, for each $i \in \mathbb{N}$, there exist $\beta_{i,0}, \ldots, \beta_{i,n-1} \in \mathbb{R}$ such that

$$C\, A^i = \sum_{j=0}^{n-1} \beta_{i,j} C\, A^j. \tag{9}$$

Note that, letting $\bar{\alpha} = \max\{|\alpha_0|, \ldots, |\alpha_{n-1}|\}$, by the Cayley-Hamilton theorem, one has that $|\beta_{i,j}| \leqslant \bar{\alpha}^i$. Hence, one has $y(t_k + \sigma, k) = \sum_{j=0}^{n-1} \sum_{i=0}^{\infty} \frac{\beta_{i,j}\, \sigma^i}{i!}\, C\, A^j\, x(t_k, k)$, for each $\sigma \in [0, \tau_M]$.

Thus, consider the series $\sum_{i=0}^{\infty} \frac{\beta_{i,j}\,\sigma^i}{i!}$. Since $|\beta_{i,j}| \leqslant \bar{\alpha}^I$, it results that $\sum_{i=0}^{\infty} \frac{|\beta_{i,j}|\,\sigma^i}{i!} \leqslant \sum_{i=0}^{\infty} \frac{\bar{\alpha}^i\,\sigma^i}{i!} = e^{\bar{\alpha}\,\sigma}$, *i.e.*, the considered series is convergent. Thus, letting $\gamma_j = \sum_{i=0}^{\infty} \frac{\beta_{i,j}\,\sigma^i}{i!}$, we have that

$$y(t_k+\sigma,k) = \sum_{j=0}^{n-1} \gamma_j(\sigma)\, C\, A^j\, x(t_k,k).$$

The proof is concluded by that fact that, by construction, if $x_0 = \xi_0$, then the monodromy state response of system (2) matches the discrete-time state response of system (6). □

By Proposition 2, the output response of the hybrid system (2), (3) can be determined from the one of the discrete-time system (6), by using a function $\Gamma : [0,\tau_M] \to \mathbb{R}^{m\times(m\,n)}$ that depends on the characteristic polynomial of the continuous-time dynamical matrix A. The following two remarks provide some additional details on the correspondence established in Proposition 2.

Remark 2. By the construction carried out in the proof of Proposition 2, the output $y(t,t_k)$ of system (2), (3) vanishes identically for all $t \in [t_k, t_{k+1}]$ if and only if the output of system (6) vanishes at k. In fact, $y(t,k) = 0$ for all $t \in [t_k, t_{k+1}]$ if and only if $C\,A^i\,x(t_k,k) = 0$, $i \in \{0,\ldots,n-1\}$, since this implies, by the Cayley-Hamilton theorem [34], that $C\,A^i\,x(t_k,k) = 0$, $i \in \mathbb{N}$, and hence that $y(t,k) = \sum_{i=0}^{\infty} C\,A^i\,x(t_k,k)\,\frac{(t-t_k)^i}{i!} = 0$.

Remark 3. In view of (5b), system (2), (3) is observable if and only if system (6) is observable. Therefore, since $x(t_{k+1},k+1) = \tilde{E}\,x(t_k,k) + F\,v(k)$ (*i.e.*, the monodromy dynamics of $x(t_k,k)$ are the same as the ones of $\xi(k)$ given in (6a)), it is not surprising that the amount of "information" that can be gathered about the state $x(t,k)$ from the continuous-time history of $y(t,k)$ is essentially the same as the one that can be gathered from $\psi(k)$.

It is worth noticing that the transfer function $\boldsymbol{H}(z)$ can be determined directly by using the hybrid transfer function $\boldsymbol{W}(\ell,z)$. In fact, by the initial value theorem [30], one has

$$\begin{aligned} C(zI-\tilde{E})^{-1}F &= \lim_{\sigma\to 0} Ce^{A\,\sigma}(zI-\tilde{E})^{-1}F \\ &= \lim_{\ell\to\infty} \ell\,\boldsymbol{W}(\ell,z). \end{aligned} \tag{10a}$$

Similarly, it results that, for each $i \in \mathbb{N}$,

$$\begin{aligned} C\,A^i(zI-\tilde{E})^{-1}F &= \lim_{\sigma\to 0} C\,\frac{d^i e^{A\sigma}}{d\sigma^i}\,(zI-\tilde{E})^{-1}F \\ &= \lim_{\ell\to\infty} \ell\left(\ell^i\,\boldsymbol{W}(\ell,z) - \sum_{j=1}^{i} \ell^{i-j}\,CA^{j-1}(zI-\tilde{E})^{-1}F\right). \end{aligned} \tag{10b}$$

This shows that the transfer function $\boldsymbol{H}(z)$ can be obtained even in the case that one does not have a state-space description of the hybrid system, but just its transfer function $\boldsymbol{W}(\ell,z)$. On the other hand, if one has a state space description of the hybrid system, then (7) is more direct than (10) to compute $\boldsymbol{H}(z)$.

The following lemma entails with the global exponential stability of system (2) in terms of the poles of the transfer function $H(z)$.

Lemma 3. *Let Assumption* 1 *hold. Then system* (2)*,* (3) *is globally exponentially stable if and only if the poles of* $\boldsymbol{H}(z)$ *are in* $\mathbb{C}_g$*.*

Proof: If system (2), (3) is strongly reachable and observable, then, by (5), system (6) is reachable and observable. Therefore, by classical results about LTI systems [32], the eigenvalues of $\tilde{E}$ are the poles of $H(z)$. Thus, the statement follows by Proposition 1. □

The main objective of the remainder of this section, is to show that by designing a controller for the discrete-time system (6) on the basis of the knowledge of $\boldsymbol{H}(z)$, it is possible to design a hybrid controller for system (2), (3). Toward this end, let

$$G(t) = \int_0^t e^{A^\top\theta}\,C^\top C\,e^{A\,\theta}\,\mathrm{d}\theta$$

be the *observability Gramian* of the continuous-time system (2a), (3) and consider the following lemma

Lemma 4. *For each* $t \in \mathbb{R}_{\geqslant 0}$*, there exists* $M(t)$ *such that* $M(t)\,G(t) = O(C,A)$*.*

Proof: By classical linear algebra results [32], one has that $\ker(G(t)) = \ker(O(C,A))$, for all $t \in \mathbb{R}_{>0}$. Therefore, by the fundamental theorem of algebra [32], it results that $(\mathrm{Im}(G^\top(t)))^\perp = (\mathrm{Im}(O^\top(C,A)))^\perp$, *i.e.*, $\mathrm{Im}(G^\top(t)) = \mathrm{Im}(O^\top(C,A))$, for all $t \in \mathbb{R}_{>0}$. Therefore, for each $t \in \mathbb{R}_{>0}$, there exists $M(t)$ such that $G^\top(t)\,M^\top(t) = O^\top(C,A)$, as to be proved. □

By using Lemma 4, let N be such that $N\,G(\tau_M) = O(C,A)$, and consider the hybrid system (whose solution are defined over the hybrid time domain $\mathcal{T}$)

$$\dot{\zeta} = -A^\top\zeta + C^\top y, \tag{11a}$$

$$\overset{+}{\zeta} = 0, \tag{11b}$$

with discrete-time output given by

$$\phi(k) = N\,e^{A^\top\tau_M}\,\zeta(t_{k+1},k). \tag{11c}$$

The following theorem establishes that the output response of the interconnection of systems (2), (3) and (11), with the discrete-time output (11c), matches the one of system (6).

Theorem 1. *Assume that* $\xi_0 = x_0$ *and consider the series interconnection of systems* (2)*,* (3) *and* (11)*. Thus, the (discrete-time) output response* $\phi(k)$ *of the interconnection of systems* (2)*,* (3)*,* (11)*, is the same as the one of system* (6)*.*

Proof: By considering that $y(t,k) = C\,e^{A\,(t-t_k)}x(t_k,k)$ for all $k \in \mathbb{N}$, it results that

$$\begin{aligned} \zeta(t,k) &= \left(\int_0^{t-t_k} e^{-A^\top(t-t_k-\theta)}C^\top C\,e^{A\,\theta}\,\mathrm{d}\theta\right)x(t_k,k) \\ &= e^{-A^\top(t-t_k)}\left(\int_0^{t-t_k} e^{A^\top\theta}C^\top C\,e^{A\,\theta}\,\mathrm{d}\theta\right)x(t_k,k) \\ &= e^{-A^\top(t-t_k)}\,G(t-t_k)\,x(t_k,k). \end{aligned}$$

Therefore, for all $k \in \mathbb{N}$, one has that

$$\zeta(t_{k+1},k) = e^{-A^\top\tau_M}\,G(\tau_M)\,x(t_k,k),$$

and hence $\phi(k) = N\,G(\tau_M)\,x(t_k,k) = O(C,A)\,x(t_k,k)$. The proof is concluded by the fact that the monodromy dynamics of $x(t_k,k)$ satisfies $x(t_{k+1},k+1) = \tilde{E}\,x(t_k,k) + F\,v(k)$, whence, if $x_0 = \xi_0$, then $x(t_k,k) = \xi(k)$. □

The following proposition characterizes the structural properties of the interconnection of systems (2), (3) and (11).

Proposition 3. *Let Assumption* 1 *hold. Then, the series interconnection of systems* (2)*,* (3) *and* (11) *is controllable and constructible.*

Proof: By Theorem 1, the hybrid dynamics of the series interconnection of systems (2), (3) and (11) are given by

$$\dot{x} = \begin{bmatrix} A & 0 \\ C^\top C & -A^\top \end{bmatrix} x, \tag{12a}$$

$$\overset{+}{x} = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} F \\ 0 \end{bmatrix} v, \tag{12b}$$

$$\phi = \begin{bmatrix} O(C,A) & 0 \end{bmatrix} x. \tag{12c}$$

Therefore, since

$$\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \exp\left(\begin{bmatrix} A & 0 \\ C^\top C & -A^\top \end{bmatrix} \tau_M \right) = \begin{bmatrix} \tilde{E} & 0 \\ 0 & 0 \end{bmatrix},$$

and, by Assumption 1 and (5),

$$\operatorname{rank}\left(\begin{bmatrix} \tilde{E} - \lambda I & 0 & F \\ 0 & -\lambda I & 0 \end{bmatrix} \right) = 2n,$$

$$\operatorname{rank}\left(\begin{bmatrix} \tilde{E} - \lambda I & 0 \\ 0 & -\lambda I \\ O(C,A) & 0 \end{bmatrix} \right) = 2n,$$

for all $\lambda \in \Lambda(\tilde{E}) \setminus \{0\}$, by [17], the series interconnection of systems (2), (3) and (11) is controllable and constructible. □

By Proposition 3 and [31, Prop. 4], a result wholly similar to Lemma 3 can be easily proved for the series interconnection of systems (2), (3) and (11).

Corollary 1. *Let Assumption* 1 *hold. Then the series interconnection of systems* (2)*,* (3) *and* (11) *is globally exponentially stable if and only if the poles of* $\boldsymbol{H}(z)$ *are in* $\mathbb{C}_g$.

By using the results given in this section, it can be easily shown that the problem of design of a hybrid controller for system (2), (3) is essentially equivalent to the problem of designing a discrete-time controller for system (6). In fact, by Proposition 2 and Lemma 3, the discrete-time system (6) comprehensively captures the hybrid dynamics of system (2), (3). Therefore, when addressing the problem of designing a controller for the hybrid system (2), it seems rather more convenient to design a controller for the discrete-time system (6). Thus, in view of Theorem 1, once such a controller has been designed, it can be readily used to control the hybrid system (2), (3) by interfacing it through system (11). In particular, given $\boldsymbol{H}(z)$, let $\boldsymbol{Q}(z)$ and $\boldsymbol{G}(z)$ be designed so that the closed loop system Σ_D depicted in Figure 1 is asymptotically stable. Such a property can be easily verified, for example, by using the Nyquist criterion for multi-input multi-output systems [35–37].

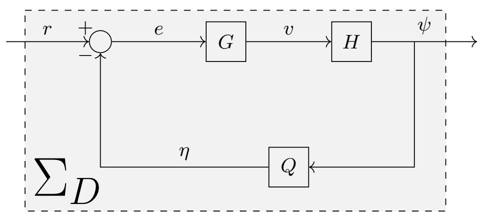

Fig. 1: A discrete-time control scheme.

Thus, in view of the correspondence established in Theorem 1, the closed loop system Σ_H depicted in Figure 2 is asymptotically stable. In particular, the response in the ψ variables of system Σ_D matches with the response in the ϕ variables of system Σ_H.

It is worth noticing that, if the closed loop system Σ_H is asymptotically stable, then, by Lemma 2, the steady state output response of the hybrid system (2) is not affected by its initial condition. Therefore, under Assumption 1, the synthesis of the controllers $\boldsymbol{Q}(z)$

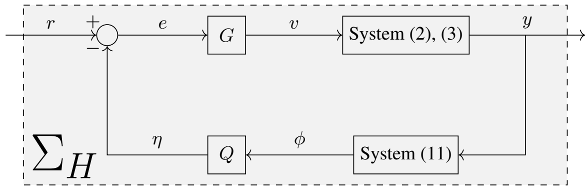

Fig. 2: An hybrid control scheme.

and $\boldsymbol{G}(z)$ can be carried out by using directly the transfer function $\boldsymbol{H}(z)$, which, in turn, can be determined directly from the hybrid transfer function $\boldsymbol{W}(\ell, z)$ by using the expressions given in (10).

In the following section, the effectiveness of the control scheme depicted in Figure 2 is highlighted through several examples.

4 Examples of applications

In this section, the results established in Section 3 are used to design controllers for several hybrid systems with impulsive inputs modeling physical processes of practical interest. In particular, it is shown that, by mean of the interfacing system (11), it is possible to design a controller for the hybrid system (2), (3) by using any strategy that allows the design of controllers for discrete-time systems.

4.1 A mass-spring mechanical system

Consider the system depicted in Figure 3, which consists of a body having mass $\mathcal{M} = 0.2\,\mathrm{Kg}$ and a spring having stiffness $\mathcal{K} = 1\,\mathrm{KN/m}$, which is periodically forced (with period $1\,\mathrm{s}$) by an impulsive force that instantaneously changes the speed of the body.

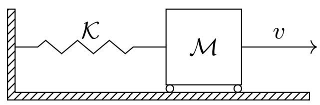

Fig. 3: A simple mechanical system.

Letting x_1 be the position of the body having mass $\mathcal{M}$, letting x_2 be its speed, and assuming that the only available measure is the speed of the body having mass $\mathcal{M}$, the dynamics of such a system can be modeled through the hybrid system (2) with

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{\mathcal{K}}{\mathcal{M}} & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}. \tag{13}$$

By using (4), one obtains that the transfer function of the hybrid system (13) is given by

$$\boldsymbol{W}(\ell, z) = \frac{z\ell - \ell\cos\left(50\sqrt{2}\right) - 50\sqrt{2}\sin\left(50\sqrt{2}\right)}{\left(\ell^2 + 5000\right)\left(z^2 - 2z\cos\left(50\sqrt{2}\right) + 1\right)},$$

whereas, by using (7), the transfer function of the discrete-time equivalent system is given by

$$\boldsymbol{H}(z) = \begin{bmatrix} \frac{z - \cos\left(50\sqrt{2}\right)}{z^2 - 2\cos\left(50\sqrt{2}\right)z + 1} \\ -\frac{50\sqrt{2}\sin\left(50\sqrt{2}\right)}{z^2 - 2\cos\left(50\sqrt{2}\right)z + 1} \end{bmatrix}.$$

By letting $\boldsymbol{G} = 1$ and $\boldsymbol{Q} = \begin{bmatrix} 0.549682 & -0.0130692 \end{bmatrix}$, it can be easily verified that the discrete-time closed-loop system Σ_D depicted in Figure 1 is asymptotically stable. Therefore, by the results given in Section 3, the hybrid control scheme given in Figure 2 is such that the closed loop system Σ_H is asymptotically stable.

A simulation has been carried out to evaluate the performance of such a control scheme, assuming $x(0) = \begin{bmatrix} 0.1\,\mathrm{m} & 0\,\mathrm{m/s} \end{bmatrix}^\top$. Figure 4 depicts the results of such a simulation.

As shown by such a figure, the proposed control scheme makes the closed-loop hybrid system asymptotically stable.

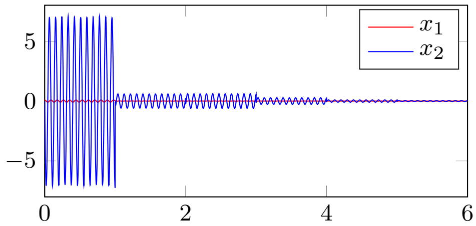

Fig. 4: State response in the x variables of Σ_H in the first example.

4.2 *Lithium ions distribution in the human body*

Lithium ions are one of the most widespread drugs for treating the manic-depressive illness and the bipolar disorder. Thus, designing a controller to regulate the concentration of such ions is a critical problem to be addressed in order improve the effectiveness of these treatments, especially due to the fact that therapeutic and toxic levels of lithium ions differ by only a factor of 2.

Consider the kinetic model of the distribution of lithium ions in the human body upon periodic oral administration obtained in [38] from experimental data. Such a model can be framed in the setting of this paper as the hybrid system (2), with $x = [\ x_1 \quad x_2 \quad x_3\ ]^\top$, where x_1, x_2, and x_3 denote the concentration (measured in $\mathrm{mmol/L}$) of lithium ions in plasma, in red blood cells, and in muscle-like cells, respectively, v is the lithium ions assumed by oral administration (measured in mmol), the (continuous-)time is measured in hours, τ_M denotes the interval between two oral administrations (assumed to be 3 h), and

$$A = \begin{bmatrix} -0.6137 & 0.1835 & 0.2406 \\ 1.2644 & -0.8 & 0 \\ 0.2054 & 0 & -0.19 \end{bmatrix}, \tag{14a}$$

$$E = I, \tag{14b}$$

$$F = \begin{bmatrix} 10.9 \\ 0 \\ 0 \end{bmatrix}. \tag{14c}$$

According to the three compartments model obtained in [38], the amount y of lithium ions excreted through urine (assumed to be the available measure to design the control) is proportional to the concentration of lithium in plasma, *i.e.*, $y = C\,x$ with

$$C = [\ 0.18 \quad 0 \quad 0\ ]. \tag{14d}$$

As in [24, 38], the objective of this section is to find a controller to designs the oral administration v of lithium ions with the aim of steering their concentration in muscle-like cells compartment to $0.65\,\mathrm{mmol/L}$. It is worth noticing that, by [39–42], since, in the continuous-time dynamics, there is not an internal model of a constant (*i.e.*, 0 is not an eigenvalue of A) and no continuous-time input is allowed, it is not possible to achieve exact regulation of the state x_3 to a constant. Therefore, the objective of the controller pursued in this section is to steer the monodromy state $x_3(t_k, k)$ to the desired value of concentration allowing a ripple between two consecutive oral administrations. Note that this objective cannot be easily pursued by using just discrete-time samples of y since the controlled output does not match with the measured one. However, in the following, it is shown that such a goal can be easily pursued by using the design strategy given in Section 3.

Toward this end, consider the observability matrix of (C, A),

$$O(C,A) = \begin{bmatrix} 0.1800 & 0 & 0 \\ -0.1105 & 0.0330 & 0.0433 \\ 0.1185 & -0.0467 & -0.0348 \end{bmatrix}.$$

Since $\mathrm{rank}(O(C,A)) = 3$, the pair (C, A) is observable, thus implying that there exists $O^{-1}(C,A)$ such that $O^{-1}(C,A)O(C,A) = I$. Hence, let L be the last row of such a matrix and let

$$\varpi(k) = L\,\phi(k), \tag{15}$$

where $\phi(k)$ is the output of the interconnection of systems (2), (3) and (11), that, by construction, is an estimate of the state x_3 obtained by using the interfacing system. By the same reasoning used to prove Theorem 1, one has that

$$\boldsymbol{H}(z) = \frac{\boldsymbol{\varpi}(z)}{\boldsymbol{v}(z)} = \frac{2.842\,z - 0.3318}{z^3 - 1.298\,z^2 + 0.3469\,z - 0.008139}, \tag{16}$$

is the transfer function of the discrete-time system

$$\overset{+}{\xi} = e^{A\,\tau_M}\,\xi + F\,v, \tag{17a}$$

$$\varpi = [\ 0 \quad 0 \quad 1\ ]\,\xi, \tag{17b}$$

which relates the amount v of lithium ions provided through oral administration and the after-administration concentration ϖ of such ions in muscle-like cells. Thus, such a transfer function can be used to design a discrete-time feedback controller that regulates the output ϖ (which, by construction, in absence of measurement noise, equals x_3) to the desired value.

A controller for the impulsive system with data (14) have been designed by using the transfer function $\boldsymbol{H}(z)$ given in (16) and classical loop-shaping techniques. In particular, consider the closed-loop system Σ_D depicted in Figure 1 with $\boldsymbol{Q}(z) = 1$ and

$$\boldsymbol{G}(z) = 0.0367\,\frac{z - 0.9382}{(z - 0.216)(z - 1)}. \tag{18}$$

By inspecting the Nyquist diagram of $\boldsymbol{H}(z)\boldsymbol{G}(z)$ (depicted in Figure 5), it can be easily derived that the closed loop system Σ_D depicted in Figure 1 is asymptotically stable.

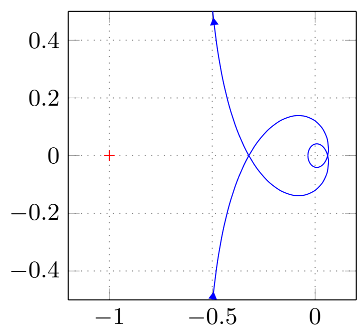

Fig. 5: Nyquist diagram of $\boldsymbol{H}(z)\boldsymbol{G}(z)$ in the second example.

Furthermore, by inspecting the step response of the closed-loop system Σ_D, whose transfer function is given by $\frac{\boldsymbol{H}(z)\boldsymbol{G}(z)}{1+\boldsymbol{H}(z)\boldsymbol{G}(z)}$, it can be easily observed that the rising time and the overshoot of such a system equal 6 time units and $8.47\,\%$, respectively (see Figure 6).

Therefore, the considered controller seems to be particularly suitable to regulate the monodromy response in the x_3 variables to the

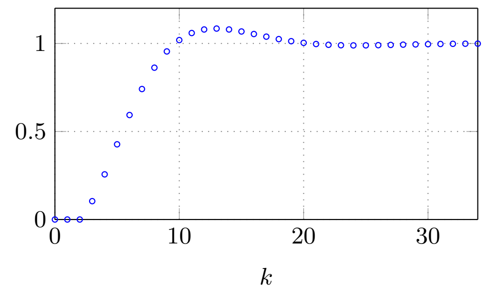

Fig. 6: Step response of $\frac{\boldsymbol{H}(z)\boldsymbol{G}(z)}{1+\boldsymbol{H}(z)\boldsymbol{G}(z)}$ in the second example.

desired target concentration. As a matter of fact, the small overshoot suggests that the considered controller can be used to achieve the regulation objective avoiding to reach critical concentrations of lithium ions (that could be toxic), whereas the small rising time suggests a fast response to the given reference signal.

In view of the correspondence established in Section 3, this controller can be readily used to regulate the monodromy response of the hybrid system with data (14). In particular, consider the control scheme depicted in Figure 2, with $\boldsymbol{Q}(z) = L$ and with the $\boldsymbol{G}(z)$ given in (18). In view of the correspondence established at the beginning of this section, the signal $\eta(k)$ equals $\varpi(k)$ for all $k \in \mathbb{N}$. Therefore, the monodromy output response of the hybrid system matches the output response of the discrete-time one. In particular, since the closed-loop system Σ_D is asymptotically stable, then, by Corollary 1, the closed-loop system Σ_H is asymptotically stable. Figure 7 depicts the state response in the x variables of the closed-loop system Σ_H to the constant reference $r(k) = 0.65$, that is the target concentration of lithium ions in muscle-like cells, and the corresponding state response of the discrete-time system (17a).

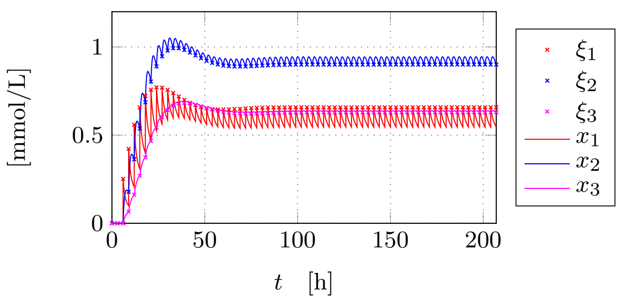

Fig. 7: State response in the x variables of Σ_H in the second example.

As shown by such a figure, the monodromy response of the hybrid system matches with the state response of the discrete-time one. Therefore, the design made at discrete-time by using standard tools can be directly used to control the hybrid system (2) by means of the interfacing system (11). It is worth stressing again that the ripple between two discrete-time inputs is unavoidable in this setting, due to the fact that the matrix A is nonsingular.

4.3 Maneuvering a spacecraft

Consider a spacecraft flying in the neighborhood of a satellite, which is traveling in a circular orbit around a planet with known angular speed ω. Assume that the control objective is to let the spacecraft track a circular, periodic reference trajectory around the satellite.

Letting $x = [\, x_1 \ \ \dot{x}_1 \ \ x_2 \ \ \dot{x}_2 \ \ x_3 \ \ \dot{x}_3 \,]^\top$, $x(t,k) \in \mathbb{R}^6$, denote the positions and velocities of the spacecraft and letting $v(k) \in \mathbb{R}^3$ denote the instantaneous change of velocities due to the periodic thrusters actions, the well-known Hill-Clohessy-Wiltshire equations [6, 43–45] can be used to model the dynamics of the spacecraft. Letting τ_M denote the period between two thrusters actions, such equations can be framed in the setting of this paper as system (2) with data

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2\omega & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega^2 & 0 \end{bmatrix},$$
$$E = I,$$
$$F = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Assuming that the spacecraft is equipped with a GPS-like system that allows to determine accurately its position [46], the output of system (2) is assumed to be given by (3), with

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Hence, consider the monodromy representation of the spacecraft dynamics given in (6), with $\tilde{E} = e^{A\,\tau_M}$. Since the pair (C, A) is observable, there exists a pseudoinverse $O^\dagger(C,A)$ of $O(C,A)$ such that $O^\dagger(C,A)\,O(C,A) = I$. Therefore, one has that

$$\varpi(k) = O^\dagger(C,A)\,\psi(k) = \xi(k), \quad k \in \mathbb{N},$$

i.e., when designing a controller for system (6), it is possible to assume that the state of system (6) is measured. The control objective is to let the monodromy position of the spacecraft track the trajectories of the following discrete-time exosystem

$$\overset{+}{\gamma} = S\,\gamma, \tag{19}$$

where $\gamma(k) \in \mathbb{R}^3$ are the sampled positions (with sampling time τ_M) of a periodic orbit with period $\frac{2\pi}{\omega^\star}$ and

$$S = \begin{bmatrix} \cos(\omega^\star \tau_M) & -\sin(\omega^\star \tau_M) & 0 \\ \sin(\omega^\star \tau_M) & \cos(\omega^\star \tau_M) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since the state ξ of system (6) and the state γ of the exosystem are both measured, it is possible to use a full information [47, 48] scheme to design a regulator for the monodromy system (6). In particular, letting K be such that $\Lambda(\tilde{E} + F\,K) \subset \mathbb{C}_g$ (such a matrix exists since the pair $(\tilde{E}, F)$ is reachable), and letting Π and Γ be the solution to the following discrete-time Francis equation

$$\begin{aligned} \Pi S &= \tilde{E}\,\Pi + F\,\Gamma, \\ C\,\Pi - I &= 0, \end{aligned}$$

consider the control scheme depicted in Figure 8, where $L = \Gamma - K\,\Pi$ and $\tilde{K} = K\,O^{\dagger}(C, A)$.

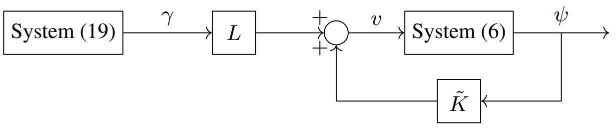

Fig. 8: Internal model full information discrete-time control scheme.

By classical results about output regulation for discrete-time linear system [47], such a control scheme is such that $\lim_{k\to\infty} C\,\xi(k) - \gamma(k) = 0$. Therefore, in view of the correspondence established in Section 3, the control scheme depicted in Figure 9 achieves monodromy regulation of the output of the hybrid system (2), (3). It is worth noticing that it is not possible to derive a full information hybrid control scheme analogous to the one depicted in Figure 9 just by considering discrete-time samples of the hybrid output y due to the fact that $\operatorname{rank}(C) = 3$ whereas $x(t, k) \in \mathbb{R}^6$.

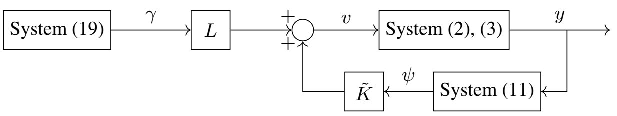

Fig. 9: Internal model full information hybrid control scheme.

A numerical simulation has been carried out to evaluate the performances of the control scheme depicted in Figure 9, letting $\omega = 0.0011\,\mathrm{rad/s}$ (corresponding to a low Earth orbit with a period of about 95 minutes), $\omega^\star = \frac{2\,\pi}{50}\,\mathrm{rad/s}$ (corresponding to a rotation of the spacecraft with a period of $50\,\mathrm{s}$ around the satellite), $\tau_M = 3\,\mathrm{s}$, $x_0 = [\; 10\,\mathrm{m} \quad 0\,\mathrm{m/s} \quad 12\,\mathrm{m} \quad 0\,\mathrm{m/s} \quad 3\,\mathrm{m} \quad 0\,\mathrm{m/s} \;]^\top$, $\gamma_0 = [\; 10\,\mathrm{m} \quad 10\,\mathrm{m} \quad 0\,\mathrm{m} \;]^\top$, and

$$K = -(I + F^\top P F)^{-1} F^\top P \tilde{E}.$$

where P is the solution to the following algebraic Riccati equation

$$P = \tilde{E}^\top P\,\tilde{E} + I - \tilde{E}^\top P\,F(I + F^\top P\,F)^{-1} F^\top P\,\tilde{E}. \qquad (20)$$

Figure 10 depicts the results of such a simulation.

As shown by such a figure, the monodromy behavior of system (2) matches with the one of the discrete-time system (6). Thus, the full information regulation control scheme depicted in Figure 9 (which has been designed just by using the discrete-time equivalent control system (6)) achieves monodromy output tracking. Note that, as in the previous example, since there is not a continuous-time internal model for the exosystem, it is not possible to achieve exact tracking at continuous-time just by exploiting discrete-time control inputs (for further details the interested reader is referred to [42]).

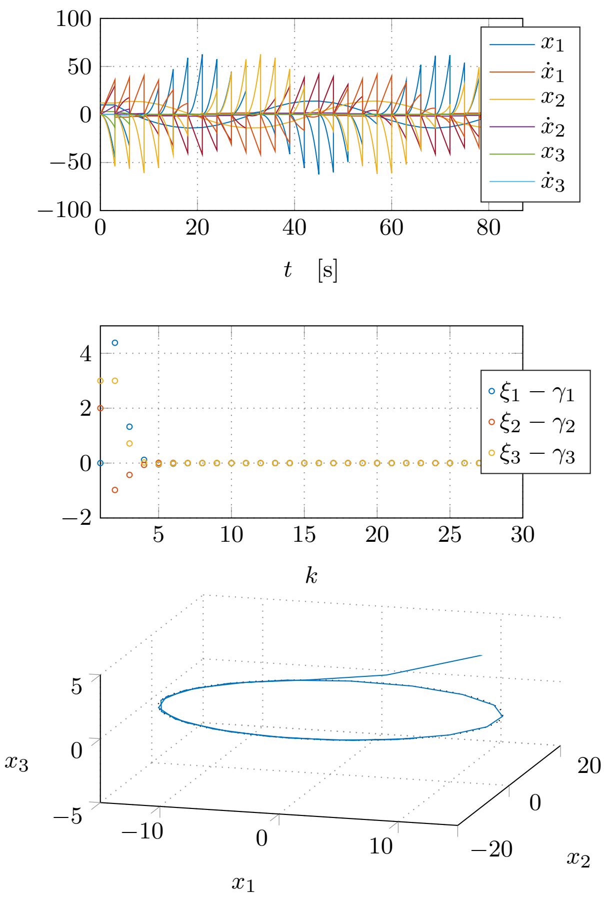

Fig. 10: Simulation of the spacecraft behavior.

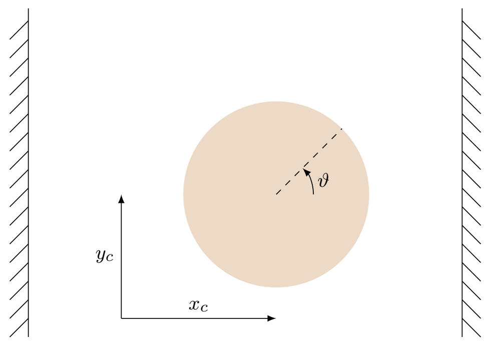

Fig. 11: A rotating disk bouncing between two walls.

4.4 Spinning and bouncing disk

Consider a disk of radius r, total mass m, and inertia $\mathcal{I}$, moving on an horizontal plane between two parallel walls, orthogonal to the plane of motion and infinitely massive. Let $l + 2r$, $l > 0$, be the distance between the two walls, let (x_c, y_c) be the coordinates of the center of mass of the disk, and let α denote the angular position of the disk (Figure 11).

Assume that $x_c(0) = 0$, that $|\dot{x}_c(t)| = |\dot{x}_c(0)| = \upsilon > 0$, and that all the impacts are elastic and occur with pre–impact conditions such that the infinitesimal interval in which the disk is in contact with the wall consists in a first interval of sliding followed by a second interval of rolling.

Under these hypotheses, and assuming that the angular velocity of the disk at impact times can be instantaneously changed, the dynamics of the mechanical system depicted in Figure 11 can be modeled through the hybrid system (2) with $\tau_M = \frac{l}{v}$, and data

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-\zeta^{-1} & 0 & -\zeta^{-1} r \\ 0 & 0 & 1 & 0 \\ 0 & -r^{-1}(1-\zeta^{-1}) & 0 & \zeta^{-1} \end{bmatrix},$$

$$F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

where $\zeta = \frac{r^2 m}{\mathcal{I}}$ and $x = [\ y_c \quad \dot{y}_c \quad \vartheta \quad \dot{\vartheta}\ ]^\top$ [49]. Assume that the only available output is $y = C\,x$, with

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The control objective is to stabilize the system, despite the presence of an unmeasurable disturbance γ generated by the exosystem

$$\overset{+}{\gamma} = S\,\gamma, \tag{21}$$

where

$$S = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},$$

that acts on the discrete-time dynamics of the system as

$$\overset{+}{x} = E\,x + F\,v + J\,\gamma,$$

where

$$J = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that, the continuous-time subsystem (2a), (3) is observable, $\mathrm{rank}(O(C,A)) = 4$, and hence, by the correspondence established in Section 3, stabilizing system (2) essentially matches with steering to zero the output ψ of the discrete-time system

$$\overset{+}{\xi} = \tilde{E}\,\xi + F\,v + J\,\gamma, \tag{22a}$$

$$\psi = O(C,A)\,\xi. \tag{22b}$$

However, since the disturbance γ is unmeasurable, it is not possible to use a full information scheme to steer the state to 0, as in Subsection 4.3. Nevertheless, by using the results given in [48], it is possible to use an error feedback scheme to achieve the regulation objective. In particular, letting $\tilde{E} = E\,e^{A\,\tau_M}$, letting K be such that

$$\Lambda(\tilde{E} + F\,K) \subset \mathbb{C}_g$$

(such a matrix exists since the pair $(\tilde{E}, F)$ is reachable), letting $Z = [\ Z_1^\top \quad Z_2^\top\ ]^\top$ be such that

$$\Lambda\left(\begin{bmatrix} \tilde{E} & J \\ 0 & S \end{bmatrix} + Z\,[\ O(C,A) \quad 0\ ]\right) \subset \mathbb{C}_g,$$

and letting Π and Γ be the solution to the following Francis equation

$$\Pi\,S = \tilde{E}\,\Pi + F\,\Gamma + J,$$
$$O(C,A)\,\Pi = 0,$$

the discrete-time controller

$$\overset{+}{\nu} = \begin{bmatrix} \tilde{E} + Z_1\,O(C,A) + F\,K & J + F(\Gamma - K\,\Pi) \\ Z_2\,O(C,A) & S \end{bmatrix}\nu - Z\,\psi, \tag{23a}$$

$$v = [\ K \quad \Gamma - K\,\Pi\ ]\nu, \tag{23b}$$

steers the output ψ of system (22) to zero. Therefore, in view of the correspondence established in Section 3, the control scheme depicted in Figure 12 achieves error feedback regulation of the considered hybrid system with discrete-time unmeasurable disturbance.

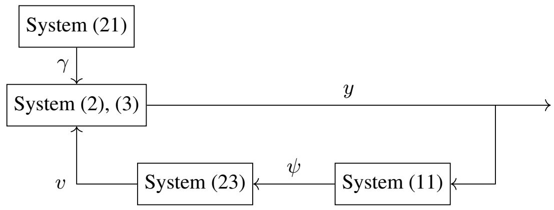

Fig. 12: Error feedback hybrid control scheme.

A numerical simulation has been carried out to evaluate the performance of the proposed control scheme assuming $r = 0.1\,\mathrm{m}$, $\tau_M = 1\,\mathrm{s}$, $m = 0.3\,\mathrm{kg}$, $\mathcal{I} = 1.5 \cdot 10^{-4}\,\mathrm{m}^4$, $\nu_0 = 0$, $\gamma_0 = [\ 0.3 \quad -0.1\ ]^\top$, and $x_0 = [\ 0.1\,\mathrm{m} \quad 0\,\mathrm{m/s} \quad -2\,\mathrm{rad} \quad 0\,\mathrm{rad/s}\ ]^\top$, whereas the gains K and Z have been designed by solving a Riccati equation wholly similar to (20). Figure 13 depicts the results of such a simulation.

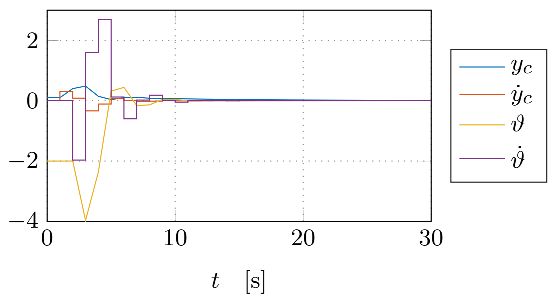

Fig. 13: State response in the x variables of the control system depicted in Figure 12.

As shown by such a figure, the control system depicted in Figure 12 is able to steer to zero the state of the hybrid system despite the presence of an unmeasurable disturbance affecting its discrete-time dynamics.

4.5 Voltage regulator

Consider the electric circuit depicted in Figure 14 that is a simple smoothing circuit controlled by a conventional digital-to-analog converter, which consists of three of capacitors, characterized by the capacitances $\mathcal{C}_1 = 2.2\,\mu\mathrm{F}$, $\mathcal{C}_2 = 4.7\,\mu\mathrm{F}$, and $\mathcal{C}_3 = 33\,\mu\mathrm{F}$, two resistors, characterized by the resistances $\mathcal{R}_1 = 2.2\,\mathrm{k}\Omega$, and $\mathcal{R}_2 = 100\,\mathrm{k}\Omega$, an operational amplifier, and two switches, which are short-circuited periodically every $\tau_M = 10\,\mathrm{ms}$.

Letting $x_i(t,k) \in \mathbb{R}$ be the voltage across the capacitance labeled with x_i and letting $v(k)$ be the voltage applied by the independent generator at $t = t_k$, the dynamics of such a system can be modeled

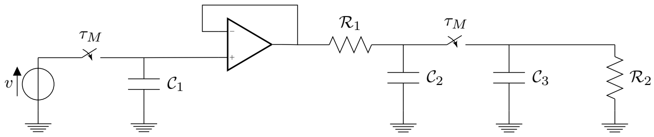

Fig. 14: A voltage smoothing circuit.

in the setting of this paper as system (2) with data

$$A = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{\mathcal{R}_1 \mathcal{C}_2} & -\frac{1}{\mathcal{R}_1 \mathcal{C}_2} & 0 \\ 0 & 0 & -\frac{1}{\mathcal{R}_2 \mathcal{C}_3} \end{bmatrix},$$

$$E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\mathcal{C}_2}{\mathcal{C}_2+\mathcal{C}_3} & \frac{\mathcal{C}_3}{\mathcal{C}_2+\mathcal{C}_3} \\ 0 & \frac{\mathcal{C}_2}{\mathcal{C}_2+\mathcal{C}_3} & \frac{\mathcal{C}_3}{\mathcal{C}_2+\mathcal{C}_3} \end{bmatrix}.$$

$$F = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

The control objective is to steer x_3 to a desired voltage. Thus, define the output matrix

$$C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

and consider the discrete-time equivalent system (6), where $\tilde{E} = E\, e^{A\,\tau_M}$. Such a system is not reachable since $0 \in \Lambda(\tilde{E})$ and

$$\operatorname{rank}\left(\begin{bmatrix} \tilde{E} & F \end{bmatrix}\right) = 2.$$

However, system (6) is controllable since

$$\operatorname{rank}\left(\begin{bmatrix} \tilde{E} - \lambda\, I & F \end{bmatrix}\right) = 3,\, \forall \lambda \in \Lambda(\tilde{E}) \setminus \{0\}.$$

On the other hand, one has $\operatorname{rank}(O(C, A)) = 1$ (*i.e.*, the continuous-time system (2a), (3) is not observable), but

$$\operatorname{rank}\left(\begin{bmatrix} \tilde{E} - \lambda\, I \\ O(C, A) \end{bmatrix}\right) = 2,\, \forall \lambda \in \Lambda(\tilde{E}) \setminus \{0\},$$

and hence system (6) is constructible. Therefore, by (5), the hybrid system (2), (3) is constructible and controllable as well, thus implying, by a trivial extension of Proposition 3, that a controller that stabilizes the discrete-time system (6) makes the hybrid system (2), (3), (11) globally exponentially stable.

Since $\operatorname{rank}(O(C, A)) = 1$, designing a controller for system (6) is essentially equivalent to design a controller for the system

$$\overset{+}{\xi} = \tilde{E}\, \xi + F\, v, \tag{24a}$$

$$\varpi = C\, \xi, \tag{24b}$$

whose transfer function is

$$\boldsymbol{H}(z) = \frac{0.07727}{z^2 - 0.9201\, z}.$$

In order to further corroborate the fact that, by using the proposed strategy, any design tool can be used to find a control law for the hybrid system (2), (3), a feedback controller for system (24) has been designed by using an automatic PID tuner (in particular, the `pidtune` algorithm implemented in `Matlab` [50]), thus obtaining

$$\boldsymbol{G}(z) = \frac{1.588\, z^2 + 0.1932\, z - 1.395}{z^2 - 0.204\, z - 0.796}.$$

Thus, by letting $L = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ and noticing that $L\, O(C, A) = C$, the controller depicted in Figure 2 with such a $\boldsymbol{G}(z)$ and with $\boldsymbol{Q}(z) = L$ has been used to control the hybrid system (2), (3). Figure 15 depicts the results of a numerical simulation in which such a control scheme has been used with $r = 5\,\mathrm{V}$. As shown by such a figure, the proposed controller scheme achieves the objective of steering the state x_3 of the hybrid system toward the reference voltage level.

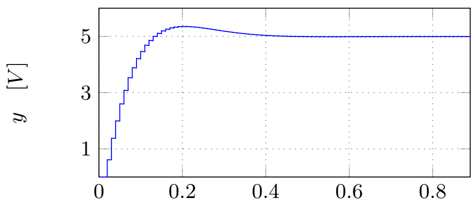

Fig. 15: Numerical simulation of the electric system.

Note that in such a simulation the voltage level are actually discontinuous due to the jumps induced by the switches. It is worth noticing that a similar results could be obtained by using the post-jump values $x_3(t_k, k)$ of the voltage across the capacitor labeled with $\mathcal{C}_3$ as discrete-time output rather than $L\, \phi(k)$, without requiring the use of the interfacing system (11).

5 Conclusions

In this paper, the problem of designing controller for hybrid systems with impulsive inputs has been addressed. It has been shown that each hybrid systems with impulsive inputs and periodic jumps can be recast into a discrete-time, linear, time-invariant plant, which, in turn, can be used to design a controller. Thus, an interfacing system has been proposed to readily use the designed discrete-time controller to control the original hybrid plant. Several examples, spanning from aerospace to biological applications, have been reported to corroborate the theoretical results.

The main innovation of the modeling strategy proposed in this paper with respect to the one given in [24] is that, while the latter presents a state space approach to obtain a discrete-time equivalent system, the former allows also to directly use discrete-time transfer functions to synthesize a controller for the hybrid system. This paves the way toward the use of frequency-domain tools for the design of controllers for hybrid systems such as automatic PID tuning and loop-shaping techniques.

It is worth noticing that, even if the results given in this paper have been presented through the modern hybrid formalism, they are strongly related to the research carried out in the context of sampled-data and periodic systems (we refer the interested reader to [51–53] for sampled–data systems and to [54–57] for periodic ones). Differently from these classical results, here it has been shown that by exploiting the hybrid nature of the system, it is possible to design a controller for the system without any assumption about the sampling time (that, in the context of this paper, is the dwell time τ_M between two consecutive jumps), provided that the hybrid system satisfies some necessary conditions about stabilizability and detectability. Furthermore, in the context of this paper, it has been shown how the discrete-time jumps that are seldom made by the system can be fully exploited in order to achieve the control objectives.

Future researches will deal with the extension of the frequency-domain tools introduced in this paper to systems having both continuous-time and discrete-time control inputs, with the design of

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controllers satisfying constraints in the state and in the control input, and with the comparison between the $\mathcal{L}_2$ gain [58] of the hybrid system (2) and of the discrete-time system (6).

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