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Global stabilization of nonlinear systems via hybrid implementation of dynamic continuous-time local controllers

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Abstract

Given a continuous-time system and a dynamic control law such that the closed-loop system satisfies standard Lyapunov conditions for *local* asymptotic stability, we propose a *hybrid implementation* of the continuous-time control law. We demonstrate that subject to certain "relaxed" conditions, the hybrid implementation yields *global* asymptotic stability properties. These conditions can be further specialized to yield local/regional asymptotic stability with an *enlarged basin of attraction* with respect to the original control law. Two illustrative numerical examples are provided to demonstrate the main results.

1 Introduction and preliminaries

Consider a nonlinear system described by the equation

$$\dot{x} = f(x, u), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ denotes the state, $u(t) \in \mathbb{R}^m$ is the control input and the mapping $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is assumed to be $\mathcal{C}^k$ for some sufficiently large $k \in \mathbb{N}$. Suppose, in addition, that $f(0, 0) = 0$, namely the origin is an equilibrium point of the unforced system.

Definition 1. *Consider the nonlinear system*

$$\dot{\xi} = \alpha(x, \xi), \qquad u = \beta(x, \xi), \tag{2}$$

with state $\xi(t) \in \mathbb{R}^s$, $s \in \mathbb{N}$, *where the mappings* $\alpha : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^s$, $\alpha(0, 0) = 0$, *and* $\beta : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^m$, $\beta(0, 0) = 0$, *are in* $\mathcal{C}^k$. *Then, system* (2) *is a* globally stabilizing controller *for* (1) *if there exists a radially unbounded, continuously differentiable, positive definite function* $V : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}_{\geqslant 0}$ *such that*

$$\dot{V} \leqslant -\rho(x, \xi), \text{ for all } (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^s, \tag{GAS}$$

along the trajectories of the closed-loop system (1)-(2), *where* $\rho : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}_{\geqslant 0}$ *is a continuously differentiable, positive-define function. Moreover, system* (2) *is instead a* locally stabilizing controller *if*

$$\dot{V} \leqslant -\rho(x, \xi), \text{ for all } (x, \xi) \in \mathcal{W}, \tag{LAS}$$

for some non-empty open neighborhood $\mathcal{W} \subset \mathbb{R}^n \times \mathbb{R}^s$ *of the origin.*

Although in practical applications it is clearly more desirable to satisfy the requirement (GAS), instead of (LAS), achieving the former objective is a rather challenging - and typically daunting - task. Several procedures have been proposed in the literature to deal with this problem, such as control Lyapunov functions (Sontag 1989), differential geometric approaches (Isidori 2013) and sliding mode tools (Shtessel et al. 2014), *etc.*.

The main objective of this note consists in providing conditions that allow to obtain global asymptotic stability of the origin for the closed-loop system by combining conditions *weaker* than those in (GAS) with a hybrid implementation of a given locally stabilizing controller.

2 Hybrid implementation of dynamic continuous-time controllers

In this section we discuss a control design technique - based on the knowledge of a locally stabilizing controller for system (1) - that guarantees global convergence. Moreover, we formalize the notion of *hybrid implementation* of the dynamic control law (2).

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Assumption 1. A *locally stabilizing controller* of the form of (2), together with the underlying functions V and ρ, is given for (1).

To provide a concise statement of the results, let

$$\mathcal{L}(x,\xi) \triangleq \tfrac{\partial V(x,\xi)}{\partial x} f(x,\beta(x,\xi)) + \tfrac{\partial V(x,\xi)}{\partial \xi}\alpha(x,\xi), \quad (3)$$

for any $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^s$, and define the sets

$$C \triangleq \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^s : \mathcal{L}(x,\xi) \leqslant -\mu\, \rho(x,\xi)\}, \quad (4a)$$
$$D \triangleq \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^s : \mathcal{L}(x,\xi) \geqslant -\mu\, \rho(x,\xi)\}. \quad (4b)$$

with $\mu \in (0,1)$. The proposed *alternative* conditions for global stabilization can be then stated as follows.

Assumption 2 (Hybrid relaxed requirements)**.** For each $(x,\xi) \in \mathcal{U} \subseteq \mathbb{R}^n \times \mathbb{R}^s$ there is $\zeta \in \mathbb{R}^s$ such that [1]

$$\mathcal{L}(x,\zeta) \leqslant -\rho(x,\zeta), \qquad \text{(HR1)}$$
$$V(x,\zeta) - V(x,\xi) \leqslant 0. \qquad \text{(HR2)}$$

Despite that the conditions (HR) resemble the Control Lyapunov Function (briefly, CLF) approach, here ζ describes the *state* of the controller, rather than the *control* input as in CLF. By comparing the inequality in (HR1) with the one in (GAS), it is evident that the former, requiring the existence of ζ associated to individual values of x, is significantly *milder* than the latter. More precisely, the (GAS) condition requires that a certain inequality holds for any possible pair of *independent* values of x and ξ, whereas the (HR) requires that, in order to satisfy the same inequality, for any x one has the possibility of selecting a different ζ within a compact set (sub-level sets of V, as entailed by (HR2)). Although conditions (HR) resemble the ones given in Seuret & Prieur (2011), Postoyan et al. (2015), Chai et al. (2017), here they are used with the aim of extending the basin of attraction of a locally stabilizing controller, rather than guaranteeing asymptotic stability of a global controller implemented in a discrete manner. Suppose now that the locally stabilizing controller of Assumption 1 satisfies the conditions (HR). Then the following *hybrid implementation* of the controller can be envisioned.

Definition 2. *The* hybrid implementation *of the controller* (2) *is defined as the hybrid system*

$$\dot{\xi} = \alpha(x,\xi), \qquad (x,\xi) \in C, \quad (5a)$$
$$\xi^+ \in \arg\min_{\zeta\in\Xi(x,\xi)} \Pi(x,\xi,\zeta), \qquad (x,\xi) \in D, \quad (5b)$$
$$u = \beta(x,\xi), \quad (5c)$$

with Π *a lower semicontinuous function and* $\Xi(x,\xi) := \{\zeta \in \mathbb{R}^s : \mathcal{L}(x,\zeta) \leqslant -\rho(x,\zeta), V(x,\zeta) \leqslant V(x,\xi)\}$.

[1] Note that, by definition of (LAS), $\mathcal{W} \subset \mathcal{U}$.

The function Π is introduced in (5b) to systematically select the *most desirable* $\xi^+ \in \Xi(x,\xi)$ according to some optimality criterion (e.g., minimum norm, minimum deviation from the current ξ, etc.). However, if for each $(x,\xi) \in \mathcal{U}$ the set $\Xi(x,\xi)$ is bounded, then it is possible to chose such a function as a constant, so that the jump dynamics (5b) read as $\xi^+ \in \Xi(x,\xi)$.

Note that, if Assumption 2 holds with $\mathcal{U} = \mathbb{R}^n \times \mathbb{R}^s$, then the set $\Xi(x,\xi)$ is not empty for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^s$. Under this assumption, in the following theorem, we show that the controller (5) renders the origin uniformly GAS for the closed-loop system (1), (5).

Theorem 1. *Let the lower semicontinuous function* $\Pi(x,\xi,\zeta)$ *be level-bounded in* ζ*, locally uniformly in* (x,ξ)*. Let* $\mathcal{U} = \mathbb{R}^n \times \mathbb{R}^s$ *and suppose that Assumptions 1 and 2 hold. Define* $\varpi(x,\xi) \triangleq \inf_{\zeta\in\Xi(x,\xi)} \Pi(x,\xi,\zeta)$ *and assume that it is locally bounded from above and continuous. Then, the origin is a* uniformly globally asymptotically stable *equilibrium point for system* (1) *in closed loop with the hybrid implementation* (5) *of the locally stabilizing controller* (2).

Proof. Consider the closed-loop system, governed by the following hybrid dynamics

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = F(x,\xi), \qquad (x,\xi) \in C, \quad (6a)$$
$$\begin{bmatrix} x^+ \\ \xi^+ \end{bmatrix} \in G(x,\xi), \qquad (x,\xi) \in D, \quad (6b)$$

where

$$F(x,\xi) = \begin{bmatrix} f(x,\beta(x,\xi)) \\ \alpha(x,\xi) \end{bmatrix}, \quad (6c)$$
$$G(x,\xi) = \begin{bmatrix} x \\ \arg\min_{\zeta\in\Xi(x,\xi)} \Pi(x,\xi,\zeta) \end{bmatrix}, \quad (6d)$$

and C and D are as defined in (4a) and (4b), respectively. First, we show that system (6) is well-posed. Since, by assumption, the functions $\mathcal{L}$ and ρ are continuous, then both the flow set C and the jump set D are closed. Moreover recall that the mappings f, g, α and β are continuous and locally bounded, which implies that the flow map $F(x,\xi)$ is outer semicontinuous and locally bounded for all $(x,\xi) \in C$ (see Corollary 5.20 of Rockafellar & Wets 2009). Furthermore, by Example 5.22 of Rockafellar & Wets (2009), the map $(x,\xi) \rightrightarrows \arg\min_{\zeta\in\Xi(x,\xi)} \Pi(x,\xi,\zeta)$ is outer semicontinuous and locally bounded, thus implying that the jump map $G(x,\xi)$ is outer semicontinuous and locally bounded. By Assumption 2, $G(x,\xi)$ is nonempty for each $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^s$. Thus, the system (6) satisfies the "hybrid basic conditions" and is well-posed (see, e.g. Theorem 6.8 of Goebel et al. 2012). Hence, since the set $\Xi(x,\xi)$ is nonempty and $C \cup D = \mathbb{R}^{n+s}$ (and hence solutions to the hybrid system (6) can neither flow nor jump outside of $C \cup D$), by Theorem S3 of Goebel et al. (2009), maximal solutions of system (6) are either complete or blow up in finite time.

We now demonstrate the stability properties of the origin of the system (6). To this end, consider then the Lyapunov function V. Since

$$\mathcal{L}(x,\xi) = \langle \nabla V(x,\xi), F(x,\xi)\rangle \leqslant -\mu\,\rho(x,\xi), \qquad (7a)$$

for all $(x,\xi) \in C$ and

$$\max_{\zeta\in(\arg\min_{\zeta\in\Xi(x,\zeta)}\Pi(x,\xi,\zeta))} V(x,\zeta) - V(x,\xi) \leqslant 0 \qquad (7b)$$

for all $(x,\xi) \in D$, the function V is non-increasing along solutions of system (6) and, therefore, its sub-level sets are positively invariant with respect to system (6). Furthermore, such sets are compact due to radial unboundedness of V. Hence, since solutions to system (6) cannot *blow up* in finite time, they are complete and the origin is uniformly globally stable for system (6).

It then remains to show uniform convergence of the trajectories to the origin. In order to prove this, we show that, given any $r \in \mathbb{R}_{>0}$ and $\varepsilon \in \mathbb{R}_{>0}$, there is a time $T \in \mathbb{R}_{>0}$ such that each solution $x(t,j)$ of system (6) starting in $r\mathbb{B}$ is such that $|x(t,j)| \leqslant \varepsilon$ for all $(t,j) \in \mathrm{dom}(x)$ such that $t+j \geqslant T$, namely, roughly speaking, any initial condition in $r\mathbb{B}$ is uniformly *shrunk* to $\varepsilon\mathbb{B}$ in finite time. Fix $r \in \mathbb{R}_{>0}$ and let $c_0 \in \mathbb{R}_{>0}$ be such that $r\mathbb{B} \subset \mathcal{S}_1 := \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^s : V(x,\xi) \leqslant c_0\}$. Fix $\varepsilon \in (0,1)$, and let $\mathcal{S}_\varepsilon := \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^s : V(x,\xi) \leqslant \varepsilon\, c_0\}$. Thus, let $\varrho := \inf_{(x,\xi)\in\mathcal{S}_1\setminus\mathcal{S}_\varepsilon} \rho(x,\xi)$, which is strictly positive since ρ is a continuous, positive definite function, $\overline{\mathcal{S}_1 \setminus \mathcal{S}_\varepsilon}$ is a compact set, and $0 \notin \overline{\mathcal{S}_1 \setminus \mathcal{S}_\varepsilon}$. Furthermore, by considering that the functions V and ρ, and the mappings α, β, f, and g are of class $\mathcal{C}^k$ for some sufficiently large k, there exists $\nu \in \mathbb{R}_{>0}$ such that, for all $(x,\xi) \in \mathcal{S}_1 \setminus \mathcal{S}_\varepsilon$, $\langle \nabla(\mathcal{L}(x,\xi) + \rho(x,\xi)), F(x,\xi)\rangle \leqslant \nu$. Hence, let

$$T \triangleq 1 + \frac{c_0(\nu + \varrho - \mu\,\varrho)}{(1-\mu)\,\mu\,\varrho^2},$$

and assume that there exists a solution (x,ξ) to system (6), with $(x(0,0),\xi(0,0)) \in r\mathbb{B}$, that stays in $\mathcal{S}_1 \setminus \mathcal{S}_\varepsilon$ for all hybrid times $(t,j) \in \mathrm{dom}(x)$ with $t+j \leqslant T$. Then, for all $k \in \mathbb{Z}_{\geqslant 1}$ such that $t_k + k \leqslant T$, it results that $\mathcal{L}(x(t_k,k),\xi(t_k,k)) + \rho(x(t_k,k),\xi(t_k,k)) \leqslant 0$. Since $\rho(x,\xi) \geqslant \varrho$ for all $(x,\xi) \in \mathcal{S}_1 \setminus \mathcal{S}_\varepsilon$ and a jump occurs at hybrid time (t_{k+1},k) only if $\mathcal{L}(x(t_{k+1},k),\xi(t_{k+1},k)) + \rho(x(t_{k+1},k),\xi(t_{k+1},k)) \geqslant (1-\mu)\,\rho(x(t_{k+1},k),\xi(t_{k+1},k)) \geqslant (1-\mu)\,\varrho$, this implies that there exists a minimum dwell-time $\tau := \frac{(1-\mu)\,\varrho}{\nu}$ between two consecutive jumps of the solution (x,ξ). Therefore, the solution (x,ξ) is non-Zeno and $(t,j) \in \mathrm{dom}(x)$ implies $j \leqslant \frac{t}{\tau} + 1$. Hence, by following Theorem 3.18 of Goebel et al. (2012), it results that

$$\begin{aligned} V(x(t,j),\xi(t,j)) &\leqslant V(x_0,\xi_0) - \mu\,\varrho\,t, \\ &\leqslant V(x_0,\xi_0) - \tfrac{\tau}{\tau+1}\,\mu\,\varrho\,(t+j-1), \\ &\leqslant c_0 - \tfrac{\tau}{\tau+1}\,\mu\,\varrho(t+j-1), \end{aligned}$$

leading to a contradiction since $(x,\xi) \in \mathcal{S}_1\setminus\mathcal{S}_\varepsilon$ if and only if $\varepsilon\,c_0 < V(x,\xi) \leqslant c_0$. Therefore, there does not exist a solution (x,ξ) to system (6) that stays in $\mathcal{S}_1 \setminus \mathcal{S}_\varepsilon$ for all hybrid times $(t,j) \in \mathrm{dom}(x)$ with $t+j \leqslant T$. Hence, since the sets $\mathcal{S}_1$ and $\mathcal{S}_\varepsilon$ are positively invariant with respect to system (6) and solutions to system (6) are complete, there exists T such that, for each solution (x,ξ) of system (6) such that $(x(0,0),\xi(0,0)) \in \mathcal{S}_1$, $(t,j) \in \mathrm{dom}(x)$ and $t+j \geqslant T$ imply $(x(t,j),\xi(t,j)) \in \mathcal{S}_\varepsilon$. Hence, by the arbitrary selection of $r \in \mathbb{R}_{>0}$ and $\varepsilon \in (0,1)$, the origin is uniformly globally asymptotically stable for system (6), thus concluding the proof. ☐

Remark 1. If Assumption 1 holds, then, by (LAS), there exists $\varepsilon^\star \in \mathbb{R}_{>0}$ such that $\varepsilon^\star\mathbb{B} \subset \mathcal{W}$ and $D \cap \varepsilon^\star\mathbb{B} = \{0\}$. By the proof of Theorem 1, this implies that solutions (x,ξ) to system (6) such that $\nexists(t,j) \in \mathrm{dom}(x,\xi)$ such that $x(t,j) = 0$ and $\xi(t,j) = 0$ have a semi-global uniform dwell-time and are eventually continuous.

By weakening the assumptions of Theorem 1, it is still possible to guarantee uniform local asymptotic stability of the origin for the closed-loop system, as shown below.

Corollary 1. *Assume that there exists an open set $\mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}^s$ containing the origin such that Assumptions 1 and 2 hold and that the function $\Pi(x,\xi,\zeta)$ is lower semicontinuous and level-bounded in ζ, locally uniformly in (x,ξ). Define $\varpi : \mathcal{U} \to \mathbb{R}$, $\varpi(x,\xi) := \inf_{\zeta\in\Xi(x,\xi)} \Pi(x,\xi,\zeta)$ for all $(x,\xi) \in \mathcal{U}$, and assume that it is locally bounded from above and continuous. Then, the origin is a* uniformly locally asymptotically stable *equilibrium point for system* (1) *in closed loop with the hybrid implementation* (5) *of the locally stabilizing controller* (2).

Proof. Let $c_0 \in \mathbb{R}_{>0}$ be such that $\mathcal{S}_1 := \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^s : V(x,\xi) \leqslant c_0\} \subset \mathcal{U}$ and consider the hybrid system

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = F(x,\xi), \qquad (x,\xi) \in C \cap \mathcal{S}_1, \qquad (8a)$$

$$\begin{bmatrix} x^+ \\ \xi^+ \end{bmatrix} \in G(x,\xi), \qquad (x,\xi) \in D \cap \mathcal{S}_1, \qquad (8b)$$

where F, G, C, and D are as in (6c)–(6d). By the same reasoning used in the proof of Theorem 1, the hybrid system (8) is well-posed. Therefore, by Theorem S3 of Goebel et al. (2009), letting ϕ be one of its maximal solutions, exactly one of the following cases holds: *(i)* ϕ

is complete; *(ii)* ϕ blows up in finite time; *(iii)* there is $(t,j) \in \mathrm{dom}(\phi)$ such that $\phi(t,j) \notin \mathcal{S}_1$. Hence, consider the function V, which satisfies (7a) for all $(x,\xi) \in C \cap \mathcal{S}_1$, and (7b) for all $(x,\xi) \in D \cap \mathcal{S}_1$, due to the fact that $\Xi(x,\xi) \neq \emptyset$ for all $(x,\xi) \in \mathcal{S}_1 \subset \mathcal{U}$. Since (7a) holds for all $(x,\xi) \in C \cap \mathcal{S}_1$, one has that $F(x,\xi) \cap T_C(x,\xi) \neq \emptyset$ for all $(x,\xi) \in (C \cap \mathcal{S}_1) \setminus D$ and hence, by Proposition 2.10 of Goebel et al. (2012), item *(iii)* holds only if $\phi(t,j)$ jumps out of $\mathcal{S}_1$. However, since (7b) holds for all $(x,\xi) \in D \cap \mathcal{S}_1$, $G(x,\xi) \in \mathcal{S}_1$ for all $(x,\xi) \in \mathcal{S}_1$, and, hence, the case of item *(iii)* cannot hold for maximal solutions starting in $\mathcal{S}_1$. Therefore, the proof follows by the same reasoning used to prove Theorem 1. □

From the constructions in the proof of Corollary 1, the estimate of the basin of attraction of the origin for the closed-loop system (obtained by using V as Lyapunov function) without hybrid implementation is a subset of the one for the closed-loop system with it. Indeed, by construction, all the points $(x,\xi) \in \mathcal{W}$ that satisfy (LAS) are such that (HR) holds with $\zeta = \xi$. The following corollary provides a *regional* interpretation of this result.

Corollary 2. *Let a bounded open set $\mathcal{E} \subset \mathbb{R}^n \times \mathbb{R}^s$ containing the origin be given and let $\mathcal{V}$ be a sublevel set of V such that $\mathcal{E} \subseteq \mathcal{V}$. Let $\mathcal{U} \triangleq \{(x,\xi) \in \mathcal{V} : \langle \nabla V(x,\xi), F(x,\xi) \rangle \geqslant 0\}$ and suppose that Assumptions 1 and 2 hold. Thus, define $\varpi : \mathcal{U} \to \mathbb{R}$, $\varpi(x,\xi) := \inf_{\zeta \in \Xi(x,\xi)} \Pi(x,\xi,\zeta)$ for all $(x,\xi) \in \mathcal{E}$, and assume that it is locally bounded from above and continuous. Then, the origin is a* uniformly asymptotically stable *equilibrium point for system* (1), (5) *with basin of attraction containing $\mathcal{E}$.*

Remark 2. Interestingly, the statement of Corollary 2 may be interpreted in a *spirit* similar to that of LaSalle's theorem. In fact, while the latter dictates to verify additional conditions on the system only in the set in which the time-derivative of the Lyapunov function is equal to zero to determine the asymptotic behavior of closed-loop trajectories, the former prescribes to verify additional conditions whenever the time-derivative is not negative, to ensure regional convergence from a given set.

3 Numerical Simulations

To corroborate the above theoretical analysis, in this section, we present two examples of application of the hybrid implementation of local controllers.

3.1 Enlarging the basin of attraction of the origin

Consider a nonlinear system described by the equation

$$\dot{x} = x + \cos(x) u \,, \tag{9}$$

with state $x(t) \in \mathbb{R}$ and input $u(t) \in \mathbb{R}$. Since $\cos(\pm\pi/2) = 0$ and, correspondingly, $\dot{x}_{|x=\pm\pi/2} = \pm\pi/2$ for any $u \in \mathbb{R}$, the largest achievable basin of attraction for the zero-equilibrium coincides with the interval $(-\pi/2, \pi/2)$. Suppose that a dynamic controller inspired by LQG strategies has been designed - based on the linearized system - to satisfy Assumption 1, namely

$$\dot{\xi} = (1 - \ell_\infty)\xi + \ell_\infty x + u \,, \quad u = -p_\infty \xi \,, \tag{10}$$

with $p_\infty = \ell_\infty = 1 + \sqrt{2}$ obtained by solving the underlying Riccati (control and filtering) equations.

The function V associated to the linearized system then provides an estimate of the basin of attraction of the zero equilibrium for the closed-loop extended system (9)-(10) as shown by the cyan line in Fig. 1, since the union of the red and green regions depicts the subset of the state-space in which $\dot{V}(x,\xi) \geqslant 0$. By relying on the arguments in Corollaries 1 and 2 and the discussion in Remark 2, the conditions (HR) are checked in the region where $\dot{V}(x,\xi) \geqslant 0$. Constructively, it is first imposed that jumps of the state ξ are allowed on the same level line of the function V, hence trivially satisfying (HR2) (corresponding to the selection $\Pi(x,\xi,\zeta) = (V(x,\xi) - V(x,\zeta))^2$). It can be verified that, instead, inequality (HR1) holds in the green region, thus reducing the set in which the relaxed conditions for stability do not hold only to the red region.

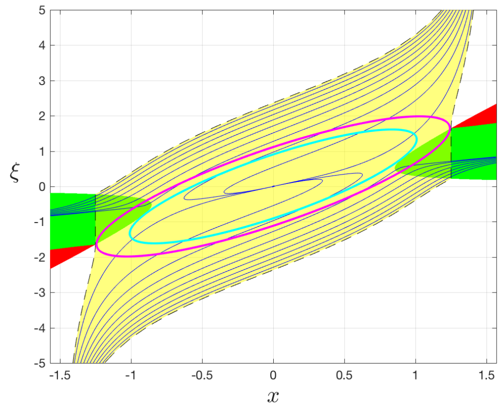

Fig. 1. Guaranteed basin of attraction of the continuous-time controller (10) (cyan line) and of its hybrid implementation (magenta line). Trajectory-based estimate of the basin for the latter controller (yellow region) and trajectories of the system (9)-(10).

Fig. 2 shows a trajectory of the closed-loop hybrid implementation of (9)-(10) corresponding to an initial condition for which the purely continuous-time evolution would diverge (see the dashed black line in the green region), while the controller is instead repeatedly reset to values of the state for which $\dot{V}(x,\xi) < 0$ every time the trajectory intersects the green subset.

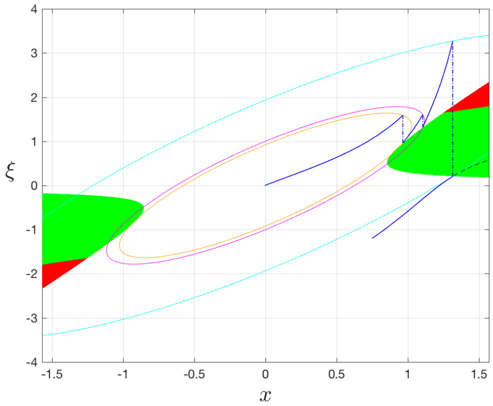

Fig. 2. Trajectory of the closed-loop hybrid implementation of (9)-(10) corresponding to an initial condition for which the purely continuous-time evolution would diverge.

3.2 Global stabilization with a local controller

Consider the nonlinear system

$$\dot{x}_1 = -2x_1 + x_2 - 3u, \tag{11a}$$
$$\dot{x}_2 = -x_1 - \frac{x_2}{x_1^2 + 1} - x_2 + 2u, \tag{11b}$$

together with the PI controller

$$\dot{\xi} = x_2, \tag{12a}$$
$$u = -5.67116\,\xi - 2.19912\,x_2, \tag{12b}$$

which has been designed by applying the Ziegler-Nichols tuning method to the linearized system, assuming that the available output is $y = x_2$. Letting

$$\begin{aligned} V(x,\xi) &= 2.23\xi^2 + 0.655\xi x_1 + 1.07\xi x_2 + 0.173x_1^2 \\ &\quad + 0.309x_2x_1 + 0.345x_2^2, \\ \rho(x,\xi) &= \tfrac{1}{2}(x_1^2 + x_2^2 + \xi^2), \end{aligned}$$

it can be checked that (LAS) holds with respect to

$$\mathcal{W} = \{(x,\xi) \in \mathbb{R}^2 \times \mathbb{R} : V(x,\xi) < 4.561\},$$

whereas (GAS) does not hold. Nevertheless, by computing the cylindrical algebraic decomposition, see *e.g.* Collins (1975), of the inequalities in (HR), it can be verified that, instead, for each $(x,\xi) \in \mathbb{R}^2 \times \mathbb{R}$ there exists $\zeta \in \mathbb{R}$ such that (HR) holds. Hence, letting

$$\Pi = 32.2\,\zeta^2 + 24.9\,\zeta x_2 + 4.84\,x_2^2,$$

which is obtained by considering the squared norm of the control input β when the state ξ of the controller (12) equals ζ (*i.e.* minimizing the control effort after the reset of the controller), by using the tools given in Menini et al. (2018), it can be checked that the function $\varpi(x,\xi) \triangleq \inf_{\zeta\in\Xi(x,\xi)} \Pi(x,\xi,\zeta)$ is locally bounded from above and continuous. Thus, the hypotheses of Theorem 1 are met and hence the hybrid implementation of the controller (12) makes the origin globally asymptotically stable for the closed-loop system. Fig. 3 depicts a trajectory of the closed-loop hybrid implementation of (11)-(12), where the post-jump value of the controller state ξ has been computed by using the numerical tools given in Calafiore & Possieri (2018).

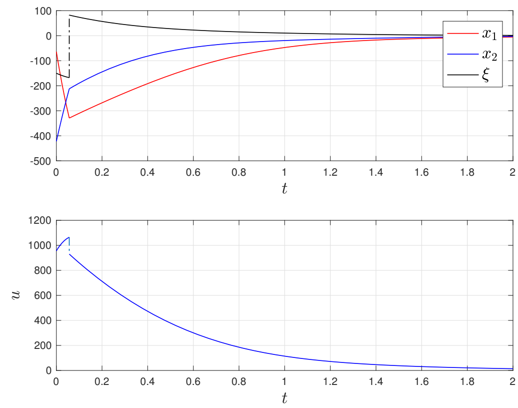

Fig. 3. Trajectory of the hybrid implementation of (11)-(12).

References

Calafiore, G. C. & Possieri, C. (2018), ‘A variation on a random coordinate minimization method for constrained polynomial optimization’, *IEEE Control Syst. Lett.* **2**(3), 531–536.

Chai, J., Casau, P. & Sanfelice, R. G. (2017), Analysis and design of event-triggered control algorithms using hybrid systems tools, *in* ‘56th IEEE Conf. Decis. Control’, IEEE, pp. 6057–6062.

Collins, G. (1975), ‘Quantifier elimination for the elementary theory of real closed fields by cylindrical algebraic decomposition. volume 33 of’, *Lect. Notes Comput. Sci.* pp. 134–183.

Goebel, R., Sanfelice, R. G. & Teel, A. R. (2009), ‘Hybrid dynamical systems’, *IEEE Control Sys.* **29**(2), 28–93.

Goebel, R., Sanfelice, R. G. & Teel, A. R. (2012), *Hybrid Dynamical Systems: modeling, stability, and robustness*, Princeton University Press.

Isidori, A. (2013), *Nonlinear control systems*, Springer.

Menini, L., Possieri, C. & Tornambe, A. (2018), ‘Algebraic methods for multiobjective optimal design of control feedbacks for linear systems’, *IEEE Trans. Autom. Control* **63**(12), 4188–4203.

Postoyan, R., Tabuada, P., Nesic, D. & Martinez, A. A. (2015), ‘A framework for the event-triggered stabilization of nonlinear systems.’, *IEEE Trans. Autom. Control* **60**(4), 982–996.

Rockafellar, R. T. & Wets, R. J.-B. (2009), *Variational analysis*, Springer, New York.

Seuret, A. & Prieur, C. (2011), Event-triggered sampling algorithms based on a lyapunov function, *in* '50th IEEE Conf. Decis. Control', pp. 6128–6133.

Shtessel, Y., Edwards, C., Fridman, L. & Levant, A. (2014), *Sliding mode control and observation*, Springer, New York.

Sontag, E. D. (1989), 'A "universal" construction of Artstein's theorem on nonlinear stabilization', *Syst. Control Lett.* **13**(2), 117–123.