Binary Operators of the Groupoids of OEIS A093112 and A093069 Numbers (Carol and Kynea Numbers)

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Here we discuss the binary operators of the sets made by the OEIS sequences of integers A093112 and A093069, also called Carol and Kynea numbers. We will see that these numbers are linked, through the binary operators, to the Mersenne and Fermat integers.

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As told in [1], there are at least three definitions of "groupoid", which are currently in use. The first type of groupoid that we can find is an algebraic structure on a set with a binary operator. The only restriction on the operator is closure. This properties means that, applying the binary operator to two elements of a given set S, we obtain a value which is itself a member of S.
Here, we consider the groupoids of the sets of the numbers given by OEIS sequences A093112 and A093069 [2,3], which are also called by Cletus Emmanuel, the Carol and Kynea numbers.

An A093112 number (Carol number) is an integer having the following form [2]:

$$C_n = 4^n - 2^{n+1} - 1 = (2^n - 1)^2 - 2$$

We can find the first numbers of the sequence A093112 in the OEIS [2]. That is: -1, 7, 47, 223, 959, 3967, 16127, 65023, 261119, 1046527, 4190207, 16769023, 67092479, 268402687, and so on.
An A093069 number (Kynea numbers) is defined as [3]:

$$K_n = (2^n + 1)^2 - 2$$

So we have [3]: 7, 23, 79, 287, 1087, 4223, 16639, 66047, 263167, 1050623, 4198399, 16785407, 67125247, 268468223, 1073807359, 4295098367, 17180131327, 68720001023, and so on.
As we did in some previous discussions (see for instance [4,5]), we can find a binary operator, which satisfy the closure, of given sets of numbers. In [4], we considered the groupoids of Mersenne, Fermat, Cullen and Woodall numbers.

Here how to find the operator for A093112 numbers. Let us use:

$$(C_m + 2)^{1/2} = (2^m - 1) = M_m \quad ; \quad (C_n + 2)^{1/2} = (2^n - 1) = M_n \quad ; \quad (C_{m+n} + 2)^{1/2} = (2^{m+n} - 1) = M_{m+n}$$

which are Mersenne numbers [4]. So we have the binary operator [4]:

$$(2^{m+n} - 1) = M_{m+n} = M_m \oplus M_n = M_m + M_n + M_m M_n = (2^m - 1)(2^n - 1) + (2^m - 1) + (2^n - 1)$$

Therefore, since $(C_{m+n} + 2)^{1/2} = (2^{m+n} - 1) = M_{m+n}$:

$$(C_{m+n}+2)^{1/2}=(C_m+2)^{1/2}(C_n+2)^{1/2}+(C_m+2)^{1/2}+(C_n+2)^{1/2}$$

We can find the binary operator for the Carol numbers as:

$$C_{m+n}=-2+[(C_m+2)^{1/2}(C_n+2)^{1/2}+(C_m+2)^{1/2}+(C_n+2)^{1/2}]^2 =$$

$$-2+(C_m+2)(C_n+2)+(C_m+2)+(C_n+2)+2(C_m+2)(C_n+2)^{1/2}+2(C_m+2)^{1/2}(C_n+2)+2(C_m+2)^{1/2}(C_n+2)^{1/2}$$

So we have:

$$C_{m+n}=6+C_m C_n+3C_m+3C_n+2(C_m+2)(C_n+2)^{1/2}+2(C_m+2)^{1/2}(C_n+2)+2(C_m+2)^{1/2}(C_n+2)^{1/2}$$

Therefore, the binary operator is defined as:

$$C_m \oplus C_n=6+C_m C_n+3C_m+3C_n+2(C_m+2)(C_n+2)^{1/2}+2(C_m+2)^{1/2}(C_n+2)+2(C_m+2)^{1/2}(C_n+2)^{1/2}$$

From this binary operation, we can have the recurrence relation: $C_{n+1}=C_n \oplus C_1$.

That is:

$$C_{n+1}=6+C_1 C_n+3C_1+3C_n+2(C_1+2)(C_n+2)^{1/2}+2(C_1+2)^{1/2}(C_n+2)+2(C_1+2)^{1/2}(C_n+2)^{1/2}$$

From $C_1=-1$, we have: 7, 47, 223, 959, 3967, 16127, 65023, 261119, 1046527, 4190207, 16769023, 67092479, 268402687, and so on.

Let us consider the Kynea numbers.

As we did before for the Carol numbers, let us use the following approach:

$$(K_m+2)^{1/2}=(2^m+1)=F_m \quad ; \quad (K_n+2)^{1/2}=(2^n+1)=F_n \quad ; \quad (K_{m+n}+2)^{1/2}=(2^{m+n}+1)=F_{m+n}$$

which are Fermat numbers [4]. So we have the binary operator [4]:

$$(2^{m+n}+1)=F_{m+n}=F_m \oplus F_n=(1-F_m)+(1-F_n)+F_m F_n=2+(2^m+1)(2^n+1)-(2^m+1)-(2^n+1)$$

$$(K_{m+n}+2)^{1/2}=(2^{m+n}+1)=F_{m+n}=2+(2^m+1)(2^n+1)-(2^m+1)-(2^n+1)$$

$$K_{m+n}=-2+[2+(2^m+1)(2^n+1)-(2^m+1)-(2^n+1)]^2=-2+[2+(K_m+2)^{1/2}(K_n+2)^{1/2}-(K_m+2)^{1/2}-(K_n+2)^{1/2}]^2$$

Then:

$$K_{m+n}=2+(K_m+2)(K_n+2)+(K_m+2)+(K_n+2) \ + \ 4(K_m+2)^{1/2}(K_n+2)^{1/2}-4(K_m+2)^{1/2}-4(K_n+2)^{1/2}$$

$$-2(K_m+2)(K_n+2)^{1/2}-2(K_m+2)^{1/2}(K_n+2)+2(K_m+2)^{1/2}(K_n+2)^{1/2}$$

Therefore, the binary operator is defined as:

$$K_m \oplus K_n = 10 + K_m K_n + 3K_m + 3K_n \;+\; 4(K_m+2)^{1/2}(K_n+2)^{1/2} - 4(K_m+2)^{1/2} - 4(K_n+2)^{1/2}$$

$$-2(K_m+2)(K_n+2)^{1/2} - 2(K_m+2)^{1/2}(K_n+2) + 2(K_m+2)^{1/2}(K_n+2)^{1/2}$$

Recurrence is given by:

$$K_{n+1} = K_n \oplus K_1 = 10 + K_n K_1 + 3K_n + 3K_1 \;+\; 4(K_1+2)^{1/2}(K_n+2)^{1/2} - 4(K_1+2)^{1/2} - 4(K_n+2)^{1/2}$$

$$-2(K_1+2)(K_n+2)^{1/2} - 2(K_1+2)^{1/2}(K_n+2) + 2(K_1+2)^{1/2}(K_n+2)^{1/2}$$

Then, starting from $K_1 = 7$, we have 23, 79, 287, 1087, 4223, 16639, 66047, 263167, 1050623, 4198399, 16785407, 67125247, 268468223, and so on.

References

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