Highly oscillatory unimodular Fourier multipliers on modulation spaces

HIGHLY OSCILLATORY UNIMODULAR FOURIER MULTIPLIERS ON MODULATION SPACES

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ABSTRACT. We study the continuity on the modulation spaces $M^{p,q}$ of Fourier multipliers with symbols of the type $e^{i\mu(\xi)}$, for some real-valued function $\mu(\xi)$. A number of results are known, assuming that the derivatives of order ≥ 2 of the phase $\mu(\xi)$ are bounded or, more generally, that the second derivatives belong to the Sjöstrand class $M^{\infty,1}$. Here we extend those results, by assuming that the second derivatives lie in the bigger Wiener amalgam space $W(\mathcal{F}L^1, L^\infty)$; in particular they could have stronger oscillations at infinity such as $\cos|\xi|^2$. Actually our main result deals with the more general case of possibly unbounded second derivatives. In that case we have boundedness on weighted modulation spaces with a sharp loss of derivatives.

1. INTRODUCTION

Fourier multipliers represent one of the main research field in Harmonic Analysis, where a number of challenging problems remain open [24]. The connections with other branches of pure and applied mathematics are uncountable (combinatorics, PDEs, signal processing, functional calculus, etc.). In this paper we consider Fourier multipliers in $\mathbb{R}^d$ of the form

$$e^{i\mu(D)}f(x) := \int_{\mathbb{R}^d} e^{2\pi i x\xi} e^{i\mu(\xi)} \hat{f}(\xi)\, d\xi \tag{1.1}$$

for some real-valued phase μ.

The prototype is given by the phase $\mu(\xi) = |\xi|^2$. In that case the operator $e^{i\mu(D)}$ is the propagator for the free Schrödinger equation, and similarly for other constant coefficient equations. Hence it is of great interest to study the continuity of such operators on several functions spaces arising in PDEs. Whereas such operators represent unitary transformations of $L^2(\mathbb{R}^d)$, their continuity on $L^p(\mathbb{R}^d)$ for $p \neq 2$ in general fails. Hence recently a number of works addressed the problem of the continuity in other function spaces. Among those, the more convenient spaces, at least in the case of the Schrödinger model, turned out to be the modulation spaces $M^{p,q}(\mathbb{R}^d)$, $1 \leq p, q \leq \infty$, widely used in Time-frequency Analysis [16, 19]. The basic reason is that the

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Schrödinger propagator is sparse with respect to Gabor frames [10], which in turn give a discrete characterization of the modulation space norms. For $p = q = 2$ we have $M^{2,2}(\mathbb{R}^d) = L^2(\mathbb{R}^d)$ and, in general, the modulation space norm gives a measure of the size of a function or temperate distribution *in phase space* or *time-frequency plane*, exactly as the Lebesgue space norms L^p provide a measure of the size of a function in the physical space. The couple of indices p, q allows one to measure the decay both in the time and frequency domain, separately. We refer to the next section for the definition and basic properties of modulation spaces.

It is known (see e.g. [27, Proposition 1.5] and [1]) that the Schrödinger propagator (hence $\mu(\xi) = |\xi|^2$ in (1.1)) is bounded $M^{p,q}(\mathbb{R}^d) \to M^{p,q}(\mathbb{R}^d)$, for every $1 \le p, q \le \infty$. This result motivated the study of the continuity of more general unimodular Fourier multipliers on modulation spaces. The recent bibliography in this connection is quite large; see e.g. [1, 2, 4, 5, 7, 9, 13, 14, 20, 21, 22, 23, 30, 31, 32]. In short, it turns out that, for unbounded (smooth enough) phases, the properties which play a key role are:

Growth and oscillations of the second derivatives $\partial^\gamma \mu$, $|\gamma| = 2$.

To put our results in context, let us just recall three basic facts.

(a) *No growth, mild oscillations* [1, Theorem 11]. *Suppose that*

$$|\partial^\gamma \mu(\xi)| \le C \quad \text{for } \xi \in \mathbb{R}^d,\ 2 \le |\gamma| \le 2(\lfloor d/2 \rfloor + 1).$$

Then $e^{i\mu(D)} : M^{p,q}(\mathbb{R}^d) \to M^{p,q}(\mathbb{R}^d)$ *is bounded for every* $1 \le p, q \le \infty$.

This result generalizes the case of the Schrödinger propagator, where the second derivatives of μ are in fact constants.

(b) *No growth, mild oscillations* [7, Lemma 2.2]. *Suppose that*

$$\partial^\gamma \mu \in M^{\infty,1}(\mathbb{R}^d) \quad \text{for } |\gamma| = 2.$$

Then $e^{i\mu(D)} : M^{p,q}(\mathbb{R}^d) \to M^{p,q}(\mathbb{R}^d)$ *is bounded for every* $1 \le p, q \le \infty$.

Actually, [7, Lemma 2.2] provides a partial but key result in this connection, from which it is easy to deduce that the symbol $\sigma(\xi) = e^{i\mu(\xi)}$ is then in the Wiener amalgam space $W(\mathcal{F}L^1, L^\infty)$ (see below for the definition), which is sufficient to conclude (see also [3, 6, 26]). The result in (b) is also a particular case of [12, Theorem 2.3] where Schrödinger equations with rough Hamiltonians were considered.

Observe that the result in (b) improves that in (a), because of the embedding $C^{d+1}(\mathbb{R}^d) \hookrightarrow M^{\infty,1}(\mathbb{R}^d)$ ([19, Theorem 14.5.3]). We also notice that $M^{\infty,1}(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$, so that here the second derivatives of μ do not grow at infinity, but they could oscillate, say, as $\cos|\xi|^\alpha$, with $0 < \alpha \le 1$ (cf. [1, Corollary 15]).

(c) *Growth at infinity, mild oscillations* [22, Theorem 1.1]. *Let* $\alpha \geq 2$, *and suppose that*

$$|\partial^\gamma \mu(\xi)| \leq C\langle\xi\rangle^{\alpha-2} \quad \text{for } 2 \leq |\gamma| \leq \lfloor d/2\rfloor + 3.$$

Then $e^{i\mu(D)} : M^{p,q}_\delta(\mathbb{R}^d) \to M^{p,q}(\mathbb{R}^d)$ *is bounded for every* $1 \leq p, q \leq \infty$ *and* $\delta \geq d(\alpha-2)|1/p - 1/2|$.

Here $M^{p,q}_\delta$ is a modulation space weighted in frequency, so that we have in fact a loss of derivatives, which is proved to be sharp.

It was proved in [1, Lemma 8] that, more generally, the operator $e^{i\mu(D)}$ is bounded on all $M^{p,q}(\mathbb{R}^d)$ for every $1 \leq p, q \leq \infty$ if its symbol $e^{i\mu(\xi)}$ belongs to the Wiener amalgam space $W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d)$ [15], whose norm is defined as

$$\|f\|_{W(\mathcal{F}L^1,L^\infty)} = \sup_{x\in\mathbb{R}^d} \|g(\cdot - x)f\|_{\mathcal{F}L^1}$$

where $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ is an arbitrary window. This suggests to look at conditions on $\mu(\xi)$ in terms of this space, rather than modulation spaces. Here is our first result in this direction.

Theorem 1.1. *(No growth, strong oscillations). Let* $\mu \in C^2(\mathbb{R}^d)$, *real-valued, satisfying*

$$\partial^\gamma \mu(\xi) \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d) \quad \text{for } |\gamma| = 2.$$

Then

$$e^{i\mu(D)} : M^{p,q}(\mathbb{R}^d) \to M^{p,q}(\mathbb{R}^d)$$

is bounded for every $1 \leq p, q \leq \infty$.

Observe that $M^{\infty,1}(\mathbb{R}^d) \subset W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ so that this result improves that in (b) above. Here the second derivatives of μ are still bounded, but they are allowed to oscillate, say, as $\cos|\xi|^2$ (cf. [1, Theorem 14]). This result is strongly inspired by [7, Lemma 2.2] and in fact the proof is similar. However, our main result deals with the case of possibly unbounded second derivatives, as stated in the following theorem.

Theorem 1.2. *(Growth at infinity, strong oscillations). Let* $\alpha \geq 2$. *Let* $\mu \in C^2(\mathbb{R}^d)$, *real-valued and such that*

$$\langle\xi\rangle^{2-\alpha}\partial^\gamma \mu(\xi) \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d) \quad \text{for } |\gamma| = 2.$$

Then

$$e^{i\mu(D)} : M^{p,q}_\delta(\mathbb{R}^d) \to M^{p,q}(\mathbb{R}^d)$$

is bounded for every $1 \leq p, q \leq \infty$ *and*

$$\delta \geq d(\alpha - 2)\left|\frac{1}{p} - \frac{1}{2}\right|. \tag{1.2}$$

The above threshold for δ agrees with that in (c), and also with the examples in [1, Theorem 16], where even stronger oscillations were considered, but only for model cases.

Theorem 1.1 is of course a particular case of Theorem 1.2 and will be used as a step in the proof of the latter.

There are a number of easy extensions that could be considered. For example, the conclusion of Theorem 1.1 still holds for a phase of the type $\mu(\xi) = \mu_1(\xi) + \mu_2(\xi)$, where μ_1 satisfies the assumption in that theorem and $\mu_2 \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d)$. Hence one could allow phases that in a compact set do not have any derivatives but only $\mathcal{F}L^1$ regularity, such as a positively homogeneous function of order $r > 0$. We leave these easy developments to the interested reader.

In short the paper is organized as follows. In Section 2 we collected a number of definitions and auxiliary results. Section 3 is devoted to the proof of Theorem 1.1, whereas in Section 4 we prove Theorem 1.2.

Notation. We define $|x|^2 = x \cdot x$, for $x \in \mathbb{R}^d$, where $x \cdot y = xy$ is the scalar product on $\mathbb{R}^d$. We set $B_R(x_0)$ for the open ball in $\mathbb{R}^d$ of centre x_0 and radius R. The space of smooth functions with compact support is denoted by $C_0^\infty(\mathbb{R}^d)$, the Schwartz class is $\mathcal{S}(\mathbb{R}^d)$, the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$. The Fourier transform is normalized to be $\hat{f}(\xi) = \mathcal{F}f(\xi) = \int f(t)e^{-2\pi it\xi}dt$. Translation and modulation operators are defined, respectively, by

$$T_x f(t) = f(t-x) \quad \text{and} \quad M_\xi f(t) = e^{2\pi i\xi t} f(t).$$

We have the formulas $(T_x f)\hat{} = M_{-x}\hat{f}$, $(M_\xi f)\hat{} = T_\xi \hat{f}$, and $M_\xi T_x = e^{2\pi i x\xi} T_x M_\xi$. The inner product of two functions $f, g \in L^2(\mathbb{R}^d)$ is $\langle f, g\rangle = \int_{\mathbb{R}^d} f(t)\overline{g(t)}\, dt$, and its extension to $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ will be also denoted by $\langle \cdot, \cdot \rangle$. The notation $A \lesssim B$ means $A \leq cB$ for a suitable constant $c > 0$ depending only on the dimension d and Lebesgue exponents $p, q, \dots$, arising in the context, whereas $A \asymp B$ means $A \lesssim B$, and $B \lesssim A$. The notation $B_1 \hookrightarrow B_2$ denotes the continuous embedding of the space B_1 into B_2.

2. Preliminary results

In this section we recall the definition and some properties of modulation and Wiener amalgam spaces, which will be used later on. We refer to [16, 17, 19, 28] for the general theory.

We consider the functions $\langle \xi \rangle^s := (1 + |\xi|^2)^{s/2}$, $s \in \mathbb{R}$, as weight functions. Then, weighted modulation spaces are defined in terms of the following time-frequency representation: the short-time Fourier transform (STFT) $V_g f$ of a function/tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ with respect to the window $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ is defined by

$$V_g f(x, \xi) = \langle f, M_\xi T_x g\rangle = \int_{\mathbb{R}^d} e^{-2\pi i\xi y} f(y)\overline{g(y-x)}\, dy, \tag{2.1}$$

i.e. the Fourier transform of $f\overline{T_x g}$.

Given a window $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$, $1 \le p, q \le \infty$ and $\delta \in \mathbb{R}$, the *modulation space* $M^{p,q}_\delta(\mathbb{R}^d)$, consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $V_g f \in L^{p,q}_{1\otimes\langle\cdot\rangle^\delta}(\mathbb{R}^{2d})$ (weighted mixed-norm Lebesgue space). The norm on $M^{p,q}_\delta$ is therefore defined as

$$\|f\|_{M^{p,q}_\delta} = \|V_g f\|_{L^{p,q}_{1\otimes\langle\cdot\rangle^\delta}} = \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |V_g f(x,\xi)|^p \langle\xi\rangle^{\delta p}\, dx\right)^{q/p} d\xi\right)^{1/p}$$

(with obvious changes when $p = \infty$ or $q = \infty$). If $p = q$, we write M^p_δ instead of $M^{p,p}_\delta$, and if $\delta = 0$ we write $M^{p,q}$ and M^p for $M^{p,q}_0$ and $M^{p,p}_0$, respectively. Then $M^{p,q}_\delta(\mathbb{R}^d)$ is a Banach space and different windows $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ give equivalent norms. For the properties of these spaces we refer to the literature quoted at the beginning of this subsection.

For $1 \le p, q \le \infty$ the Wiener amalgam space $W(\mathcal{F}L^p, L^q)(\mathbb{R}^d)$ consists of the temperate distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{W(\mathcal{F}L^p, L^q)} = \left(\int_{\mathbb{R}^d} \|g(\cdot - x) f\|^q_{\mathcal{F}L^p}\, dx\right)^{1/q} < \infty,$$

where $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ is an arbitrary window (with obvious changes if $q = \infty$). It is easy to show that the Fourier transform establishes an isomorphism

$$\mathcal{F} : M^{p,q}(\mathbb{R}^d) \to W(\mathcal{F}L^p, L^q)(\mathbb{R}^d).$$

The duality theory goes as expected, namely $(M^{p,q}_\delta)^* = M^{p',q'}_{-\delta}$, with $1 \le p, q < \infty$, p', q' being the conjugate exponents.

Here is the basic complex interpolation result (see e.g. [15, 16, 17] and [29, Theorem 2.3] for a direct proof).

Proposition 2.1. *Let $0 < \theta < 1$, $p_j, q_j \in [1, \infty]$ and $\delta_j \in \mathbb{R}$, $j = 1, 2$. Set*

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \quad \delta = (1-\theta)\delta_1 + \theta\delta_2.$$

Then

$$(M^{p_1,q_1}_{\delta_1}(\mathbb{R}^d), M^{p_2,q_2}_{\delta_2}(\mathbb{R}^d))_{[\theta]} = M^{p,q}_\delta(\mathbb{R}^d).$$

We now recall the dilation properties. For $(1/p, 1/q) \in [0,1] \times [0,1]$, we define the subsets

$$I_1 : \max(1/p, 1/p') \le 1/q, \qquad I_1^* : \min(1/p, 1/p') \ge 1/q,$$

$$I_2 : \max(1/q, 1/2) \le 1/p', \qquad I_2^* : \min(1/q, 1/2) \ge 1/p',$$

$$I_3 : \max(1/q, 1/2) \le 1/p, \qquad I_3^* : \min(1/q, 1/2) \ge 1/p,$$

as shown in Figure 1 below.

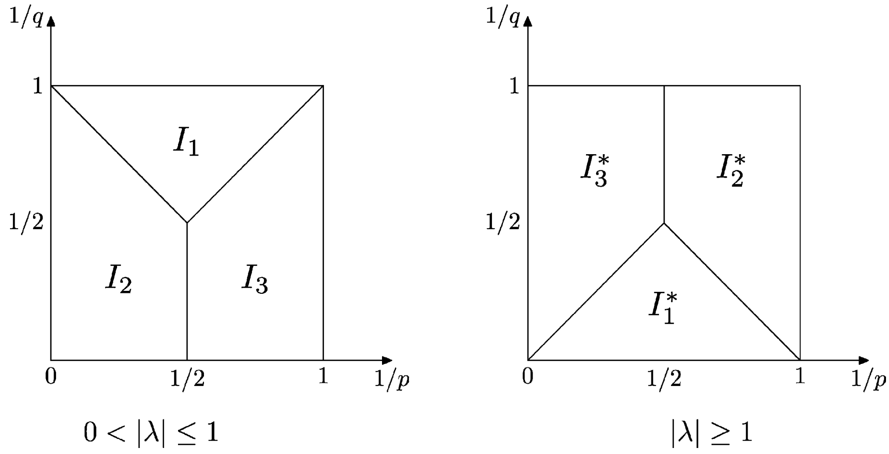

FIGURE 1. The index sets

We introduce the indices:

$$\mu_1(p,q)=\begin{cases}-1/p & \text{if} \quad (1/p,1/q)\in I_1^*,\\ 1/q-1 & \text{if} \quad (1/p,1/q)\in I_2^*,\\ -2/p+1/q & \text{if} \quad (1/p,1/q)\in I_3^*,\end{cases} \tag{2.2}$$

and

$$\mu_2(p,q)=\begin{cases}-1/p & \text{if} \quad (1/p,1/q)\in I_1,\\ 1/q-1 & \text{if} \quad (1/p,1/q)\in I_2,\\ -2/p+1/q & \text{if} \quad (1/p,1/q)\in I_3.\end{cases} \tag{2.3}$$

Here is the main result about the behaviour of the dilation operator in modulation spaces. Set $U_\lambda f(x) := f(\lambda x)$, $\lambda \neq 0$.

Theorem 2.2. [25, Theorem 3.1] *Let* $1 \leq p,q \leq \infty$, *and* $\lambda \neq 0$.

(i) We have

$$\|U_\lambda f\|_{M^{p,q}} \lesssim |\lambda|^{d\mu_1(p,q)}\|f\|_{M^{p,q}}, \qquad \forall\ |\lambda|\geq 1,\ \forall f\in M^{p,q}(\mathbb{R}^d).$$

Conversely, if there exists $\alpha\in\mathbb{R}$ *such that*

$$\|U_\lambda f\|_{M^{p,q}} \lesssim |\lambda|^{\alpha}\|f\|_{M^{p,q}}, \qquad \forall\ |\lambda|\geq 1,\ \forall f\in M^{p,q}(\mathbb{R}^d),$$

then $\alpha \geq d\mu_1(p,q)$.

(ii) We have

$$\|U_\lambda f\|_{M^{p,q}} \lesssim |\lambda|^{d\mu_2(p,q)}\|f\|_{M^{p,q}}, \qquad \forall\ 0<|\lambda|\leq 1,\ \forall f\in M^{p,q}(\mathbb{R}^d).$$

Conversely, if there exists $\beta\in\mathbb{R}$ *such that*

$$\|U_\lambda f\|_{M^{p,q}} \lesssim |\lambda|^{\beta}\|f\|_{M^{p,q}}, \qquad \forall\ 0<|\lambda|\leq 1,\ \forall f\in M^{p,q}(\mathbb{R}^d),$$

then $\beta \leq d\mu_2(p,q)$.

By a conjugation with the Fourier transform one deduces the following dilation property for Wiener amalgam spaces.

Corollary 2.3. [8, Corollary 3.2] *With the above notation,*

$$\|U_\lambda f\|_{W(\mathcal{F}L^1,L^\infty)} \lesssim \|f\|_{W(\mathcal{F}L^1,L^\infty)} \quad \forall\, 0<|\lambda|\leq 1, \forall f\in W(\mathcal{F}L^1,L^\infty)(\mathbb{R}^d).$$

The following proposition can be deduced, via Fourier transform, from the convolution properties of modulation spaces.

Proposition 2.4. [8, Proposition 2.5] *For every* $1\leq p,q\leq\infty$ *we have*

$$\|fu\|_{W(\mathcal{F}L^p,L^q)} \lesssim \|f\|_{W(\mathcal{F}L^1,L^\infty)}\|u\|_{W(\mathcal{F}L^p,L^q)}.$$

We conclude this preliminary section with a known result (see e.g. [1, Lemma 8]), which we recall together with a shorter proof for the benefit of the reader.

Lemma 2.5. *Let* $\sigma\in W(\mathcal{F}L^1,L^\infty)$*. Then,*

$$\sigma(D): M^{p,q}\to M^{p,q}$$

is bounded, for every $1\leq p,q\leq\infty$*.*

Proof. We can write $\sigma(D)=\mathcal{F}^{-1}\circ A_\sigma\circ\mathcal{F}$, where $A_\sigma f(\xi)=\sigma(\xi)f(\xi)$. Using Proposition 2.4 we have

$$\begin{aligned}\|A_\sigma f\|_{W(\mathcal{F}L^p,L^q)} &= \|\sigma f\|_{W(\mathcal{F}L^p,L^q)}\\ &\lesssim \|\sigma\|_{W(\mathcal{F}L^1,L^\infty)}\|f\|_{W(\mathcal{F}L^p,L^q)},\end{aligned}$$

so that $A_\sigma: W(\mathcal{F}L^p,L^q)\to W(\mathcal{F}L^p,L^q)$ is bounded, for every $1\leq p,q\leq\infty$. Hence, since the Fourier transform establishes an isomorphism $\mathcal{F}: M^{p,q}\to W(\mathcal{F}L^p,L^q)$, we see that $\sigma(D): M^{p,q}\to M^{p,q}$ is bounded too. □

3. No growth, strong oscillations

This section is devoted to the proof of Theorem 1.1. We begin with a preliminary result which is strongly inspired by [7, Lemmas 2.1 and 2.2], where a similar investigation is carried on in the framework of modulation spaces (as opposite to the Wiener amalgam spaces considered here).

Lemma 3.1. *Let* $f\in W(\mathcal{F}L^1,L^\infty)(\mathbb{R}^d)$ *and* $\chi\in C_0^\infty(B)$*, where* B *is an open ball with center at the origin. Let*

$$g_{x_0}(x)=\chi(x-x_0)\int_0^1(1-t)f(t(x-x_0)+x_0)\,dt,$$

for some $x_0\in\mathbb{R}^d$*.*

Then $g_{x_0}\in W(\mathcal{F}L^1,L^\infty)(\mathbb{R}^d)$*, and for some constant* C *independent of* x_0 *and* f *we have*

$$\|g_{x_0}\|_{W(\mathcal{F}L^1,L^\infty)}\leq C\|f\|_{W(\mathcal{F}L^1,L^\infty)}.$$

Proof. Using Proposition 2.4 and Corollary 2.3 we have

$$\begin{aligned}
\|g_{x_0}(x)\|_{W(\mathcal{F}L^1,L^\infty)} &= \left\|\chi(x-x_0)\int_0^1(1-t)f(t(x-x_0)+x_0)dt\right\|_{W(\mathcal{F}L^1,L^\infty)}\\
&\lesssim \|\chi(x-x_0)\|_{W(\mathcal{F}L^1,L^\infty)}\left\|\int_0^1(1-t)f(tx+(1-t)x_0)dt\right\|_{W(\mathcal{F}L^1,L^\infty)}\\
&\le \|\chi(x-x_0)\|_{W(\mathcal{F}L^1,L^\infty)}\int_0^1(1-t)\,\|f(tx+(1-t)x_0)\|_{W(\mathcal{F}L^1,L^\infty)}\,dt\\
&= \|\chi\|_{W(\mathcal{F}L^1,L^\infty)}\int_0^1(1-t)\,\|f(tx)\|_{W(\mathcal{F}L^1,L^\infty)}\,dt\\
&\lesssim \|\chi\|_{W(\mathcal{F}L^1,L^\infty)}\int_0^1(1-t)\,\|f\|_{W(\mathcal{F}L^1,L^\infty)}\,dt\\
&\lesssim \|f\|_{W(\mathcal{F}L^1,L^\infty)}\,.
\end{aligned}$$

□

Lemma 3.2. *Assume that $B\subset\mathbb{R}^n$ is an open ball, $\mu\in C^2(\mathbb{R}^d)$ is real-valued and satisfies $\partial^\gamma\mu\in W(\mathcal{F}L^1,L^\infty)(\mathbb{R}^d)$ for all multi-indices γ with $|\gamma|=2$ and that $f\in M^1(\mathbb{R}^d)\cap\mathcal{E}'(B)$. Then $fe^{i\mu}\in M^1(\mathbb{R}^d)$ and for some constant C which only depends on d and the radius of the ball B we have*

$$\|fe^{i\mu}\|_{M^1}\le C\|f\|_{M^1}\exp\Big(C\sum_{|\gamma|=2}\|\partial^\gamma\mu\|_{W(\mathcal{F}L^1,L^\infty)}\Big).$$

Proof. We may assume that B is the unit ball which is centered at the origin. By Taylor expansion it follows that $\mu=\psi_1+\psi_2$, where

$$\psi_1(x)=\mu(0)+\langle\nabla\mu(0),x\rangle,\quad \psi_2(x)=\sum_{|\gamma|=2}\frac{2}{\gamma!}\int_0^1(1-t)\partial^\gamma\mu(tx)dt\,x^\gamma.$$

Since modulations do not affect the modulation space norms we have $\|fe^{i\psi_1}\|_{M^1}=\|f\|_{M^1}$. Furthermore, if $\chi\in C_0^\infty(\mathbb{R}^d)$ satisfies $\chi(x)=1$ on B, then it follows from the previous Lemma that, for some constant $C_1>0$,

$$\|\chi\psi_2\|_{W(\mathcal{F}L^1,L^\infty)}\le C_1\sum_{|\gamma|=2}\|\partial^\gamma\mu\|_{W(\mathcal{F}L^1,L^\infty)}.$$

Hence, by Proposition 2.4, for some $C_2\ge 1$ we have

$$\begin{aligned}
\|e^{i\chi\psi_2}\|_{W(\mathcal{F}L^1,L^\infty)} &= \left\|\sum_{n=0}^\infty\frac{(\chi\psi_2)^n}{n!}\right\|_{W(\mathcal{F}L^1,L^\infty)}\le\sum_{n=0}^\infty\frac{C_2^{n-1}}{n!}\|\chi\psi_2\|^n_{W(\mathcal{F}L^1,L^\infty)}\\
&\le\exp\Big(C_2\|\chi\psi_2\|_{W(\mathcal{F}L^1,L^\infty)}\Big)\\
&\le\exp\Big(C_1C_2\sum_{|\gamma|=2}\|\partial^\gamma\mu\|_{W(\mathcal{F}L^1,L^\infty)}\Big).
\end{aligned}$$

Using $M^1 = W(\mathcal{F}L^1, L^\infty)$ and Proposition 2.4 again, this gives

$$\begin{aligned}\|fe^{i\mu}\|_{M^1} = \|fe^{i\psi_1}e^{i\chi\psi_2}\|_{M^1} &\lesssim \|fe^{i\psi_1}\|_{M^1}\|e^{i\chi\psi_2}\|_{W(\mathcal{F}L^1,L^\infty)} \\ &\leq C\|f\|_{M^1}\exp\Big(C\sum_{|\gamma|=2}\|\partial^\gamma\mu\|_{W(\mathcal{F}L^1,L^\infty)}\Big).\end{aligned}$$

□

Proof of Theorem 1.1. Let us first show that $e^{i\mu(x)} \in W(\mathcal{F}L^1, L^\infty)$. We know that there exists $\chi \in C_0^\infty(\mathbb{R}^d)$ (cf. [15, 16]) such that

$$\|e^{i\mu(x)}\|_{W(\mathcal{F}L^1,L^\infty)} = \sup_{k\in\mathbb{Z}^d}\{\|\chi(x-k)e^{i\mu(x)}\|_{\mathcal{F}L^1}\} \asymp \sup_{k\in\mathbb{Z}^d}\{\|\chi(x-k)e^{i\mu(x)}\|_{M^1}\}$$

where the last equivalence follows from that fact the for functions supported in a ball the $\mathcal{F}L^1$ and M^1 norms are equivalent, with constants depending only on the radius of the ball.

Hence, using Lemma 3.2 we can continue our estimate as

$$\begin{aligned}&\leq \sup_{k\in\mathbb{Z}^d}\Big\{C\|\chi(x-k)\|_{M^1}\exp\Big(C\sum_{|\gamma|=2}\|\partial^\gamma\mu\|_{W(\mathcal{F}L^1,L^\infty)}\Big)\Big\} \\ &= C\|\chi\|_{M^1}\exp\Big(C\sum_{|\gamma|=2}\|\partial^\gamma\mu\|_{W(\mathcal{F}L^1,L^\infty)}\Big).\end{aligned}$$

Hence $e^{i\mu(x)} \in W(\mathcal{F}L^1, L^\infty)$ and by Lemma 2.5 we deduce that $e^{i\mu(D)} : M^{p,q} \to M^{p,q}$ is bounded, for every $1 \leq p, q \leq \infty$. □

4. Growth at infinity, strong oscillations

In this section we are going to prove Theorem 1.2.

We begin with the following preliminary result.

Lemma 4.1. *Let $\mu(\xi)$ be a real-valued C^2 function, satisfying*

$$\langle\xi\rangle^{2-\alpha}\partial^\gamma\mu \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d) \text{ for } |\gamma| = 2. \tag{4.1}$$

Then

(i) $\langle\xi\rangle^{-\alpha}\mu \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d)$,

(ii) $\langle\xi\rangle^{1-\alpha}\partial^\gamma\mu \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d)$ *for* $|\gamma| = 1$.

Proof. To prove (i), consider a Taylor expansion

$$\mu(\xi) = \mu(0) + \langle\nabla\mu(0), \xi\rangle + \sum_{|\gamma|=2}\frac{2}{\gamma!}\int_0^1(1-t)\partial^\gamma\mu(t\xi)dt\,\xi^\gamma.$$

Hence

$$\begin{aligned}\langle\xi\rangle^{-\alpha}\mu(\xi) = \mu(0)\langle\xi\rangle^{-\alpha} &+ \langle\nabla\mu(0), \xi\rangle\langle\xi\rangle^{-\alpha} \\ &+ \sum_{|\gamma|=2}\frac{2}{\gamma!}\int_0^1(1-t)\partial^\gamma\mu(t\xi)dt\,\xi^\gamma\langle\xi\rangle^{-\alpha}\end{aligned} \tag{4.2}$$

where

$$\mu(0)\langle\xi\rangle^{-\alpha} \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d), \quad \langle\nabla\mu(0), \xi\rangle\langle\xi\rangle^{-\alpha} \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d),$$

because $\alpha \geq 2$. Here we used that fact that the functions $\langle\xi\rangle^{-\alpha}$ and $\xi_j\langle\xi\rangle^{-\alpha}$ are bounded together their derivatives of every order, so that they belong to $M^{\infty,1}(\mathbb{R}^d)$ ([19, Theorem 14.5.3]) and hence to $W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d)$ as well.

Let us show that the last summation in (4.2) belongs to $W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d)$ too. We have

$$\begin{aligned}
&\left\| \int_0^1 (1-t)\partial^\gamma\mu(t\xi)dt\xi^\gamma\langle\xi\rangle^{-\alpha} \right\|_{W(\mathcal{F}L^1,L^\infty)} \\
&= \left\| \int_0^1 (1-t)\partial^\gamma\mu(t\xi)\langle t\xi\rangle^{2-\alpha}\langle t\xi\rangle^{-2+\alpha}dt\,\xi^\gamma\langle\xi\rangle^{-\alpha} \right\|_{W(\mathcal{F}L^1,L^\infty)} \\
&\lesssim \left\| \int_0^1 (1-t)\langle t\rangle^{-2+\alpha}\partial^\gamma\mu(t\xi)\langle t\xi\rangle^{2-\alpha}dt\,\xi^\gamma\langle\xi\rangle^{-\alpha}\langle\xi\rangle^{-2+\alpha} \right\|_{W(\mathcal{F}L^1,L^\infty)}.
\end{aligned}$$

Using Proposition 2.4 and Corollary 2.3 we can continue the above estimate as

$$\begin{aligned}
&\lesssim \int_0^1 (1-t)\langle t\rangle^{-2+\alpha} \left\|\partial^\gamma\mu(t\xi)\langle t\xi\rangle^{2-\alpha}\right\|_{W(\mathcal{F}L^1,L^\infty)} dt \left\|\xi^\gamma\langle\xi\rangle^{-2}\right\|_{W(\mathcal{F}L^1,L^\infty)} \\
&\lesssim \int_0^1 (1-t)\langle t\rangle^{-2+\alpha}dt \left\|\partial^\gamma\mu(\xi)\langle\xi\rangle^{2-\alpha}\right\|_{W(\mathcal{F}L^1,L^\infty)} \left\|\xi^\gamma\langle\xi\rangle^{-2}\right\|_{W(\mathcal{F}L^1,L^\infty)} \\
&\lesssim \left\|\partial^\gamma\mu(\xi)\langle\xi\rangle^{2-\alpha}\right\|_{W(\mathcal{F}L^1,L^\infty)} \left\|\xi^\gamma\langle\xi\rangle^{-2}\right\|_{W(\mathcal{F}L^1,L^\infty)}.
\end{aligned}$$

This concludes the proof of (i) because, arguing as above, we have $\xi^\gamma\langle\xi\rangle^{-2} \in M^{\infty,1} \subset W(\mathcal{F}L^1, L^\infty)$, whereas $\partial^\gamma\mu(\xi)\langle\xi\rangle^{2-\alpha} \in W(\mathcal{F}L^1, L^\infty)$ by assumption.

To prove (ii), consider the Taylor expansion of $\partial^\gamma\mu$, for $|\gamma| = 1$

$$\partial^\gamma\mu(\xi) = \partial^\gamma\mu(0) + \sum_{|\beta|=1}\int_0^1 \partial^{\gamma+\beta}\mu(t\xi)dt\,\xi^\beta,$$

so that

$$\langle\xi\rangle^{1-\alpha}\partial^\gamma\mu(\xi) = \partial^\gamma\mu(0)\langle\xi\rangle^{1-\alpha} + \sum_{|\beta|=1}\int_0^1 \partial^{\gamma+\beta}\mu(t\xi)dt\,\xi^\beta\langle\xi\rangle^{1-\alpha}.$$

Now $\partial^\gamma \mu(0)\langle\xi\rangle^{1-\alpha} \in W(\mathcal{F}L^1, L^\infty)$, because $\alpha \geq 2$, and arguing as above

$$\begin{aligned}&\left\| \int_0^1 \partial^{\gamma+\beta}\mu(t\xi)dt\,\xi^\beta\langle\xi\rangle^{1-\alpha}\right\|_{W(\mathcal{F}L^1,L^\infty)}\\ &= \left\| \int_0^1 \partial^{\gamma+\beta}\mu(t\xi)\langle t\xi\rangle^{2-\alpha}\langle t\xi\rangle^{-2+\alpha}dt\,\xi^\beta\langle\xi\rangle^{1-\alpha}\right\|_{W(\mathcal{F}L^1,L^\infty)}\\ &\lesssim \left\|\partial^{\gamma+\beta}\mu(\xi)\langle\xi\rangle^{2-\alpha}\right\|_{W(\mathcal{F}L^1,L^\infty)}\left\|\xi^\beta\langle\xi\rangle^{-1}\right\|_{W(\mathcal{F}L^1,L^\infty)},\end{aligned}$$

where $\xi^\beta\langle\xi\rangle^{-1} \in M^{\infty,1}(\mathbb{R}^d) \subset W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d)$ because $|\beta| = 1$, and moreover $\partial^{\gamma+\beta}\mu(\xi)\langle\xi\rangle^{2-\alpha} \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d)$ by assumption, because $\gamma + \beta = 2$. □

We observe that, by complex interpolation of weighted modulation spaces, namely Proposition 2.1, it suffices to prove the conclusion of Theorem 1.2 when (p, q) is one of the four vertices of the interpolation square, $(1, 1)$, $(1,\infty)$, $(\infty, 1)$, (∞,∞), with $\delta = d(\alpha - 2)/2$, as well as for the points $(2, 1)$, $(2,\infty)$ with $\delta = 0$. To this end, we reduce matters to the case of unweighted modulation spaces by means of the following easy lemma.

Lemma 4.2. *A multiplier $\sigma(D)$ is bounded from $M^{p,q}_\delta(\mathbb{R}^d)$ to $M^{p,q}(\mathbb{R}^d)$ if and only if the multiplier $\sigma(D)\langle D\rangle^{-\delta}$ is bounded on $M^{p,q}(\mathbb{R}^d)$.*

Proof. We know e.g. from [27, Theorem 2.2, Corollary 2.3] that $\langle D\rangle^t$ defines an isomorphism $M^{p,q}_s(\mathbb{R}^d) \to M^{p,q}_{s-t}(\mathbb{R}^d)$ for every $s, t \in \mathbb{R}$, so that the conclusion is immediate. □

Therefore we may work with the operator

$$Tf(x) = \int_{\mathbb{R}^d} e^{2\pi i x\xi} e^{i\mu(\xi)}\langle\xi\rangle^{-\delta}\hat{f}(\xi)d\xi.$$

We have to prove that T is bounded on $M^{1,1}$, $M^{1,\infty}$, $M^{\infty,1}$, $M^{\infty,\infty}$ for $\delta = \frac{d(\alpha-2)}{2}$, and on $M^{2,1}$ and $M^{2,\infty}$ for $\delta = 0$.

Boundedness on $M^{1,1}$ and $M^{\infty,1}$ for $\delta = \frac{d(\alpha-2)}{2}$. We will need the following lemma (cf. [11, 25]).

Lemma 4.3. *Let χ be a smooth function supported where $B_0^{-1} \leq |\xi| \leq B_0$ for some $B_0 > 0$. Then, for $1 \leq p \leq \infty$,*

$$\sum_{j=1}^{\infty} \|\chi(2^{-j}D)f\|_{M^{p,1}} \leq C\|f\|_{M^{p,1}}.$$

Proof. We will use the following characterization of the $M^{p,q}$ norm [28]: let $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that $\varphi(\xi) \geq 0$, $\sum_{m\in\mathbb{Z}^d}\varphi(\xi - m) = 1$, $\forall\xi \in \mathbb{R}^d$. Then

$$\|f\|_{M^{p,q}} \asymp \Big(\sum_{m\in\mathbb{Z}^d}\|\varphi(D - m)f\|^q_{L^p}\Big)^{1/q}.$$

Hence it turns out

$$\sum_{j=1}^{\infty}\|\chi(2^{-j}D)f\|_{M^{p,1}}\asymp\sum_{j=1}^{\infty}\sum_{m\in\mathbb{Z}^d}\|\varphi(D-m)\chi(2^{-j}D)f\|_{L^p}$$
$$=\sum_{m\in\mathbb{Z}^d}\sum_{j=1}^{\infty}\|\chi(2^{-j}D)\varphi(D-m)f\|_{L^p}.$$

Now, the number of indices $j\geq 1$ for which $\operatorname{supp}\chi(2^{-j}\cdot)\cap\operatorname{supp}\varphi(\cdot-m)\neq 0$ is finite for every m, and even uniformly bounded with respect to m. Hence the last expression is

$$\lesssim\sum_{m\in\mathbb{Z}^d}\sup_{j\geq 1}\|\chi(2^{-j}D)\varphi(D-m)f\|_{L^p}.$$

Since the operators $\chi(2^{-j}D)$ are uniformly bounded on L^p we can continue the estimate as

$$\lesssim\sum_{m\in\mathbb{Z}^d}\|\varphi(D-m)f\|_{L^p}\asymp\|f\|_{M^{p,1}}.$$

□

Consider now a Littlewood-Paley decomposition of the frequency domain. Namely, fix a smooth function ψ_0 such that $\psi_0(\xi)=1$ for $|\xi|\leq 1$ and $\psi_0(\xi)=0$ for $|\xi|\geq 2$. Set $\psi(\xi)=\psi_0(\xi)-\psi_0(2\xi)$. Then $\psi_j(\xi):=\psi(2^{-j}\xi)$ for $j\geq 1$ is supported where $2^{j-1}\leq|\xi|\leq 2^{j+1}$. We can write

$$T=T^{(0)}+\sum_{j=1}^{\infty}T^{(j)}\tag{4.3}$$

where $T^{(j)}$ is the Fourier multiplier with symbol $\sigma_j(\xi):=e^{i\mu(\xi)}\psi_j(\xi)\langle\xi\rangle^{-\delta}$, $j\geq 0$.

Now, $T^{(0)}$ is bounded on $M^{p,q}$ for every $1\leq p,q\leq\infty$ as a consequence of Lemma 2.5, because $\sigma_0\in M^1\subset W(FL^1,L^\infty)$ by Lemma 3.2.

Consider now the above sum over $j\geq 1$.

Let

$$\lambda_j=2^{-\frac{\alpha-2}{2}j},$$

and consider the operators $\tilde{T}^{(j)}$ defined by

$$T^{(j)}=U_{\lambda_j}\tilde{T}^{(j)}U_{\lambda_j^{-1}},\tag{4.4}$$

where $U_\lambda f(x)=f(\lambda x)$, $\lambda>0$, is the dilation operator. In other terms,

$$\tilde{T}^{(j)}f(x)=\int_{\mathbb{R}^d}e^{2\pi ix\xi}e^{i\mu(\lambda_j\xi)}\psi_j(\lambda_j\xi)\langle\lambda_j\xi\rangle^{-\delta}\hat{f}(\xi)d\xi.$$

Let $\chi_j(\chi) := \chi(2^{-j}\xi)$ with $\chi \in C_0^\infty(\mathbb{R}^d)$ supported where $\frac{1}{4} \leq |\xi| \leq 4$ and $\chi(\xi) = 1$ on the support of ψ, so that $\chi_j(\xi) = 1$ on the support of ψ_j. We can therefore write

$$\tilde{T}^{(j)} f(x) = \int_{\mathbb{R}^d} e^{2\pi i x\xi} e^{i\chi_j(\lambda_j\xi)\mu(\lambda_j\xi)} \psi_j(\lambda_j\xi)\langle\lambda_j\xi\rangle^{-\delta} \hat{f}(\xi) d\xi,$$

hence

$$\tilde{T}^{(j)} = A_j B_j, \tag{4.5}$$

where

$$A_j = e^{i(\chi_j\mu)(\lambda_j D)}, \qquad B_j = \psi_j(\lambda_j D)\langle\lambda_j D\rangle^{-\delta}.$$

Taking into account that on the support of $\psi_j(\lambda_j\xi)$ we have $\lambda_j|\xi| \asymp 2^j$ and $\delta = d(\alpha-2)/2$, the following estimate is easily verified:

$$|\partial^\gamma(\psi_j(\lambda_j\xi)\langle\lambda_j\xi\rangle^{-\delta})| \lesssim 2^{-\frac{d(\alpha-2)}{2}j}, \quad \forall \gamma \in \mathbb{Z}_+^d.$$

Then, by the classical boundedness results of pseudodifferential operators on modulation spaces (see e.g. [19, Theorems 14.5.2, 14.5.2]) we have

$$\|B_j\|_{M^{p,q}\to M^{p,q}} \lesssim 2^{-\frac{d(\alpha-2)}{2}j}, \tag{4.6}$$

for every $1 \leq p, q \leq \infty$.

Let us now prove that

$$\|A_j\|_{M^{p,q}\to M^{p,q}} \lesssim 1, \tag{4.7}$$

for all $j \geq 1$ and for every $1 \leq p, q \leq \infty$.

Using Theorem 1.1 it is sufficient to check that

$$\|\partial^\gamma[\chi_j(\lambda_j\xi)\mu(\lambda_j\xi)]\|_{W(\mathcal{F}L^1, L^\infty)} \lesssim 1,$$

for $|\gamma| = 2$ and all $j \geq 1$ (we are in fact using the fact that the operator norm of the multiplier in Theorem 1.1 is bounded when $\partial^\gamma\mu$, $|\gamma| = 2$, belong to a bounded subset of $W(\mathcal{F}L^1, L^\infty)$).

For $|\gamma| = 2$, we have

$$\partial^\gamma[\chi_j(\lambda_j\xi)\mu(\lambda_j\xi)] = \lambda_j^2 \partial^\gamma[\chi_j\mu](\lambda_j\xi),$$

and by Leibniz' formula it is enough to prove that

$$\lambda_j^2 \|(\partial^\gamma\chi_j)\mu\|_{W(\mathcal{F}L^1, L^\infty)} \lesssim 1 \qquad |\gamma| = 2 \tag{4.8}$$

$$\lambda_j^2 \|\partial^\gamma\chi_j \partial^\beta\mu\|_{W(\mathcal{F}L^1, L^\infty)} \lesssim 1 \qquad |\gamma| = |\beta| = 1 \tag{4.9}$$

$$\lambda_j^2 \|\chi_j \partial^\gamma\mu\|_{W(\mathcal{F}L^1, L^\infty)} \lesssim 1 \qquad |\gamma| = 2. \tag{4.10}$$

First, let us prove (4.8). Using Lemma 4.1 (i), Proposition 2.4 and the embeddings $C^{d+1}(\mathbb{R}^d) \hookrightarrow M^{\infty,1}(\mathbb{R}^d) \hookrightarrow W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d)$ ([19, Theorem 14.5.3]) we can estimate

$$\begin{aligned}\lambda_j^2 \|(\partial^\gamma\chi_j)\mu\|_{W(\mathcal{F}L^1, L^\infty)} &\lesssim \lambda_j^2 \|\langle\xi\rangle^{-\alpha}\mu\|_{W(\mathcal{F}L^1, L^\infty)} \|\langle\xi\rangle^\alpha \partial^\gamma\chi_j\|_{W(\mathcal{F}L^1, L^\infty)} \\ &\lesssim \lambda_j^2 \sum_{\beta \leq d+1} \|\partial^\beta[\langle\xi\rangle^\alpha \partial^\gamma\chi_j]\|_{L^\infty}.\end{aligned}$$

On the other hand,

$$
\begin{aligned}
|\partial^\beta[\langle\xi\rangle^\alpha\partial^\gamma\chi_j(\xi)]| &= \left|\sum_{\nu\leq\beta}\binom{\beta}{\nu}\partial^\nu\langle\xi\rangle^\alpha\partial^{\gamma+\beta-\nu}\chi_j(\xi)\right| \\
&\lesssim \sum_{\nu\leq\beta}\binom{\beta}{\nu}\langle\xi\rangle^{\alpha-|\nu|}2^{-j|\gamma+\beta-\nu|}\left|(\partial^{\gamma+\beta-\nu}\chi)(2^{-j}\xi)\right| \\
&\lesssim \sum_{\nu\leq\beta}\binom{\beta}{\nu}2^{(\alpha-|\nu|)j}2^{-2j}\lesssim 2^{j(\alpha-2)},
\end{aligned}
$$

because on the support of χ_j, $|\xi|\asymp 2^j$ and $|\gamma+\beta-\nu|\geq 2$. Thus

$$
\lambda_j^2\|(\partial^\gamma\chi_j)\mu\|_{W(\mathcal{F}L^1,L^\infty)}\lesssim\lambda_j^2 2^{j(\alpha-2)}=1.
$$

Now, let us prove (4.9), using Lemma 4.1 (ii) and arguing as above we write

$$
\begin{aligned}
\lambda_j^2\|\partial^\gamma\chi_j\partial^\beta\mu\|_{W(\mathcal{F}L^1,L^\infty)} &\lesssim \lambda_j^2\|\langle\xi\rangle^{1-\alpha}\partial^\beta\mu\|_{W(\mathcal{F}L^1,L^\infty)}\|\langle\xi\rangle^{\alpha-1}\partial^\gamma\chi_j\|_{W(\mathcal{F}L^1,L^\infty)} \\
&\lesssim \lambda_j^2\sum_{\beta\leq d+1}\|\partial^\beta[\langle\xi\rangle^{\alpha-1}\partial^\gamma\chi_j]\|_{L^\infty}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
|\partial^\beta[\langle\xi\rangle^{\alpha-1}\partial^\gamma\chi_j(\xi)]| &\lesssim \sum_{\nu\leq\beta}\binom{\beta}{\nu}\langle\xi\rangle^{\alpha-1-|\nu|}2^{-j|\gamma+\beta-\nu|}\left|(\partial^{\gamma+\beta-\nu}\chi)(2^{-j}\xi)\right| \\
&\lesssim \sum_{\nu\leq\beta}\binom{\beta}{\nu}2^{(\alpha-1-|\nu|)j}2^{-j}\lesssim 2^{j(\alpha-2)},
\end{aligned}
$$

because now $|\gamma+\beta-\nu|\geq 1$. Thus

$$
\lambda_j^2\|\partial^\gamma\chi_j\partial^\beta\mu\|_{W(\mathcal{F}L^1,L^\infty)}\lesssim\lambda_j^2 2^{j(\alpha-2)}=1.
$$

Finally, let's us prove (4.10), using the hypothesis $\langle\xi\rangle^{2-\alpha}\partial^\gamma\mu(\xi)\in W(\mathcal{F}L^1,L^\infty)(\mathbb{R}^d)$, $|\gamma|=2$, we have

$$
\begin{aligned}
\lambda_j^2\|\chi_j\partial^\gamma\mu\|_{W(\mathcal{F}L^1,L^\infty)} &\lesssim \lambda_j^2\|\langle\xi\rangle^{2-\alpha}\partial^\gamma\mu\|_{W(\mathcal{F}L^1,L^\infty)}\|\langle\xi\rangle^{\alpha-2}\chi_j\|_{W(\mathcal{F}L^1,L^\infty)} \\
&\lesssim \lambda_j^2\sum_{\beta\leq d+1}\|\partial^\beta[\langle\xi\rangle^{\alpha-2}\chi_j]\|_{L^\infty}.
\end{aligned}
$$

Moreover, arguing as above

$$
\begin{aligned}
|\partial^\beta[\langle\xi\rangle^{\alpha-2}\chi_j(\xi)]| &\lesssim \sum_{\nu\leq\beta}\binom{\beta}{\nu}\langle\xi\rangle^{\alpha-2-|\nu|}2^{-j|\beta-\nu|}\left|(\partial^{\beta-\nu}\chi)(2^{-j}\xi)\right| \\
&\lesssim \sum_{\nu\leq\beta}\binom{\beta}{\nu}2^{(\alpha-2-|\nu|)j}2^0\lesssim 2^{j(\alpha-2)}.
\end{aligned}
$$

Thus

$$
\lambda_j^2\|\chi_j\partial^\gamma\mu\|_{W(\mathcal{F}L^1,L^\infty)}\lesssim\lambda_j^2 2^{j(\alpha-2)}=1.
$$

Hence, by (4.5), (4.6) and (4.7) we have

$$\|\tilde{T}^{(j)}f\|_{M^{p,q}} = \|A_jB_jf\|_{M^{p,q}} \lesssim 2^{-\frac{d(\alpha-2)}{2}j}\|f\|_{M^{p,q}}, \tag{4.11}$$

for every $1 \le p, q \le \infty$.

We now combine this estimate with those for the dilation operator, given in Theorem 2.2. For $p = 1, \infty$ and $q = 1$ they read

$$\|U_{\lambda_j}f\|_{M^{1,1}} \lesssim 2^{\frac{d(\alpha-2)}{2}j}\|f\|_{M^{1,1}},$$

$$\|U_{\lambda_j}f\|_{M^{\infty,1}} \lesssim \|f\|_{M^{\infty,1}},$$

and

$$\|U_{\lambda_j^{-1}}f\|_{M^{1,1}} \lesssim \|f\|_{M^{1,1}},$$

$$\|U_{\lambda_j^{-1}}f\|_{M^{\infty,1}} \lesssim 2^{\frac{d(\alpha-2)}{2}j}\|f\|_{M^{\infty,1}}.$$

Therefore we obtain, for $p = 1, \infty$,

$$\|T^{(j)}f\|_{M^{p,1}} \lesssim 2^{-\frac{d(\alpha-2)}{2}j}2^{\frac{d(\alpha-2)}{2}j}\|f\|_{M^{p,1}} = \|f\|_{M^{p,1}}.$$

Finally, to sum over $j \ge 1$ these last estimates we take advantage of the fact we are working with functions which are localized in shells of the frequency domain. Precisely, let χ as before, namely a smooth function satisfying $\chi(\xi) = 1$ for $1/2 \le |\xi| \le 2$ and $\chi(\xi) = 0$ for $|\xi| \le 1/4$ and $|\xi| \ge 4$ (so that $\chi\psi = \psi$). With $\chi_j(\xi) = \chi(2^{-j}\xi)$ and $p = 1, \infty$ we have

$$\|T^{(j)}f\|_{M^{p,1}} = \|T^{(j)}(\chi(2^{-j}D)f)\|_{M^{p,1}} \lesssim \|\chi(2^{-j}D)f\|_{M^{p,1}},$$

so that Lemma 4.3 gives us

$$\left\|\sum_{j\ge1} T^{(j)}f\right\|_{M^{p,1}} \le \sum_{j\ge1} \|T^{(j)}f\|_{M^{p,1}} \lesssim \|f\|_{M^{p,1}}.$$

Boundedness on $M^{1,\infty}$ and $M^{\infty,\infty}$ for $\delta = \frac{d(\alpha-2)}{2}$. We first establish the following lemma (cf. [26, Proposition 1.4] and [11, 25]).

Lemma 4.4. *For $k \ge 0$, let $f_k \in \mathcal{S}(\mathbb{R}^d)$ satisfy* $\operatorname{supp} \hat{f}_0 \subset B_2(0)$ *and*

$$\operatorname{supp} \hat{f}_k \subset \{\xi \in \mathbb{R}^d : \ 2^{k-1} \le |\xi| \le 2^{k+1}\}, \quad k \ge 1.$$

Then, if the sequence f_k is bounded in $M^{p,\infty}(\mathbb{R}^d)$ for some $1 \le p \le \infty$, the series $\sum_{k=0}^{\infty} f_k$ converges in $M^{p,\infty}(\mathbb{R}^d)$ and

$$\left\|\sum_{k=0}^{\infty} f_k\right\|_{M^{p,\infty}} \lesssim \sup_{k\ge0} \|f_k\|_{M^{p,\infty}}. \tag{4.12}$$

Proof. The convergence of the series $\sum_{k=0}^{\infty} f_k$ in $M^{p,\infty}(\mathbb{R}^d)$ is straightforward. We now prove the desired estimate.

Choose a window function g with $\operatorname{supp}\hat{g} \subset B_{1/2}(0)$. We can write

$$V_g(f_k)(x,\xi) = (\hat{f}_k * M_{-x}\hat{\overline{g}})(\xi).$$

Hence, $\operatorname{supp} V_g(f_0) \subset B_{5/2}(0) \subset B_{2^2}(0)$, and

$$\begin{aligned}\operatorname{supp} V_g(f_k) &\subset \{(x,\xi) \in \mathbb{R}^{2d} :\ 2^{k-1} - 2^{-1} \leq |\xi| \leq 2^{k+1} + 2^{-1}\} \\ &\subset \{(x,\xi) \in \mathbb{R}^{2d} :\ 2^{k-2} \leq |\xi| \leq 2^{k+2}\},\end{aligned}$$

for $k \geq 1$. Hence, for each ξ, there are at most four nonzero terms in the sum $\sum_{k=0}^{\infty} \|V_g(f_k)(\cdot,\xi)\|_{L^p}$. Using this fact we obtain

$$\begin{aligned}\|\sum_{k=0}^{\infty} f_k\|_{M^{p,\infty}} &\asymp \|\sum_{k=0}^{\infty} V_g(f_k)\|_{L^{p,\infty}} \leq \sup_{\xi\in\mathbb{R}^d} \sum_{k=0}^{\infty} \|V_g(f_k)(\cdot,\xi)\|_{L^p} \\ &\leq 4 \sup_{k\geq 0} \sup_{\xi\in\mathbb{R}^d} \|V_g(f_k)(\cdot,\xi)\|_{L^p} = 4 \sup_{k\geq 0} \|V_g(f_k)\|_{L^{p,\infty}} \\ &\asymp \sup_{k\geq 0} \|f_k\|_{M^{p,\infty}}.\end{aligned}$$

□

We now consider the same decomposition as above, namely (4.3), and the operators $\tilde{T}^{(j)}$ in (4.4), $j \geq 1$. From (4.11) for $q = \infty$ we have the following estimate:

$$\|\tilde{T}^{(j)} f\|_{M^{p,\infty}} \leq 2^{-\frac{d(\alpha-2)}{2}j} \|f\|_{M^{p,\infty}}.$$

We then combine this estimate with those for the dilation operator which here read

$$\|U_{\lambda_j} f\|_{M^{1,\infty}} \lesssim 2^{d(\alpha-2)j} \|f\|_{M^{1,\infty}},$$

$$\|U_{\lambda_j} f\|_{M^{\infty,\infty}} \lesssim 2^{\frac{d(\alpha-2)}{2}j} \|f\|_{M^{\infty,\infty}},$$

and

$$\|U_{\lambda_j^{-1}} f\|_{M^{1,\infty}} \lesssim 2^{-\frac{d(\alpha-2)}{2}j} \|f\|_{M^{1,\infty}},$$

$$\|U_{\lambda_j^{-1}} f\|_{M^{\infty,\infty}} \lesssim \|f\|_{M^{\infty,\infty}}.$$

Therefore we obtain, for $p = 1, \infty$,

$$\|T^{(j)} f\|_{M^{p,\infty}} \lesssim 2^{-\frac{d(\alpha-2)}{2}j} 2^{\frac{d(\alpha-2)}{2}j} \|f\|_{M^{p,\infty}} = \|f\|_{M^{p,\infty}}.$$

We finally conclude by applying Lemma 4.4: for $p = 1, \infty$,

$$\left\|\sum_{j=1}^{\infty} T^{(j)} f\right\|_{M^{p,\infty}} \lesssim \sup_{j\geq 1} \|T^{(j)} f\|_{M^{p,\infty}} \lesssim \|f\|_{M^{p,\infty}}.$$

Boundedness on $M^{2,1}$ and $M^{2,\infty}$ for $\delta = 0$. Indeed, we will prove boundedness on $M^{2,q}$ for every $1 \leq q \leq \infty$ and $\delta = 0$. This is a special case of the following result.

Proposition 4.5. *Any Fourier multiplier T with symbol $\sigma \in L^\infty$ is bounded on $M^{2,q}$ for every $1 \leq q \leq \infty$.*

Proof. The desired result follows at once from the estimates $\|\sigma(D)f\|_{L^2} \leq \|\sigma\|_{L^\infty}\|f\|_{L^2}$ and [18, Theorem 17 (3)].

We provide and direct proof for the benefit of the reader. Namely

$$\begin{aligned} \|Tf\|_{M^{2,q}} &= \|\|M_x \hat{\bar{g}} * \left(\sigma \hat{f}\right)\|_{L^2_x}\|_{L^q} \\ &= \|\|\int e^{2\pi i x(\xi - y)}\hat{\bar{g}}(\xi - y)\sigma(y)\hat{f}(y)dy\|_{L^2_x}\|_{L^q_\xi} \\ &= \|\|\sigma \hat{f} T_\xi \hat{\bar{g}}\|_{L^2}\|_{L^q_\xi}, \end{aligned}$$

where we used Parseval's formula. In particular, this computation with $\sigma \equiv 1$ gives $\|f\|_{M^{2,q}} = \|\|\hat{f} T_\xi \hat{g}\|_{L^2}\|_{L^q_\xi}$, so we deduce at once the desired estimate

$$\|Tf\|_{M^{2,q}} \lesssim \|\sigma\|_{L^\infty}\|f\|_{M^{2,q}}.$$

□

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