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On the Equivalence of Displacement-Based Third-Order Shear Deformation Plate Theories

Marco DI SCIUVA
Department of Mechanical and Aerospace Engineering-Politecnico di Torino (I)
Corso Duca degli Abruzzi, 24-10129 Torino (I)
marco.disciuva@polito.it

Abstract

Based on the literature review of the assumed kinematics in the so-called higher-order displacement-based shear deformation theories, a generalization of this kinematics is first proposed and used to formulate variationally consistent field equations and boundary conditions for bending and vibration of a flat plate of rectangular platform. Second, attention is focused on the displacement-based polynomial shear deformation plate theories. It is shown that all the kinematics of polynomial third-order theories ({3,0}-order polynomial) proposed in the open literature are special cases of the present theory. Furthermore, it is concluded that the {3,0}-order polynomial kinematics of all the theories is the same when the maximum transverse shear strain is used as generalized displacement coordinates. A deep analysis of the static and dynamic behavior of simply supported rectangular plates in cylindrical bending is performed in order to substantiate the general conclusion that all the {3,0}-order polynomial displacement-based shear deformation theories give the same numerical results, i.e., they are kinematically equivalent, although not all are statically equivalent.

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Key words

Beams and plate theories; higher-order shear-deformation theory; polynomials theories.

Introduction

There is a huge open literature on the single and multilayer beam/plate/shells theories that account for transverse shear strains and stresses and provide various degrees of refinement to the classical beam/plate/shell theories. These theories are known in the literature as higher-order shear deformation theories. The interested reader can usefully refer, among others, the recent reviews by Hu et al. 2008, Asadi et al. 2012, Khandan et al. 2012, Viola et al. 2013, Sayyad et al. 2015, 2017, Abrate et al. 2017.

Although the open literature has recently registered an increasing interest also on theories where the higher-order contribution to the in-plane displacement are given through trigonometric or, in general, hyperbolic and exponential functions (see, for example, Viola et al. 2013, Sayyad et al. 2015, 2017, Abrate et al. 2017), the polynomial displacement-based theories, are certainly still the most used in the open literature. In this paper, the focus is on those displacement-based theories which assume a third-order polynomial expansion for the in-plane displacements and a zero-order expansion for the transverse displacement along the thickness, the most commonly used in the open literature, Abrate et al. 2017. Hereafter they will be named {3,0}-order polynomial displacement-based theories, in order to distinguishe them form other displacement-based third-order theories where also the transverse displacement is a function of the thickness co-ordinate.

Each author generally claims that his proposed theory is a new theory and, obviously, better than the other theories.

With very few exceptions, the kinematics of various displacement-based third-order theories are perceived by many researchers as being different from one another and thus giving rise to different plate theories.

To the best author's knowledge, the first researcher to draw attention on the equivalence among various kinematical hypotheses used in the literature has been Jemielita 1990, *"To set the record straight, I wish to point out that the kinematical hypothesis used by the aforementioned authors, as well as by other contributors, was a starting point in the Vlasov's theory dated from 1957."*

At the same conclusion arrived Reddy 1990 "*The displacement field used by all the authors is the same, except for the choice of variables.* Omissis. *Thus, the works of Levinson (1980)*, Murthy (1981)*, Bhimarrady and Stevens (1984)*, and Senthilnathan et al (1987)* are essentially duplicates of previously existing works*." Kapuria et al 2004, after an accurate comparative analysis of the governing equations of two theories concluded "*The equations of motion and boundary conditions of Ray's theory, Ray (2003)*, are mathematically equivalent to those of the dynamic version of Reddy's theory.*" Recently, Challamel et al 2013 performed a plate buckling analysis of simply supported rectangular flat plate and derived an analytical formula for the buckling load that is common to all higher-order shear plate models. They concluded "*It is shown that cubic-based interpolation models for the displacement field are kinematically equivalent, and lead to the same buckling load results. This conclusion concerns for instance the plate models of Reddy J [J. appl. Mech. **51**(1984) 745**]** or the one of Shi [Int. J. Solids Struct. **44** (2007) 4299] event though these models are statically distinct (leading to* *different stress calculation along the cross-section).*" More recently, Nguyen et al 2016 presented a unified approach within the general framework of high-order deformation plate theories and assessed the validity of the proposed approach. The authors concluded "*..omissis..In the current HSDT framework, the deflection and stresses obtained based on the generalized displacement field are not affected by the linear combinations of transverse shear functions. For instant, the KPR Model (Kaczkowski 1968, Reissner 1975, Panc 1975)* and LMR (Levinson 1980, Reddy 1984, Murthy 1981)* are the linear combination of Ambartsumian 1960*..omissis... As a consequence, the static results of the three models are exactly identical .. omissis..*".

Note that the starred references in the quoted sentences refer to those used in the present paper.

It is apparent from the previous review that there is an interest to substantiate within a general framework the conclusions drawn by these researchers on the equivalence of {3,0}-kinematics and the limit of this equivalence, at least to restrict the profileration of claimed new {3,0}-kinematics.

The purpose of this article is twofold. First, starting from a fairly general assumed kinematics encompassing many of the higher-order shear deformation theories (HSDT) appeared in the open literature, derive the equations of motion and the variationally consistent boundary conditions. Secondly, with specific reference to the {3,0}-order polynomial theories, substantiate the full equivalence of the various kinematics, also those so-called three- and four-variable plate theories. For this purpose, a generalized {3,0}-order polynomial kinematics is first formulated which contains, as special cases, all the {3,0}-order polynomial kinematics, and it is shown that all of these kinematics satisfying the zero transverse shear strain on the bottom and top surfaces of the plate are equivalent when the transverse shear strain in the middle plane of the plate is assumed as generalized displacement co-ordinate.

Subsequently, in order to substantiate the general conclusion, the general equations are particularized to the cylindrical bending of simply supported plates and it is shown that, at least for the two problems investigated (**P1**) bending under transverse load, (**P2**) natural frequencies, the results are independent of the parameters entering the generalized {3,0}-order polynomial kinematics, i.e., all the theories give the same results.

Geometry and reference frame

We consider a rectangular flat plate of constant thichness, *h*, length *a* and width *b*. The point of 3-D plate are referred to a right handed rectangular Cartesian coordinate system, (x, y, z), where (x, y) denotes the reference plane of the plate (here selected to coincide with the mid-surface of the plate), and $z \in \left[-\frac{h}{2}, \frac{h}{2}\right]$ the thickness coordinate. So, the edges of the plate are *x=0,a* and *y=0,b*, respectively, and the top and bottom surfaces of the plate are placed at $z = \pm\frac{h}{2}$ (see, fig. 1).

The plate is subjected to a transverse load $\bar{p}_z$ applied on the top surface of the plate, and to uniformly distributed in-plane edge loads for unit length, $\bar{P}_{xx}, \bar{P}_{yy}$ and $\bar{P}_{xy}$ applied along the edges *x=0,a* and *y=0,b*, respectively (see, fig. 1).

In what follows, a comma followed by subscripts is used to denote partial differentiation with respect to the subscripts; for example, $w_{,x}^{(0)} = \partial w^{(0)} / \partial x$, and so on. The overdot indicates differentiation with respect to time, *t*. Unless otherwise stated, Greek indices range fram 1 to 2, respectively, with $1 \equiv x -$ axis and $2 \equiv y -$axis. Furthermore, if not otherwise stated, repeated indices imply the summation over the range of variation of those indices.

Kinematics

The displacement field corresponding to the plate theories under consideration may be expressed in its general form as follows

$$\begin{aligned} &u_{\alpha}(x,y,z;t) = u_{\alpha}^{(0)}(x,y;t) + f^{(b)}(z) w_{,\alpha}^{(0)}(x,y;t) + f^{(s)}(z) g_{\alpha}(x,y;t) \quad (\alpha = x, y) \\ &u_{z}(x,y.z;t) = w^{(0)}(x,y;t) \end{aligned} \tag{1}$$

where $u_{\alpha}(x,y,\mathrm{z};t)$ and $u_{z}(x,y,z;t)$ are the displacements along the α- and $z-$axes, respectively, $u_{\alpha}^{(0)}(x,y;t)$ is the uniform in-plane displacements, $w^{(0)}(x,y;t)$ is the deflection, $g_{\alpha}(x,y;t)$ represents a generalized in-plane displacement component; $f^{(b)}(z)$ and $f^{(s)}(z)$ are functions of the thickness coordinate, z. Note that if $f^{(b)}(z)$ is a non-linear function of z (more precisely, $f^{(b)}(z) \neq -z$), then also $w_{,x}^{(0)}(x,y;t)$ and $w_{,y}^{(0)}(x,y;t)$ will contribute to the transverse shear deformations, as can be argued from Eq. (2).

Strains

Corresponding to the displacement field given by Eq. (1) and within the realm of the linear plate theory, the following expressions for the components of the strain tensor hold,

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{(0)} + f^{(b)} \boldsymbol{\varepsilon}^{(b)} + f^{(s)} \boldsymbol{\varepsilon}^{(s)}; \qquad \boldsymbol{\gamma} = \left(1 + f_{,z}^{(b)}\right) \boldsymbol{\gamma}^{(b)} + f_{,z}^{(s)} \boldsymbol{\gamma}^{(s)} \tag{2}$$

where

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} = 2\varepsilon_{xy} \end{Bmatrix} = \begin{Bmatrix} u_{x,x} \\ u_{y,y} \\ u_{x,y} + u_{y,x} \end{Bmatrix}, \quad \boldsymbol{\gamma} = \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} 2\varepsilon_{xz} \\ 2\varepsilon_{yz} \end{Bmatrix} = \begin{Bmatrix} u_{x,z} + w^{(0)}_{,x} \\ u_{y,z} + w^{(0)}_{,y} \end{Bmatrix} \tag{3}$$

and

$$\begin{gathered}
\boldsymbol{\varepsilon}^{(0)} = \begin{Bmatrix} \varepsilon^{(0)}_{xx} \\ \varepsilon^{(0)}_{yy} \\ \gamma^{(0)}_{xy} \end{Bmatrix} = \begin{Bmatrix} u^{(0)}_{x,x} \\ u^{(0)}_{y,y} \\ u^{(0)}_{x,y} + u^{(0)}_{y,x} \end{Bmatrix}, \ \boldsymbol{\varepsilon}^{(b)} = \begin{Bmatrix} \varepsilon^{(b)}_{xx} \\ \varepsilon^{(b)}_{yy} \\ \gamma^{(b)}_{xy} \end{Bmatrix} = \begin{Bmatrix} w^{(0)}_{,xx} \\ w^{(0)}_{,yy} \\ 2w^{(0)}_{,xy} \end{Bmatrix}, \ \boldsymbol{\varepsilon}^{(s)} = \begin{Bmatrix} \varepsilon^{(s)}_{xx} \\ \varepsilon^{(s)}_{yy} \\ \gamma^{(s)}_{xy} \end{Bmatrix} = \begin{Bmatrix} g_{x,x} \\ g_{y,y} \\ g_{x,y} + g_{y,x} \end{Bmatrix}, \\
\boldsymbol{\gamma}^{(b)} = \begin{Bmatrix} \gamma^{(b)}_{xz} \\ \gamma^{(b)}_{yz} \end{Bmatrix} = \begin{Bmatrix} w^{(0)}_{,x} \\ w^{(0)}_{,y} \end{Bmatrix}, \ \boldsymbol{\gamma}^{(s)} = \begin{Bmatrix} \gamma^{(s)}_{xz} \\ \gamma^{(s)}_{yz} \end{Bmatrix} = \begin{Bmatrix} g_{x} \\ g_{y} \end{Bmatrix}
\end{gathered} \tag{4}$$

Stresses

Within the assumptions that (i) the material is linearly elastic and orthotropic, with a plane of elastic symmetry parallel to the reference plane, (ii) the transverse normal stress $\sigma_z = 0$, the constitutive relations in the material principal axes of orthotropy take on the form

$$\boldsymbol{\sigma} = \mathbf{Q}_{\sigma}\boldsymbol{\varepsilon}; \quad \boldsymbol{\tau} = \mathbf{Q}_{\tau}\boldsymbol{\gamma} \tag{5}$$

In expanded form,

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}; \quad \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \begin{bmatrix} Q_{44} & 0 \\ 0 & Q_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \tag{6}$$

where σ and τ stand for the normal and shear stress components, $Q_{ij}\left(i, j=1,2,6\right)$ and $Q_{ii}\left(i=4,5\right)$ denote the in-plane and transverse shear elastic reduced stiffness coefficients, respectively (see, Reddy 2004).

Equations of motion and variationally consistent boundary conditions

The equations of motion and the variationally consistent boundary conditions for the plate under consideration are derived using the D'Alembert's principle (**V**ariational **E**quation of **D**ynamics (**VED**)) (see, Appendix).

Equations of Motion

$$\delta u_{\alpha}^{(0)} \Rightarrow N_{\alpha\beta,\beta} = m^{(0)}\ddot{u}_{\alpha}^{(0)} + m^{(b)}\ddot{w}_{,\alpha}^{(0)} + m^{(s)}\ddot{g}_{\alpha} \tag{7}$$

$$\delta w^{(0)} \Rightarrow -V_{z\alpha,\alpha}^{(b)} = \overline{p}_z - m^{(0)}\ddot{w}^{(0)} + m^{(b)}\ddot{u}_{\alpha,\alpha}^{(0)} + m^{(bb)}\ddot{w}_{,\alpha\alpha}^{(0)} + m^{(sb)}\ddot{g}_{\alpha,\alpha} \tag{8}$$

$$\delta g_{\alpha} \Rightarrow -V_{z\alpha}^{(s)} = m^{(s)}\ddot{u}_{\alpha}^{(0)} + m^{(bs)}\ddot{w}_{,\alpha}^{(0)} + m^{(ss)}\ddot{g}_{\alpha} \tag{9}$$

where we have posed

$$V_{z\alpha}^{(b)} = -\left(R_{\beta\alpha,\beta}^{(b)} - R_{\alpha z}^{(b)}\right);\; V_{z\alpha}^{(s)} = -\left(R_{\beta\alpha,\beta}^{(s)} - R_{\alpha z}^{(s)}\right). \tag{10}$$

Boundary conditions

The variationally consistent boundary conditions for the present theory are of the form

Specify,

$\underline{x = 0, a)}$

$$
\begin{aligned}
&\text{either } u_\alpha^{(0)} = \bar{u}_\alpha^{(0)} \quad \text{or} \quad N_{x\alpha} = \bar{P}_{x\alpha} \\
&w^{(0)} = \bar{w}^{(0)} \quad \text{or} \quad V_{zx} + m^{(b)}\ddot{u}^{(0)} + m^{(bb)}\ddot{w}_{,x}^{(0)} + m^{(sb)}\ddot{g}_x = 0 \\
&w_{,\alpha}^{(0)} = \bar{w}_{,\alpha}^{(0)} \quad \text{or} \quad R_{x\alpha}^{(b)} = 0 \\
&g_\alpha = \bar{g}_\alpha \quad \text{or} \quad R_{x\alpha}^{(s)} = 0
\end{aligned}
\tag{11}
$$

$\underline{y = 0, b)}$

$$
\begin{aligned}
&\text{either } u_\alpha^{(0)} = \bar{u}_\alpha^{(0)} \quad \text{or} \quad N_{y\alpha} = \bar{P}_{y\alpha} \\
&w^{(0)} = \bar{w}^{(0)} \quad \text{or} \quad V_{zy} + m^{(b)}\ddot{v}^{(0)} + m^{(bb)}\ddot{w}_{,y}^{(0)} + m^{(sb)}\ddot{g}_y = 0 \\
&w_{,\alpha}^{(0)} = \bar{w}_{,\alpha}^{(0)} \quad \text{or} \quad R_{\alpha y}^{(b)} = 0 \\
&g_\alpha = \bar{g}_\alpha \quad \text{or} \quad R_{\alpha y}^{(s)} = 0
\end{aligned}
\tag{12}
$$

Equations of motion in terms of generalized displacements

The above equations of motion are given in terms of force and moment stress resultants. To express them in terms of generalized displacements, we use the plate constitutive equations, Eq. (97), and the strain-displacement relations, Eqs. (2)-(4). The result is

$$
\delta u_\alpha^{(0)})\ L_{\alpha\beta}^{A} u_\beta^{(0)} + {}^{(b)B}L_{\alpha+3} w^{(0)} + {}^{(s)B}L_{\alpha\beta}\, g_\beta = m^{(0)}\ddot{u}_\alpha^{(0)} + m^{(b)}\ddot{w}_{,\alpha}^{(0)} + m^{(s)}\ddot{g}_\alpha \tag{13}
$$

$$
\begin{aligned}
\delta w^{(0)})\ &{}^{(b)B}L_{\alpha+3} u_\alpha^{(0)} + {}^{(b)D}L_{6} w^{(0)} + {}^{(bs)D}L_{\alpha+3}\, g_\alpha - {}^{(bs)}A_{\alpha+3,\alpha+3}\left(g_{\alpha,\alpha} + w_{,\alpha\alpha}^{(0)}\right) = \\
&= \bar{p}_z - m^{(0)}\ddot{w}^{(0)} + m^{(b)}\ddot{u}_{\alpha,\alpha}^{(0)} + m^{(bb)}\ddot{w}_{,\alpha\alpha}^{(0)} + m^{(sb)}\ddot{g}_{\alpha,\alpha}
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
\delta g_\alpha)\ &{}^{(s)B}L_{\alpha\beta} u_\beta^{(0)} + {}^{(bs)D}L_{3+\alpha} w^{(0)} - {}^{(bb)}A_{\alpha+3,\alpha+3} w_{,\alpha}^{(0)} + {}^{(s)D}L_{\alpha\beta}\, g_\beta^{(0)} - {}^{(bs)}A_{\alpha+3,\alpha+3}\, g_\alpha = \\
&= m^{(s)}\ddot{u}_\alpha^{(0)} + m^{(bs)}\ddot{w}_{,\alpha}^{(0)} + m^{(ss)}\ddot{g}_\alpha
\end{aligned}
\tag{15}
$$

where the differential operators L are defined as follows

$$
\begin{aligned}
&L_{11}^{\tau} = \tau_{11}(.)_{,xx} + \tau_{66}(.)_{,yy};\ \ L_{12}^{\tau} = L_{21}^{\tau} = \left(\tau_{12} + \tau_{66}\right)(.)_{,xy};\ \ L_{22}^{\tau} = \tau_{66}(.)_{,xx} + \tau_{22}(.)_{,yy}; \\
&L_{4}^{\tau} = \tau_{11}(.)_{,xxx} + \left(\tau_{12} + 2\tau_{66}\right)(.)_{,xyy};\ \ L_{5}^{\tau} = \left(\tau_{12} + 2\tau_{66}\right)(.)_{,xyy} + \tau_{22}(.)_{,yyy}; \\
&L_{6}^{\tau} = \tau_{11}(.)_{,xxxx} + 2\left(\tau_{12} + 2\tau_{66}\right)(.)_{,xxyy} + \tau_{22}(.)_{,yyyy};
\end{aligned}
\tag{16}
$$

On the choice of the functions $f^{(b)}(z)$ and $f^{(s)}(z)$

In the previous Sections, we derived the equations of motion and the variationally consistent boundary condition using as starting point the general unified kinematics given by Eq. (1). As stated in the Introduction, this kinematics is very general, i.e., many of the assumed kinematics in higher-order shear deformation beam/plate/shell theories actually can be obtained by appropriately choosing the functions $f^{(b)}(z)$ and $f^{(s)}(z)$.

In this Section, we will focus our attention on the so-called {3,0}-*order polynomial displacement-based theories*, with the aim to show that all these theories satisfying zero transverse shear strain on the bottom and top surfaces of the plate are kinematically equivalent.

For this purpose, let us first compute the zero (mean value) and first-moment (rotation) of the in-plane displacement, i.e.,

$$U_\alpha = \frac{1}{h}\left\langle u_\alpha \right\rangle = \frac{1}{h}\int_{-h/2}^{+h/2} u_\alpha dz \tag{17}$$

$$\phi_\alpha = \frac{1}{h}\left\langle z u_\alpha \right\rangle = \frac{1}{h}\int_{-h/2}^{+h/2} z u_\alpha dz \tag{18}$$

Substituting Eq. (1) into Eq. (17), yields

$$U_\alpha = u_\alpha^{(0)} + \frac{1}{h}\left\langle f^{(b)}(z)\right\rangle w_{,\alpha}^{(0)} + \frac{1}{h}\left\langle f^{(s)}(z)\right\rangle g_\alpha \tag{19}$$

Then, for $u_\alpha^{(0)}(x,y;t)$ be the uniform in-plane displacement components along the *x*- and *y*-axis, respectively, the following kinematical constraints must be satisfied,

$$\left\langle f^{(b)}(z)\right\rangle = 0; \qquad \left\langle f^{(s)}(z)\right\rangle = 0 \tag{20}$$

Let us consider the transverse shearing strain at the top and bottom bounding surfaces of the plate. From Eq. (2)-(4), it follows

$$\gamma_{\alpha z}\left(\pm\frac{h}{2}\right) = \left(1 + f_{,z}^{(b)}\left(\pm\frac{h}{2}\right)\right) w_{,\alpha}^{(0)} + f_{,z}^{(s)}\left(\pm\frac{h}{2}\right) g_\alpha \tag{21}$$

So, for the transverse shearing strain (and stress) vanishes on the top and bottom bounding surfaces, the following relations must hold

$$f_{,z}^{(b)}\left(\pm\frac{h}{2}\right)=-1; \quad f_{,z}^{(s)}\left(\pm\frac{h}{2}\right)=0 \tag{22}$$

Constraints on the functions $f^{(b)}(z)$ and $f^{(s)}(z)$ as given by Eqs. (20) and (22) are listed in the first two rows of Table 1. A cursory examination of these constraints highlights that they allow for a very large class of higher-order theories to be built-up all satisfying the free-boundary conditions of the transverse shearing stresses on the top and bottom faces. Obviously, in general, different choices for these functions will result in different kinematics. As remarked in the Introduction, it is quite common belief that these kinematics are different from one another. We will discuss this in detail in the next Sections.

As the commonly used higher-order shear deformation plate theories belong to the so-called {3,0}-order polynomial shear deformation theories, in the following we will focus our attention on this class of theories with the functions $f^{(b)}(z)$ and $f^{(s)}(z)$ satisfying the constraints listed in the first two rows of Table 1.

{3,0}-order polynomial kinematics.

Let us expand $f^{(b)}(z)$ and $f^{(s)}(z)$ in a cubic power series of z, i.e.,

$$f^{(b)}(z)=f_0^{(b)}+f_1^{(b)}z+f_2^{(b)}z^2+f_3^{(b)}z^3 \tag{23}$$

$$f^{(s)}(z)=f_0^{(s)}+f_1^{(s)}z+f_2^{(s)}z^2+f_3^{(s)}z^3 \tag{24}$$

By satisfying the constraints listed in the first and second row of Table1, we obtain

$$f_0^{(b)}=0, \quad f_2^{(b)}=0, \quad f_1^{(b)}+\frac{3}{4}h^2 f_3^{(b)}=-1 \tag{25}$$

$$f_0^{(s)} = 0, \quad f_2^{(s)} = 0, \quad f_1^{(s)} + \frac{3}{4}h^2 f_3^{(s)} = 0 \tag{26}$$

Substituting these results into Eqs. (23) and (24), yields

$$f^{(b)}(z) = f_1^{(b)} z\left(1 - \frac{4}{3h^2}z^2\right) - \frac{4}{3h^2}z^3 = f_1^{(b)} F(z) - c_1 z^3 = f_1^{(b)} z - (1 + f_1^{(b)})\frac{4}{3h^2}z^3 \tag{27}$$

$$f^{(s)}(z) = f_1^{(s)} z\left(1 - \frac{4}{3h^2}z^2\right) = f_1^{(s)} F(z) \tag{28}$$

where, following Reddy 2004, we have posed

$$F(z) = z\left(1 - c_1 z^2\right) \quad \text{with} \quad c_1 = \frac{4}{3h^2}. \tag{29}$$

Substituting Eqs. (27) and (28) into Eq. (1), yields

$$\begin{aligned} u_\alpha &= u_\alpha^{(0)} + \left(f_1^{(b)} z\left(1 - \frac{4}{3h^2}z^2\right) - \frac{4}{3h^2}z^3 \right) w_{,\alpha}^{(0)} + f_1^{(s)} z\left(1 - \frac{4}{3h^2}z^2\right) g_\alpha \\ &= u_\alpha^{(0)} + \left(f_1^{(b)} F(z) - c_1 z^3 \right) w_{,\alpha}^{(0)} + f_1^{(s)} F(z) g_\alpha \end{aligned} \tag{30}$$

Table 2 gives a (not exhaustive) list of the explicit expressions for the functions $f^{(b)}(z)$ and $f^{(s)}(z)$ of well-known {3,0}-order polynomial plate theories. It should be stressed that the physical meaning of g_α $(\alpha = x, y)$ in Table 2 depends on the displacement field employed.

It is clearly shown that *all the listed displacement fields are special cases of that given in Eq. (30), i.e., they are obtained from Eq. (30) by giving specific values to the arbitrary parameters* $f_1^{(b)}$ *and* $f_1^{(s)}$.

Let us return back to the strain expressions, Eqs. (2)-(4). By taking into account Eqs. (27) and (28), we obtain

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{(0)} + f^{(b)}\boldsymbol{\varepsilon}^{(b)} + f^{(s)}\boldsymbol{\varepsilon}^{(s)} = \boldsymbol{\varepsilon}^{(0)} + \left(f_1^{(b)} F(z) - c_1 z^3 \right)\boldsymbol{\varepsilon}^{(b)} + f_1^{(s)} F(z)\boldsymbol{\varepsilon}^{(s)} \tag{31}$$

$$\boldsymbol{\gamma} = \left(1 + f_{,z}^{(b)}\right)\boldsymbol{\gamma}^{(b)} + f_{,z}^{(s)}\boldsymbol{\gamma}^{(s)} = F_{,z}(z)\left(\left(1 + f_1^{(b)}\right)\boldsymbol{\gamma}^{(b)} + f_1^{(s)}\boldsymbol{\gamma}^{(s)}\right) = F_{,z}\boldsymbol{\gamma}^{(0)} \tag{32}$$

where

$$\boldsymbol{\gamma}^{(0)} = \left(1 + f_1^{(b)}\right)\boldsymbol{\gamma}^{(b)} + f_1^{(s)}\boldsymbol{\gamma}^{(s)} \tag{33}$$

is the transverse shear strain on the reference plane, $z = 0$. Table 3 gives the espressions of $\gamma_{\alpha z}^{(0)}$ of the considered displacement-based third-order plate theories as special cases of Eq. (33). Following many authors, let us re-write Eqs. (31) and (32) as follows,

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{(0)} + z\boldsymbol{\varepsilon}^{(1)} + z^3\boldsymbol{\varepsilon}^{(3)} \tag{34}$$

$$\boldsymbol{\gamma} = \boldsymbol{\gamma}^{(0)} + z^2\boldsymbol{\gamma}^{(2)} \tag{35}$$

A comparison of Eqs. (34) and (35) with Eqs. (31) and (32), yields

$$\begin{aligned}
&\boldsymbol{\varepsilon}^{(1)} = \left(f_1^{(b)}\boldsymbol{\varepsilon}^{(b)} + f_1^{(s)}\boldsymbol{\varepsilon}^{(s)}\right) = \left(\left(1 + f_1^{(b)}\right)\boldsymbol{\varepsilon}^{(b)} + f_1^{(s)}\boldsymbol{\varepsilon}^{(s)} - \boldsymbol{\varepsilon}^{(b)}\right) = \nabla\boldsymbol{\gamma}^{(0)} - \boldsymbol{\varepsilon}^{(b)} \\
&\boldsymbol{\varepsilon}^{(3)} = -c_1\left(\left(1 + f_1^{(b)}\right)\boldsymbol{\varepsilon}^{(b)} + f_1^{(s)}\boldsymbol{\varepsilon}^{(s)}\right) = -c_1\nabla\boldsymbol{\gamma}^{(0)} \\
&\boldsymbol{\gamma}^{(2)} = -3c_1\boldsymbol{\gamma}^{(0)}
\end{aligned} \tag{36}$$

with $\boldsymbol{\varepsilon}^{(b)}$ and $\boldsymbol{\varepsilon}^{(s)}$ given by Eqs. (4) and

$$\nabla = \begin{bmatrix} (.)_{,x} & 0 \\ 0 & (.)_{,y} \\ (.)_{,y} & (.)_{,x} \end{bmatrix}.$$

So, if we take $\boldsymbol{\gamma}^{(0)}$ as a generalized variable, all the components of the strain will be independent from $f_1^{(b)}$ and $f_1^{(s)}$ (see, Table 4). At this point, let us go a step further by making a change of the generalized displacement, i.e., substitute $g_\alpha(x, y; t)$ in Eq. (30) with $\gamma_{\alpha z}^{(0)}(x, y; t)$, using Eq. (33). The result is

$$u_\alpha(x, y, z; t) = u_\alpha^{(0)}(x, y; t) - zw_{,\alpha}^{(0)}(x, y; t) + F(z)\gamma_{\alpha z}^{(0)}(x, y; t) \tag{37}$$

This means that all the {3,0}-order polynomial kinematics given in Table 2 reduce to the one given by Eq. (37) when the corresponding $\gamma_{\alpha z}^{(0)}$ *is used as a generalized displacement***.** Column 2 of Table 3 shows in detail this statement.

Comparing expression (37) with those listed in Table 2, it is concluded that this displacement model is that proposed in Reissner 1975, Panc 1975, Bhimaraddi and Stevens 1984, Reddy 1984, 1990, when g_α is given the meaning of transverse shearing strain at the reference plane of the plate. By taking into account Eq. (33), this is equivalent to imposing the additional constraints

$$f_1^{(b)} = -1, \quad f_1^{(s)} = 1. \tag{38}$$

or, in an equivalent manner, satisfying the additional constraints listed in the third row of Table 1.

In summary, *all the {3,0}-order polynomial kinematics listed in Table 2 can be reduced to the one given by Eq. (37) with* $\gamma_{\alpha z}^{(0)}$ *given by Eq. (48).* This means assuming in Eq. (1)

$$f^{(b)}(z) = -z, \qquad f^{(s)}(z) = z\left(1 - \frac{4}{3h^2}z^2\right) = F(z) \tag{39}$$

and $\gamma_{\alpha z}^{(0)}$ given by Eq. (33).

The previous conclusion holds also for another group of {3,0}-order polynomial beam/plate theories, the so-called *three-variables beams theories and four-variables plates and shells theories*. In what follows we will consider two of these theories.

The first one is (Bisplinghoff et al 1957, Krishna Murthy 1984, Senthilnathan et al 1987, Benachour et al 2011, Thai et al 2013),

$$\begin{aligned} u_\alpha(x,y,z;t) &= u_\alpha^{(0)}(x,y;t) - z w_{b,\alpha}^{(0)}(x,y;t) + z\left(\frac{1}{4} - \frac{5}{3h^2}z^2\right) w_{s,\alpha}^{(0)}(x,y;t) \\ u_z(x,y,z;t) &= w_b^{(0)}(x,y;t) + w_s^{(0)}(x,y;t) \end{aligned} \tag{40}$$

where $w_b^{(0)}(x,y;t)$ and $w_s^{(0)}(x,y;t)$ are the transverse displacements due to bending strains and transverse shearing strains, respectively.

Let us compute the transverse shearing strain,

$$\gamma_{\alpha z}=\frac{5}{4}\left(1-\frac{4}{h^2}z^2\right)w_{s,\alpha}^{(0)} \quad\Rightarrow\quad \gamma_{\alpha z}^{(0)}=\frac{5}{4}w_{s,\alpha}^{(0)} \quad\Rightarrow\quad w_{s,\alpha}^{(0)}=\frac{4}{5}\gamma_{\alpha z}^{(0)} \tag{41}$$

From Eq. (40_2) we obtain

$$w_b^{(0)}+w_s^{(0)}=w^{(0)} \quad\Rightarrow\quad w_{b,\alpha}^{(0)}=w_{,\alpha}^{(0)}-w_{s,\alpha}^{(0)}=w_{,\alpha}^{(0)}-\frac{4}{5}\gamma_{\alpha z}^{(0)} \tag{42}$$

Substituting for $w_{s,\alpha}^{(0)}$ and $w_{b,\alpha}^{(0)}$ in Eq. (40) their expression in terms of $w_{,\alpha}^{(0)}$ and $\gamma_{\alpha z}^{(0)}$, as given by Eqs. (41) and (42), yields

$$\begin{aligned} u_\alpha &= u_\alpha^{(0)}-zw_{,\alpha}^{(0)}+z\left(1-\frac{4}{3h^2}z^2\right)\gamma_{\alpha z}^{(0)} \\ u_z &= w^{(0)} \end{aligned} \tag{43}$$

that is, exactly the displacement model of Reissner 1975, Panc 1975, Bhimaraddi and Stevens 1984, Ren and Hinton 1986, Reddy 1990.

Apparently, Thai et al 2013 do not seem to be aware of this equivalence, although all numerical results listed in their tables show that the two theories are perfectly coincident (at least for the boundary conditions considered in the paper).

As a second example, we consider the following displacement field (apparently first proposed by Senthilnathan et al 1987),

$$\begin{aligned} u_\alpha(x,y,z;t) &= u_\alpha^{(0)}(x,y;t)-zw_{b,\alpha}^{(0)}(x,y;t)-\frac{4}{3h^2}z^3w_{s,\alpha}^{(0)}(x,y;t) \quad (\alpha=x,y) \\ u_z(x,y,z;t) &= w_b^{(0)}(x,y;t)+w_s^{(0)}(x,y;t) \end{aligned} \tag{44}$$

As before, let us compute the transverse shearing strain,

$$\gamma_{\alpha z} = \left(1 - \frac{4}{h^2} z^2\right) w_{s,\alpha}^{(0)} \quad \Rightarrow \quad \gamma_{\alpha z}^{(0)} = w_{s,\alpha}^{(0)} \tag{45}$$

From Eq. (44 $_2$), it follows

$$u_z = w_b^{(0)} + w_s^{(0)} = w^{(0)} \quad \Rightarrow \quad w_{b,\alpha}^{(0)} = w_{,\alpha}^{(0)} - w_{s,\alpha}^{(0)} = w_{,\alpha}^{(0)} - \gamma_{\alpha z}^{(0)} \tag{46}$$

Substituting for $w_{s,\alpha}^{(0)}$ and $w_{b,\alpha}^{(0)}$ in Eq. (44) their expression in terms of $w_{,\alpha}^{(0)}$ and $\gamma_{\alpha z}^{(0)}$, as given by Eqs. (45) and (46), yields

$$\begin{aligned} u_\alpha &= u_\alpha^{(0)} - z w_{,\alpha}^{(0)} + z\left(1 - \frac{4}{3h^2} z^2\right)\gamma_{\alpha z}^{(0)} \\ u_z &= w^{(0)} \end{aligned} \tag{47}$$

that is, as before, exactly the displacement model of Reissner 1975, Panc 1975, Bhimaraddi and Stevens 1984, Ren and Hinton 1986, Reddy 1990 (see, Table 5).

To this end, refer also to Kant and Swaminathan 2001, 2002. In these papers, the authors compare, among the others, Senthilnathan et al 1987 and Reddy 1990 models. As before, apparently, the authors appear to be not aware that the two theories should give the same numerical results.

It should be emphasized that these conclusions hold for the simple support boundary conditions investigated in the quoted papers. It is evident that four-variables theories provide at the clamped end a nonzero thickness-wise distribution of $\gamma_{\alpha z}$ (and, as a consequence, of $\tau_{\alpha z}$), unlike the classical ones listed in Table 2. *It follows that for the case of clamped boundary conditions, these theories are cinematically equivalent, but not statically equivalent.* In a recent paper by Nguyen et al. 2017, a novel three-variable shear deformation plate theory is formulated, starting from the four variable Senthilnathan theory, Senthilnathan et al. 1987, and eliminating the variable $w_s^{(0)}$. Following the approach outlined in this Section, it is easy to show that also this novel kinematics reduces to that of Reissner 1975, Panc 1975, Bhimaraddi and Stevens 1984, Ren and Hinton

1986, Reddy 1984, 1990. Obviously, as before, this would not mean that all the five, four and three variable kinematics are statically equivalent and perform in the same way when they are used as basis for the formulation of approximate numerical methods, such as finite element method, see Nguyen et al. 2017. For example, three and four variable kinematics do not suffer of the drawback of estimating zero transverse shear strain (and stress when evaluated from the constitutive equation) at clamped edge.

Cylindrical bending in the (x, z) −plane

In this Section an assessment of the general conclusions drawn in the previous Section is made.

In order to make the problem tractable from an analytical point of view, i.e., to obtain an exact closed form solution, and without lack of generality in the conclusions, we will study the problem of elastodynamic behavior of a plate extended infinitely along the y −axis, that is, the problem known as the cylindrical bending in the (x, z) plane. We consider a monolayer plate made of homogeneous and orthotropic material in cylindricasl bending in the (x, z) plane, of length $a = L$, simple-supported on both the edges $x = 0, L$.

We will investigate two problems:

Problem **P1**) Elastostatic analysis of the plate subjected to a tranverse sinusoidal load $\bar{p}_z(x; t) = \bar{p}_0 sin\lambda x$ with $\lambda = \frac{\pi}{L}$. (48)

Problem **P2**) Natural frequencies and mode shapes.

For these cases study, the in-plane behavior is uncoupled from the transverse behavior. So, we study only this last one.

The governing equations and boundary condition are readily derived from the general ones and are summarized here below.

Equations of motion

$$\left({}^{(b)}D_{11} w^{(0)}_{,xx} - {}^{(bb)}A_{44} w^{(0)} \right)_{,xx} + \left({}^{(bs)}D_{11} g_{x,xx} - {}^{(bs)}A_{44} g_x \right)_{,x} = RHS_w \tag{49}$$

$$\left({}^{(bs)}D_{11} w^{(0)}_{,xx} - {}^{(bs)}A_{44} w^{(0)} \right)_{,x} + \left({}^{(s)}D_{11} g_{x,xx} - {}^{(ss)}A_{44} g_x \right) = RHS_g \tag{50}$$

where

Problem **P1**) $RHS_w = \bar{p}_z \; ; \quad RHS_g = 0$ (51)

Problem **P2**) $RHS_w = -m^{(0)}\ddot{w}^{(0)} + m^{(bb)}\ddot{w}^{(0)}_{,xx} + m^{(sb)}\ddot{g}_{x,x} \; ; \; RHS_g = m^{(bs)}\ddot{w}^{(0)}_{,x} + m^{(ss)}\ddot{g}_x$ (52)

Boundary conditions on $\mathbf{x = 0, L}$

$$w^{(0)} = w^{(0)}_{,xx} = g_{x,x} = 0 \text{ on } x = 0, x = L. \tag{53}$$

Eqs. (53) follow from Eqs. (26) when Eqs. (14) and (25) are taken into account.

As a first step in solving the system of equations (49) and (50), we note that we can write

$$\begin{aligned} &{}^{(b)}D_{11} = {}^{(b)}\hat{D}_{11}; \quad {}^{(s)}D_{11} = f_1^{(s)2}\, {}^{(s)}\hat{D}_{11}; \quad {}^{(bs)}D_{11} = f_1^{(s)}\, {}^{(bs)}\hat{D}_{11} \\ &{}^{(bb)}A_{44} = {}^{(bb)}\hat{A}_{44}; \quad {}^{(bs)}A_{44} = f_1^{(s)}\, {}^{(bs)}\hat{A}_{44}; \quad {}^{(ss)}A_{44} = f_1^{(s)2}\, {}^{(ss)}\hat{A}_{44} \end{aligned} \tag{54}$$

where the quantities with hat are independent of $f_1^{(s)}$,

$$
\begin{aligned}
{}^{(b)}\hat{D}_{11} &= f_1^{(b)2}\mathcal{D}_{11} - f_1^{(b)}(2+f_1^{(b)})c_1\mathcal{F}_{11} + c_1^2 H_{11}\\
{}^{(s)}\hat{D}_{11} &= \mathcal{D}_{11} - c_1\mathcal{F}_{11}\\
{}^{(bs)}\hat{D}_{11} &= f_1^{(b)}\mathcal{D}_{11} - \left(1+f_1^{(b)}\right)c_1\mathcal{F}_{11}\\
{}^{(bb)}\hat{A}_{44} &= (1+f_1^{(b)})^2\left(\mathcal{A}_{44} - 3c_1\mathcal{D}_{44}\right)\\
{}^{(bs)}\hat{A}_{44} &= (1+f_1^{(b)})\left(\mathcal{A}_{44} - 3c_1\mathcal{D}_{44}\right)\\
{}^{(ss)}\hat{A}_{44} &= \mathcal{A}_{44} - 3c_1\mathcal{D}_{44}
\end{aligned} \tag{55}
$$

In the same way, we write the inertia terms as

$$
m^{(0)} = I_0 = \hat{m}^{(0)} \ ; \qquad \begin{Bmatrix} m^{(bb)} \\ m^{(ss)} \\ m^{(bs)} \end{Bmatrix} = \begin{Bmatrix} f_1^{(b)2}J_2 - f_1^{(b)}(2+f_1^{(b)})J_4 + c_1^2 I_6 \\ f_1^{(s)2}\left(J_2 - c_1 J_4\right) \\ f_1^{(b)} f_1^{(s)} J_2 - f_1^{(s)}(1+f_1^{(b)})c_1 J_4 \end{Bmatrix} = \begin{Bmatrix} \hat{m}^{(bb)} \\ f_1^{(s)2}\hat{m}^{(ss)} \\ f_1^{(s)}\hat{m}^{(bs)} \end{Bmatrix} \tag{56}
$$

Eqs. (55) and (56) follow from Eqs. (15) and (21) by remembering Eqs. (27) and (28), and taking into account the following definitions

$$
(D_{ij};\ E_{ij};\ F_{ij};\ H_{ij}) = \left\langle Q_{ij}\left(z^2;\ z^3;\ z^4;\ z^6\right)\right\rangle \ \left(i,\, j = 1, 2, 6\right) \tag{57}
$$

$$
\left(D_{ij},\ F_{ij}\right) = \left\langle Q_{ij}\left(1;\ z^2;\ z^4\right)\right\rangle \ \left(i,\, j = 4, 5\right) \tag{58}
$$

$$
\mathcal{D}_{ij} = D_{ij} - c_1 F_{ij} \ ; \quad \mathcal{F}_{ij} = F_{ij} - c_1 H_{ij} \quad \left(i,\, j = 1, 2, 6\right) \tag{59}
$$

$$
\mathcal{A}_{ij} = A_{ij} - 3c_1 D_{ij};\quad \mathcal{D}_{ij} = D_{ij} - 3c_1 F_{ij} \quad \left(i,\, j = 4, 5\right) \tag{60}
$$

and

$$
I_i = \left\langle \rho\ z^i \right\rangle \ \text{ and } \ J_i = I_i - c_1 I_{i+2}\,. \tag{61}
$$

So, Eqs. (49) and (50) read

$$
\left({}^{(b)}\hat{D}_{11} w^{(0)}_{,xx} - {}^{(bb)}\hat{A}_{44} w^{(0)}\right)_{,xx} + f_1^{(s)}\left({}^{(bs)}\hat{D}_{11} g_{x,xx} - {}^{(bs)}\hat{A}_{44} g_x\right)_{,x} = RHS_w \tag{62}
$$

$$
f_1^{(s)}\left({}^{(bs)}\hat{D}_{11} w^{(0)}_{,xx} - {}^{(bs)}\hat{A}_{44} w^{(0)}\right)_{,x} + f_1^{(s)2}\left({}^{(s)}\hat{D}_{11} g_{x,xx} - {}^{(ss)}\hat{A}_{44} g_x\right) = RHS_g \tag{63}
$$

where for Problem **P2**

$$RHS_w = -m^{(0)}\ddot{w}^{(0)} + \hat{m}^{(bb)}\ddot{w}^{(0)}_{,xx} + f_1^{(s)}\hat{m}^{(sb)}\ddot{g}_{x,x}$$
$$RHS_g = f_1^{(s)}\hat{m}^{(bs)}\ddot{w}^{(0)}_{,x} + f_1^{(s)2}\hat{m}^{(ss)}\ddot{g}_x \tag{64}$$

Problem P1: Elastostatic analysis

The following assumed set of functions will satisfy the boundary conditions (53),

$$w^{(0)}(x) = A_w sin\lambda x, \quad g_x(x) = A_g cos\lambda x,\ \lambda = \frac{\pi}{L} \tag{65}$$

Sustituting Eqs. (65) into Eqs. (62) and (63), and taking into account Eq. (50), yields

$$[K]\{A\} = \{P\} \tag{66}$$

where

$$[K] = \begin{bmatrix} \lambda^2\hat{K}_{ww} & \lambda f_1^{(s)}\hat{K}_{wg} \\ \lambda f_1^{(s)}\hat{K}_{wg} & f_1^{(s)2}\hat{K}_{gg} \end{bmatrix}; \quad \{A\} = \begin{Bmatrix} A_w \\ A_g \end{Bmatrix}; \quad \{P\} = \begin{Bmatrix} \bar{p}_0 \\ 0 \end{Bmatrix} \tag{67}$$

and

$$\begin{aligned}\hat{K}_{ww} &= {}^{(b)}\hat{D}_{11}\lambda^2 + {}^{(bb)}\hat{A}_{44} \\ &= \lambda^2\left(f_1^{(b)2}\mathcal{D}_{11} - f_1^{(b)}(2+f_1^{(b)})c_1\mathcal{F}_{11} + c_1^2 H_{11}\right) + (1+f_1^{(b)})^2\left(\mathcal{A}_{44} - 3c_1\mathcal{D}_{44}\right)\end{aligned} \tag{68}$$

$$\begin{aligned}\hat{K}_{wg} &= {}^{(bs)}\hat{D}_{11}\lambda^2 + {}^{(bs)}\hat{A}^{13}_{13} \\ &= \lambda^2\left(f_1^{(b)}\mathcal{D}_{11} - \left(1+f_1^{(b)}\right)c_1\mathcal{F}_{11}\right) + (1+f_1^{(b)})\left(\mathcal{A}_{44} - 3c_1\mathcal{D}_{44}\right)\end{aligned} \tag{69}$$

$$\begin{aligned}\hat{K}_{gg} &= {}^{(s)}\hat{D}_{11}\lambda^2 + {}^{(ss)}\hat{A}_{44} \\ &= \lambda^2\left(\mathcal{D}_{11} - c_1\mathcal{F}_{11}\right) + \mathcal{A}_{44} - 3c_1\mathcal{D}_{44}\end{aligned} \tag{70}$$

In writing Eqs. (68)-(70), we have taken into account relations (55).

Solving Eqs. (66), we obtain

$$A_w = \frac{1}{\lambda^2}\frac{\hat{K}_{gg}}{D\hat{E}N}\bar{p}_0 = \frac{1}{\lambda^2}\frac{\left({}^{(s)}\hat{D}_{11}\lambda^2 + {}^{(ss)}\hat{A}_{44}\right)}{D\hat{E}N}\bar{p}_0 \tag{71}$$

$$A_g = -\frac{1}{\lambda}\frac{\hat{K}_{wg}}{f_1^{(s)} D\hat{E}N}\,\overline{p}_0 = -\frac{1}{\lambda}\frac{\left({}^{(bs)}\hat{D}_{11}\lambda^2 + {}^{(bs)}\hat{A}_{44}\right)}{f_1^{(s)} D\hat{E}N}\,\overline{p}_0 \tag{72}$$

where

$$\begin{aligned} D\hat{E}N &= \det[\hat{K}] = \det\begin{bmatrix} \hat{K}_{ww} & \hat{K}_{wg} \\ \hat{K}_{wg} & \hat{K}_{gg} \end{bmatrix} = \\ &= \left({}^{(b)}\hat{D}_{11}\lambda^2 + {}^{(bb)}\hat{A}_{44}\right)\left({}^{(s)}\hat{D}_{11}\lambda^2 + {}^{(ss)}\hat{A}_{44}\right) - \left({}^{(bs)}\hat{D}_{11}\lambda^2 + {}^{(bs)}\hat{A}_{44}\right)^2 \end{aligned} \tag{73}$$

Using Eqs. (71) and (72) into Eq. (33), yields

$$\gamma_{xz}^{(0)} = \frac{1}{\lambda}\frac{N\hat{U}M}{D\hat{E}N}\,\overline{p}_0 \tag{74}$$

where

$$N\hat{U}M = \left((1+f_1^{(b)})\,{}^{(s)}\hat{D}_{11} - {}^{(bs)}\hat{D}_{11}\right)\lambda^2 + \left((1+f_1^{(b)})\,{}^{(ss)}\hat{A}_{44} - {}^{(bs)}\hat{A}_{44}\right) \tag{75}$$

So, let us calculate A_w and A_g. Taking into account Eqs. (68) ÷ (70), after some lenghtly but straightforward calculations, we obtain

$$\begin{aligned} D\hat{E}N &= \lambda^4 c_1^2\left(H_{11}\left(\mathcal{D}_{11} - c_1\mathcal{F}_{11}\right) - \mathcal{F}_{11}\mathcal{F}_{11}\right) + \lambda^2\left(\mathcal{A}_{44} - 3c_1\mathcal{D}_{44}\right)D_{11} \\ &= \lambda^4\left(D_{11}\left(\mathcal{D}_{11} - c_1\mathcal{F}_{11}\right) - \mathcal{D}_{11}\mathcal{D}_{11}\right) + \lambda^2\left(\mathcal{A}_{44} - 3c_1\mathcal{D}_{44}\right)D_{11} \\ &= \lambda^4 c_1^2\left(D_{11}H_{11} - F_{11}F_{11}\right) + \lambda^2\left(\mathcal{A}_{44} - 3c_1\mathcal{D}_{44}\right)D_{11} \end{aligned} \tag{76}$$

$$\hat{N}UM = \lambda^2\mathcal{D}_{11} \tag{77}$$

Eqs. (70), (76) and (77) show that the transverse deflection, Eq. (71), and the maximum transverse shearing strain, Eq. (72) , are independent from the parameters $f_1^{(b)}$ and $f_1^{(s)}$, that appear in the various kinematics (Table 2). In other words, *the deflection and the maximum transverse shearing strain are the same for all the {3,0}-order polynomial beam/plate models.*

Let us make a step further and calculate the axial and shearing strain (stress) distribution along the thickness. Remembering Eqs. (34) ÷ (36),we obtain

$$\varepsilon_{xx}(x,z) = \left(\varepsilon_{xx}^{(1)} z + \varepsilon_{xx}^{(3)} c_1 z^3\right) \sin \lambda x \tag{78}$$

$$\gamma_{xz} = (1 - 3c_1 z^2)\gamma_{xz}^{(0)} \cos \lambda x \tag{79}$$

where

$$\varepsilon_{xx}^{(1)} = -\lambda\left(\lambda f_1^{(b)} A_w + f_1^{(s)} A_g\right); \quad \varepsilon_{xx}^{(3)} = \lambda\left(\lambda(1 + f_1^{(b)}) A_w + f_1^{(s)} A_g\right) \tag{80}$$

By remembering that (see, Eq. (33))

$$\gamma_{xz}^{(0)} = \lambda(1 + f_1^{(b)}) A_w + f_1^{(s)} A_g. \tag{81}$$

Eqs. (80) read,

$$\varepsilon_{xx}^{(1)} = -\lambda\left(\gamma_{xz}^{(0)} - \lambda A_w\right); \qquad \varepsilon_{xx}^{(3)} = \lambda\left(\lambda(1 + f_1^{(b)}) A_w + f_1^{(s)} A_g\right) = \lambda \gamma_{xz}^{(0)} \tag{82}$$

which confirm the general conclusions given by Eqs. (34)÷(36).

Eqs. (78) and (79) , when Eqs. (74) and (82) are taken into account, show clearly that also *the thickness distribution of the transverse shear strain (and stress) and of the axial strain (and stress) is the same for all the {3,0}-order polyomial beam/plate models.*

Problem P1) Natural frequencies and mode shapes .

In this case, the solving equation (characteristic equation or frequency equation) is obtained by imposing

$$\det\left[[K] - \omega^2 [M]\right] = 0 \tag{83}$$

where matrix $[K]$ is given by Eq. (67) and

$$[M] = \begin{bmatrix} \hat{M}_{ww} & \lambda f_1^{(s)} \hat{M}_{wg} \\ \lambda f_1^{(s)} \hat{M}_{wg} & f_1^{(s)2} \hat{M}_{gg} \end{bmatrix} \tag{84}$$

with

$$\hat{M}_{ww} = \hat{m}^{(0)} + \hat{m}^{(bb)}\lambda^2; \quad \hat{M}_{wg} = \hat{m}^{(sb)}; \quad \hat{M}_{gg} = \hat{m}^{(ss)} \tag{85}$$

and

$$\hat{m}^{(0)} = m^{(0)} = I_0; \qquad \begin{Bmatrix} \hat{m}^{(bb)} \\ \hat{m}^{(ss)} \\ \hat{m}^{(bs)} \end{Bmatrix} = \begin{Bmatrix} f_1^{(b)2} J_2 - f_1^{(b)}(2 + f_1^{(b)})c_1 J_4 + c_1^2 I_6 \\ J_2 - c_1 J_4 \\ f_1^{(b)} J_2 - (1 + f_1^{(b)})c_1 J_4 \end{Bmatrix} \tag{86}$$

First, we note that

$$\begin{aligned} \det\left([K] - \omega^2 [M]\right) &= \det[K] + \omega^4 \det[M] - \omega^2 \left(\det\begin{bmatrix} K_{ww} & K_{wg} \\ M_{wg} & M_{gg} \end{bmatrix} + \det\begin{bmatrix} M_{ww} & M_{wg} \\ K_{wg} & K_{gg} \end{bmatrix} \right) \\ &= f_1^{(2)} \left(\det\left[\hat{K}\right] + \omega^4 \det\left[\hat{M}\right] - \omega^2 \left(\lambda^2 \hat{K}_{ww}\hat{M}_{gg} + \hat{M}_{ww}\hat{K}_{gg} - 2\lambda^2 \hat{K}_{wg}\hat{M}_{wg} \right) \right) \end{aligned} \tag{87}$$

After some lenghtly but straightforward calculations, we obtain the following results,

$$\det\left[\hat{M}\right] = \left[I^{(0)} + \lambda^2 \left[c_1^2 I_6 \left(J_2 - c_1 J_4 \right) - c_1^2 J_4^2 \right] \right] \tag{88}$$

$$\begin{aligned} \lambda^2 \hat{K}_{ww}\hat{M}_{gg} + \hat{M}_{ww}\hat{K}_{gg} - 2\lambda^2 \hat{K}_{wg}\hat{M}_{wg} = {} & \lambda^2 \left(\lambda^2 c_1^2 H_{11} + \mathcal{A}_{44} - 3c_1 \mathcal{D}_{44} \right)\left(J_2 - c_1 J_4 \right) + \\ & + \left[\lambda^2 \left(\mathcal{D}_{11} - c_1 \mathcal{F}_{11} \right) + \mathcal{A}_{44} - 3c_1 \mathcal{D}_{44} \right]\left(I^{(0)} + c_1^2 I_6 \right) + \\ & - 2\lambda^2 \left[\lambda^2 c_1 \mathcal{F}_{11} - \left(\mathcal{A}_{44} - 3c_1 \mathcal{D}_{44} \right) \right] c_1 J_4 \end{aligned} \tag{89}$$

Substitution of the Eqs. (73), (77), (88) and (89) into Eq. (87), yields the frequency equation independent of $f_1^{(b)}$ and $f_1^{(s)}$. This means that also *the natural frequencies and mode shapes are the same for all the {3,0}-order polyomial beam/plate models.* It is worthwile to note that this result, here obtained for the specific case of the simply supported beam, is also substantiated by the numerical results obtained by Qu et al 2013, tables 7 and 8, for different boundary conditions (F-F, F-C. S-S, C-S, C-C; F=free, C=clampled, S=simply supported) , using the kinematics of Reissner 1945, and Giavotto 1969 and that of Kaczkowski 1968, Reissner 1975, Panc 1975, etc.

Table 6 aummarizes the main results obtained in this Section.

In closing this Section, we note that, as observed by one Reviewer, it should be sufficed to write the general equations of these two problems in terms of the maximum transverse shear strain to show that they do not depend on the polynomial coefficients and thus on the specific kinematics, without computing the results. The choice to write the governing equations starting from the kinematics in the first natural way (Eq. (1)) , i.e., to derive them as particular case of the general equations (28)-(30), was dictated by the aim to use the set of general governing equations in which the coefficients $f_1^{(b)}$ and $f_1^{(s)}$ appear explicitly.

Concluding remarks

Based on a few suggestions in the open literature (Jemielita 1990, Reddy 1990, Kapuria et al 2004, Challamel et al 2013, Nugyen et al 2016), first a generalization of the assumed kinematics in the so-called {3,0}-order polynomial displacement-based shear deformation theories is derived. Second, based on this general {3,0}-order polynomial displacement-based kinematics, the equations of motion and variationally consistent boundary conditions for a rectangular flat plate made of orthotropic material are derived. Third, it is shown that all the {3,0}-order polynomial theories proposed in the open literature are special cases of the general kinematics developed in this paper. Fourth, it is shown that the {3,0}-order polynomial kinematics satisfying the zero transverse shear strain on the bottom and top surfaces of the plate is the same when the maximum transverse shear strain is used as generalized displacement co-ordinate., also for the so-called three- and four-variable plate theories. In other words, they are kinematically equivalent. This conclusion apply also to the so-called four and three-variable plate theories. Obviously, it does not apply to {3,0}-order polynomial kinematics not satisfying the zero transverse shear strain on the bottom and top surfaces of the plate. Fifth, in order to substantiate the general conclusion that all the {3,0}-order polynomial displacement-based shear deformation theories are kinematically equivalent, we performed a deep analysis of the static and dynamic behavior of simply supported rectangular plates in cylindrical bending. All the {3,0}-order polynomial displacement-based shear deformation theories give the same numerical results. So, at least for the investigated problems, all the {3,0}-order polynomial displacement-based shear deformation theories are not only kinematically, but also statically equivalent, i.e., they give the same numerical results. As quoted in the Introdution, Challamel et al. 2013 using a different approach has studied the buckling loads of simply supported rectangular flat plate under uni-axial compression load using the Reddy 1984 and Shi 2007 theories. The authors reached the same conclusions, that is, they found the same buckling loads. Nguyen et al. 2016 studied the bending (deflection and stresses) of simply supported laminated plates within the general framework of high-order deformation plate theories and found that the static results (deflection and stresses) of KPR model (Kaczkowski 1968, Reissner 1975, Panc 1975), of LMR model (Levinson 1980, Reddy 1984, Murthy 1981) and Ambartsumian model (Ambartsumian 1960) are exactly identical.

Of course, this conclusion does not hold in general for the class of kinematics studied in this paper: they are kinematically equivalent, altought not all are statically equivalent.

For example, static equivalence does not hold true for all type of boundary conditions. It is well-known that three- and four-variable kinematics do not suffer of the drawback of estimating zero transverse shear strain (and stress when evaluated from the constitutive equation) at clamped edge. Furthermore, different plate theories behave differently when used as a basis for the formulation of numerical approaches, such as the finite element method.

As mentioned in the Introduction, recently the literature has shown a wide interest in theories in which the higher-order contribution to the in-plane displacement are given through trigonometric or, in general, hyperbolic and exponential functions. Previous conclusions on the kinematic equivalence of all the {3,0}-order polynomial theories do not apply to the kinematics used in these theories.

As a concluding comment, the author hopes that the results obtained may stimulate further research aimed at establishing the equivalences and differences of the various plate theories of which literature is very rich, in order to avoid the proliferation of new theories that could be kinematically and statically equivalent to those already existing.

Appendix Derivation of equations of motion and the variationally consistent boundary conditions.

In order to derive the equations of motion and the variationally consistent boundary conditions for the plate under consideration, we make use of the D'Alembert's principle (**V**ariational **E**quation of **D**ynamics (**VED**)).

Let $\delta\Phi$ be the internal virtual work (virtual variation of the strain energy), δW the virtual variation of the work done by the applied loads, and δW_{in} the virtual work done by the inertia forces, where δ denotes the variational operator.

The **V**ariational **E**quation of **D**ynamics (**VED**) states that

$$\delta\Phi - \delta W = \delta W_{\text{in}} \tag{90}$$

where

$$\delta\Phi = \int_0^a \int_0^b \left\langle \boldsymbol{\sigma}^T \delta\boldsymbol{\varepsilon} + \boldsymbol{\tau}^T \delta\boldsymbol{\gamma} \right\rangle dxdy \tag{91}$$

$$\delta W = \int_0^a \int_0^b \bar{p}_z \delta w^{(0)} dxdy + \int_0^a \left[\bar{P}_{xy} \delta u_x^{(0)} + \bar{P}_{yy} \delta u_y^{(0)} \right]_0^b dx + \int_0^b \left[\bar{P}_{xx} \delta u_x^{(0)} + \bar{P}_{xy} \delta u_y^{(0)} \right]_0^a dy \tag{92}$$

$$\delta W_{\text{in}} = -\int_0^a \int_0^b \left\langle \rho \ \left(\ddot{u}_\alpha \delta u_\alpha + \ddot{w}^{(0)} \delta w^{(0)} \right) \right\rangle dxdy \tag{93}$$

and

$$\langle \dots \rangle \equiv \int_{-h/2}^{+h/2} (\dots) dz \tag{94}$$

$(.)^T$ stands for the transpose of a matrix; ρ is the material mass density of the plate.

Making use of the assumed displacement field, Eqs. (1), of the strain-displacement relations, Eqs. (2)÷(4), and the stress-strain relations, Eq. (5), yields

$$\delta \Phi = \int_0^a \int_0^b \left(\left\lfloor \mathbf{N}^T \quad \mathbf{R}_\sigma^{(b)T} \quad \mathbf{R}_\sigma^{(s)T} \right\rfloor \delta \begin{Bmatrix} \boldsymbol{\varepsilon}^{(0)} \\ \boldsymbol{\varepsilon}^{(b)} \\ \boldsymbol{\varepsilon}^{(s)} \end{Bmatrix} + \left\lfloor \mathbf{R}_\tau^{(b)T} \quad \mathbf{R}_\tau^{(s)T} \right\rfloor \delta \begin{Bmatrix} \boldsymbol{\gamma}^{(b)} \\ \boldsymbol{\gamma}^{(s)} \end{Bmatrix} \right) dxdy \tag{95}$$

In Eq. (95), the following generalized stress resultants have been introduced

$$\left(\mathbf{N},\ \mathbf{R}_\sigma^{(b)},\ \mathbf{R}_\sigma^{(s)} \right) \equiv \left(\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix},\ \begin{Bmatrix} R_{xx}^{(b)} \\ R_{yy}^{(b)} \\ R_{xy}^{(b)} \end{Bmatrix},\ \begin{Bmatrix} R_{xx}^{(s)} \\ R_{yy}^{(s)} \\ R_{xy}^{(s)} \end{Bmatrix} \right) = \left\langle \left(1,\ f^{(b)},\ f^{(s)} \right) \boldsymbol{\sigma} \right\rangle$$

(96)

$$\left(\mathbf{R}_\tau^{(b)},\ \mathbf{R}_\tau^{(s)} \right) \equiv \left(\begin{Bmatrix} R_{xz}^{(b)} \\ R_{yz}^{(b)} \end{Bmatrix},\ \begin{Bmatrix} R_{xz}^{(s)} \\ R_{yz}^{(s)} \end{Bmatrix} \right) = \left\langle \left(\left(1 + f_{,z}^{b} \right),\ f_{,z}^{s} \right) \boldsymbol{\tau} \right\rangle$$

Substituting in Eq. (96) the stress-strain relation (5) and the strain-displacement relation (2), yields the plate constitutive equations

$$\begin{Bmatrix} \mathbf{N} \\ \mathbf{R}_\sigma^{(b)} \\ \mathbf{R}_\sigma^{(s)} \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & {}^{(b)}\mathbf{B} & {}^{(s)}\mathbf{B} \\ {}^{(b)}\mathbf{B} & {}^{(b)}\mathbf{D} & {}^{(bs)}\mathbf{D} \\ {}^{(s)}\mathbf{B} & {}^{(bs)}\mathbf{D} & {}^{(s)}\mathbf{D} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon}^{(0)} \\ \boldsymbol{\varepsilon}^{(b)} \\ \boldsymbol{\varepsilon}^{(s)} \end{Bmatrix}; \quad \begin{Bmatrix} \mathbf{R}_\tau^{(b)} \\ \mathbf{R}_\tau^{(s)} \end{Bmatrix} = \begin{bmatrix} {}^{(bb)}\mathbf{A}_\tau & {}^{(bs)}\mathbf{A}_\tau \\ {}^{(bs)}\mathbf{A}_\tau & {}^{(ss)}\mathbf{A}_\tau \end{bmatrix} \begin{Bmatrix} \boldsymbol{\gamma}^{(b)} \\ \boldsymbol{\gamma}^{(s)} \end{Bmatrix} \tag{97}$$

where,

$$
\begin{aligned}
&(\mathbf{A};\,{}^{(b)}\mathbf{B};\,{}^{(b)}\mathbf{D}) = \left\langle \left(1;\ f^{(b)};\ f^{(b)2}\right)\mathbf{Q}_{\sigma} \right\rangle;\quad ({}^{(s)}\mathbf{B};\quad {}^{(bs)}\mathbf{D};\quad {}^{(s)}\mathbf{D}) = \left\langle f^{(s)}\left(1;\ f^{(b)};\ f^{(s)}\right)\mathbf{Q}_{\sigma} \right\rangle \\
&\left({}^{(bb)}\mathbf{A}_{\tau},\,{}^{(bs)}\mathbf{A}_{\tau},\,{}^{(ss)}\mathbf{A}_{\tau}\right) = \left\langle \left(\left(1+f^{b}_{,z}\right)^{2},\left(1+f^{b}_{,z}\right)f^{(s)}_{,z},\,f^{(s)2}_{,z}\right)\mathbf{Q}_{\tau} \right\rangle
\end{aligned} \tag{98}
$$

are the generalized plate stiffnesses.

By taking into account Eqs. (4) and (95), a tedious, but straightforward application of the Green's theorem wherever feasible, yields the following expression for the virtual variation of the strain energy,

$$
\begin{aligned}
\delta\Phi = &-\int_0^a\int_0^b N_{\alpha\beta,\beta}\,\delta u^{(0)}_{\alpha}\,dxdy - \int_0^a\int_0^b \left(R^{(b)}_{\alpha z} - R^{(b)}_{\beta\alpha,\beta}\right)_{,\alpha}\delta w^{(0)}\,dxdy + \\
&+\int_0^a\int_0^b \left(R^{(s)}_{\alpha\beta,\beta} - R^{(s)}_{\alpha z}\right)\delta g_{\alpha}\,dxdy + BT_{\Phi}
\end{aligned} \tag{99}
$$

with the boundary term

$$
\begin{aligned}
BT_{\Phi} = &\int_0^b \left[N_{x\alpha}\delta u^{(0)}_{\alpha} - \left(R^{(b)}_{x\alpha,\alpha} - R^{(b)}_{xz}\right)\delta w^{(0)} + R^{(b)}_{x\alpha}\delta w^{(0)}_{,\alpha} + R^{(s)}_{x\alpha}\delta g_{\alpha}\right]_0^a dy + \\
&+\int_0^a \left[N_{y\alpha}\delta u^{(0)}_{\alpha} - \left(R^{(b)}_{y\alpha,\alpha} - R^{(b)}_{yz}\right)\delta w^{(0)} + R^{(b)}_{y\alpha}\delta w^{(0)}_{,\alpha} + R^{(s)}_{y\alpha}\delta g_{\alpha}\right]_0^b dx
\end{aligned} \tag{100}
$$

Moreover,

$$
\begin{aligned}
&\delta W_{\text{in}} = -\int_0^a\int_0^b \left(m^{(0)}\ddot{u}^{(0)}_{\alpha} + m^{(b)}\ddot{w}^{(0)}_{,\alpha} + m^{(s)}\ddot{g}_{\alpha}\right)\delta u^{(0)}_{\alpha}\,dxdy \\
&+\int_0^a\int_0^b \left(-m^{(0)}\ddot{w}^{(0)} + m^{(b)}\ddot{u}^{(0)}_{\alpha,\alpha} + m^{(bb)}\ddot{w}^{(0)}_{,\alpha\alpha} + m^{(sb)}\ddot{g}_{\alpha,\alpha}\right)\delta w^{(0)}\,dxdy \\
&-\int_0^a\int_0^b \left(m^{(s)}\ddot{u}^{(0)}_{\alpha} + m^{(bs)}\ddot{w}^{(0)}_{,\alpha} + m^{(ss)}\ddot{g}_{\alpha}\right)\delta g_{\alpha}\,dxdy + BT_{in}
\end{aligned} \tag{101}
$$

with the boundary term

$$
\begin{aligned}
BT_{in} = &\int_0^a \left[\left(m^{(b)}\ddot{u}^{(0)}_{y} + m^{(bb)}\ddot{w}^{(0)}_{,y} + m^{(sb)}\ddot{g}_{y}\right)\delta w^{(0)}\right]_0^b dx \\
&+\int_0^b \left[\left(m^{(b)}\ddot{u}^{(0)}_{x} + m^{(bb)}\ddot{w}^{(0)}_{,x} + m^{(sb)}\ddot{g}_{x}\right)\delta w^{(0)}\right]_0^a dy
\end{aligned} \tag{102}
$$

In Eqs. (101) and (102)

$$
\left(m^{(0)},\ m^{(b)},\ m^{(s)},\ m^{(bb)}, m^{(ss)}, m^{(bs)}\right) = \left\langle \rho\ \left(1, f^{(b)}, f^{(s)}, f^{(b)2}, f^{(s)2}, f^{(b)}f^{(s)}\right)\right\rangle \tag{103}
$$

are the generalized inertia resultants.

Upon substitution of the Eqs. (92), (99) and (101) into Eq.(90), using the stationary conditions leads to the field equations and the variationally consistent boundary conditions in term of generalized force and stress resultants.

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Table 1. Conditions to be satisfied by $f^{(b)}(z)$ and $f^{(s)}(z)$.

Conditions on $f^{(b)}(z)$	Conditions on $f^{(s)}(z)$
$\left\langle f^{(b)}(z)\right\rangle = 0$	$\left\langle f^{(s)}(z)\right\rangle = 0$
$f_{,z}^{(b)}(\pm\frac{h}{2}) = -1$	$f_{,z}^{(s)}(\pm\frac{h}{2}) = 0$
$f_{,z}^{(b)}(0) = -1$	$f_{,z}^{(s)}(0) = 1$

Note: Conditions listed in the first row state that $u_{\alpha}^{(0)}(x,y;t)$ are the uniform in-plane displacement components; those listed in the second row state that the shear strain and stress vanish on the top and bottom bounding surfaces. Those listed in the third row state that $g_{\alpha}(x,y;t)$ in Eq. (1) is $\gamma_{\alpha z}^{(0)}(x,y;t)$ $(\alpha \equiv x,y)$, the transverse shearing strain at the reference plane of the plate.

Table 2. Relationship of the thickness-wise distribution of the in-plane displacement of well-known $3,0$ -order polynomial shear-deformation plate theories.

Reference	In-plane displacement field	
Present theory	$u_\alpha = u_\alpha^{(0)} + \left(f_1^{(b)} z \left(1 - \frac{4}{3h^2} z^2\right) - \frac{4}{3h^2} z^3 \right) w_{,\alpha}^{(0)} + f_1^{(s)} z \left(1 - \frac{4}{3h^2} z^2\right) g_\alpha$ $= u_\alpha^{(0)} + \left(f_1^{(b)} F(z) - c_1 z^3 \right) w_{,\alpha}^{(0)} + f_1^{(s)} F(z) g_\alpha$	
Note that in all the following theories $f_1^{(b)} = -1$.		
Reissner 1945, Giavotto 1969	$u_\alpha = u_\alpha^{(0)} - z w_{,\alpha}^{(0)} + \frac{5}{4} F(z) g_\alpha$	$\left(f_1^{(b)} = -1, \quad f_1^{(s)} = \frac{5}{4} \right)$
Kromm 1953, 1955; Schmidt 1977	$u_\alpha = u_\alpha^{(0)} - z w_{,\alpha}^{(0)} + \frac{3}{2} F(z) g_\alpha$	$\left(f_1^{(b)} = -1, \quad f_1^{(s)} = \frac{3}{2} \right)$
Ambartsumyan 1960, 1964	$u_\alpha = u_\alpha^{(0)} - z w_{,\alpha}^{(0)} - \frac{h^2}{8} F(z) \frac{g_\alpha}{G}$	$\left(f_1^{(b)} = -1, \quad f_1^{(s)} = -\frac{h^2}{8G} \right)$
Kaczkowski 1968, Reissner 1975, Panc 1975, Bhimaraddi and Stevens 1984 Reddy 1984, 1990; Ren and Hinton 1986 Soldatos 1986, 1988	$u_\alpha = u_\alpha^{(0)} - z w_{,\alpha}^{(0)} + F(z) g_\alpha$	$\left(f_1^{(b)} = -1, \quad f_1^{(s)} = 1 \right)$
Krishna Murty 1977	$u_\alpha = u_\alpha^{(0)} - z w_{,\alpha}^{(0)} - \frac{3}{4h} F(z) g_\alpha$	$\left(f_1^{(b)} = -1, \quad f_1^{(s)} = -\frac{3}{4h} \right)$
Note that in all the following theories $f_1^{(s)} = 1 + f_1^{(b)}$.		
Vlasov 1957, Jemielita 1975, Levinson 1980, Bickford 1982, Reddy 1984, 1986	$u_\alpha = u_\alpha^{(0)} - c_1 z^3 w_{,\alpha}^{(0)} + F(z) g_\alpha$	$\left(f_1^{(b)} = 0, \quad f_1^{(s)} = 1 \right)$
Murthy 1981, Shi 2007, Shi and Voyiadji 2011	$u_\alpha = u_\alpha^{(0)} + \left(\frac{1}{4} F(z) - c_1 z^3 \right) w_{,\alpha}^{(0)} + \frac{5}{4} F(z) g_\alpha$	$\left(f_1^{(b)} = \frac{1}{4}, \quad f_1^{(s)} = \frac{5}{4} \right)$
In the previous equations, $c_1 = \frac{4}{3h^2}$, $F(z) = z\left(1 - \frac{4}{3h^2} z^2\right) = z\left(1 - c_1 z^2\right)$		

Table 3. Relation between different kinematic quantities.

Reference	Transverse shear stress $\gamma_{\alpha z}=F_{,z}\gamma_{\alpha z}^{(0)}$ Eq. (47)
Present	$\gamma_{\alpha z}^{(0)}=\left(1+f_1^{(b)}\right)w_{,\alpha}^{(0)}+f_1^{(s)}g_\alpha$
Note that in the following theories $\gamma_{\alpha z}^{(0)}=f_1^{(s)}g_\alpha$	
Reissner 1945, Giavotto 1969	$\gamma_{\alpha z}^{(0)}=\frac{5}{4}g_\alpha$
Kromm 1953, 1955; Schmidt 1977	$\gamma_{\alpha z}^{(0)}=\frac{3}{2}g_\alpha$
Ambartsumyan 1960, 1964	$\gamma_{\alpha z}=\frac{h^2}{8}g_\alpha$
Kaczkowski 1968, Reissner 1975, Panc 1975, Bhimaraddi and Stevens 1984 Reddy 1984, 1990; Ren and Hinton 1986 Soldatos 1986, 1988	$\gamma_{\alpha z}^{(0)}=g_\alpha$
Krishna Murty 1977	$\gamma_{\alpha z}^{(0)}=-\frac{3}{4h}g_\alpha$
Note that in the following theories $\gamma_{\alpha z}^{(0)}=f_1^{(s)}(w_{,\alpha}^{(0)}+g_\alpha)$ where $f_1^{(s)}=1+f_1^{(b)}$.	
Vlasov 1957, Jemielita 1975, Levinson 1980, Bickford 1982, Reddy 1984, 1986	$\gamma_{\alpha z}^{(0)}=w_{,\alpha}^{(0)}+g_\alpha$
Murthy 1981, Shi 2007, Shi and Voyiadjis 2011	$\gamma_{\alpha z}^{(0)}=\frac{5}{4}(w_{,\alpha}^{(0)}+g_\alpha)$

Table 4. Summary of the main kinematic relations of the present $\{3,0\}$ -order polynomial plate theory.

In-plane displacement field	$u_\alpha = u_\alpha^{(0)} + \left(f_1^{(b)} z \left(1 - \frac{4}{3h^2} z^2\right) - \frac{4}{3h^2} z^3 \right) w_{,\alpha}^{(0)} + f_1^{(s)} z \left(1 - \frac{4}{3h^2} z^2\right) g_\alpha$ $= u_\alpha^{(0)} + \left(f_1^{(b)} F(z) - c_1 z^3 \right) w_{,\alpha}^{(0)} + f_1^{(s)} F(z) g_\alpha$
Strain field	$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{(0)} + z\boldsymbol{\varepsilon}^{(1)} + z^3 \boldsymbol{\varepsilon}^{(3)}$ $\boldsymbol{\gamma} = \boldsymbol{\gamma}^{(0)} + z^2 \boldsymbol{\gamma}^{(2)}$ $\boldsymbol{\varepsilon}^{(1)} = \left(f_1^{(b)} \boldsymbol{\varepsilon}^{(b)} + f_1^{(s)} \boldsymbol{\varepsilon}^{(s)} \right) = \left(\left(1 + f_1^{(b)}\right) \boldsymbol{\varepsilon}^{(b)} + f_1^{(s)} \boldsymbol{\varepsilon}^{(s)} - \boldsymbol{\varepsilon}^{(b)} \right) = \nabla \boldsymbol{\gamma}^{(0)} - \boldsymbol{\varepsilon}^{(b)}$ $\boldsymbol{\varepsilon}^{(3)} = -c_1 \left(\left(1 + f_1^{(b)}\right) \boldsymbol{\varepsilon}^{(b)} + f_1^{(s)} \boldsymbol{\varepsilon}^{(s)} \right) = -c_1 \nabla \boldsymbol{\gamma}^{(0)}$ $\boldsymbol{\gamma}^{(2)} = -3c_1 \boldsymbol{\gamma}^{(0)}$ $\boldsymbol{\gamma}^{(0)} = \left(1 + f_1^{(b)}\right) \boldsymbol{\gamma}^{(b)} + f_1^{(s)} \boldsymbol{\gamma}^{(s)}$

Table 5. Relations between four variable and five variable {3,0}-order polynomial kinematics.

Four-variable	Five-variable
Bisplinghoff et al 1957, Krishna Murthy 1984, Senthilnathan et al 1987, Benachour et al 2011, Thai et al 2013	Reissner 1975, Panc 1975, Bhimaraddi and Stevens 1984, Ren 1986, Reddy 1990
$u_\alpha = u_\alpha^{(0)} - z w_{b,\alpha}^{(0)} + z\left(\frac{1}{4} - \frac{5}{3h^2}\right) w_{s,\alpha}^{(0)}$ $u_z = w_b^{(0)} + w_s^{(0)}$ $w_{s,\alpha}^{(0)} = \frac{4}{5}\gamma_{\alpha z}^{(0)}$; $w_{b,\alpha}^{(0)} = w_{,\alpha}^{(0)} - \frac{4}{5}\gamma_{\alpha z}^{(0)}$	$u_\alpha = u_\alpha^{(0)} - z w_{,\alpha}^{(0)} + z\left(1 - \frac{4}{3h^2} z^2\right)\gamma_{\alpha z}^{(0)}$ $u_z = w^{(0)}$
Senthilnathan et al 1987	
$u_\alpha = u_\alpha^{(0)} - z w_{b,\alpha}^{(0)} - \frac{4}{3h^2} z^3 w_{s,\alpha}^{(0)}$ $u_z = w_b^{(0)} + w_s^{(0)}$ $w_{s,\alpha}^{(0)} = \gamma_{\alpha z}^{(0)}$; $w_{b,\alpha}^{(0)} = w_{,\alpha}^{(0)} - \gamma_{\alpha z}^{(0)}$	

Table 6. Summary of the equations for cylindrical bending

Equations of motion	
$\left(\left(f_1^{(b)2}\mathcal{D}_{11} - f_1^{(b)}(2+f_1^{(b)})c_1\mathcal{F}_{11} + c_1^2 H_{11}\right)w^{(0)}_{,xx} - \left((1+f_1^{(b)})^2\left(\mathcal{A}_{44}-3c_1\mathcal{D}_{44}\right)\right)w^{(0)}\right)_{,xx} +$ $+ f_1^{(s)}\left(\left(f_1^{(b)}\mathcal{D}_{11} - \left(1+f_1^{(b)}\right)c_1\mathcal{F}_{11}\right)g_{x,xx} - (1+f_1^{(b)})\left(\mathcal{A}_{44}-3c_1\mathcal{D}_{44}\right)g_x\right)_{,x} = RHS_w$ $f_1^{(s)}\left(\left(f_1^{(b)}\mathcal{D}_{11} - \left(1+f_1^{(b)}\right)c_1\mathcal{F}_{11}\right)w^{(0)}_{,xx} - (1+f_1^{(b)})\left(\mathcal{A}_{44}-3c_1\mathcal{D}_{44}\right)w^{(0)}\right)_{,x} +$ $+ f_1^{(s)2}\left(\left(\mathcal{D}_{11}-c_1\mathcal{F}_{11}\right)g_{x,xx} - \left(\mathcal{A}_{44}-3c_1\mathcal{D}_{44}\right)g_x\right) = RHS_g$	
Problem **P1**)	$RHS_w = \bar{p}_z\,;\quad RHS_g = 0$
Problem **P2**)	$RHS_w = -I^{(0)}\ddot{w}^{(0)} + \left(f_1^{(b)2}J_2 - f_1^{(b)}(2+f_1^{(b)})J_4 + c_1^2 I_6\right)\ddot{w}^{(0)}_{,xx} +$ $+\left(f_1^{(b)}f_1^{(s)}J_2 - f_1^{(s)}(1+f_1^{(b)})c_1 J_4\right)\ddot{g}_{x,x};$ $RHS_g = \left(f_1^{(b)}f_1^{(s)}J_2 - f_1^{(s)}(1+f_1^{(b)})c_1 J_4\right)\ddot{w}^{(0)}_{,x} + f_1^{(s)2}\left(J_2 - c_1 J_4\right)\ddot{g}_x$
$\mathcal{D}_{ij} = D_{ij} - c_1 F_{ij}\,;\quad \mathcal{F}_{ij} = F_{ij} - c_1 H_{ij}\,;\ (D_{ij};\ F_{ij}\,;\ H_{ij}) = \left\langle Q_{ij}\left(z^2;\ z^4;\ z^6\right)\right\rangle\ \ (i,j=1,2,6)$	
$\mathcal{A}_{ij} = A_{ij} - 3c_1 D_{ij};\quad \mathcal{D}_{ij} = D_{ij} - 3c_1 F_{ij}\,;\ \left(A_{ij},\ D_{ij},\ F_{ij}\right) = \left\langle Q_{ij}\left(1;\ z^2;\ z^4\right)\right\rangle\ \ (i,j=4,5)$	
$J_i = I_i - c_1 I_{i+2}\ ;\ I_i = \left\langle \rho\ z^i\right\rangle\quad (i=1-6)$	
Boundary conditions on $x = 0, L$: $w^{(0)} = w^{(0)}_{,xx} = g_{x,x} = 0.$	
Assumed solution: $w^{(0)}(x) = A_w sin\lambda x,\quad g_x(x) = A_g cos\lambda x,\ \lambda = \frac{\pi}{L}.$	
Problem **P1**) Elastostatic analysis	
$A_w = \frac{1}{\lambda^2}\frac{\lambda^2\left(\mathcal{D}_{11}-c_1\mathcal{F}_{11}\right)+\mathcal{A}_{44}-3c_1\mathcal{D}_{44}}{D\hat{E}N}\bar{p}_0\,;\quad A_g = -\frac{1}{\lambda}\frac{\left({}^{(bs)}\hat{D}_{11}\lambda^2 + {}^{(bs)}\hat{A}_{44}\right)}{f_1^{(s)}D\hat{E}N}\bar{p}_0;\ \gamma^{(0)}_{xz} = \frac{1}{\lambda}\frac{N\hat{U}M}{D\hat{E}N}\bar{p}_0$	
Problem **P2**) Frequency equation	
$D\hat{E}N + \omega^4\left(I^{(0)} + \lambda^2\left[c_1^2 I_6\left(J_2 - c_1 J_4\right) - c_1^2 J_4^2\right]\right) + \omega^2\lambda^2\left(\lambda^2 c_1^2 H_{11} + \mathcal{A}_{44} - 3c_1\mathcal{D}_{44}\right)\left(J_2 - c_1 J_4\right) +$ $+\omega^2\left(\left[\lambda^2\left(\mathcal{D}_{11}-c_1\mathcal{F}_{11}\right)+\mathcal{A}_{44}-3c_1\mathcal{D}_{44}\right]\left(I^{(0)}+c_1^2 I_6\right) - 2\lambda^2\left[\lambda^2 c_1\mathcal{F}_{11} - \left(\mathcal{A}_{44}-3c_1\mathcal{D}_{44}\right)\right]c_1 J_4\right) = 0$	
$D\hat{E}N = \lambda^4 c_1^2\left(D_{11}H_{11} - F_{11}F_{11}\right) + \lambda^2\left(\mathcal{A}_{44} - 3c_1\mathcal{D}_{44}\right)D_{11};\ \ N\hat{U}M = \lambda^2\mathcal{D}_{11}$	

Figure caption:

Fig. 1 - Plate geometry, coordinate system and in-plane loads.