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Patrizia Semeraro
Dipartimento di Scienze Matematiche G. L. Lagrange
Dipartimento di eccellenza 2018-2019
Politecnico di Torino
Corso Duca degli Abruzzi, 24, Torino, ITALY
e-mail address: patrizia.semeraro@polito.it

Abstract

In this note we use multivariate subordination to introduce a multivariate extension of the generalized asymmetric Laplace motion. The class introduced provides a unified framework for several multivariate extensions of the popular variance gamma process. We also show that the associated time one distribution extends the multivariate generalized asymmetric Laplace distributions proposed in the statistical literature.
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Introduction

The generalized Laplace distribution (Laplace (1774)) is an infinitely divisible distribution which can account for heavier than Gaussian tails. For this reason, it has become popular in applications and was extended to asymmetric and multivariate settings. The multivariate generalized asymmetric Laplace distribution studied in Kozubowski et al. (2013) is introduced by its characteristic function

$$\psi_{\boldsymbol{Y}}(\boldsymbol{z}) = \left(1 - \frac{\left(i\boldsymbol{z}^T\mu - \frac{1}{2}\boldsymbol{z}^T\Sigma\boldsymbol{z}\right)}{\beta}\right)^{-a}, \ \ \boldsymbol{z} \in \mathbb{R}^n,$$

where $\boldsymbol{\mu} \in \mathbb{R}^n$, $\boldsymbol{\Sigma} = (\sigma_{ij})$ is a $n \times n$ non-negative definite matrix and $a > 0, \beta > 0$. The generalized asymmetric Laplace distribution is infinitely divisible, thus it is the time one distribution of a multivariate Lévy process, that is the generalized asymmetric Laplace motion. The generalized asymmetric Laplace motion can be interpreted as a subordinated Brownian motion, with a univariate gamma subordinator.

Generalized asymmetric Laplace distributions and their associated Lévy motions are widely used for multivariate modelling in several areas, such as engineering and finance (Kotz et al. (2001)). For instance, Madan and Seneta (1990) proposed the generalized asymmetric Laplace motion, with the name of variance gamma process, as a model of asset returns with the interesting interpretation of the gamma subordinator as a random economic time. Since then, the generalized asymmetric Laplace motion has become very popular in finance, especially for derivative pricing, for its simple analytical tractability as well as fitting properties. The first multivariate version of the variance gamma process was constructed by subordination of a multivariate Brownian motion with a common gamma subordinator, which represented market time (see Leoni and Schoutens (2008) and Luciano and Schoutens (2006)). The variance gamma process was further generalized to account for cross sectional properties of asset returns and trades, using the multivariate subordination in Barndorff-Nielsen et al. (2001). A multivariate gamma factor-based subordinator, the α-gamma process, composed of a common component and an idiosyncratic component was introduced in Semeraro (2008) to construct the α-variance gamma process. The factor structure of the α-gamma subordinator reflects the factor structure of trades, empirically investigated in Lo and Wang (2000). The same subordinator was employed in Luciano and Semeraro (2010) to construct the more general $\rho\alpha$-variance gamma process. The α-gamma subordinator was also used in Jevtić et al. (2017) to propose a multivariate generalized asymmetric Laplace motion, which is the result of multivariate subordination of a Brownian motion with dependent components.

Using the same technique and the multivariate gamma subordinator introduced in Pérez-Abreu and Stelzer (2014) we introduce the class of multivariate gamma-Laplace motions which includes as subcases the above mentioned processes. The associated time one multivariate distribution generalizes the multivariate versions of the generalized asymmetric Laplace distribution in the literature, introduced in Kotz et al. (2001), Kozubowski and Podgorski (2000) and Kozubowski et al. (2013).

The paper is organized as follows. After some preliminary notions recalled in Section 1, the multivariate gamma-Laplace distribution is defined in Section 2. We show that the multivariate distribution introduced includes as subcases several processes widely used in finance. In Section 3, we further propose a generalization of the $\rho\alpha$-variance gamma process. Some examples conclude.

1 Preliminaries

We refer to Sato (1999) for infinitely divisible distributions, Lévy processes and one dimensional subordination. We refer to Barndorff-Nielsen et al. (2001) and Pedersen and Sato (2003) for multivariate subordination. We recall the notion of $\mathbb{R}^d_+$-parameter Lévy process, which is an important building block of multivariate subordination. Consider the multiparameter $\boldsymbol{s} = (s_1, ..., s_d)^T \in \mathbb{R}^d_+$ and the partial order on $\mathbb{R}^d_+$

$$\boldsymbol{s}^1 \preceq \boldsymbol{s}^2 \;\;\Leftrightarrow\;\; \boldsymbol{s}^1_j \leq \boldsymbol{s}^2_j,\, j = 1, \ldots, d.$$

Let $\boldsymbol{L}(\boldsymbol{s}) = (L_1(\boldsymbol{s}), L_2(\boldsymbol{s}), \ldots, L_n(\boldsymbol{s}))^T$ be a process with parameters in $\mathbb{R}^d_+$ and values in $\mathbb{R}^n$. It is called an $\mathbb{R}^d_+$-parameter Lévy process on $\mathbb{R}^n$ if the following hold:

- for any $m \geq 3$ and for any choice of $\boldsymbol{s}^1 \preceq ... \preceq \boldsymbol{s}^m$, $\boldsymbol{L}(\boldsymbol{s}^j) - \boldsymbol{L}(\boldsymbol{s}^{j-1})$, $j = 2, ..., m$, are independent,
- for any $\boldsymbol{s}^1 \preceq \boldsymbol{s}^2$ and $\boldsymbol{s}^3 \preceq \boldsymbol{s}^4$ satisfying $\boldsymbol{s}^2 - \boldsymbol{s}^1 = \boldsymbol{s}^4 - \boldsymbol{s}^3$, $\mathcal{L}(\boldsymbol{L}(\boldsymbol{s}^2) - \boldsymbol{L}(\boldsymbol{s}^1)) = \mathcal{L}(\boldsymbol{L}(\boldsymbol{s}^4) - \boldsymbol{L}(\boldsymbol{s}^3))$ where $\mathcal{L}(\cdot)$ denotes the law of the random variable,
- $\boldsymbol{L}(0) = 0$ almost surely, and
- almost surely, $\boldsymbol{L}(\boldsymbol{s})$ is right continuous with left limits in $\boldsymbol{s}$ in the partial ordering of $\mathbb{R}^d_+$.

The notation $\mathcal{M}_{n\times d}$ indicates the class of $n \times d$ matrices with real entries. We denote the unit sphere in $\mathbb{R}^n$ by $S^{n-1} := \{\boldsymbol{w} \in \mathbb{R}^n : |\boldsymbol{w}| = 1\}$.

The class of multivariate gamma distributions introduced in Pérez-Abreu and Stelzer (2014) extends the gamma distribution to the multivariate case. In this work, we consider multivariate gamma distributions defined on the positive cone $\mathbb{R}^d_+$. A random vector $\boldsymbol{X}$ on $\mathbb{R}^d_+$ has a *d-dimensional gamma distribution $\gamma_{\boldsymbol{X}}$ with parameters α and β*, denoted as $\Gamma_d(\alpha, \beta)$, if there exists a finite measure α on $S^{d-1}_+ := \{\boldsymbol{w} \in \mathbb{R}^n_+ : |\boldsymbol{w}| = 1\}$ and a measurable function $\beta : S^{d-1}_+ \to \mathbb{R}_+$ such that $\boldsymbol{X}$ has the following characteristic function:

$$\psi_{\boldsymbol{X}}(\boldsymbol{z}) = exp\{\int_{S_+^{d-1}} \int_{\mathbb{R}_+} (e^{is\langle \boldsymbol{w}, \boldsymbol{z}\rangle} - 1) \frac{e^{-\beta(\boldsymbol{w})s}}{s} ds \alpha(d\boldsymbol{w})\}, \ \forall \boldsymbol{z} \in \mathbb{R}^d.$$

We denote with $\phi_{\boldsymbol{X}}(\boldsymbol{z}) := \ln\{\psi_{\boldsymbol{X}}(\boldsymbol{z})\}$ the characteristic exponent of $\boldsymbol{X}$. If β is constant $\boldsymbol{X}$ is called homogeneous. A one dimensional homogeneous gamma distribution with parameters α and β is a gamma distribution. In fact, if $\mathcal{L}(\boldsymbol{X}) = \Gamma_1(\alpha, \beta)$, then $\mathcal{L}(\boldsymbol{X}) = \Gamma(a, b)$, where $a = \alpha(\{1\})$ and $b = \beta(1)$.

A d-dimensional gamma distribution has the following Lévy measure:

$$\nu_{\boldsymbol{X}}(E) = \int_{S_+^{d-1}} \int_{\mathbb{R}_+} \mathbf{1}_E(s\boldsymbol{w}) \frac{e^{-\beta(\boldsymbol{w})s}}{s} ds \alpha(d\boldsymbol{w}). \tag{1.1}$$

Pérez-Abreu and Stelzer (2014) proved that (1.1) defines a Lévy measure if and only if $\int_{S_+^{d-1}} \ln(1 + \frac{1}{\beta(\boldsymbol{w})})\alpha(d\boldsymbol{w}) < \infty$ and in this case $\int_{\mathbb{R}_+^d} (|\boldsymbol{w}| \wedge 1)\nu_{\boldsymbol{X}}(d\boldsymbol{w}) < \infty$. They also proved the following closure properties, which are necessary for our construction:

a) Let $\mathcal{L}(\boldsymbol{X}) = \Gamma_d(\alpha, \beta)$ and $c > 0$. Then $\mathcal{L}(c\boldsymbol{X}) = \Gamma_d(\alpha, \beta/c)$.

b) Let $\mathcal{L}(\boldsymbol{X}_1) = \Gamma_d(\alpha_1, \beta)$ and $\mathcal{L}(\boldsymbol{X}_2) = \Gamma_d(\alpha_2, \beta)$ be two independent gamma variables. Then $\mathcal{L}(\boldsymbol{X}_1 + \boldsymbol{X}_2) = \Gamma_d(\alpha_1 + \alpha_2, \beta)$.

Since a multivariate gamma distribution is infinitely divisible, it is the time one distribution of a multivariate subordinator. We call *multivariate gamma subordinator* with parameters α and β, denoted as $S\Gamma_d(\alpha, \beta)$, the subordinator $\{\boldsymbol{X}(t), t \geq 0\}$ associated with a multivariate gamma distribution $\Gamma_d(\alpha, \beta)$. It follows from the closure properties of d-dimensional gamma distributions that if a random vector $\boldsymbol{X}$ has multivariate gamma distribution $\Gamma_d(\alpha, \beta)$ then the associated Lévy process $\{\boldsymbol{X}(t), t \geq 0\}$ satisfies $\mathcal{L}(\boldsymbol{X}(t)) = \Gamma_d(t\alpha, \beta)$, $t \in \mathbb{R}_+$. Since $\mathcal{L}(\boldsymbol{X}) = \mathcal{L}(\boldsymbol{X}(1))$ we simply write $\boldsymbol{X}$ instead of $\boldsymbol{X}(1)$.

2 Multivariate gamma-Laplace motion

This section introduces the multivariate gamma-Laplace motion, defined by subordination of an $\mathbb{R}_+^d$-parameter Brownian motion with a multivariate gamma subordinator.

We recall the definition of $\mathbb{R}_+^d$-parameter Brownian motion (see Semeraro (2008)). Let $\{B_i^I(t), t \geq 0\}$, $i = 1, \ldots, d$ be independent Brownian motions with drift μ_i and standard deviation σ_i and let $\{\boldsymbol{B}^I(\boldsymbol{s})) := (B_1(s_1), \ldots, B_d(s_d))^T, \boldsymbol{s} \in \mathbb{R}_+^d\}$ be the associated multiparameter Lévy process. Let $\boldsymbol{A} \in \mathcal{M}_{n \times d}$, we can define the process

$$\boldsymbol{B}(s) := \boldsymbol{A}\boldsymbol{B}^I(\boldsymbol{s}), \ \boldsymbol{s} \in \mathbb{R}_+^d. \tag{2.1}$$

The process $\{\boldsymbol{B}(\boldsymbol{s}), \boldsymbol{s} \in \mathbb{R}_+^d\}$ is an $\mathbb{R}_+^d$*-parameter Brownian motion* on $\mathbb{R}^n$ with parameters $\boldsymbol{\mu}, \boldsymbol{A}, \boldsymbol{\Sigma}$. It is denoted as $BM(\boldsymbol{\mu}, \boldsymbol{A}, \boldsymbol{\Sigma})$.

A natural extension of the generalized asymmetric Laplace motion to the multivariate setting can be obtained by subordinating a $\mathbb{R}_+^d$-parameter Brownian motion on $\mathbb{R}^n$ with a multivariate gamma subordinator.

Definition 2.1. *Let $\{\boldsymbol{B}(\boldsymbol{s}),\, \boldsymbol{s} \in \mathbb{R}_+^d\}$ be a $\mathbb{R}_+^d$-parameter Brownian motion and let $\{\boldsymbol{X}(t),\, t > 0\}$ be a multivariate gamma subordinator $S\Gamma_d(\alpha, \beta)$, independent of $\boldsymbol{B}(t)$. The process $\{\boldsymbol{Y}(t),\, t > 0\}$ defined by*

$$\boldsymbol{Y}(t) := \boldsymbol{B}(\boldsymbol{X}(t)),\, t \geq 0, \tag{2.2}$$

is a multivariate gamma-Laplace motion, denoted as $M\Gamma L_d^n(\boldsymbol{\mu}, \boldsymbol{A}, \boldsymbol{\Sigma}, \alpha, \beta)$. If β is constant, the process $\boldsymbol{Y}$ is called homogeneous.

We also denote as $M\Gamma L_d^n(\boldsymbol{\mu}, \boldsymbol{A}, \boldsymbol{\Sigma}, \alpha, \beta)$ the time one distribution $\lambda_{\boldsymbol{Y}}$ of a multivariate gamma-Laplace motion $\boldsymbol{Y}$. It satisfies:

$$\lambda_{\boldsymbol{Y}}(A) = \int_{\mathbb{R}_+^d} \lambda_{\boldsymbol{s}}(B)\gamma_{\boldsymbol{X}}(d\boldsymbol{s}), A \in \mathcal{B}(\mathbb{R}^n),$$

where $\lambda_{\boldsymbol{s}} := \mathcal{L}(\boldsymbol{B}(\boldsymbol{s}))$, $\boldsymbol{s} \in \mathbb{R}_+^d$, and $\gamma_{\boldsymbol{X}}$ is the time one distribution of the subordinator $\boldsymbol{X}(t)$. The Lévy triplet $(\boldsymbol{\gamma}_{\boldsymbol{Y}}, \boldsymbol{\Sigma}_{\boldsymbol{Y}}, \nu_{\boldsymbol{Y}})$ of $\boldsymbol{Y}$ is a direct consequence of Theorem 4.7 in Barndorff-Nielsen et al. (2001),

$$\begin{gathered}\boldsymbol{\Sigma}_{\boldsymbol{Y}} = \boldsymbol{0},\\ \nu_{\boldsymbol{Y}}(A) = \textstyle\int_{\mathbb{R}_+^d} \lambda_{\boldsymbol{s}}(A)\nu_{\boldsymbol{X}}(d\boldsymbol{s}),\\ \boldsymbol{\gamma}_{\boldsymbol{Y}} = \textstyle\int_{\mathbb{R}_+^d} \nu_{\boldsymbol{X}}(d\boldsymbol{s}) \int_{|\boldsymbol{x}|\leq 1} \boldsymbol{x}\lambda_{\boldsymbol{s}}(d\boldsymbol{x}),\end{gathered}$$

where $\lambda_{\boldsymbol{s}} = \mathcal{L}(\boldsymbol{B}(\boldsymbol{s}))$, $\boldsymbol{s} \in \mathbb{R}_+^d$, $\boldsymbol{x} = (x_1, ..., x_n)^T$; and $A \in \mathbb{R}^n \setminus \{0\}$ and

$$\nu_{\boldsymbol{X}}(E) = \int_{S_+^{d-1}} \int_{\mathbb{R}_+} \mathbf{1}_E(s\boldsymbol{w}) \frac{e^{-\beta(\boldsymbol{w})s}}{s} ds\alpha(d\boldsymbol{w})$$

is the Lévy measure of the subordinator. The multivariate gamma-Laplace motion has bounded variations, as does its one dimensional counterpart.

Proposition 2.1. Let $\boldsymbol{Y}(t)$ be a $M\Gamma L_d^n(\boldsymbol{\mu}, \boldsymbol{A}, \boldsymbol{\Sigma}, \alpha, \beta)$ process. Then $\int_{|\boldsymbol{x}|\leq 1} |\boldsymbol{x}|\nu_{\boldsymbol{Y}}(\boldsymbol{x}) < \infty$, i.e. $\boldsymbol{Y}(t)$ has bounded variation.

Proof. It is sufficient to show that

$$\int_{|\boldsymbol{x}|\leq 1} |\boldsymbol{x}|^{1/2}\nu_{\boldsymbol{X}}(\boldsymbol{x}) < \infty,$$

where $\nu_{\boldsymbol{X}}$ is the Lévy measure of $\boldsymbol{X}$. Then the assertion follows from Theorems 3.3 and 4.7 in Barndorff-Nielsen et al. (2001). It holds that

$$\begin{aligned}\int_{|\boldsymbol{x}|\leq 1} |\boldsymbol{x}|^{1/2}\nu_{\boldsymbol{X}}(\boldsymbol{x}) &= \int_{S_+^{d-1}} \int_0^1 e^{-\beta(w)r} r^{1/2} dr\alpha(d\boldsymbol{w}) \leq \int_{S_+^{d-1}} \alpha(d\boldsymbol{w}) \int_0^1 e^{-\beta(\boldsymbol{w})r} dr\\ &= \int_{S_+^{d-1}} \frac{1 - e^{-\beta(\boldsymbol{w})}}{\beta(\boldsymbol{w})} \alpha(d\boldsymbol{w}) \leq \alpha(S_+^{d-1}) < \infty,\end{aligned}$$

since it holds $1 - e^x \leq x$, for any $x \in \mathbb{R}_+$. □

Let $\boldsymbol{Y}(t)$ be a $M\Gamma L_d^n(\boldsymbol{\mu}, \boldsymbol{A}, \boldsymbol{\Sigma}, \alpha, \beta)$. By (2.1), it follows that

$$\boldsymbol{Y}(t) =_{\mathcal{L}} \boldsymbol{A}\boldsymbol{Y}^I(t), \tag{2.3}$$

where

$$\boldsymbol{Y}^I(t) = \boldsymbol{B}^I(\boldsymbol{X}(t)) = \begin{pmatrix} B_1^I(X_1(t)) \\ .... \\ B_d^I(X_d(t)) \end{pmatrix}, \ t \geq 0, \tag{2.4}$$

is a $M\Gamma L_d^d(\boldsymbol{\mu}, \boldsymbol{I}, \boldsymbol{\Sigma}, \alpha, \beta)$ motion. Notice that the process $\boldsymbol{Y}^I(t)$ is constructed by componentwise subordination, thus $d = n$. We simply write $M\Gamma L_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \alpha, \beta)$ instead of $M\Gamma L_n^n(\boldsymbol{\mu}, \boldsymbol{I}, \boldsymbol{\Sigma}, \alpha, \beta)$. We provide the characteristic function of a gamma-Laplace motion in two cases: $\boldsymbol{X}$ homogeneous and $\boldsymbol{X}$ non-homogeneous, but α has assumed to have countable support.

Proposition 2.2. The time one characteristic function of a multivariate gamma-Laplace motion $\boldsymbol{Y}(t)$ is given by

$$\psi_{\boldsymbol{Y}}(\boldsymbol{z}) = exp\Big(\int_{S_+^{d-1}} \ln\big(\frac{\beta}{\beta - \sum_{j=1}^n w_j(i\mu_j \sum_{k=1}^n a_{kj}z_k - \frac{1}{2}\sigma_j^2(\sum_{k=1}^n a_{kj}z_k)^2)}\big)\alpha(d\boldsymbol{w})\Big) \tag{2.5}$$

for all $\boldsymbol{z} \in \mathbb{R}^n$.

Proof. **From Theorem 4.7 in Barndorff-Nielsen et al. (2001),**

$$\psi_{\boldsymbol{Y}(t)}(\boldsymbol{z}) = \exp\{t\Psi_{\boldsymbol{X}}(\ln \psi_{\boldsymbol{B}}(\boldsymbol{z}))\}, \tag{2.6}$$

where $\ln(\psi_{\boldsymbol{B}}(\boldsymbol{z})) := (\ln \psi_1(\boldsymbol{z}), \ldots, \ln \psi_d(\boldsymbol{z}))$, $\boldsymbol{z} \in \mathbb{R}^n$ **and** $\psi_j(\boldsymbol{z})$ **are the characteristic functions of** $\boldsymbol{L}(\boldsymbol{\delta}_j)$ **and** $\boldsymbol{\delta}_j = (\delta_{j1}, \ldots, \delta_{jd})$ **with Kronecker's** δ_{jk}**.**

The characteristic exponent of $\boldsymbol{X}$, $\Psi_{\boldsymbol{X}}$, is given by

$$\Psi_{\boldsymbol{X}}(\boldsymbol{w}) = \int_{\mathbb{S}_+^{d-1}} \ln\left(\frac{\beta}{\beta - \langle \boldsymbol{w}, \boldsymbol{z}\rangle}\right)\alpha(d\boldsymbol{w}), \tag{2.7}$$

where ln is the main branch of the complex logarithm (see Pérez-Abreu and Stelzer (2014) for derivation of (2.7)**). Therefore,**

$$\psi_{\boldsymbol{Y}}(\boldsymbol{z}) = exp\Big(\int_{S_+^{d-1}} \ln\big(\frac{\beta}{\beta - \langle \boldsymbol{w}, \ln \psi_{\boldsymbol{B}}(\boldsymbol{z})\rangle}\big)\alpha(d\boldsymbol{w})\Big) \tag{2.8}$$

for all $\boldsymbol{z} \in \mathbb{R}^n$.

From Proposition 3.3 in Jevtić et al. (2017) we have

$$\psi_j(\boldsymbol{z}) = \exp\left\{i\mu_j \sum_{k=1}^n a_{kj}z_k - \frac{1}{2}\sigma_j^2\Big(\sum_{k=1}^n a_{kj}z_k\Big)^2\right\}, \tag{2.9}$$

for $j = 1, \ldots, d$**. Hence, sustituiting** (2.9) **in** (2.8) **we have** (2.5)**.**

□

If we consider the case of a homogeneous $M\Gamma L_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \alpha, \beta)$ process $\boldsymbol{Y}^I(t)$, we have:

$$\psi_{\boldsymbol{Y}^I}(\boldsymbol{z}) = exp\big(\int_{S_+^{n-1}} \ln(\frac{\beta}{\beta - \langle \boldsymbol{w}, \ln \psi_{B^I}(\boldsymbol{z})\rangle})\alpha(d\boldsymbol{w})\big)$$
$$= exp\big(\int_{S_+^{n-1}} \ln(\frac{\beta}{\beta - \sum_{j=1}^n w_j(i\mu_j z_j - \frac{1}{2}\sigma_j^2 z_j^2)})\alpha(d\boldsymbol{w})\big) \text{ for all } \boldsymbol{z} \in \mathbb{R}^n,$$

where $\psi_{\boldsymbol{B}^I}(z) = (\psi_1(z_1), \ldots, \psi_d(z_n))$ and $\psi_i(z_i)$, $i = 1, \ldots, n$ are the characteristic functions of the one dimensional independent Brownian motions $\{B_i^I(t), t \geq 0\}$, i.e. $\ln \psi_i(z_i) = i\mu_j z_j - \frac{1}{2}\sigma_j^2 z_j^2$, $i = 1, \ldots, n$.

Proposition 2.3. *If α has countable support $\mathcal{S} = \{\boldsymbol{w}_k \in S_+^{d-1}, k \in \mathbb{N}\}$, then the time one characteristic function of a multivariate gamma-Laplace motion $\boldsymbol{Y}(t)$ is:*

$$\psi_{\boldsymbol{Y}}(\boldsymbol{z}) = \prod_{\boldsymbol{w}_k \in \mathcal{S}} \big(1 - \frac{\sum_{j=1}^d w_{kj}(i\mu_j \sum_{h=1}^n a_{hj} z_h - \frac{1}{2}\sigma_1^2(\sum_{h=1}^n a_{hj} z_h)^2)}{b_h}\big)^{-a_k} \textit{ for all } \boldsymbol{z} \in \mathbb{R}^n. \tag{2.10}$$

Proof. If α has countable support $\mathcal{S}$, then the characteristic function of $\boldsymbol{X}$ is (Pérez-Abreu and Stelzer (2014)):

$$\psi_{\boldsymbol{X}}(\boldsymbol{z}) = \prod_{\boldsymbol{w}_k \in \mathcal{S}} \big(1 - i\frac{(\sum_{j=1}^d w_{kj} z_j)}{b_k}\big)^{-a_k} \text{ for all } \boldsymbol{z} \in \mathbb{R}^n, \tag{2.11}$$

where $a_k = \alpha(\{\boldsymbol{w}_k\})$ and $b_k = \beta(\boldsymbol{w}_k)$. Thus, **from equations** (2.9)**,** (2.11)**, and** (2.6) we find the characteristic function of $\boldsymbol{Y}^I$ by composition. □

The characteristic function of $\boldsymbol{Y}^I$ follows easily:

$$\psi_{\boldsymbol{Y}^I}(\boldsymbol{z}) = \prod_{\boldsymbol{w}_k \in \mathcal{S}} \big(1 - \frac{\sum_{j=1}^n w_{kj}(i\mu_j z_j - \frac{1}{2}\sigma_j^2 z_j^2)}{b_k}\big)^{-a_k} \text{ for all } \boldsymbol{z} \in \mathbb{R}^d.$$

Notice that the j-th marginal law of $\boldsymbol{Y}^I$ has characteristic function:

$$\psi_j^I(z_j) = \prod_{k \in \mathbb{N}} \big(1 - \frac{w_{kj}(i\mu_j z_j - \frac{1}{2}\sigma_j^2 z_j^2)}{b_k}\big)^{-a_k} \text{ for all } \boldsymbol{z} \in \mathbb{R},$$

which is the product of the characteristic functions of one dimensional generalized asymmetric Laplace distributions. Thus we have proved the following proposition.

Proposition 2.4. *Let $\boldsymbol{Y}^I(t)$ be a $M\Gamma L_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \alpha, \beta)$ process. If α has countable support $\mathcal{S} = \{\boldsymbol{w}_k, k \in \mathbb{N}\}$, then the -time one- one dimensional marginal distributions of $\boldsymbol{Y}^I(t)$ are countable convolutions of one dimensional generalized asymmetric Laplace distributions.*

A direct consequence of the representation in (2.3) is that the - time one - one dimensional margins of $\boldsymbol{Y}(t)$ are - in law - linear combinations of the one dimensional margins of $\boldsymbol{Y}^I(t)$.

2.1 Submodels

The multivariate gamma-Laplace motion is a generalization of several multivariate extensions of the popular variance gamma process. In this section, we show that it generalizes the multivariate variance gamma processes with a one dimensional subordinator introduced in Madan and Seneta (1990), Leoni and Schoutens (2008) and Luciano and Schoutens (2006). Furthermore, we show that its time one distribution includes as subcases the generalized Laplace distributions reported in Kotz et al. (2001), Kozubowski and Podgorski (2000) and Kozubowski et al. (2013). It also includes as submodels the Laplace marked Poisson process introduced in Jevtić et al. (2017), the α-variance gamma process introduced in Semeraro (2008), and the multivariate variance gamma process via linear transformation in Ballotta and Bonfiglioli (2012).

2.1.1 One dimensional gamma subordinator

Consider a one dimensional $S\Gamma_1(a,b)$ subordinator $Z(t)$ with time one distribution $\gamma_Z = \Gamma_1(\alpha,\beta) = \Gamma(a,b)$, where $a = \alpha(\{1\})$ and $b = \beta(1)$. Let $\boldsymbol{Y}(t)$ be a multivariate gamma-Laplace $M\Gamma L_1^n(\boldsymbol{\mu}, \boldsymbol{A}, \boldsymbol{\Sigma}, a, b)$ motion. The time one characteristic function of $\boldsymbol{Y}$ becomes

$$\psi_{\boldsymbol{Y}}(\boldsymbol{z}) = \left(1 - \frac{\left(i\boldsymbol{z}^T\boldsymbol{\mu}_1 - \frac{1}{2}\boldsymbol{z}^T\boldsymbol{\Sigma}_1\boldsymbol{z}\right)}{b}\right)^{-a}, \quad \boldsymbol{z} \in \mathbb{R}_+^n,$$

where $\boldsymbol{\Sigma}_1 = \boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{A}^T$ and $\boldsymbol{\mu}_1 = \boldsymbol{A}\boldsymbol{\mu}$. The process $\boldsymbol{Y}(t)$ is, in law, the multivariate variance gamma process in Leoni and Schoutens (2008) and Luciano and Schoutens (2006). If $\boldsymbol{\mu} = \boldsymbol{0}$ we find the symmetric variance gamma process introduced in Madan and Seneta (1990). The time one distribution of $\boldsymbol{Y}(t)$ is the multivariate generalized asymmetric Laplace distribution introduced in Kozubowski et al. (2013) and the subcase $a = \beta = 1$ is the multivariate asymmetric Laplace distribution in Kozubowski and Podgorski (2000) and Kotz et al. (2001).

2.1.2 Laplace marked Poisson process and α-variance gamma process

An important building block of the α-variance gamma process and of the Laplace marked Poisson process is the α-gamma subordinator in Semeraro (2008).

Definition 2.2. *Let X_j and Z be independent random variables distributed according to gamma laws $\mathcal{L}(X_j) = \Gamma\left(\frac{1}{c_j} - a, \frac{1}{c_j}\right)$, $j = 1, ..., n$, and $\mathcal{L}(Z) = \Gamma(a, 1)$. A multidimensional subordinator $\{\boldsymbol{G}(t), t \geq 0\}$ defined by*

$$\boldsymbol{G}(t) := \boldsymbol{X}(t) + \boldsymbol{c}Z(t) = (X_1(t) + c_1 Z(t), ..., X_n(t) + c_n Z(t))^T, \tag{2.12}$$

where $0 < c_j < \frac{1}{a}$, $j = 1, ..., n$, is called α-gamma subordinator.

The α-gamma subordinator has one dimensional gamma distributions, i.e. $\mathcal{L}(G_j) = \Gamma\left(\frac{1}{c_j}, \frac{1}{c_j}\right), \quad j = 1, ..., n$. We show that the α-gamma process belongs to the class of multivariate gamma subordinators.

Proposition 2.5. *Let $\boldsymbol{G}(t)$ be an α-gamma subordinator with parameters c_i, $i = 1, \ldots, n$ and a. Then $\boldsymbol{G}$ has distribution $\Gamma_n(\alpha_G, \frac{1}{|\boldsymbol{c}|})$, where α_G has support $\mathcal{S} = \{\boldsymbol{e}_i, i = 1, \ldots, n\} \cup \{\frac{\boldsymbol{c}}{|\boldsymbol{c}|}\}$, $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ is the canonical basis of $\mathbb{R}^n$, and β is defined on $\mathcal{S}$. It holds $\alpha(\boldsymbol{e}_j) = \frac{1}{c_j} - a$, $\alpha(\frac{\boldsymbol{c}}{|\boldsymbol{c}|}) = a$; $\beta(\boldsymbol{e}_j) = \frac{1}{c_j}$ and $\beta(\frac{\boldsymbol{c}}{|\boldsymbol{c}|}) = \frac{1}{|\boldsymbol{c}|}$.*

We need the following two lemmas.

Lemma 2.1. *Let Z have distribution $\Gamma(a, b)$ and $\boldsymbol{c} \in \mathbb{R}^n_+$. The vector $\boldsymbol{c}Z := (c_1 Z, \ldots, c_n Z)^T$ has homogeneous multivariate gamma distribution $\Gamma_d(\alpha_a, \frac{b}{|\boldsymbol{c}|})$, where α_a is a finite measure such that $\alpha_a(\{w\}) = 0$ if $w \in S^{n-1}_+$, $w \neq \frac{\boldsymbol{c}}{|\boldsymbol{c}|}$ and $\alpha_a(\{\frac{\boldsymbol{c}}{|\boldsymbol{c}|}\}) = a$.*

Proof. The Lévy measure of $\boldsymbol{c}Z$ can be determined as in Semeraro (2008). Consider a set $B \in \mathcal{B}(\mathbb{R}^n)$ and $\Delta_{\mathbf{c}} = \{(c_1 s, ..., c_n s)^T : s \in \mathbb{R}_+\}$. Define $B^c_j = \pi_j(B \cap \Delta_{\mathbf{c}}), \quad j = 1, ..., n$, where $\pi_j, i = 1, ..., n$ are the projection of B on the coordinate axes. Obviously, $\frac{B^c_j}{c_j} = \{s \in \mathbb{R} : c_j s \in B^c_j\}$, observe that, by construction, $\frac{B^c_j}{c_j} = \frac{B^c_k}{c_k} := B_\Delta$ for each $j, k = 1, ..., n$. Finally, $B^* = \times_{j=1}^n B_j$. Semeraro (2008) proved that:

$$\nu_{\boldsymbol{c}Z}(B) = \nu_{\boldsymbol{c}Z}(B \cap \Delta_{\mathbf{c}}) = \nu_{\boldsymbol{c}Z}(B^* \cap \Delta_{\mathbf{c}}) = \nu_{\boldsymbol{c}Z}(B^*) = \nu_Z(B_\Delta).$$

Consider now the characteristic exponent of $\boldsymbol{c}Z$:

$$\begin{aligned}
\phi_{\boldsymbol{c}Z}(\boldsymbol{z}) &= \int_{\mathbb{R}^n_+} (e^{i\langle \boldsymbol{z}, \boldsymbol{w} \rangle} - 1)\nu_{cZ}(d\boldsymbol{w}) = \int_{\mathbb{R}^n_+} (e^{\sum_{j=1}^n i w_j z_j} - 1)\nu_{cZ}(d\boldsymbol{w}) \\
&= \int_{\mathbb{R}^n_+ \cap \Delta_{\mathbf{c}}} (e^{\sum_{j=1}^n i w_j z_j} - 1)\nu_{cZ}(d\boldsymbol{w}) \\
&= \int_{\mathbb{R}_+} (e^{is\sum_{j=1}^n c_j z_j} - 1)\nu_Z(ds) \\
&= \int_{\mathbb{R}_+} (e^{is\sum_{j=1}^n c_j z_j} - 1)a\frac{e^{-bs}}{s}ds.
\end{aligned} \tag{2.13}$$

Let now $\boldsymbol{X}$ have distribution $\Gamma_d(\alpha_a, \beta)$, with $\beta(\boldsymbol{w}) = \frac{b}{|\boldsymbol{c}|}$, $\forall \boldsymbol{w} \in S^{n-1}_+$. Since, by assumption, the measure α_a satisfies $\alpha_a(\boldsymbol{w}) = 0$ if $w \in S^{n-1}_+$ and $w \neq \frac{\boldsymbol{c}}{|\boldsymbol{c}|}$ and $\alpha_a(\{\frac{\boldsymbol{c}}{|\boldsymbol{c}|}\}) = a$, we have

$$\begin{aligned}
\phi_{\boldsymbol{X}}(\boldsymbol{z}) &= \int_{S^{n-1}_+} \int_{\mathbb{R}_+} (e^{is\langle \boldsymbol{w}, \boldsymbol{z} \rangle} - 1)\frac{e^{-\beta(\boldsymbol{w})\boldsymbol{s}}}{\boldsymbol{s}} d\boldsymbol{s}\alpha_a(d\boldsymbol{w}) \\
&= \int_{\mathbb{R}_+} (e^{is\sum_{i=1}^d \alpha_i z_i \frac{1}{|\boldsymbol{c}|}} - 1)a\frac{e^{-\frac{b}{|\boldsymbol{c}|}s}}{s}ds \\
&= \int_{\mathbb{R}_+} (e^{iu\sum_{i=1}^n \alpha_i z_i} - 1)a\frac{e^{-bu}}{u}du,
\end{aligned}$$

where the last equality follows imposing $\frac{s}{|\boldsymbol{c}|} = u$. The assertion follows by observing that $\phi_{\boldsymbol{X}}(\boldsymbol{z}) = \phi_{\boldsymbol{c}Z}(\boldsymbol{z})$. □

Lemma 2.2. *Let $\boldsymbol{X}(t)$ be a gamma process with $\mathcal{L}(\boldsymbol{X}) = \Gamma_n(\alpha, \beta)$ and let $\boldsymbol{X} = (X_1, \ldots, X_n)$, $j = 1, ..., d$, have independent components. Then, $\mathcal{L}(X_j) = \Gamma(a_i, b_i)$, where $a_i := \alpha(\{\boldsymbol{e}_i\})$, $b_i := \beta(\boldsymbol{e}_i)$ and $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$ is the canonical basis of $\mathbb{R}^n$.*

Proof. Since independent Lévy processes never jump together, $\nu_{\boldsymbol{X}}$ is supported on the union of the coordinate axis (see, for example, Sato (1999), E 12.10). Thus, it holds that

$$\nu_{\boldsymbol{X}}(E) = \sum_{j=1}^{n} \nu_{\boldsymbol{X}}(E_j),$$

where $E \in \mathcal{B}(\mathbb{R}^n \setminus \{0\})$, $E_j = E \cap A_j$ and $A_j = \{x \in \mathbb{R}^n : x_k = 0, k \neq j, k = 1, ..., n\}$. Thus,

$$\begin{aligned} \nu_{\boldsymbol{X}}(E) &= \int_{S_+^{n-1}} \int_{\mathbb{R}_+} \mathbf{1}_{E_j}(s\boldsymbol{w}) \frac{e^{-\beta(w)s}}{s} d\boldsymbol{s}\alpha(d\boldsymbol{w}) \\ &= \sum_{j=1}^{n} \int_{\mathbb{R}_+} a_j \mathbf{1}_{E_j}(s\boldsymbol{e}_j) \frac{e^{-b_j s}}{s} ds, \end{aligned} \tag{2.14}$$

where $a_j = \alpha(\{\boldsymbol{e}_j\})$ and $\beta(\boldsymbol{e}_j) = b_j$.

From (2.14), we have

$$\nu_{\boldsymbol{X}}(E_j) = \nu_j(E_j) = \int_{\mathbb{R}_+} a_j \mathbf{1}_E(se_j) \frac{e^{-b_j s}}{s} ds,$$

where ν_j is the Lévy measure of X_j. Thus for each $j = 1, \ldots, n$,

$$\nu_j(dx) = a_j \frac{e^{-b_j s}}{s} \mathbf{1}_{s>0} ds,$$

which is the Lévy measure of a $\Gamma(a_j, b_j)$ distribution and the assertion is proved.

□

Notice that if $b_i = b$, $i = 1, \ldots, n$, $\boldsymbol{X}$ is homogeneous.

Proof of Proposition 2.5. Let $\boldsymbol{G}(t)$ be an α-gamma subordinator. Let $\beta : S^{n-1} \to \mathbb{R}_+$ defined on $\mathcal{S}$ by: $\beta(\boldsymbol{e}_i) = \frac{1}{c_i}$, $i = 1, \ldots, n$, $\beta(\frac{\boldsymbol{c}}{|\boldsymbol{c}|}) = \frac{1}{|\boldsymbol{c}|}$. Let α have support on the canonical basis of $\mathbb{R}^n$ and $\alpha(\{\boldsymbol{e}_i\}) = \frac{1}{c_i} - a$. Then, from Lemma 2.2, $\boldsymbol{X}$ has distribution $\Gamma_n(\alpha, \beta)$. Since α_a has support $\frac{\boldsymbol{c}}{|\boldsymbol{c}|}$, from Lemma 2.1, Z has distribution $\Gamma(\alpha_a, \beta)$. The assertion follows by convolution closure properties of gamma distributions.

□

Let $\boldsymbol{Y}(t)$ be the multivariate gamma-Laplace motion in Definition 2.1. If $\boldsymbol{X}(t)$ in (2.2) is an α-gamma subordinator, $\boldsymbol{Y}(t)$ becomes in law the multivariate Laplace marked Poisson process in Jevtić et al. (2017). The process $\boldsymbol{Y}^{\boldsymbol{I}}(t)$ in (2.4) becomes in law the α-variance gamma process introduced in Semeraro (2008).

2.1.3 Multivariate variance gamma via linear transformation

The multivariate variance gamma process proposed in Ballotta and Bonfiglioli (2012) is also a multivariate gamma-Laplace motion. Let $\boldsymbol{X}$ be a multivariate gamma subordinator on $\mathbb{R}_+^{n+1}$ with independent components and let $\mathcal{L}(X_j) = \Gamma(\frac{1}{\nu_j}, \frac{1}{\nu_j})$. Let $\boldsymbol{A} \in \mathcal{M}_{n\times(n+1)}$ be such that:

$$\boldsymbol{A} := \begin{pmatrix} 1 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & \cdots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & 1 & a_n \end{pmatrix}.$$

Define

$$\boldsymbol{Y}^{\boldsymbol{I}}(t) := \boldsymbol{B}^{\boldsymbol{I}}(\boldsymbol{X}(t)) = \begin{pmatrix} B_1^I(X_1(t)) \\ .... \\ B_{n+1}^I(X_{n+1}(t)) \end{pmatrix}, \, t \geq 0.$$

The process $\boldsymbol{Y}^{\boldsymbol{I}}(t)$ has independent variance gamma margins with parameters (μ_j, σ_j, ν_j).

Let now:

$$\boldsymbol{Y}(t) := \boldsymbol{A}\boldsymbol{B}^{\boldsymbol{I}}(\boldsymbol{X}(t)) = \begin{pmatrix} Y_1^I(t) + a_1 Y_{n+1}^I(t) \\ .... \\ Y_n^I(t) + a_n Y_{n+1}^I(t) \end{pmatrix}, \, t \geq 0.$$

Up to constraints on the parameters, the process $\boldsymbol{Y}$ is in law the multivariate variance gamma process introduced in Ballotta and Bonfiglioli (2012).

3 Factor-based Laplace motions

This section proposes a multivariate model which generalizes the $\rho\alpha$-variance gamma process introduced in Luciano and Semeraro (2010).

Let $\{B_i^I(t), \, t \geq 0\}$, $i = 1, \ldots, d$, be independent Brownian motions with drift μ_i and standard deviation σ_i and $\{B_1(t), \ldots, B_n(t)\}$ be a multivariate Brownian motion with drift $\boldsymbol{\mu}^c := (\mu_1 c_1, \ldots, \mu_n c_n)$ and covariance matrix $\boldsymbol{\Sigma}^c = (\Sigma_{ij}^c)$, where $\Sigma_{ij} = \rho_{ij}\sigma_i\sigma_j\sqrt{c_i c_j}$ and ρ_{ij} is the linear correlation between $B_i(t)$ and $B_j(t)$. Let $\boldsymbol{B}^{\boldsymbol{I}}(t)$ and $\boldsymbol{B}(t)$ be independent.

Definition 3.1. *Let $\boldsymbol{X}(t)$ be a $S\Gamma_n(\alpha,\beta)$ subordinator and let $Z(t)$ be a $S\Gamma(a,1)$ subordinator and let them be independent. Let $\boldsymbol{B}^I(t)$ and $\boldsymbol{B}(t)$ be the Brownian motions defined above and let them be independent of $\boldsymbol{X}(t)$ and $Z(t)$. Define the $\mathbb{R}^n$-valued process $\boldsymbol{Y}$ by*

$$\boldsymbol{Y}(t) =: \boldsymbol{Y}_{\boldsymbol{X}}(t) + \boldsymbol{Y}_Z(t) = \begin{pmatrix} B_1^I(X_1(t)) + B_1(Z(t)) \\ .... \\ B_n^I(X_n(t)) + B_n(Z(t)) \end{pmatrix}, t \geq 0, \tag{3.1}$$

The process $\boldsymbol{Y}$ is called ***factor-based Laplace*** *$(FBL_n(\boldsymbol{\mu},\boldsymbol{\sigma},\rho,\alpha,\beta,\boldsymbol{c}))$* ***motion****.*

A factor-based Laplace motion $\boldsymbol{Y}(t)$ is the sum of two independent processes, $\boldsymbol{Y}_{\boldsymbol{X}}(t)$ and $\boldsymbol{Y}_Z(t)$. The former is the multivariate gamma-Laplace motion $M\Gamma L_n(\boldsymbol{\mu},\boldsymbol{\Sigma},\alpha,\beta)$ introduced in (2.4), the latter is a multivariate Laplace motion with a one-dimensional subordinator. Therefore, by applying Theorems 4.3 and 4.7 in Barndorff-Nielsen et al. (2001), we can fully characterize $\boldsymbol{Y}(t)$. The process $\boldsymbol{Y}(t)$ is a Lévy process and its Lévy triplet $(\boldsymbol{\gamma}_{\boldsymbol{Y}},\boldsymbol{\Sigma}_{\boldsymbol{Y}},\nu_Y)$ is as follows:

$$\begin{gathered} \boldsymbol{\Sigma}_Y = \boldsymbol{0}, \\ \nu_{\boldsymbol{Y}}(B) = \int_{\mathbb{R}_+^n} \lambda_{\boldsymbol{s}}^I(B)\nu_{\boldsymbol{X}}(d\boldsymbol{s}) + \int_{\mathbb{R}_+} \lambda_t(B)\nu_Z(dt), \\ \boldsymbol{\gamma}_{\boldsymbol{Y}} = \int_{\mathbb{R}_+^n} \nu_{\boldsymbol{X}}(d\boldsymbol{s}) \int_{|\boldsymbol{x}|\leq 1} \boldsymbol{x}\lambda_{\boldsymbol{s}}^I(d\boldsymbol{x}) + \int_{\mathbb{R}_+} \nu_Z(dt)\int_{|x|\leq 1} x\lambda_t(dx), \end{gathered}$$

where $B \subseteq \mathbb{R}^n \backslash \{0\}$, ν_Z is the Lévy measure of the one dimensional $S\Gamma(a,1)$ subordinator, and $\nu_{\boldsymbol{X}}(E)$ is the Lévy measure of the $S\Gamma_n(\alpha,\beta)$ subordinator in (1.1).

If $\boldsymbol{X}$ has independent one dimensional margins, the factor-based Laplace motion has the same dependence structure as the $\rho\alpha$-variance gamma process in Luciano and Semeraro (2010). The proof of the following proposition is omitted, since is it is an adaptation of the proof of Theorem 5.1 in Luciano and Semeraro (2010).

Proposition 3.1. *Let $\boldsymbol{Y}(t)$ be a $FBL_n(\boldsymbol{\mu},\boldsymbol{\sigma},\rho,\alpha,\beta,\boldsymbol{c})$ motion. Then*

$$Y_j(t) =_{\mathcal{L}} B_j(G_j(t)), t \geq 0,$$

where B_j are one dimensional Brownian motions with drift μ_j and standard deviation σ_j and $G_j(t)$ are the one dimensional marginal processes of the subordinator $\boldsymbol{G}(t) := \boldsymbol{X}(t) + \boldsymbol{c}Z(t)$, which we call factor-based subordinator.

The knowledge of the distribution of the factor-based subordinator $\boldsymbol{G}(t)$ in Proposition 3.1 is necessary to specify the properties of the process $\boldsymbol{Y}(t)$. Since the distribution of $\boldsymbol{G}$ is the convolution of two multivariate gamma distributions we now discuss its law and provide convolution conditions for $\boldsymbol{G}$ to have multivariate gamma distribution too. When the marginal distributions of $\boldsymbol{G}$ are known, we can derive the marginal distributions of $\boldsymbol{Y}$.

Proposition 3.2. *Let $\boldsymbol{X}$ have homogeneous multivariate distribution $\Gamma_n(\alpha,\frac{b}{|\boldsymbol{c}|})$ and let Z be independent of $\boldsymbol{X}$ and have one dimensional distribution $\Gamma(a,b)$. Then $\boldsymbol{G} := \boldsymbol{X} + \boldsymbol{c}Z$ has distribution $\Gamma_n(\alpha+\alpha_a,\frac{b}{|\boldsymbol{c}|})$.*

Proof. The assertion follows from Lemma 2.1 and convolution closure properties of gamma distributions.

□

Let $\boldsymbol{Y}(t)$ be a $FBL_n(\boldsymbol{\mu}, \boldsymbol{\sigma}, \rho, \alpha, \beta, \boldsymbol{c})$ motion. If $\boldsymbol{X}$ has a homogeneous multivariate distribution $\Gamma_n(\alpha, \frac{1}{|\boldsymbol{c}|})$, its characteristic function is of the form

$$\psi_{\boldsymbol{Y}}(z) = exp(\int_{S_+^{n-1}} \ln(\frac{\frac{1}{|\boldsymbol{c}|}}{\frac{1}{|\boldsymbol{c}|} - \sum_{j=1}^n w_j(i\mu_j z_j - \frac{1}{2}\sigma_j^2 z_j^2)})\alpha(dw)) \left(1 - \left(i\boldsymbol{z}^T\mu^c - \frac{1}{2}\boldsymbol{z}^T\Sigma^c\boldsymbol{z}\right)\right)^{-a},$$

for any $z \in \mathbb{R}_+^n$.

The above proposition holds under weaker assumptions if α has countable or finite support. In fact we do not have to require that $\boldsymbol{X}$ is homogeneous. In this case we also specify the one dimensional marginal distributions of $\boldsymbol{G}$ and $\boldsymbol{Y}$.

Proposition 3.3. *Let $\boldsymbol{X}$ have gamma distribution $\Gamma_n(\alpha, \beta)$. If α has countable support $\mathcal{S} = \{\boldsymbol{w}_k \in S_+^{n-1},\, k \in \mathbb{N}\}$, then*

$$\boldsymbol{X} =_{\mathcal{L}} \sum_{k\in\mathbb{N}} \boldsymbol{w}_k Z_k = \sum_{k\in\mathbb{N}} (w_{k1}Z_k, \ldots, w_{kd}Z_k)^T,$$

where Z_k are independent and have distribution $\Gamma(a_k, b_k)$, where $a_k = \alpha(\{\boldsymbol{w}_k\})$ and $b_k = \beta(\boldsymbol{w}_k)$.

Proof. If $\boldsymbol{X}$ has gamma distribution $\Gamma_n(\alpha, \beta)$, and α has countable support, its characteristic exponent is:

$$\begin{aligned}
\phi_{\boldsymbol{X}}(z) &= \sum_{k\in\mathbb{N}} \int_{\mathbb{R}_+} (e^{is\langle \boldsymbol{w}_k, \boldsymbol{z}\rangle} - 1)\frac{e^{-\beta(\boldsymbol{w}_k)s}}{s} ds\alpha(\boldsymbol{w}_k) \\
&= \sum_{k\in\mathbb{N}} \int_{\mathbb{R}_+} a_k(e^{is\langle \boldsymbol{w}_k, \boldsymbol{z}\rangle} - 1)\frac{e^{-b_k s}}{s} ds \\
&= \sum_{k\in\mathbb{N}} \int_{\mathbb{R}_+} (e^{iu\sum_{i=1}^n w_{ki}z_i} - 1)a_k\frac{e^{-b_k u}}{u} du.
\end{aligned}$$

By equation (2.13) in proof of Proposition 2.1, $\phi_{\boldsymbol{X}}(z) = \sum_{k\in\mathbb{N}} \phi_k(\boldsymbol{z})$, where $\phi_k(\boldsymbol{z}) = \int_{\mathbb{R}_+} (e^{iu\sum_{i=1}^n w_{ki}z_i} - 1)a_k\frac{e^{-b_k u}}{u} du$ is the characteristic exponent of $\boldsymbol{w}_k Z_k(t)$, where Z_k have distribution $\Gamma(a_k, b_k)$. The assertion follows. □

Proposition 3.4. *Let Z have one dimensional distribution $\Gamma(a, b)$ and $\boldsymbol{c} \in \mathbb{R}_+^n$. Let $\boldsymbol{X}$ have multivariate distribution $\Gamma_n(\alpha, \beta)$ with countable (finite) support $\mathcal{S}$ and let it be independent of Z. If $\beta(\frac{\boldsymbol{c}}{|\boldsymbol{c}|}) = \frac{b}{|\boldsymbol{c}|}$ or $\frac{b}{|\boldsymbol{c}|} \notin \mathcal{S}$ then $\boldsymbol{G} := \boldsymbol{X} + \boldsymbol{c}Z$ has distribution $\Gamma_n(\alpha + \alpha_a, \beta)$ and $\alpha + \alpha_a$ has support $\mathcal{S}' = \mathcal{S} \cup \frac{c}{|\boldsymbol{c}|}$.*

Proof. Since $\boldsymbol{X}$ and $\boldsymbol{c}Z$ are independent, $\phi_{\boldsymbol{G}}(z) = \phi_{\boldsymbol{X}}(z) + \phi_{\boldsymbol{c}Z}(z)$. We set $\alpha(\frac{\boldsymbol{c}}{|\boldsymbol{c}|}) = 0$ if $\frac{\boldsymbol{c}}{|\boldsymbol{c}|} \notin \mathcal{S}$. By assumption we have $\beta(\frac{\boldsymbol{c}}{|\boldsymbol{c}|}) = \frac{b}{|\boldsymbol{c}|}$. Thus we can write:

$$\phi_{\boldsymbol{X}}(\boldsymbol{z}) = \sum_{\boldsymbol{w}_k \in \mathcal{S}} \int_{\mathbb{R}_+} (e^{iu\langle \boldsymbol{w}_k, \boldsymbol{z}\rangle} - 1)\alpha(\boldsymbol{w}_k)\frac{e^{-b_k u}}{u}du + \int_{\mathbb{R}_+} (e^{iu\langle \frac{\boldsymbol{c}}{|\boldsymbol{c}|}, \boldsymbol{z}\rangle} - 1)\alpha(\frac{b}{|\boldsymbol{c}|})\frac{e^{-\frac{b}{|\boldsymbol{c}|}u}}{u}du,$$

we have

$$\begin{aligned}
\phi_{\boldsymbol{G}}(z) &= \sum_{\boldsymbol{w}_k \in \mathcal{S}} \int_{\mathbb{R}_+} (e^{iu\langle \boldsymbol{w}_k, \boldsymbol{z}\rangle} - 1)\alpha(\boldsymbol{w}_k)\frac{e^{-b_k u}}{u}du + \int_{\mathbb{R}_+} (e^{iu\langle \frac{\boldsymbol{c}}{|\boldsymbol{c}|}, \boldsymbol{z}\rangle} - 1)\alpha(\frac{b}{|\boldsymbol{c}|})\frac{e^{-\frac{b}{|\boldsymbol{c}|}u}}{u}du \\
&\quad + \int_{\mathbb{R}_+} (e^{is\langle \frac{\boldsymbol{c}}{|\boldsymbol{c}|}, \boldsymbol{z}\rangle} - 1)\frac{e^{-\frac{b}{|\boldsymbol{c}|}s}}{s}ds\alpha_a(\frac{b}{|\boldsymbol{c}|}) \\
&= \sum_{\boldsymbol{w}_k \in \mathcal{S}} \int_{\mathbb{R}_+} (e^{iu\langle \boldsymbol{w}_k, \boldsymbol{z}\rangle} - 1)\alpha(\boldsymbol{w}_k)\frac{e^{-b_k u}}{u}du + \int_{\mathbb{R}_+} (e^{iu\langle \frac{\boldsymbol{c}}{|\boldsymbol{c}|}, \boldsymbol{z}\rangle} - 1)(\alpha + \alpha_a)(\frac{\boldsymbol{c}}{|\boldsymbol{c}|})\frac{e^{-\frac{b}{|\boldsymbol{c}|}u}}{u}du \\
&= \sum_{\boldsymbol{w}_k \in \mathcal{S}'} \int_{\mathbb{R}_+} (e^{iu\langle \boldsymbol{w}_k, \boldsymbol{z}\rangle} - 1)(\alpha + \alpha_a)(\boldsymbol{w}_k)\frac{e^{-b_k u}}{u}du,
\end{aligned}$$

where the last equality holds since $\alpha_a(\boldsymbol{w}_k) = 0$ if $\boldsymbol{w}_k \neq \frac{\boldsymbol{c}}{|\boldsymbol{c}|}$. The assertion is proved. □

If $\mathcal{L}(\boldsymbol{X}) = \Gamma(\alpha, \beta)$ and α has countable support it is easy to prove the following proposition using characteristic functions:

Proposition 3.5. *Let $\boldsymbol{G} = \boldsymbol{X} + \boldsymbol{c}Z$ have distribution $\Gamma_n(\alpha + \alpha_a, \beta)$ and let α have countable support $\mathcal{S}$. Then, the one dimensional marginal distributions of G_j, $j = 1, \ldots, n$ are countable convolutions of gamma distributions.*

We now introduce a multivariate Laplace motion using the subordinator $\boldsymbol{G}(t)$ associated to $\boldsymbol{G}$ in Proposition 3.5.

Definition 3.2. Let $\boldsymbol{Y}(t)$ be the factor-based Laplace motion in (3.1). Assume that Z has one dimensional distribution $\Gamma(a, 1)$ and $\boldsymbol{c} \in \mathbb{R}^n_+$. Assume further that $\boldsymbol{X}$ has multivariate distribution $\Gamma_n(\alpha, \beta)$ with countable (finite) support $\mathcal{S}$ and is independent of Z. If $\beta(\frac{\boldsymbol{c}}{|\boldsymbol{c}|}) = \frac{1}{|\boldsymbol{c}|}$ or $\frac{1}{|\boldsymbol{c}|} \notin \mathcal{S}$, the process $\boldsymbol{Y}(t)$ is called factor-based gamma-Laplace ($FB\Gamma L_n(\boldsymbol{\mu}, \boldsymbol{\sigma}, \rho, \alpha, \boldsymbol{c})$) motion.

The characteristic function of $\boldsymbol{Y}(t)$ is given by:

$$\psi_{\boldsymbol{Y}}(z) = \prod_{w_k \in \mathcal{S}} \left(1 - (\sum_{j=1}^{n} w_{kj}(i\mu_j z_j - \frac{1}{2}\sigma_j^2 z_j^2))\right)^{-a_k} \left(1 - \left(i\boldsymbol{z}^T \mu^c - \frac{1}{2}z^T \Sigma^c z\right)\right)^{-a}, \quad (3.2)$$

for all $\boldsymbol{z} \in \mathbb{R}^n$.

Remark 1. *If the support of α is the canonical basis $\boldsymbol{e}_j, j = 1, \ldots, n$, then $\boldsymbol{X} = \sum_{k=1}^{n} \boldsymbol{e}_k Z_k$ and it has independent components. In this case, we have*

$$\psi_{\boldsymbol{X}}(\boldsymbol{z}) = \prod_{k=1}^{n} \big(1 - i(\frac{\sum_{j=1}^{n} e_{kj} z_j}{\frac{1}{|\boldsymbol{c}|}})\big)^{-a_k} = \prod_{k=1}^{n} \big(1 - i(|\boldsymbol{c}| z_k)\big)^{-a_k} \text{ for all } \boldsymbol{z} \in \mathbb{R}^n.$$

Furthermore, if we set $a_k = \frac{1}{c_k} - a$, then $\boldsymbol{G}(t)$ is in law the α-gamma subordinator and we recover the $\rho\alpha$-variance gamma process.

We have proved that the one-dimensional marginal distributions are subordinated Brownian motions (Proposition 3.1) and that the time one distribution of the subordinator is the convolution of gamma distributions (Proposition 3.5). A case of interest for applications is the factor-based gamma-Laplace motion with α with finite support, say $\mathcal{S} = \{\boldsymbol{w}_k \in S_+^{n-1},\ k = 0, \ldots h\}$. In this case, from Proposition 3.3 we have

$$\boldsymbol{X} = \boldsymbol{W}^T \boldsymbol{Z} = \sum_{k=1}^{h} \boldsymbol{w}_k Z_k, \tag{3.3}$$

where $\mathcal{L}(Z_k) = \Gamma(a_k, b_k)$ and $\boldsymbol{W} \in \mathcal{M}_{n \times h}$. Therefore, the subordinator has a factor structure, which, for instance, is consistent with interpretation of economic time as trading activity in financial applications. We conclude this section with some examples. We specify the subordinator by properly choosing $\boldsymbol{X}$.

3.1 Examples

In this section we give some examples in the bivariate case. We consider some possible specifications of the multivariate gamma subordinator with a parsimonious factor structure.

Example 1 As a first step we specify $\boldsymbol{X}$ and, as a second step, $\boldsymbol{Y}$.

Step 1. Let Z_i be such that $\mathcal{L}(Z_i) = \Gamma(a_i, 1)$. Let $\boldsymbol{D} = (d_{ij})$ be a matrix of order two and $\boldsymbol{d}_i = (d_{i1}, d_{i2})^T$. Define

$$\boldsymbol{X} := \boldsymbol{D}^T \boldsymbol{Z} = \begin{pmatrix} d_{11} Z_1(t) + d_{21} Z_2(t) \\ d_{12} Z_1(t) + d_{22} Z_2(t) \end{pmatrix}. \tag{3.4}$$

It is easy to see that $\mathcal{L}(\boldsymbol{X}) = \Gamma(\alpha, \beta)$, where α and β have finite support $\mathcal{S} = \{\frac{\boldsymbol{d}_i}{|\boldsymbol{d}_i|},\ i = 1, 2\}$ and $\alpha(\frac{\boldsymbol{d}_1}{|\boldsymbol{d}_1|}) = a_1$, $\alpha(\frac{\boldsymbol{d}_2}{|\boldsymbol{d}_2|}) = a_2$. Also β is defined on $\mathcal{S}$ and $\beta(\frac{\boldsymbol{d}_1}{|\boldsymbol{d}_1|}) = \frac{1}{|\boldsymbol{d}_1|}$ and $\beta(\frac{\boldsymbol{d}_2}{|\boldsymbol{d}_2|}) = \frac{1}{|\boldsymbol{d}_2|}$. In this case the vector $\boldsymbol{X}$ has a non-homogeneous distribution. The characteristic function of $\boldsymbol{X}$ is given by

$$\psi_{\boldsymbol{X}}(\boldsymbol{z}) = \big(1 - i(d_{11} z_1 + d_{21} z_2)\big)^{-a_1} \big(1 - i(d_{12} z_1 + d_{22} z_2)\big)^{-a_2} \text{ for all } z \in \mathbb{R}^n.$$

Notice that $\mathcal{L}(d_{ij}X_i) = \Gamma(a_i, \frac{1}{|\boldsymbol{d}_i|})$. Therefore, the marginal distributions of $\boldsymbol{X}$ are convolution of two gamma distributions.

Step 2. Let $\boldsymbol{X}(t)$ be the subordinator associated to $\boldsymbol{X}$ and let $\mathcal{L}(Z) = \Gamma(a, 1)$. Let $\boldsymbol{Y}(t)$ be the factor-based Laplace motion in (3.1), where $B_j^I(t)$ is a Brownian motion with parameters $\mu_i, 1$ and $\boldsymbol{B}(t)$ is a bivariate Brownian motion with parameters $\boldsymbol{\mu}^c, \boldsymbol{\Sigma}^c$, where $\boldsymbol{\mu}^c = (\mu_1 c_1, \mu_2 c_2)^T$ and $\boldsymbol{\Sigma}^c = (\rho_{ij}\sqrt{c_i c_j})$. The time one characteristic function of $\boldsymbol{Y}(t)$ is

$$\psi_{\boldsymbol{Y}}(\boldsymbol{z}) = \left(1 - (d_{11}(i\mu_1 z_1 - \frac{1}{2}z_1^2) + d_{21}(i\mu_2 z_j - \frac{1}{2}z_2^2))\right)^{-a_1} \cdot \\ \left(1 - (d_{12}(i\mu_1 z_1 - \frac{1}{2}z_1^2) + d_{22}(i\mu_2 z_j - \frac{1}{2}z_2^2))\right)^{-a_2} \left(1 - \left(i\boldsymbol{z}^T \mu^c - \frac{1}{2}\boldsymbol{z}^T \Sigma^c \boldsymbol{z}\right)\right)^{-a}, \tag{3.5}$$

for all $\boldsymbol{z} \in \mathbb{R}^n$. The parameters of the model are $\mu_1, \mu_2, \beta, c_1, c_2, a_1, a_2, \rho$ and a.

The correlation of the process $\boldsymbol{Y}(t)$ is

$$\rho = \frac{\mu_1\mu_2(d_{11}d_{12}a_1 + d_{21}d_{22}a_2 + c_1c_2a) + \rho\sqrt{c_1c_2}a}{\sqrt{V(Y_1)V(Y_2)}}. \tag{3.6}$$

If $\boldsymbol{D}$ is diagonal, the process has the same dependence structure of a $\rho\alpha$-variance gamma process (without convolution conditions necessary to preserve variance gamma one dimensional marginal processes), and with parameters $\mu_1, \mu_2, d_{11}, d_{22}, c_1, c_2, a_1, a_2, a, \rho$. In this case the correlation reduces to

$$\rho = \frac{\mu_1\mu_2c_1c_2a + \rho\sqrt{c_1c_2}a}{\sqrt{V(Y_1)V(Y_2)}}. \tag{3.7}$$

Comparing equations (3.6) and (3.7) we notice that in the $\rho\alpha$-case the covariance of the process is lower and does not depend on the parameters of the gamma process $\boldsymbol{X}(t)$. The following example is constructed to make this evident, i.e. we require conditions so that the two processes have the same margins and consider two cases: the case where $\boldsymbol{X}$ has maximal dependence and the case where $\boldsymbol{X}$ has independent components.

Example 2. Let $\mathcal{L}(X) = \Gamma(\tilde{a}, 1)$ and let $\boldsymbol{X}_1 = (c_1X, c_2X)^T$. Let $\boldsymbol{X}_2 = (X_1^I, X_2^I)$ have independent components with $\mathcal{L}(X_{2i}) = \Gamma(\tilde{a}, \frac{1}{c_i})$. The gamma vectors $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ have the same marginal distributions.

Let us define the subordinated processes $\boldsymbol{Y}_1(t)$ and $\boldsymbol{Y}_2(t)$ as in (3.1), using the subordinators $\boldsymbol{X}_1(t)$ and $\boldsymbol{X}_2(t)$ associated to $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$, respectively. The two processes $\boldsymbol{Y}_1(t)$ and $\boldsymbol{Y}_2(t)$ have the same marginal distributions by Proposition 3.1, but they have different correlations. In particular, the process $\boldsymbol{Y}_2(t)$ has the same dependence structure of a $\rho\alpha$-model. The correlations of $\boldsymbol{Y}_1(t)$ and $\boldsymbol{Y}_2(t)$ are, respectively,

$$\rho_1 = \frac{\mu_1\mu_2 c_1 c_2(\tilde{a}+a) + \rho\sqrt{c_1c_2}a}{\sqrt{V(Y_{11})V(Y_{12})}},$$

$$\rho_2 = \frac{\mu_1\mu_2 c_1 c_2 a + \rho\sqrt{c_1c_2}a}{\sqrt{V(Y_{21})V(Y_{22})}}.$$

Since, by construction, the two processes have the same -in law- one dimensional marginal processes, it holds that $V(Y_{1j}) = V(Y_{2j})$, $j = 1, 2$, and

$$\rho_1 = \frac{\mu_1\mu_2 c_1 c_2 \tilde{a}}{\sqrt{V(Y_{11})V(Y_{12})}} + \rho_2.$$

We conclude with one example of multivariate Laplace distribution with variance gamma margins, which generalizes the $\rho\alpha$-variance gamma process.

Example 3. Let $\mathcal{L}(Z_0) = \Gamma(\tilde{a}, 1)$, $\mathcal{L}(Z_i) = \Gamma(\frac{1}{c_i} - \tilde{a} - a, 1)$, $i = 1, 2$ and let them be independent. Let $\boldsymbol{Z} = (Z_0, Z_1, Z_2)^T$ and $\boldsymbol{X} = \boldsymbol{D}^T\boldsymbol{Z}$, where

$$\boldsymbol{D}^T = (\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3) = \begin{pmatrix} c_1 & c_1 & 0 \\ c_2 & 0 & c_2 \end{pmatrix}.$$

It is straightforward to show that $\boldsymbol{X} = \Gamma_2(\alpha, \beta)$, where $\mathcal{S} = \{\frac{\boldsymbol{d}_i}{|d_i|}, i = 1, 2, 3\}$ and $\alpha(\frac{\boldsymbol{d}_1}{|\boldsymbol{d}_1|}) = \tilde{a}$, $\alpha(\frac{\boldsymbol{d}_i}{|\boldsymbol{d}_i|}) = \frac{1}{c_i} - \tilde{a} - a$, $i = 2, 3$. Also, β is defined on $\mathcal{S}$ and $\mathcal{L}(X_i) = \Gamma(\frac{1}{c_i} - a, \frac{1}{c_i})$. Therefore, the vector $\boldsymbol{G} = \boldsymbol{X} + cZ$ has marginal distributions $\mathcal{L}(G_i) = \Gamma(\frac{1}{c_i}, \frac{1}{c_i})$, $i = 1, 2$.

By Proposition 3.1, the factor-based process $\boldsymbol{Y}(t)$ has variance gamma margins with parameters μ_i, σ_i, c_i. Therefore, we can compare the correlation of this process with the correlation of the $\rho\alpha$-variance gamma process with parameters μ_i, σ_i, c_i, a, which is obtained as a subcase for $\tilde{a} = 0$.

So let $\boldsymbol{Y}_1(t)$ be a $FBL_2(\mu_i, \sigma_i, c_i, \rho, a, \tilde{a})$ and $\boldsymbol{Y}_2(t)$ be a $\rho\alpha$-model with parameters $(\mu_i, \sigma_i, c_i, a, \rho)$. We have

$$\rho_1 = \frac{\mu_1\mu_2 c_1 c_2(\tilde{a}_1 + a) + \rho\sqrt{c_1c_2}a}{\sqrt{(\sigma_1^2 + \mu_1^2 c_1)(\sigma_2^2 + \mu_2^2 c_1)}}, \;\; 0 < a + \tilde{a} \leq \min\{\frac{1}{c_i}\}, \tag{3.8}$$

where the constraint on $\tilde{a}$ and a is necessary to have $\boldsymbol{G}$ with gamma marginal distribution. This constraint is binding for the improvement with respect to the $\rho\alpha$-models, but allows us to isolate the contribution of the multivariate gamma variable $\boldsymbol{X}$ to correlation. In fact, $\boldsymbol{Y}_1(t)$ has the same variance gamma marginal distributions of the $\rho\alpha$-variance gamma $\boldsymbol{Y}_2(t)$, whose correlation is:

$$\rho_2 = \frac{\mu_1\mu_2 c_1 c_2 + \rho\sigma_i\sigma_j\sqrt{c_1c_2}}{\sqrt{(\sigma_1^2 + \mu_1^2 c_1)(\sigma_2^2 + \mu_2^2 c_1)}}a, \;\; 0 < a \leq \min\{\frac{1}{c_i}\}. \tag{3.9}$$

Since $V(Y_{1j}) = V(Y_{2j})$, $j = 1, 2$, we have

$$\rho_1 = \frac{\mu_1\mu_2 c_1 c_2 \tilde{a}}{\sqrt{(\sigma_1^2+\mu_1^2 c_1)(\sigma_2^2+\mu_2^2 c_1)}} + \rho_2. \tag{3.10}$$

Therefore, the correlation of $\boldsymbol{Y}_1(t)$ has one one more term, which is linked to the dependence of the one dimensional marginal processes of $\boldsymbol{X}(t)$.

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