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# Some Remarks on the Geometry of Circle Maps With a Break Point

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**Abstract.** We study circle homeomorphisms $f \in C^2(\mathbb{S}^1 \setminus \{x_b\})$ whose rotation number $\rho_f$ is irrational, with a single break point $x_b$ at which $f'$ has a jump discontinuity. We prove that the behavior of the ratios of the lengths of any two adjacent intervals of the dynamical partition depends on the size of break and on the continued fraction decomposition of $\rho_f$. We also prove a result analogous to Yoccoz's lemma on the asymptotic behaviour of the lengths of the intervals of trajectories of the renormalization transformation $R_n(f)$.

## 1. Introduction and Statement of Results

One of the modern problems in circle dynamics is the study of the smoothness of the conjugacy between two circle maps, which a priori are only topologically equivalent. This problem is called the *rigidity problem*. It has been intensively studied for circle diffeomorphisms with a break point at which the first derivative has a jump discontinuity (see [9, 11, 12, 14, 15]). The analysis of the intervals of dynamical partitions of circle maps plays an important role in establishing $C^{1+\beta}$ or $C^1$ smoothness of conjugacy for two $C^{2+\nu}$- smooth circle homeomorphisms with a break point, which provides the exponential convergence of corresponding renormalizations. In case of critical circle maps, that is circle homeomorphisms which are smooth everywhere except at one point where the first derivative vanishes, the comparability of adjacent intervals of dynamical partition hold true for a set of irrational rotation numbers of full Lebesgue measure, and this property has been used to prove rigidity of the conjugacy (see [4, 15]).

To achieve completeness of rigidity results,there are examples by authors for the absence of a smooth conjugacy (rigidity) by varying the smoothness conditions or rotation numbers of circle maps ([1, 4, 5, 10, 17, 20, 22]). As we noted above, one of the main tools in proving rigidity results is convergence of corresponding renormalizations. Our second main result is an analogue of a famous Lemma of Yoccoz which describes the asymptotic behaviour of the lengths of the intervals of trajectories for the renormalization transformation. It could be useful in the future construction of conjugacy for $C^2$- smooth circle maps with a break point.

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We proceed with precise definitions and the formulation of the main results. Let $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ with induced orientation, metric, Lebesgue measure and the operation of addition be the unit circle. Every circle homeomorphism $f : \mathbb{S}^1 \to \mathbb{S}^1$ is defined by the homeomorphism $F : \mathbb{R} \to \mathbb{R}$ with property $F(x+1) = F(x)+1$, connected with $f$ by the relation $f(x) = F(x)(\mathrm{mod}1)$. The homeomorphism $F$ is called the *lift* of the homeomorphism $f$ and is defined up to an integer term.

The most important arithmetic characteristic of the circle homeomorphism $f$ is the rotation number $\rho(f) = \lim_{n\to\infty} F^n(x)/n(\mathrm{mod}1)$, where the limit exists for all $x \in \mathbb{R}$ and is independent of $x$. Here and later on $F^n$ denotes the $n$-th iterate of $F$. Let the rotation number $\rho(f)$ be irrational. Then it can be represented as an infinite continued fraction, i.e. $\rho(f) = [k_1, k_2, ..., k_n, ...] = 1/(k_1 + 1/(k_2 + ... + 1/(k_n + ...)))$. The positive integers $k_n$ are the *partial quotients* of $\rho(f)$. They give rise to a sequence of *return times* for $f$, recursively defined by $q_{n+1} = k_{n+1}q_n + q_{n-1}$, with initial conditions $q_0 = 1$, $q_1 = k_1$.

Fix a point $x_0 \in \mathbb{S}^1$ and consider the marked trajectory $x_i = f^i(x_0)$, $i \geq 0$. The subsequence $\{x_{q_n}\}_{n\geq 0}$ indexed by the denominators of the sequence of rational convergents of the rotation number $\rho$, will be called the sequence of *dynamical convergents*. It follows from the simple arithmetic properties of the rational convergents that the sequence of dynamical convergents $\{x_{q_n}\}_{n\geq 0}$ for the rigid rotation $f_\rho(x) = x + \rho(\mathrm{mod}\ 1)$ has the property that its subsequence with $n$ odd approaches $x_0$ from the left and the subsequence with $n$ even approaches $x_0$ from the right. Since all circle homeomorphisms with the same irrational rotation number are combinatorially equivalent, the order of their dynamical convergents is the same.

The intervals $[x_{q_n}, x_0]$ for $n$ odd and $[x_0, x_{q_n}]$ for $n$ even will be denoted by $\Delta_0^{(n)}$ and called the $n$-th *renormalization segment* associated to $x_0$. The $n$-th renormalization segment associated to $x_i$ will be denoted by $\Delta_i^{(n)}$, that is $\Delta_i^{(n)} = f^i\Delta_0^{(n)}$. We also have the following important property: the only points of the orbit $\{x_i : 0 < i \leq q_{n+1}\}$ that belong to $\Delta_0^{(n-1)}(x_0)$ are $\{x_{q_{n-1}+iq_n} : 0 \leq i \leq k_{n+1}\}$.

Certain images of $\Delta_0^{(n-1)}$ and $\Delta_0^{(n)}$ under the iterates of $f$ with rotation number $\rho$ cover the whole circle without overlapping besides the end points and form the $n$-th *dynamical partition* of the circle

$$\xi_n := \left\{\Delta_i^{(n-1)},\ 0 \leq i < q_n;\ \Delta_j^{(n)},\ 0 \leq j < q_{n-1}\right\}$$

The intervals $\Delta_0^{(n)}$ and $\Delta_0^{(n-1)}$ will be called the *fundamental intervals* for $\xi_n$ of ranks $n$ and $n-1$, respectively. Obviously the partition $\xi_{n+1}$ is a refinement of the partition $\xi_n$: indeed, the intervals $\Delta_j^{(n)}$, $0 \leq j < q_{n-1}$ of rank $n$ belong to $\xi_{n+1}$ and each interval $\Delta_i^{(n-1)} \in \xi_n$ $0 \leq i < q_n$, is partitioned into $k_{n+1}+1$ intervals belonging to $\xi_{n+1}$ such that

$$\Delta_i^{(n-1)} = \Delta_i^{(n+1)} \cup \bigcup_{s=0}^{k_{n+1}-1} \Delta_{i+q_{n-1}+sq_n}^{(n)}. \tag{1}$$

This paper concerns circle maps $f : \mathbb{S}^1 \to \mathbb{S}^1$ with a break, i.e. there exists a point $x_b \in \mathbb{S}^1$ such that:
(*i*) $f \in C^2(\mathbb{S}^1 \setminus \{x_b\})$;
(*ii*) $\inf_{x\neq x_b} f'(x) > 0$;
(*iii*) *$f$ has one-sided derivatives* $f'(x_b - 0) \neq f'(x_b - 0)$.

We refer to $x_b$ as the *break point* and to $\sigma := \sqrt{f'(x_b-0)/f'(x_b+0)} \neq 1$ as the *size of break*. In the following, we only consider the dynamical partition of the marked point $x_0 = x_b$.

Denote by $\mathbb{H}^2(\sigma, \rho)$ the class of circle homeomorphisms $f$, with irrational rotation number $\rho = \rho(f)$ and satisfying the conditions (*i*) – (*iii*).

From now on we shall denote by $C$ any constant that depend only on $f$ and, with some abuse of notation, we denote by the same $C$ any sum of a finite number of such constants and when multiplied by a universal constant.

Denote by $|I|$ the length of the interval $I$. The intervals $I$ and $J$ are said to be *comparable*, if the ratio of their lengths is uniformly bounded, that is, there exists a constant $C > 1$, such that the relation $C^{-1}|J| \leq |I| \leq C|J|$ is fulfilled, which will be denoted by $|I| \asymp |J|$.

One of the remarkable features of critical circle homeomorphisms is that any two adjacent intervals of the $n$ -th dynamical partition of a critical point are $C$-comparable for some $C > 1$ (see, for instance, [4, 6, 19]). In this case, one says that $f$ has *nearby bounded geometry* (see [7]).

We will show that for homeomorphisms of the circle with a break point the behavior of the ratio of the lengths of adjacent intervals belonging to the dynamical partitions is quite different from the case of a diffeomorphism or a critical homeomorphism.

Let $I$ and $J$ be any two adjacent intervals of the dynamical partition $\xi_n$. If their ranks are the same, then one of them will be the image of the other one under $f^{q_{n-1}}$ or $f^{q_n}$ and Denjoy's inequality implies that they are $e^v$- comparable. If their ranks are different and $I,\ J \subset \mathbb{S}^1 \setminus V_n(x_0)$, then they will be some images of fundamental intervals of rank $n$ and $n-1$. In the latter case, using Lemma 2.4 it is not difficult to show that

$$e^{-2v}\frac{|\Delta_0^{(n)}|}{|\Delta_0^{(n-1)}|} \le \frac{|I|}{|J|} \le e^{2v}\frac{|\Delta_0^{(n)}|}{|\Delta_0^{(n-1)}|}. \tag{2}$$

Next we formulate our first main result.

**Theorem 1.1.** *Let $f \in \mathbb{H}^2(\sigma, \rho)$. Assume that $\sigma > 1$. Then there exists constants $C > 1$, $\tau_1, \tau_2 > 0$, depending only on $f$, such that for $n$ large enough the following inequalities are fulfilled*

$$C^{-1} \le \frac{|\Delta_0^{(2n-1)}|}{|\Delta_0^{(2n-2)}|} \le C, \tag{3}$$

$$C^{-1} \cdot e^{-\tau_2 k_{2n+1}} \le \frac{|\Delta_0^{(2n)}|}{|\Delta_0^{(2n-1)}|} \le C \cdot e^{-\tau_1 k_{2n+1}}. \tag{4}$$

*Suppose $0 < \sigma < 1$. Then for $n$ large enough we have*

$$C^{-1} \le \frac{|\Delta_0^{(2n)}|}{|\Delta_0^{(2n-1)}|} \le C, \qquad C^{-1} \cdot e^{-\tau_2 k_{2n}} \le \frac{|\Delta_0^{(2n-1)}|}{|\Delta_0^{(2n-2)}|} \le C \cdot e^{-\tau_1 k_{2n}}.$$

Note that a result analogous to Theorem 1.1 has been obtained in [14] for $C^{2+\nu}$- smooth circle maps with a break point.

Let $l$ be a positive integer and $\Delta_1,\ \Delta_2,\ ....,\ \Delta_{l+1}$ be consecutive intervals of the real line or of the circle. By an *almost parabolic map* of length $l$ and fundamental domains $\Delta_j,\ 1 \le j \le l$, one means a negative-Schwarzian diffeomorphism

$$f : \Delta_1 \cup \Delta_2 \cup .... \cup \Delta_l \longrightarrow \Delta_2 \cup \Delta_3 \cup .... \cup \Delta_{l+1},$$

such that $f(\Delta_j) = \Delta_{j+1}$.

The basic geometric estimate for almost parabolic maps obtained by J.-C. Yoccoz is the following:

**Yoccoz's Lemma.** ([4, 21]) *Let $f : I \to J$ be an almost parabolic map of length $l$ and fundamental domains $\Delta_j,\ 1 \le j \le l$. If $|\Delta_1| \ge \tau|I|$ and $|\Delta_l| \ge \tau|I|$, for some $0 < \tau < 1$, then*

$$\frac{1}{C_\tau}\frac{|I|}{\min\{j,\ l-j\}^2} \le |\Delta_j| \le C_\tau \frac{|I|}{\min\{j,\ l-j\}^2}$$

*where the constant $C_\tau > 1$ does not depend on $f$.*

Notice that Yoccoz's lemma plays an important role in the rigidity theory for critical circle maps (see for instance [4]). We will prove an analogue of Yoccoz's result for the renormalizations of a circle homeomorphism with a break point. First, we recall a few basic concepts of the renormalization method for circle maps with irrational rotation number.

Consider the $n-$th dynamical partition $\xi_n$ and the $n$-th *renormalization neighborhood* $V_n = \Delta_0^{(n-1)} \cup \Delta_0^{(n)}$. We define the Poincaré map $\pi_n = (f^{q_n}, f^{q_{n-1}}) : V_n \to V_n$ as follows

$$\pi_n(x) = \begin{cases} f^{q_n}(x), & \text{if} \quad x \in \Delta_0^{(n-1)} \setminus \{x_0\}, \\ f^{q_{n-1}}(x), & \text{if} \quad x \in \Delta_0^{(n)}. \end{cases}$$

The main idea of the renormalization method is to study the behavior of the Poincaré map $\pi_n$ as $n \to \infty$. To this end, usually one uses suitable re-scaled coordinates. So, we define the *renormalized coordinate* $z$ on $V_n$ by $x = x_0 + z(x_0 - x_{q_{n-1}})$ and we denote by $a_n$ and $(-b_n)$ the new coordinates of the points $x_{q_n}$ and $x_{q_n+q_{n-1}}$, respectively, i.e.

$$a_n = \frac{x_{q_n} - x_0}{x_0 - x_{q_{n-1}}}, \quad b_n = \frac{x_0 - x_{q_n+q_{n-1}}}{x_0 - x_{q_{n-1}}}. \tag{5}$$

It is clear, that the coordinate $z$ varies from -1 to 0, when $x$ is varying from $x_{q_{n-1}}$ to $x_0$ and it varies from 0 to $a_n$ when $x$ is varying from $x_0$ to $x_{q_n}$. With respect to the renormalized coordinate, the Poincaré map $\pi_n$ is represented by $(f_n, g_n)$, where $f_n$ and $g_n$ are defined as follows

$$f_n(z) = \frac{f^{q_n}(x_0 + z(x_0 - x_{q_{n-1}})) - x_0}{x_0 - x_{q_{n-1}}}, \quad z \in [-1,\ 0),$$

$$g_n(z) = \frac{f^{q_{n-1}}(x_0 + z(x_0 - x_{q_{n-1}})) - x_0}{x_0 - x_{q_{n-1}}}, \quad z \in [0,\ a_n].$$

The pair of functions $(f_n, g_n)$ is called the $n$-th *renormalization pair* of the initial homeomorphism $f$ at the point $x_0$ and denoted by $R_n(f)$. The renormalization $R_n(f)$ defined for all $n \geq 0$ if and only if $\rho$ is irrational; otherwise, $n$ is less than or equal to the length of the continued fraction expansion of $\rho$.

Recall that the interval $[x_{q_{n-1}}, x_0]$ is partitioned into intervals belonging to the next partition $\xi_{n+1}$, with end points $x = x_{q_{n-1}+sq_n},\ 0 \leq s \leq k_{n+1}$. Denote by $z_s^{(n)}$ the renormalized coordinates of $x_{q_{n-1}+sq_n}$, i.e.

$$z_s^{(n)} = \frac{x_{q_{n-1}+sq_n} - x_0}{x_0 - x_{q_{n-1}}}.$$

From the construction of the dynamical partition it follows that

$$z_0^{(n)} = -1 < z_1^{(n)} < z_2^{(n)} < ... < z_{k_{n+1}}^{(n)} < 0, \ \text{ where } \ z_{s+1}^{(n)} = f_n(z_s^{(n)}),\ 0 \leq s \leq k_{n+1} - 1.$$

Put $I_s^{(n)} := [z_{s-1}^{(n)}, z_s^{(n)}],\ s = 1, 2, ..., k_{n+1}$, then it is clear that $I_{s+1}^{(n)} = f_n(I_s^{(n)}),\ s = 1, 2, ..., (k_{n+1}-1)$. As mentioned above, the interval $[x_{q_{n-1}}, x_0]$ is partitioned by the intervals of $\xi_{n+1}$ with end points $x_{q_{n-1}+sq_n},\ 0 \leq s \leq k_{n+1}$. The corresponding renormalized coordinates $z_s^{(n)}$ of the points $x_{q_{n-1}+sq_n}$ give rise to a partition of the interval $[-1, z_{k_{n+1}}^{(n)}]$ into the intervals $I_s^{(n)} = [z_{s-1}^{(n)}, z_s^{(n)}],\ s = 1, ..., k_{n+1}$.

Our second main result concerns the behavior of the lengths of the intervals $I_s^{(n)}$, which are joining consecutive points of the $f_n$- orbits of the point $z_0^{(n)} = -1$.

**Theorem 1.2.** *Let $f \in \mathbb{H}^2(\sigma, \rho)$. Assume $\sigma > 1$. Then there exist $n_0 = n_0(f) \in \mathbb{N}$, such that for any $n \geq n_0$ and for all $1 \leq s \leq k_{2n} - 1$, the following inequalities hold true:*

$$\frac{C^{-1}}{\min\{s,\ k_{2n} - s\}^2} \leq |I_s^{(2n-1)}| \leq \frac{C}{\min\{s,\ k_{2n} - s\}^2}.$$

*Suppose $0 < \sigma < 1$. Then for any $n \geq n_0$ and for all $1 \leq s \leq k_{2n+1} - 1$, the following inequalities hold true:*

$$\frac{C^{-1}}{\min\{s,\ k_{2n+1} - s\}^2} \leq |I_s^{(2n)}| \leq \frac{C}{\min\{s,\ k_{2n+1} - s\}^2}$$

*where the constant $C > 1$ does not depend on $f$.*

It is worth to stress that the convexity of the map $R_n(f)$ plays a crucial role in the proof of this result.

Note that a result analogous to Theorem 1.2 has been used in [5, 20], to restrict the examples of pairs of infinitely smooth circle maps with a break point, which are $C^1$- smoothly conjugate but not $C^{1+\beta}$- smoothly conjugate for any $\beta > 0$.

The paper is organized as follows. In Section 2 we illustrate the basic notions and we collect from the existing literature a few classical facts about homeomorphisms of the circle with a break point. In Section 3 we present results about the approximation of the renormalization maps of a circle homeomorphism to certain linear-fractional maps. Then, we will prove the asymptotically linear dependence of the renormalization parameters $a_n$, $b_n$, $c_n$, $m_n$ as $n \to \infty$, a fact that plays an important role in the proof of the first main Theorem. The last Section is devoted to the proof the two main results.

## 2. Preliminaries and Notations

Let $f$ be an orientation preserving homeomorphism of the circle with lift $F$ and irrational rotation number $\rho(f)$. Consider any point $x_0 \in \mathbb{S}^1$ and its $n$-th dynamical partition $\xi_n$ with *fundamental intervals* $\Delta_0^{(n-1)}$ and $\Delta_0^{(n)}$. The following Lemma plays a key role for studying the metrical properties of the homeomorphism $f$ :

**Lemma 2.1.** *Suppose that the one-sided derivatives $f'(x_b-0)$, $f'(x_b+0) > 0$ do exist, that $f$ belongs to $C^1([x_b, x_b+1))$ and, in addition, assume that $var_{[x_b,x_b+1)} \log f' = \overline{v} < \infty$. Then, putting*

$$v = \overline{v} + |\log f'(x_b - 0) - \log f'(x_b + 0)| = \overline{v} + 2|\log \sigma|,$$

*the following inequality is fulfilled*

$$e^{-v} \leq \prod_{i=0}^{q_n-1} f'(y_i) \leq e^{v} \tag{6}$$

*for any $n$ and for any $y_0$ such that $y_i \neq x_b$, $i = 0, 1, 2, ...$*

The inequality (6) is called the *Denjoy's inequality*. We set $\lambda = (1 + e^{-v})^{-1/2} < 1$.

**Lemma 2.2.** *Let $f$ be a circle maps satisfying the conditions of Lemma 2.1. Then*

*(a) for any $x_0 \in S^1$:*

$$|\Delta_0^{(n)}(x_0)| \leq \lambda^{2k} |\Delta_0^{(n-2k)}(x_0)|, \ 0 \leq 2k \leq n;$$

*(b) if $\Delta^{(n)} \in \xi_n$, $\Delta^{(m)} \in \xi_m$, $n - m \geq 2$ and $\Delta^{(n)} \subset \Delta^{(m)}$, then*

$$\frac{|\Delta^{(n)}|}{|\Delta^{(m)}|} \leq C \cdot \lambda^{n-m}.$$

The proofs of Lemma 2.1 and Lemma 2.2 are more or less the same as those of similar assertions in [16]. Recall the following definition introduced in [8]:

**Definition 2.3.** An interval $I = (\tau, t)$ is called $q_n$-small and its endpoints $\tau, t$ are $q_n$-close if the system of intervals $f^i(I)$, $0 \leq i < q_n$, are disjoint.

One checks easily that the interval $I = (\tau, t)$ is $q_n$-small if, depending on the parity of n, either $t \leq \tau \leq f^{q_{n-1}}(t)$ or $f^{q_{n-1}}(\tau) \leq t \leq \tau$.

**Lemma 2.4.** *Suppose that a circle homeomorphism $f$ satisfies the conditions of Lemma 2.1 and $x, y \in \mathbb{S}^1$ are $q_n$-close. Then for any $0 \leq k \leq q_n$, we have*

$$e^{-v} \leq \frac{Df^k(x)}{Df^k(y)} \leq e^{v}. \tag{7}$$

*Proof.* Take any two $q_n$-close points $x, y \in \mathbb{S}^1$ and $0 \le k \le q_n - 1$. Denote by $I$ the open interval with endpoints $x$ and $y$. Since the intervals $f^i(I)$, $0 \le i < q_n$ are disjoint, we obtain

$$|\log Df^k(x) - \log Df^k(y)| \le \sum_{s=0}^{q_n-1} |\log Df(f^s(x)) - \log Df(f^s(y))| \le v.$$

From this, we deduce immediately the inequality (7). □

Denote by $\mathbb{H}^{KO}(\sigma, \rho)$ the class of circle homeomorphisms $f$ with irrational rotation number $\rho = \rho(f)$ satisfying the assumptions (*ii*) and (*iii*) stated in the introduction and the further condition

(*iv*) $f'(x)$ *is absolutely continuous and* $f''/f' \in \mathbb{L}_p(S^1, d\ell)$ *for some* $p > 1$.

This latter condition is known in the literature as the *Katznelson and Ornstein smoothness conditions* [8]. Suppose that $f \in \mathbb{H}^{KO}(\sigma, \rho)$, then using the dynamical partition $\xi_n$, we define a sequence $\{\Phi_n\}$ of step functions on the circle by

$$\Phi_n(x) = \frac{1}{|\Delta^{(n)}|} \int_{\Delta^{(n)}} \frac{f''(y)}{f'(y)} dy, \text{ if } x \in \Delta^{(n)}, \ \Delta^{(n)} \in \xi_n(x_0), \ n \ge 1.$$

We set $\Phi_0(x) \equiv \log \sigma^2$, $x \in \mathbb{S}^1$. Denote by $\{\Xi_n\}$ the sequence of algebras generated by the dynamical partitions $\xi_n$. A simple calculation shows that the sequence of $\{\Phi_n\}$ is a martingale with respect to $\{\Xi_n\}$. Moreover, using Hölder's inequality we obtain

$$\|\Phi_n\|_p^p = \int_{S^1} |\Phi_n(x)|^p dx = \sum_{\Delta^{(n)} \in \xi_n} \int_{\Delta^{(n)}} |\Phi_n(x)|^p dx$$

$$= \sum_{\Delta^{(n)} \in \xi_n} \frac{1}{|\Delta^{(n)}|^{p-1}} \Big| \int_{\Delta^{(n)}} \frac{f''(x)}{f'(x)} dx \Big|^p \le \sum_{\Delta^{(n)} \in \xi_n} \int_{\Delta^{(n)}} \Big| \frac{f''(x)}{f'(x)} \Big|^p dx = \Big\| \frac{f''}{f'} \Big\|_p^p.$$

Hence, $\{\Phi_n\}$ is an $\mathbb{L}_p$-bounded martingale.

Following Katznelson and Ornstein, one can define the difference of martingales [8], that is $h_n(x) = \Phi_n(x) - \Phi_{n-1}(x)$, $n \ge 1$. From this one can deduce the following

**Lemma 2.5.** *Let* $f \in \mathbb{H}^{KO}(\sigma, \rho)$. *Then we have*

$$\frac{f''}{f'} - 2\log\sigma = \sum_{n=1}^{\infty} h_n, \quad (in \ \mathbb{L}_1 - norm).$$

The Lemma 2.5 can be proved easily, using the properties of the dynamical partition.

## 3. Asymptotically Linearly Dependence of Renormalization Parameters

The renormalization behavior of circle diffeomorphisms has been studied by Sinai and Khanin in [13] and Stark in [18] and plays a key role in their proof of smoothness of the corresponding conjugacy classes. Now we state Sinai and Khanin's result.

**Theorem 3.1.** ([13]) *Let* $f \in C^{2+\nu}(\mathbb{S}^1)$, $\nu > 0$, *be a diffeomorphism of the circle with irrational rotation number* $\rho = \rho(f)$. *Then for all* $n \ge 1$

$$\|f_n - \widetilde{F}_n\|_{C^1([-1,0])} \le C \cdot \lambda^{n\nu}, \quad \|g_n - \widetilde{G}_n\|_{C^1([0,a_n])} \le C \cdot a_n^{-1} \lambda^{n\nu},$$

*where* $\widetilde{F}_n(z) = z + a_n$, $\widetilde{G}_n(z) = z - 1$ *and the constant* $\lambda \in (0, 1)$ *depends only on* $f$.

We define two linear-fractional functions $F_n$ and $G_n$ as follows:

$$F_n(z) := \frac{a_n + (a_n + b_n m_n)z}{1 + (1 - m_n)z}, \quad G_n(z) := \frac{-a_n c_n + (c_n - b_n m_n)z}{a_n c_n + (m_n - c_n)z}, \tag{8}$$

where $a_n$ and $b_n$ are defined as in (5) and

$$c_n := \sigma^{(-1)^{n-1}}, \quad m_n := \exp\left\{(-1)^{n-1}\sum_{i=0}^{q_n-1}\int_{\Delta_i^{(n-1)}}\frac{f''(y)}{2f'(y)}dy\right\}. \tag{9}$$

The renormalization behavior of the diffeomorphisms of the circle with one break point $x_b$ was first studied by Khanin and Vul in [16]. One of the most significant results proved in [16] is the following

**Theorem 3.2.** ([16]) *Let $f \in C^{2+\nu}(\mathbb{S}^1\setminus\{x_b\})$, $\nu > 0$ be a circle homeomorphism with a break point $x_0 = x_b$ and irrational rotation number $\rho = \rho(f)$. Then, for all $n \geq 1$, we have*

$$\|f_n - F_n\|_{C^2([-1,0])} \leq C \cdot \lambda^{n\nu}, \quad \|g_n - G_n\|_{C^2([0,a_n])} \leq \frac{C \cdot \lambda^{n\nu}}{a_n},$$

*where $\lambda \in (0,1)$ is a constant depending only on $f$.*

Notice that that the proofs of Theorems 3.1 and 3.2 are based on the analysis of the dynamical partitions $\xi_n$. Recently, this result has been extended to the wider class $\mathbb{H}^{KO}(\sigma,\rho)$. It can be formulated as follows

**Theorem 3.3.** ([2]) *Let $f \in \mathbb{H}^{KO}(\sigma,\rho)$. Then for all $n \geq 1$, we have*

$$\|f_n - F_n\|_{C^1([-1,0])} \leq C \cdot \eta_n, \quad \|f_n'' - F_n''\|_{L_1([-1,0],d\ell)} \leq C \cdot \eta_n,$$

$$\|g_n - G_n\|_{C^1([0,a_n])} \leq C \cdot \eta_n, \quad \|g_n'' - G_n''\|_{L_1([0,a_n],d\ell)} \leq C \cdot \eta_n,$$

*where the sequence of positive numbers $\{\eta_n\}$ belongs to $l_2$ and depends only on $f$.*

Note that in [3] a similar result for circle homeomorphisms $f \in \mathbb{H}^{KO}(\sigma,\rho)$ with rational rotation numbers has been proved. It is interesting to note that the proof of Theorem 3.3 makes use of martingales (Lemma 2.5) which in the context of circle dynamics have been used also by Katznelson and Ornstein in [8].

Now we formulate a lemma which will be used in the sequel. Assume $n$ is even. For every $0 \leq i \leq q_n$ we define relative coordinates $z_i$ for points $x$ in the intervals $\Delta_i^{(n-1)}$ by setting $x = x_i - z_i(x_i - x_{q_{n-1}+i})$. The point $x \in \Delta_i^{(n-1)}$ is mapped by $f$ to the point $f(x) \in \Delta_{i+1}^{(n-1)}$ with relative coordinate $z_{i+1}$. Next we put for $x \in \Delta_i^{(n-1)}$

$$\alpha_i := \alpha_i(n) = x_{i+q_{n-1}}, \ \gamma_i := \gamma_i(n) = x_i, \ \beta_i := \beta_i(n) = f^i(x) \in \Delta_i^{(n-1)}, \tag{10}$$

$$A_i := A_i(n) = -\frac{\frac{1}{f'(\alpha_i)(\beta_i-\alpha_i)}\int_{\alpha_i}^{\beta_i} f''(y)(y-\alpha_i)dy + \frac{1}{f'(\alpha_i)(\gamma_i-\beta_i)}\int_{\beta_i}^{\gamma_i} f''(y)(\gamma_i - y)dy}{1 + \frac{1}{f'(\alpha_i)(\gamma_i-\alpha_i)}\int_{\alpha_i}^{\gamma_i} f''(y)(\gamma_i - y)dy}, \tag{11}$$

$$B_i := B_i(n) = \int_{\alpha_i}^{\gamma_i}\frac{f''(y)}{2f'(y)}dy, \ \ \psi_i(z_0) := -B_i - \log\left(\frac{1 + A_i z_i}{1 + A_i(z_i - 1)}\right), \tag{12}$$

$$m(j) := m_n(j) = \exp\{\sum_{i=0}^{j-1} B_i\}, \ \tau^{(j)}(z_0) := \sum_{i=0}^{j-1}\psi_i(z_0). \tag{13}$$

The next Lemma shows that $z_j$ is an almost linear-fractional function of $z_0$.

**Lemma 3.4.** ([2]) *Suppose that the circle homeomorphism $f$ satisfies the conditions of Theorem 3.3. Then, for every $1 \leq j \leq q_n$, we have*

$$z_j = \frac{z_0 m(j) e^{\tau^{(j)}(z_0)}}{1 + z_0(m(j)e^{\tau^{(j)}(z_0)} - 1)}.$$

*In addition, $\tau^{(q_n)}$ and its derivatives satisfy the following bounds*

$$\max_{0\leq z_0\leq 1} |\tau^{(q_n)}(z_0)| \leq C \cdot \eta_n, \quad \max_{0\leq z_0\leq 1} |(z_0 - z_0^2)\frac{d\tau^{(q_n)}(z_0)}{dz_0}| \leq C \cdot \eta_n,$$

$$\int_0^1 |\frac{d\tau^{(q_n)}(z_0)}{dz_0}| dz_0 \leq C \cdot \eta_n, \quad \int_0^1 |(z_0 - z_0^2)|\frac{d^2\tau^{(q_n)}(z_0)}{dz_0^2}| dz_0 \leq C \cdot \eta_n,$$

*where the sequence of positive numbers $\{\eta_n\}$ belongs to $l_2$ and depends only on $f$.*

Using Theorem 2.5, Theorem 3.3 and Lemma 3.4 we will prove the following

**Theorem 3.5.** *Let $a_n, b_n, m_n, c_n$ be the parameters in the linear-fractional functions $F_n$ and $G_n$ defined as in (8). Suppose that $f \in \mathbb{H}^{KO}(\sigma, \rho)$. Then for all $n \geq 1$ we have*

$$|a_n + b_n m_n - c_n| \leq C \cdot a_n \cdot \eta_n,$$

*where the sequence of positive numbers $\{\eta_n\}$ belongs to $l_2$ and depends only on $f$.*

*Proof.* We show that the parameters $a_n, b_n, m_n, c_n$ in Theorem 3.5 get asymptotically linearly dependent as $n \to \infty$. Let us assume $n$ to be odd. We put

$$\Theta_i := \int_{x_{i+q_{n-2}}}^{x_i} \frac{f''(y)}{2f'(y)} dy, \quad \overline{\Theta}_i := \int_{x_{i+q_n}}^{x_i} \frac{f''(y)}{2f'(y)} dy, \quad 0 \leq i \leq q_{n-1}.$$

It is easy to see that

$$\exp\{\sum_{i=0}^{q_{n-1}-1} \overline{\Theta}_i\} \cdot m_n = \exp \int_{S^1} \frac{f''(y)}{2f'(y)} dy = \sigma, \quad c_n = \sigma$$

and consequently, $m_n = c_n \cdot \exp\{-\sum_{i=0}^{q_{n-1}-1} \overline{\Theta}_i\}$.

We need the following Lemma (see [2], Lemma 5.1).

**Lemma 3.6.** ([2]) *The numbers $m_n$ satisfy*

$$m_n = c_n(1 + a_n a_{n-1}(m_{n-1} - 1)) \exp(\chi_n),$$

*where $|\chi_n| \leq C \cdot a_n a_{n-1} \eta_n$ and the sequence $\{\eta_n\}$ belongs to $l_2$.*

We are now in a position to continue the proof of Theorem 3.5. Put $r_n := a_n + b_n m_n - c_n$. Using Lemma 3.6 we prove that

$$r_n = -c_n a_n r_{n-1} + \delta_n, \quad \text{where } \{\delta_n\} \in l_2. \tag{14}$$

Actually, the relation

$$b_n = \frac{f_{n-1}(-a_n a_{n-1})}{a_{n-1}} = \frac{F_{n-1}(-a_n a_{n-1})}{a_{n-1}} + \frac{\nu_n}{a_{n-1}},$$

with $\nu_n = f_{n-1}(-a_n a_{n-1}) - F_{n-1}(-a_n a_{n-1})$, implies that

$$r_n = a_n + b_n m_n - c_n = a_n + c_n[1 - (a_{n-1} + b_{n-1}m_{n-1}a_n)\exp(\chi_n)] +$$

$$+\frac{\nu_n}{a_{n-1}} - c_n = c_n\left(\frac{1}{c_n} - a_{n-1} - b_{n-1}m_{n-1}\right) +$$

$$+[c_n a_n(a_{n-1} + b_{n-1}m_{n-1}) - c_n](1 - \exp(\chi_n)) + \frac{\nu_n m_n}{a_{n-1}} = -c_n a_n r_{n-1} + \delta_n,$$

where

$$\delta_n = [c_n a_n(a_{n-1} + b_{n-1}m_{n-1}) - c_n](1 - \exp(\chi_n)) + \frac{\nu_n m_n}{a_{n-1}}.$$

Using Theorem 3.3, we obtain

$$|\frac{\nu_n m_n}{a_{n-1}}| = m_n|\frac{f_{n-1}(-a_n a_{n-1}) - F_{n-1}(-a_n a_{n-1})}{a_{n-1}}| \le C \cdot a_n \cdot \eta_n,\ \{\eta_n\} \in l_2.$$

This implies the relation (14). Iterating the identity (14), we get

$$r_n = -c_n a_n r_{n-1} + \delta_n = -c_n a_n(-c_{n-1}a_{n-1}r_{n-2} + \delta_{n-1}) + \delta_n =$$

$$= -c_n a_n(-c_{n-1}a_{n-1})r_{n-2} - c_n a_n \delta_{n-1} + \delta_n = r_1 \prod_{j=2}^{n}(-c_j a_j) + \sum_{i=2}^{n}\delta_i \prod_{j=i+1}^{n}(-c_j a_j).$$

On the other hand, Lemma 2.2 implies that

$$|\prod_{j=i}^{n}(-c_j a_j)| \le\ C \cdot a_n \lambda^{n-i},$$

where $\lambda \in (0,1)$. Therefore $|r_n| \le C \cdot a_n \sum\limits_{i=2}^{n} \lambda^{n-i}|\delta_i|$. Put $S_n = \sum\limits_{i=1}^{n} \lambda^{n-i}|\delta_i|$ and $k = [\frac{n}{2}]$. We estimate $S_n$ as follows:

$$S_n = \sum_{i=1}^{n} \lambda^{n-i}|\delta_i| = \sum_{i=1}^{n-k-1} \lambda^{n-i}|\delta_i| + \sum_{i=n-k}^{n} \lambda^{n-i}|\delta_i| \le$$

$$\le \max_{1\le i\le n-k-1} |\delta_i| \cdot (\lambda^{n-1} + ... + \lambda^{k+1}) + |\delta_{n-k}|(\lambda^k + ... + 1) \le C_1\left(\lambda^{k+1} + |\delta_{n-k}|\right) \le$$

$$\le C\left(\lambda^{[\frac{n}{2}]+1} + |\delta_{[\frac{n}{2}]}|\right) \le \eta_n,$$

where we assumed w.l.o.g.$\delta_{[\frac{n}{2}]} \neq 0$. This completes the proof of Theorem 3.5. □

We need the following estimate (see [9] )

$$\max_{\Delta_1, \Delta_2 \in \xi_{n-k}} \left( \frac{\sum_{j:\ \Delta_j^{(n-1)} \subset \Delta_1} |\Delta_j^{(n-1)}|}{|\Delta_1|} - \frac{\sum_{j:\ \Delta_j^{(n-1)} \subset \Delta_2} |\Delta_j^{(n-1)}|}{|\Delta_2|} \right) \le Const \cdot \lambda_1^{\sqrt{k}}, \tag{15}$$

where $0 < \lambda_1 < 1$. The estimate (15) means that one can define an approximate density of the intervals $\Delta_j^{(n-1)},\ 0 \le j < q_n$. This density is equal to $p_n = \sum\limits_{j=0}^{q_n-1} |\Delta_j^{(n-1)}|$. It follows that

$$m_n = \exp\left((-1)^{n-1}\widetilde{p}_n \int_{\mathbb{S}^1} \frac{f''(y)}{2f'(y)}\right) = e^{(-1)^{n-1}\widetilde{p}_n \log\sigma}, \tag{16}$$

where $|p_n - \widetilde{p}_n| \leq Const \cdot \lambda_2^{\sqrt{n}}$, $0 < \lambda_2 < 1$.

It is easy to see that, there exists a constant $p$ such that the inequality $0 < p < p_n$ holds for $n$ large enough. Then relation (16) implies that for $n$ large enough:

$$m_{2n-1} > 1, \text{ and } m_{2n} < 1, \text{ when } \sigma > 1, \tag{17}$$

$$m_{2n-1} < 1, \text{ and } m_{2n} > 1, \text{ when } 0 < \sigma < 1.$$

## 4. Proofs of Main Results

In this section we assume that $f \in \mathbb{H}^2(\sigma, \rho)$. In this case, the estimate for $f_n - F_n$ and the relation for $a_n$, $c_n$ and $m_n$ (see [2]) can be improved as follows

$$\|f_n - F_n\|_{C^2([-1,0])} \leq C \cdot \eta_n, \tag{18}$$

$$a_n + \phi_n a_n = c_n - b_n m_n, \tag{19}$$

where $|\phi_n| \leq C\eta_n$, and $\{\eta_n\} \in l_2$. It is easy to see that the second order derivative of $F_n$ satisfies

$$F_n''(z) = \frac{2m_n(m_n - 1)(a_n + b_n)}{(1 + (1 - m_n)z)^3}, \ z \in [-1, 0]. \tag{20}$$

Relations (18), (17) and (20) imply the following

**Corollary 4.1.** *Let $f \in \mathbb{H}^2(\sigma, \rho)$ and $\sigma > 1$. Then for n large enough, the following assertions are fulfilled*
- *$f_{2n-1}''(x) > 0$, for $\forall x \in [-1, 0]$ and consequently $f_{2n-1}'$ is increasing on $[-1, 0]$;*
- *$f_{2n}''(x) < 0$, for $\forall x \in [-1, 0]$ and consequently $f_{2n}'$ is decreasing on $[-1, 0]$.*

In the proofs of our main results we use essentially Corollary 4.1.

**Proof of Theorem 1.1**. We will prove Theorem 1.1 in the case $\sigma > 1$. The case $0 < \sigma < 1$ can be treated analogously. Recall that $a_n = \frac{|\Delta_0^{(n)}|}{|\Delta_0^{(n-1)}|}$. It is easy to see that

$$a_n = \frac{|\Delta_0^{(n)}|}{|\Delta_0^{(n+1)}| \cup \bigcup_{s=0}^{k_{n+1}-1} |\Delta_{q_{n-1}+sq_n}^{(n)}|} = \left(a_{n+1} + \sum_{s=0}^{k_{n+1}-1} \frac{|\Delta_{q_{n-1}+sq_n}^{(n)}|}{|\Delta_0^{(n)}|}\right)^{-1} =$$

$$= (a_{n+1} + Df^{q_{n-1}}(\widetilde{y}_0) + Df^{q_{n-1}}(\widetilde{y}_0) \cdot Df^{q_n}(\widetilde{y}_1) + ...+$$

$$+Df^{q_{n-1}}(\widetilde{y}_0) \cdot Df^{q_n}(\widetilde{y}_1) \cdot ... \cdot Df^{q_n}(\widetilde{y}_{k_{n+1}-2}))^{-1} \tag{21}$$

where $\widetilde{y}_s \in \Delta_{q_{n-1}+sq_n}^{(n)}$, $0 \leq s \leq k_{n+1} - 2$. Using Denjoy's inequality in (21), we get the following inequality for all $n$:

$$a_n \geq \frac{1}{1 + e^v + e^{2v} + ... + e^{(k_{n+1}-1)v}} = \frac{e^v - 1}{e^{k_{n+1}v} - 1} \geq (e^v - 1)e^{-k_{n+1}v}.$$

This gives the bound from below in (4). It is easy to see that the relation (21) implies also the upper bound in (3).

Now we will show the remaining bounds in (3) and (4). Consider the odd level dynamical partition $\xi_{2n-1}$ and the renormalization map $f_{2n-1}$. Due to (17) and (18), for sufficiently large values of $n$, we obtain

$$f_{2n-1}'(0) = F_{2n-1}'(0) + f_{2n-1}'(0) - F_{2n-1}'(0) \geq$$
$$\geq \sigma + (m_{2n-1} - 1 - \phi_{2n-1})a_{2n-1} - C \cdot \eta_{2n-1} \geq \sigma > 1. \tag{22}$$

Suppose that $f'_{2n-1}(-1) < 1$. Then there exist $\widetilde{\varsigma}_0 \in [-1, 0]$, such that $f'_{2n-1}(\widetilde{\varsigma}_0) = 1$. Due to Taylor's formula we get

$$f_{2n-1}(z) = f_{2n-1}(\widetilde{\varsigma}_0) + (z - \widetilde{\varsigma}_0) + \frac{f''_{2n-1}(\widetilde{\varsigma}_1)}{2}(z - \widetilde{\varsigma}_0)^2.$$

Next, using (18) we obtain

$$a_{2n-1} + \widetilde{\psi}_1 = f_{2n-1}(0) = f_{2n-1}(\widetilde{\varsigma}_0) - \widetilde{\varsigma}_0 + \frac{f''_{2n-1}(\widetilde{\varsigma}_1)}{2}\widetilde{\varsigma}_0^2,$$

$$1 - b_{2n-1} + \widetilde{\psi}_2 = f_{2n-1}(-1) + 1 = f_{2n-1}(\widetilde{\varsigma}_0) - \widetilde{\varsigma}_0 + \frac{f''_{2n-1}(\widetilde{\varsigma}_2)}{2}(\widetilde{\varsigma}_0 + 1)^2,$$

where $|\widetilde{\psi}_i| \le C \cdot \eta_{2n-1}$ and $\widetilde{\varsigma}_i \in (-1, 0)$, $i = 1, 2$. The last two equalities together with the relation

$$1 - b_{2n-1} = 1 - \frac{x_0 - x_{q_{2n-1}+q_{2n}}}{x_0 - x_{q_{2n-2}}} = \frac{f^{q_{2n-2}}(x_{q_{2n-1}}) - f^{q_{2n-2}}(x_0)}{x_{q_{2n-1}} - x_0} a_{2n-1} = Df^{q_{2n-2}}(\omega_0) a_{2n-1},$$

where $\omega_0 \in \Delta_0^{(2n-1)}$ with $e^{-v} \le Df^{q_{2n-2}}(\omega_0) \le e^v$ imply that

$$a_{2n-1}\,(1 + Df^{q_{2n-2}}(\omega_0)) = 2(f_{2n-1}(\widetilde{\varsigma}_0) - \widetilde{\varsigma}_0) + \frac{f''_{2n-1}(\widetilde{\varsigma}_1)}{2}\widetilde{\varsigma}_0^2 + \frac{f''_{2n-1}(\widetilde{\varsigma}_2)}{2}(\widetilde{\varsigma}_0 + 1)^2 - (\widetilde{\psi}_1 + \widetilde{\psi}_2). \tag{23}$$

Since the rotation number $\rho(f)$ is irrational, the map $f_{2n-1}(z)$ has no fixed point and we have $f_{2n-1}(z) > z$. Hence, the relation (23) and Corollary 4.1 imply that

$$a_{2n-1} \ge \frac{C}{(1 + Df^{q_{2n-2}}(\omega_0))} \ge \frac{C}{(1 + e^v)} = C_1.$$

To complete the proof of the bound from below in the inequality (3) it hence suffices to show that for $n$ large enough the inequality $f'_{2n-1}(-1) < 1$ is fulfilled. Actually, assume the contrary, i.e. $f'_{2n-1}(-1) \ge 1$. Then, since $f'_{2n-1}(z)$ is increasing, we get $f'_{2n-1}(z) \ge 1$, $z \in [-1, 0]$. Consider the renormalized coordinates $z_s^{(2n-1)}$ of the points $x_{q_{2n-2}+sq_{2n-1}}$, $0 \le s \le k_{2n}$, and the intervals $I_s^{(2n-1)} := [z_s^{(2n-1)}, z_{s+1}^{(2n-1)}]$, where $I_{s+1}^{(2n-1)} = f_{2n-1}(I_s^{(2n-1)})$. Then

$$1 \ge \sum_{s=0}^{k_{2n}-1} |I_s^{(2n-1)}| = |I_0^{(2n-1)}|(1 + f'_{2n-1}(\widetilde{t}_1) + ... + f'_{2n-1}(\widetilde{t}_1)...f'_{2n-1}(\widetilde{t}_{k_{2n}-1})) \ge$$

$$\ge k_{2n}(z_1^{(2n-1)} - z_0^{(2n-1)}) = k_{2n}(f_{2n-1}(-1) + 1) = k_{2n}(Df^{q_{2n-2}}(\omega_0) a_{2n-1} + \widetilde{\psi}_2),$$

where $|\widetilde{\psi}_2| \le C \cdot \eta_{2n}$ and $\widetilde{t}_i \in I_{i-1}^{(2n-1)}$, $i = 1, 2, ..., k_{2n} - 1$.

Using the last relations we get

$$a_{2n-1} \le \frac{1}{k_{2n} Df^{q_{2n-2}}(\omega_0)} - \frac{\widetilde{\psi}_2}{Df^{q_{2n-2}}(\omega_0)} \le \frac{e^v}{k_{2n}} + |\widetilde{\psi}_2 e^v|.$$

Hence $\lim_{n\to\infty} a_{2n-1} = 0$, if $k_{2n} \to \infty$. Then we have $\lim_{n\to\infty} b_{2n-1} = 1$ and $\lim_{n\to\infty} m_{2n-1} = \sigma$. For sufficiently large $n$ we obtain

$$f'_{2n-1}(-1) = F'_{2n-1}(-1) + \widetilde{\psi}_2 = \frac{a_{2n-1} + b_{2n-1}}{m_{2n-1}} + \widetilde{\psi}_2 < \frac{\sigma + 1}{2\sigma} < 1$$

But, the last inequality contradicts the assumptions. If on the other hand $k_{2n} \le C_1$, then using Denjoy's inequality and relation (21), we obtain $a_{2n-1} \ge C$. The inequality (3) is completely proved.

Consider next the even level dynamical partition $\xi_{2n}$ and the renormalization map $f_{2n}$. First we prove that

$$a_{2n} \le \frac{C}{k_{2n+1}}, \quad n \ge 1, \tag{24}$$

where the constant $C > 0$ doesn't depend on $n$ and $k_{2n+1}$. By Corollary 4.1 $f_{2n}^{'}(z)$ is decreasing on $[-1, 0]$ Here are possible the following cases:

Then the following three cases can arise:

– either there exists a point $\widetilde{\varsigma_0} \in (-1, 0)$, such that $f_{2n}'(\widetilde{\varsigma_0}) = 1$;

– or $f_{2n}'(-1) = 1$ respectively $f_{2n}'(0) = 1$;

– or $f_{2n}'(z) > 1,\ \forall z \in [-1, 0]$ respectively $f_{2n}'(-1) < 1,\ \forall z \in [-1, 0]$.

We prove the estimate (24) for the first case only, the other case can be treated similarly. Put $\tilde{x_0} = x_0 + \widetilde{\varsigma_0}(x_0 - x_{q_{2n-1}})$. It is obvious that $\tilde{x_0} \in \Delta^{(2n)}_{q_{2n-1}+lq_{2n}}$, for some $0 \le l \le k_{2n+1}$, and $Df^{q_{2n}}(\tilde{x_0}) = 1$.

We have $Df^{q_{2n}}(x_{q_{2n-1}}) > Df^{q_{2n}}(\tilde{x_0}) = 1 > Df^{q_{2n}}(\tilde{x_0})$. Notice that $Df^{q_{2n}}(x)$ is decreasing on $[x_{q_{2n-1}}, x_0]$. Consequently, $Df^{q_{2n}}(x) > 1$ on $[x_{q_{2n-1}}, \tilde{x_0})$ and $Df^{q_{2n}}(x) < 1$ on $(\tilde{x_0}, x_0]$. Using these relations we obtain:

$$|\Delta^{(2n)}_{q_{2n-1}}| < |\Delta^{(2n)}_{q_{2n-1}+q_{2n}}| < ... < |\Delta^{(2n)}_{q_{2n-1}+lq_{2n}}|, \tag{25}$$

$$|\Delta^{(2n)}_{q_{2n-1}+(l+1)q_{2n}}| > \Delta^{(2n)}_{q_{2n-1}+(l+2)q_{2n}}| > ... > |\Delta^{(2n)}_{q_{2n-1}+(k_{2n+1}-1)q_{2n}}|. \tag{26}$$

Using the Denjoe inequality it is easy to show that

$$C_1^{-1}|\Delta_0^{(2n)}| \le |\Delta^{(2n)}_{q_{2n-1}}|,\ |\Delta^{(2n)}_{q_{2n-1}+(k_{2n+1}-1)q_{2n}}| \le C_1|\Delta_0^{(2n)}|, \tag{27}$$

where the constant $C_1 > 1$ depend on $f$ only. Using the last relations we obtain:

$$a_{2n} = \frac{|\Delta_0^{(2n)}|}{|\Delta_0^{(2n+1)}| \cup \bigcup_{s=0}^{k_{2n+1}-1} |\Delta^{(2n)}_{q_{2n-1}+sq_{2n}}|} = \left(a_{2n+1} + \sum_{s=0}^{k_{2n+1}-1} \frac{|\Delta^{(2n)}_{q_{2n-1}+sq_{2n}}|}{|\Delta_0^{(2n)}|}\right)^{-1} \le$$

$$\left(a_{2n+1} + \sum_{s=0}^{k_{2n+1}-1} C_1^{-1}\right)^{-1} \le \frac{C_2}{k_{2n+1}},$$

where the constant $C_2 > 0$ doesn't depend on $n$ and $k_{2n+1}$. The last relation, together with (18), (19) and (17) imply that

$$|b_{2n} - 1| \le \frac{C_3}{k_{2n+1}},\quad |m_{2n} - \sigma^{-1}| \le \frac{C_3}{k_{2n+1}}, \tag{28}$$

where the constant $C_2 > 0$ doesn't depend on $n$ and $k_{2n+1}$. It is easy to check that $F^{'}(-1) = \frac{a_{2n}+b_{2n}}{m_{2n}}$ and $F^{'}(0) = (a_{2n} + b_{2n})m_{2n}$. For sufficiently large $n$ and $k_{2n+1}$, using the obtained estimates for $a_{2n}$, $b_{2n}$, $m_{2n}$ and (18), can be showed

$$f_{2n}'(0) < \frac{\sigma + 1}{2\sigma} < 1,\quad f_{2n}'(-1) > \frac{\sigma + 1}{2} > 1. \tag{29}$$

It follows that there exist a point $t_0 := b\widetilde{\tau_0} \in [-1, 0]$ such that $f_{2n}'(\widetilde{\tau_0}) = 1$. Thus, one of the intervals $[-1, t_0]$ and $[t_0, 0]$ contains more than $\left[\frac{k_{2n+1}}{2}\right]$ intervals of the system $\{I_s^{(2n)}\}_{s=1}^{k_{2n+1}}$. Let $[-1, t_0]$ be that interval. Next we take a point $t_1 \in [-1, t_0)$ such that $f_{2n}'(t_1) = \lambda_1 > 1$. It is clear that

$$f_{2n}'(t_1) - f_{2n}'(-1) = f_{2n}''(t_2)(1 + t_1),\ \ t_2 \in (-1, t_1).$$

Hence, according to Corollary 4.1 this implies that the interval $[-1, t_1]$ has the length of the order of a constant.

Now we consider the interval $[-1,t_0)=[-1,t_1]\cup(t_1,t_0)$. Let $I_1^{(2n)},...,I_{l_1}^{(2n)}$ be the intervals lying on $[-1,t_1)$ and let $I_{l_1+1}^{(2n)}$ be the interval that contains the point $t_1$. Let $I_{l_1+2}^{(2n)},...,I_{l_0}^{(2n)}$ be the intervals lying on $(t_1,t_0)$, where $\frac{k_{2n+1}}{2}\le l_0\le k_{2n+1}$. Since $f_{2n}'(z)\ge\lambda_1,\ z\in[-1,t_1]$, we get

$$1\le\frac{\sum\limits_{s=1}^{l_1}|I_s^{(2n)}|}{|I_{l_1}^{(2n)}|}=1+\frac{1}{f_{2n}'(\widetilde{t}_{l_1-1})}+...+\frac{1}{f_{2n}'(\widetilde{t}_{l_1-1})\cdots f_{2n}'(\widetilde{t}_1)}\le\sum_{s=0}^{l_1-1}\lambda_1^{-s}\le C.$$

This means that the intervals $I_{l_1}^{(2n)}$ and $\bigcup\limits_{s=1}^{l_1}I_s^{(2n)}$ are comparable. Since the length of $[-1,t_1]$ is of the order of a constant, it follows from the comparability of the intervals $I_{l_1}^{(2n)}$ and $\bigcup\limits_{s=1}^{l_1}I_s^{(2n)}$, that the interval $I_{l_1}^{(2n)}$ has the length of the order of a constant. Since $f_{2n}'(z)$ is strictly monotone, one can easily show that the number of intervals lying in $(t_1,t_0)$ is bounded, i.e. $l_0-l_1\le C$. It is clear that

$$|I_{l_0}^{(2n)}|=\frac{|f^{q_{2n-1}+(l_0-1)q_{2n}}\Delta_0^{(2n)}|}{|\Delta_0^{(2n-1)}|}=\frac{|\Delta_0^{(2n)}|}{|\Delta_0^{(2n-1)}|}\cdot\frac{|f^{q_{2n-1}}\Delta_0^{(2n)}|}{|\Delta_0^{(2n)}|}\cdot\frac{|f^{q_{2n-1}+q_{2n}}\Delta_0^{(2n)}|}{|f^{q_{2n-1}}\Delta_0^{(2n)}|}\cdot...\cdot$$

$$\cdot\frac{|f^{q_{2n-1}+(l_0-1)q_{2n}}\Delta_0^{(2n)}|}{|f^{q_{2n-1}+(l_0-2)q_{2n}}\Delta_0^{(2n)}|}=a_{2n}\cdot Df^{q_{2n-1}}(\zeta_0)\cdot f_{2n}'(\zeta_1)\cdot...\cdot f_{2n}'(\zeta_{l_0-1}),$$

where $\zeta_0\in\Delta_0^{(2n)}$, and $\zeta_s\in I_s^{(2n)},\ s=1,2,...,l_0-1$. Then, using Denjoy's inequality and $f_{2n}'(z)\ge\lambda_1,\ z\in[-1,t_1]$, we obtain

$$a_{2n}\le|I_{l_0}^{(2n)}|\cdot e^v\cdot\lambda_1^{-(l_0-1)}. \tag{30}$$

On the other hand,

$$|I_{l_0}^{(2n)}|=|I_{l_1}^{(2n)}|\cdot\frac{|f^{q_{2n-1}+l_1q_{2n}}\Delta_0^{(2n)}|}{|f^{q_{2n-1}+(l_1-1)q_{2n}}\Delta_0^{(2n)}|}\cdot...\cdot\frac{|f^{q_{2n-1}+(l_0-1)q_{2n}}\Delta_0^{(2n)}|}{|f^{q_{2n-1}+(l_0-2)q_{2n}}\Delta_0^{(2n)}|}=$$

$$=|I_{l_1}^{(2n)}|\cdot Df^{q_{2n-1}}(\zeta_{l_1})\cdot...\cdot Df^{q_{2n}}(\zeta_{l_0-1}),$$

where $\zeta_i\in f^{q_{2n-1}+(i-1)q_{2n}}\Delta_0^{(2n)},\ i=l_1,...,l_0-1$. Since the interval $I_{l_1}^{(2n)}$ has the length of the order of a constant and $l_0-l_1\le C$, Denjoy's inequality implies that $|I_{l_0}^{(2n)}|\le C$. Hence, using (30) and the inequality $\frac{k_{2n+1}}{2}\le l_0\le k_{2n+1}$, we obtain the upper bound in (4). This completes the proof of Theorem 1.1.

**Proof of Theorem 1.2**. In the proof we focus on the map $f_{2n-1}$ when $\sigma>1$. The same arguments can be used to check that the same properties are also true for the map $f_{2n}$ when $0<\sigma<1$. By Corollary 4.1, the function $f_{2n-1}$ is convex, i.e. $f_{2n-1}''(z)\ge C>0,\ z\in[-1,0]$.

We now use the convexity of $f_{2n-1}$ to *squeeze* its graph between the graphs of two Möbus functions. Then, the required estimate for $f_{2n-1}$ will follow from the corresponding estimate for Möbus transformations, which we now state and prove.

Consider the function $v_{2n-1}(t)=-f_{2n-1}(-t),\ 0\le t\le1$. In what follows, for simplicity we omit the index of $v_{2n-1}$. It is clear that

$$v''(t)=-f_{2n-1}''(-t)\le C<0 \tag{31}$$

Since the function $f_{2n-1}$ has no fixed points and $f_{2n-1}(-1)=-b_{2n-1},\ 0<b_{2n-1}<1,\ f_{2n-1}(0)=a_{2n-1},\ 0<a_{2n-1}<1$, its graph lies strictly above the line $y=z$. This implies that $v(t)<t,\ 0\le t\le1$. It is clear, there exists a point $t_0$ such that $v'(t_0)=1$. Since the function $v'(t)$ is decreasing, then $v'(t)>1$ on $[0,t_0)$ and $v'(t)<1$

on $(t_0, 1]$. The lengths of the intervals $[0, t_0]$ and $[t_0, 1]$ are of the order of a constant. Using Taylor's formula and inequality (31) it is not difficult to show that $t_0 - v(t_0) < t - v(t)$, for any $t \in [0, 1]$.

We put $\varepsilon = t_0 - v(t_0)$ and $G(t) = t - v(t)$, $t \in [t_0, 1]$. To be specific, we take the interval $[t_0, 1]$. Since our further arguments do not depend on the choice of affine coordinates, we may replace without loss of generality the interval $[t_0, 1]$ with $[0, 1]$. We then have

$$G(0) < G(t) \le t, \text{ and } 0 < G(0) = \varepsilon,\ G'(0) = 1, \quad \forall t \in [0, 1].$$

We put

$$\chi = \frac{1}{2} \min_{t\in[0,1]}(-G''(t)), \quad \lambda_0 = \max_{t\in[0,1]} \frac{t - (G(t) - G(0))}{t(G(t) - G(0))},$$

$$A_\lambda(t) = \frac{t}{1 + \lambda t} + G(0), \quad B_\mu(t) = \frac{t}{1 + \mu t} + G(0), \quad \lambda > 0,\ \mu > 0.$$

We need the following

**Lemma 4.2.** *(1) For each $0 < \mu < \chi$ the following inequality holds true*

$$G(t) \le B_\mu(t), \quad t \in [0, 1]. \tag{32}$$

*(2) For each $\lambda \ge \lambda_0$, we have*

$$A_\lambda(t) \le G(t), \quad t \in [0, 1]. \tag{33}$$

*Proof.* At first we prove (32). From the inequality

$$\mu = \max_{0\le t\le 1} \frac{\mu}{1 + \mu t} \le \frac{1}{2} \min_{0\le t\le 1}(-G''(t)) = \chi$$

it follows that for any $\xi \in [0, 1]$

$$\frac{G''(\xi)}{2} \le -\frac{\mu}{1 + \mu t}, \text{ for any } t \in [0, 1].$$

Thus, using $G(t) = G(0) + t + \frac{G''(\xi)}{2} t^2$, $\xi \in (0, t)$, we deduce the inequality (32).

Now we turn to the inequality (33). Take $\lambda \ge \lambda_0$, then

$$\lambda \ge \frac{t - (G(t) - G(0))}{t(G(t) - G(0))}, \text{ for any } t \in [0, 1].$$

Hence, from the explicit form of $A_\lambda$, we deduce at once the inequality (33). This concludes the proof of Lemma 4.2. □

To finish the proof of Theorem 1.2, we present the proof of the assertions of Theorem 1.2 for Möbus functions. These results are also presented in [4] which we follow very closely. In [4] the authors use the negative Schwarzian property of $f$ to squeeze it between two Möbus functions. In our case, we essentially use the convexity of the renormalizations of the map $f$.

Consider the fractional linear transformation $T(y) = \frac{y}{1+y}$. For given $\varepsilon > 0$, let $T_\varepsilon(y) = T(y) - \varepsilon$ and $y_n = T_\varepsilon^n(y_0)$, $y_0 = 1$.

**Lemma 4.3.** ([4]) *Let $N \ge 1$ be such that $y_{N+1} \le 0 < y_N$ for some $N \ge 1$. Then we have $N \asymp 1/\sqrt{\varepsilon}$ and moreover $y_n - y_{n+1} \asymp 1/n^2$ for $n = 0, 1, ..., N$.*

Let $A_\lambda$ and $B_\mu$ be the Möbus functions defined in Lemma 4.2. For simplicity of notations everywhere below we omit the indices of $A_\lambda$ and $B_\mu$.

**Lemma 4.4.** ([4]) *Let $t \in [0,1]$ and $k > 0$ be such that $A(t) < B^k(t)$. Then $k \leq 1 + \lambda/\mu$.*

By construction the points $z_s^{(2n-1)}$, $0 \leq s \leq k_{2n}$ determine the partition of the interval $[-1, z_{k_{2n}}^{(2n-1)}]$ associated to the orbit of the point $z_0^{(2n-1)} = -1$ under the map $f_{2n-1}$. There is an analogous partition of the interval $[0,1]$ determined by the orbit of the map $G = G(t) = t - v_{2n-1}(t)$ with $y_i = G^i(y_0)$ and $y_0 = 1$. Denote by $\Delta_i = [y_i, y_{i+1}]$ the elements of this partition. In this setting we want to prove that $|\Delta_i| \asymp 1/i^2$ for all $i$.

Now, let us write $\alpha_n = A^n(y_0)$ and $\beta_n = B^n(y_0)$. By lemma 4.4, the number of $\beta_i$'s inside each interval of the form $[\alpha_{n+1}, \alpha_n]$ is bounded independently of $n$. Moreover, since $\alpha_n < y_n < \beta_n$ for all $n$, the number of $y_i$'s inside each interval of the form $[\alpha_{n+1}, \alpha_n]$ is bounded independently $n$. To prove that $|\Delta_i| \asymp 1/i^2$, we proceed as follows. Let $\ell > 0$ be such that $\beta_{\ell+1} \leq y_i \leq \beta_\ell \leq y_{i-1}$. Then Lemma 4.4 says that $\ell \leq Ci$, and using Lemma 4.2 we have

$$|\beta_{i+1} - \beta_i| < |B(y_{i-1}) - y_{i-1}| < |y_i - y_{i-1}|.$$

Since by Lemma 4.3 we have $|\beta_{\ell+1} - \beta_\ell| \asymp \frac{1}{\ell^2} \geq \frac{1}{Ci^2}$, it follows that $|\Delta_i| = |y_i - y_{i-1}| \geq 1/Ci^2$. To prove an inequality in the opposite direction, let $m$ be the largest integer such that $\alpha_m > x_{i-1}$. Then, again by Lemma 4.4, we have $i \leq Cm$. Since $A(t) < G(t) < t$ for all $t$, we also have $\Delta_i \subset [\alpha_{m+2}, \alpha_m]$. Using Lemma 4.3 once more, we deduce that $|\Delta_i| \leq \frac{C}{m^2} \leq \frac{1}{i^2}$. It is clear, that the lengths of intervals $|\Delta_i|$ and $I_i^{(2n-1)}$ are $C_1-$comparable (the constant $C_1 > 1$ doesn't depend on $i$). This completes the proof of Theorem 1.2.

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