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Abstract. This work investigates the free vibration of plate structures subjected to geometric nonlinear equilibrium states. The proposed method extends 2D (plate) higher-order finite elements based on the Carrera Unified Formulation (CUF) to deal with large displacements. Hence, full Green-Lagrange strains are employed in a total Lagrangian scenario along with a path following method for investigating modes and frequency change with no loss of generality. In fact, according to CUF, the accuracy of the analysis – and thus the computational costs – can be set as input, depending on the problem complexity. The results demonstrate that refined models are needed for most of the analyses in which stress states do not meet the hypotheses of classical plate theories.

Introduction

Modal behaviour of structures is evidently a property of the equilibrium state. As a matter of fact, the natural frequencies can be affected by pre-stress states and large displacements/rotations states in the case of thin plates, for example. Thin-film solar sails for deep-space propulsion, high-altitude unmanned surveillance aircraft, lightweight solar-powered high-endurance aircraft, rotorcraft, bridges, pipelines, and towers all possess structural components subject to a variety of loading conditions which can affect their non-trivial vibration characteristics [1].

The methodology introduced in this work is based on the Carrera Unified Formulation (CUF) [2, 3], according to which any theory of structures can degenerate into a generalized kinematics that makes use of an arbitrary expansion of the generalized viariables. In this manner, the nonlinear governing equations and the related finite element arrays of the generic geometrically-exact beam theory are written in terms of *fundamental nuclei* (FNs). These fundamental nuclei represent the basic building blocks that, when opportunely expanded, allow for the straightforward generation of low- and high-order finite elements. The formulation has been recently employed for the static [4] and dynamic [5] geometric nonlinear analysis of beams. This work, in contrast, further extends CUF for the free vibration analysis of *plate structures* subjected to in-plane loadings, along moderate- and large-displacement equilibrium states.

Unified plate element

Consider a plate laying on the xy-plane of a Cartesian coordinate system. According to CUF, the three-dimensional displacement field $\mathbf{u}$ can be expressed as

$$\mathbf{u}(x, y, z) = F_\tau(z)\mathbf{u}_\tau(x, y), \qquad \tau = 1, 2, ..., M \tag{1}$$

where F_τ are the thickness functions of the coordinate z, $\mathbf{u}_\tau$ is the vector of the *generalized* displacements on the mid-plane, M stands for the number of the terms used in the expansion, and the repeated subscript τ indicates summation. The choice of F_τ determines the class of the 2D CUF model that is required and subsequently to be adopted. Accordingly, Taylor, Lagrange and Legendre-type polynomials have been used in the domain of CUF to formulate high-order theories, see [3].

Note that, if the Finite Element Method (FEM) is adopted, the generalized displacements can be further approximated by discretizing the problem domain on the mid-plane by using classical shape functions N_i to give

$$\mathbf{u}_\tau(x, y) = N_i(x, y)\mathbf{q}_{\tau i} \qquad i = 1, 2, \dots, p + 1 \tag{2}$$

where p is the order of the shape functions and i stands for summation. $\mathbf{q}_{\tau i}$ is the vector of the FE nodal parameters.

Free vibration of nonlinear plates

Equations of motion of an elastic body undergoing undamped free vibration can be obtained by imposing the equality of the virtual variation of the internal energy and the virtual variation of the work of the inertia loads.

$$\delta L_{\text{int}} - \delta L_{\text{ine}} = 0 \tag{3}$$

By using CUF, FEM, the Green-Lagrange strain-displacement relations and assuming a linear elastic material, the virtual variation of the strain energy can be written as [4]

$$\delta L_{\text{int}} =< \delta \boldsymbol{\epsilon}^T \boldsymbol{\sigma} >= \delta \mathbf{q}_{\tau i}^T \, \mathbf{K}_S^{ij\tau s} \, \mathbf{q}_{sj} \tag{4}$$

where $< (\cdot) >= \int_V (\cdot)\ \mathrm{d}V$. $\mathbf{K}_S^{ij\tau s}$ is the fundamental nucleus (FN) of the second-order nonlinear, secant stiffness matrix. It is a 3×3 matrix that, given the theory approximation order – i.e., given the cross-sectional functions ($F_\tau = F_s$, for $\tau = s$) and the shape functions ($N_i = N_j$, for $i = j$)–, can be expanded by using the indexes $\tau, s = 1, ..., M$ and $i, j = 1, ..., p+1$ in order to obtain the element stiffness matrices of any arbitrarily refined plate model. In other words, by opportunely choosing the plate kinematics, classical to higher-order plate theories and related stiffness array can be implemented in an automatic manner by exploiting the index notation of CUF. In a similar manner, the FN of the linear mass matrix can be obtained from the virtual variation of the inertial loadings as follows:

$$\delta L_{\text{ine}} =< \delta \mathbf{q}^T \boldsymbol{\rho}\, \ddot{\mathbf{q}} >= \delta \mathbf{q}_{sj}^T \mathbf{M}^{ij\tau s}\, \ddot{\mathbf{q}}_{\tau i} \tag{5}$$

where $\mathbf{M}^{ij\tau s}$ is the FN of the mass matrix and $\boldsymbol{\rho}$ is the material density.
It is fairly obvious that the modal behaviour of any system is a property of the equilibrium. Inherently, free vibration analysis needs to be made about a linearized equilibrium state along the equilibrium path. For this purpose, Eq. (3) needs to be properly linearized in order to obtain the modal behaviour of the structure about given states of the equilibrium path. Assuming as linear the virtual variation of the inertial work, we need to linearize the virtual variation of the nonlinear, internal strain energy to obtain the *tangent stiffness matrix*.

$$\delta(\delta L_{\text{int}}) =< \delta(\delta \boldsymbol{\epsilon}^T \boldsymbol{\sigma}) >=< \delta \boldsymbol{\epsilon}^T \delta \boldsymbol{\sigma} > + < \delta(\delta \boldsymbol{\epsilon}^T) \boldsymbol{\sigma} >= \delta \mathbf{q}_{\tau i}^T \mathbf{K}_T^{ij\tau s} \delta \mathbf{q}_{sj} \tag{6}$$

The FN of the tangent stiffness matrix $\mathbf{K}_T^{ij\tau s}$ is made of different contributions, including $\mathbf{K}_0$ (linear stiffness) and $\mathbf{K}_\sigma$ (geometric stiffness). $\mathbf{K}_\sigma$, in particular, comes from the linearization of the nonlinear form of the strain-displacement equations and it takes into account the effect of internal stresses on the nonlinear equilibrium state. By using Eqs. (6) and (5) into the linearization of Eq. (3) and assuming harmonic displacements, the equations of motion for free vibrations hold the form of a classical eigenvalue problem, which in unified form reads:

$$(\mathbf{K}_T^{ij\tau s} - \omega^2 \mathbf{M}^{ij\tau s})\, \bar{\mathbf{q}}_{\tau i} = 0 \tag{7}$$

where ω is a natural period and $\bar{\mathbf{q}}_{\tau i}$ the related amplitude eigenvector. In Eq. (7), the tangent stiffness matrix is evaluated at a given point of the nonlinear equilibrium, see Fig. 1. If linear stiffness matrix is used instead, Eq. (7) gives the natural frequencies of the system around the trivial equilibrium state.

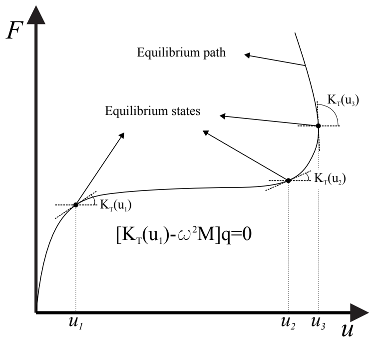

Figure 1: Evaluation of natural frequencies about non-trivial equilibrium states of the nonlinear equilibrium path.

Conclusions

In this paper, higher-order plate elements based on the Carrera Unified Formulation (CUF) have been extended to deal with large displacement analysis. Numerical results will show that the free vibration (natural frequencies and modes) of plates can be severely affected as the equilibrium state is far from the trivial solution, in a geometrically nonlinear sense. Buckling can be found inherently as a particular state in which the tangent stiffness matrix is singular. Also, the results will demonstrate that classical models are not sufficiently accurate in the case of complex stress states. The reason is that the geometrical stiffness plays a fundamental role in this kind of analyses.

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