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A Calderon Regularized Symmetric Formulation for the Electroencephalography Forward Problem

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Abstract

The symmetric formulation of the electroencephalography (EEG) forward problem is a well-known and widespread equation thanks to the high level of accuracy that it delivers. However, this equation is first kind in nature and gives rise to ill-conditioned problems when the discretization density or the brain conductivity contrast increases, resulting in numerical instabilities and increasingly slow solutions. This work addresses and solves this problem by proposing a new regularized symmetric formulation. The new scheme is obtained by leveraging on Calderon identities which allow to introduce a dual symmetric equation that, combined with the standard one, results in a second kind operator which is both stable and well-conditioned under all the above mentioned conditions. The new formulation presented here can be easily integrated into existing EEG imaging packages since it can be obtained with the same computational technology required by the standard symmetric formulation. The performance of the new scheme is substantiated by both theoretical developments and numerical results which corroborate the theory and show the practical impact of the new technique.

Keywords: EEG, BEM, Calderon Solvers.

1. Introduction

Functional brain imaging based on high-resolution scalp Electroencephalographies (EEGs) is characterized by a high temporal resolution and, as such, it provides an unmatched overview on the underlying brain activity [1, 2, 3, 4]. This technique relies on the key task, referred to as the EEG inverse problem, of recovering the brain electric current sources responsible for a measured potential at the EEG scalp electrodes [5, 6]. The EEG inverse problem requires multiple solutions of the EEG forward problem, i.e. the computation of the scalp potential starting from the source currents [7, 8]. It has been widely studied and reported that the accuracy of EEG forward problem solvers has a direct impact on EEG inverse solution procedures [9, 10, 11, 12]. For this reason, any advancement of the state of the art in EEG forward solution technologies will have a direct impact on the overall high-resolution EEG imaging process.

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10 When realistic head models are used [11, 13, 14, 15, 16, 17], the solution of the EEG forward problem can only
11 be obtained numerically. Classical strategies to obtain this numerical solution are the Finite Element Method (FEM),
12 the Finite Difference Method (FDM) or the Boundary Element Method (BEM) [18]. Both FDM and FEM leverage on
13 a volume discretization of the considered head model. This allows these methods to account for the inhomogeneity
14 and anisotropy of the head’s conductivity at the cost, however, of a higher computational demand. Previous works
15 have shown that by using transfer matrices the computation time of the FEM formulations can be reduced [19, 20].
16 The use of these transfer matrices in FEM formulations yield similar computational times as BEM formulations for
17 comparable accuracies. [21].

18 When the conductivity of the head is modelled as piecewise homogeneous, BEM can be easily used to compute
19 the solution of the EEG forward problem. In other words, the main limitation of the BEM formulation resides in its
20 inability to model anisotropies. However, this method has the advantage that it requires only the discretization of the
21 interface between regions with different conductivities [18, 22]. Several studies (for example [12, 23, 24]) focused
22 on the impact of the head model simplifications in recovering the electric brain sources from the measurement of
23 scalp potential. In particular, when computing the EEG forward problem, [25, 26, 27] have shown the importance of
24 modelling correctly the skull anisotropy. However, when the anisotropic conductivity values are not known, it can be
25 preferable to model this region as isotropic, as explained in [25]. Moreover, the anisotropic conductivity of the skull
26 is due to its layered structure, **a cancellous bone between two compact bones**. This means that when those three layers
27 are available, the skull can accurately be modelled with three isotropic layers instead of one anisotropic layer as [28]
28 shows.

29 The relevant computational savings which the use of BEM strategies can lead to, explain the attention the technique
30 has received by the community, resulting in a continuous series of advances [18, 29, 30, 31, 32, 33]. Among them
31 a method, published in [34] and referred to as the “symmetric formulation”, became quite popular and impacted
32 several EEG based imaging tools [35, 36, 37, 38]. The peculiarity of the BEM method proposed in [34] is the quite
33 higher level of accuracy that it can achieve when compared to previously existing schemes. However these beneficial
34 properties are obtained at the cost of using a first kind formulation (while the majority of standard strategies relies
35 on second kind formulations). The computational consequence of this fact is that, when the “symmetric formulation”
36 is discretized to be solved numerically, the condition number of the resulting BEM matrix (the ratio of the largest
37 over the smallest singular value of the matrix) will grow as a function of the discretization density (the number of
38 boundary elements used to discretize the structure) [39]. Similarly, a condition number growth is observed in the
39 symmetric formulation also when the conductivity contrast between two regions of the head is increased (a case of
40 practical interest given that the conductivity of the skull is often modeled with a much smaller value with respect to the
41 conductivity of the brain [11, 40, 41, 42]). In several cases, especially when handling models issued of high resolution
42 Magnetic Resonance Imaging (MRI) [43], the solution of the EEG forward problem is obtained iteratively [18, 44].
43 A low and stable condition number is desirable since, on the one hand, the number of iterations of an iterative solver
44 is growing with the condition numbers [45] and, on the other hand, the condition number controls the amplification

45 in the solution of any initial error in the sources [45]. In other words, the higher the condition number, the longer the
 46 time needed to compute the solution, and the less correct the solution will be.

47 The purpose of this work is to address the ill-conditioning problems of the symmetric formulation. Given the favor
 48 that the formulation has found in the community and the fact that it is already implemented in several neuroimaging
 49 packages, a particular attention will be devoted to develop a solution strategy that will be conservative, in the sense
 50 that will not require the change of previous implementations of the symmetric formulation but will just require the
 51 addition of some extra steps to it. This will be achieved by developing a purely multiplicative preconditioner based
 52 on Calderon formulas, i.e. we will design a preconditioning matrix that is spectrally equivalent to the inverse of the
 53 symmetric formulation. After left multiplication of this matrix with the symmetric formulation matrix, the resulting
 54 linear system will, on the one hand, keep the accuracy the symmetric formulation is well known for and, on the
 55 other hand, will provide a stable condition number both when the mesh is refined and when the conductivity contrast
 56 between two adjacent domains increases. **The reader should notice that Calderón strategies have been successfully
 57 applied to the regularization of other integral operators in the context of full-wave vector electromagnetic problems
 58 scattered by metallic [46, 47, 48, 49, 50, 51, 52, 53] and penetrable [54, 55, 56, 57] objects. The regularization
 59 presented here, however, does not automatically follow from none of the strategies above due to the peculiar nature,
 60 both in terms of frequency and operator structure/properties, of the symmetric formulation under consideration in this
 61 work. It should also be noted that the problem of simulating high contrast dielectric materials has been addressed
 62 for high frequency formulations with Calderon strategies both in [55] and [57]. Both of these two clever approaches,
 63 designed for the vector case, are not applicable in our scalar scenario where the spectral high-contrast scaling expresses
 64 itself as a block scaling issue due to the decoupling of electric and magnetic quantities characterizing a static problem.**

65 This paper is organized as follows: Section 2 provides the reader with some necessary background material and
 66 notation used in the following developments. Section 3 presents the new Calderon preconditioner proposed in this
 67 work, while Section 4 focuses on its discretization and on the solution of the preconditioned symmetric formulation
 68 system. Section 5 complements the paper’s theoretical developments with numerical results which will show the
 69 efficiency and effectiveness of the new approach. Partial results from this work has been presented in the conference
 70 contribution [58].

71 2. Background on the EEG Forward Problem

72 This section will briefly review the relevant formulations, currently available in literature, used to solve the EEG
 73 forward problem. The treatment will be synthetic and for the sole purpose of setting up the notation. The reader
 74 interested in a more profuse treatment should refer, for example to [34, 59] and to references therein.

75 2.1. The EEG Problem

76 Let $\Omega = \bigcup_{i=1}^N \Omega_i$ be a nested domain with Lipschitz boundaries $\partial\Omega_i = (\bar{\Omega}_{i-1} \cap \bar{\Omega}_i) \cup (\bar{\Omega}_i \cap \bar{\Omega}_{i+1})$ as in Fig. 1. We
 77 denote with \mathbf{n}_i the outward going normal to the surface Γ_i , where $\Gamma_i = \bar{\Omega}_i \cap \bar{\Omega}_{i+1}$.

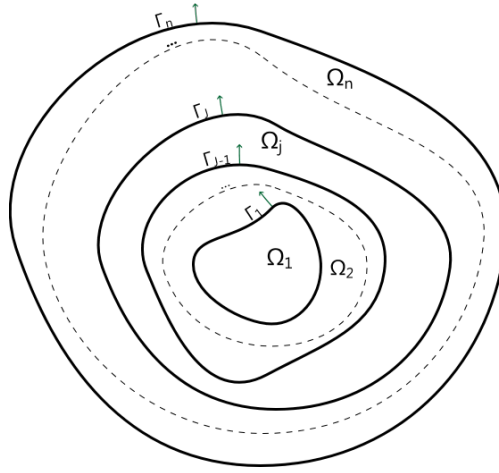


Figure 1. Geometry under consideration.

78 Solving the EEG forward problem amounts to computing the potential V at given electrodes' positions when the
 79 active brain current sources are known. Under quasi-static assumptions and isotropic conductivity, the EEG forward
 80 problem reads [60]:

$$\sigma \Delta V = \nabla \cdot \mathbf{j} \quad \text{in each } \Omega_i \quad (1)$$

81 where $\Delta = \nabla \cdot \nabla$ is the Laplace operator, σ is the conductivity and \mathbf{j} the current sources. The conductivity is assumed
 82 to be piecewise isotropic and homogeneous: in Ω_i , $\sigma = \sigma_i$. In the exterior domain, the conductivity is assumed to
 83 be 0. The current sources, as it is customary in literature [61], are assumed to be dipolar in nature. Hence, denoting
 84 with $f_i = \nabla \cdot \mathbf{j}$ the electric source in Ω_i , we have $f_i = q_i \cdot \nabla \delta_{\mathbf{r}_i}$ with q_i the electric dipole moment and \mathbf{r}_i its position.
 85 Furthermore, the symbol $[g]_i = g^- - g^+$, will refer to the jump of the function g at the interface Γ_i , with g^\mp the inner
 86 and outer trace of g at Γ_i respectively. Then, the solvability of (1) is assured under the following boundary conditions
 87 [60]:

$$[V]_i = 0 \quad \forall i \leq N \quad (2a)$$

$$[\sigma \mathbf{n} \cdot \nabla V]_i = 0 \quad \forall i \leq N \quad (2b)$$

89 that enforce the continuity of the potential and the current between the different layers of the domain Ω .

90 The Green's function associated with (1) reads [39]

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} \quad (3)$$

for which we can derive Green's representation theorem using the integral operators [39]

$$S\Psi(r) = \int_{\partial\Omega} G(r, r')\Psi(r')dr' \quad (4a)$$

$$D\Phi(r) = p.v. \int_{\partial\Omega} \partial_{n'}G(r, r')\Phi(r')dr' \quad (4b)$$

$$D^*\Psi(r) = p.v. \int_{\partial\Omega} \partial_n G(r, r')\Psi(r')dr'. \quad (4c)$$

$$N\Phi(r) = f.p. \int_{\partial\Omega} \partial_n \partial_{n'}G(r, r')\Phi(r')dr' \quad (4d)$$

91 In the above equation and *p.v.* and *f.p.* stand for Cauchy principal value and Hadamard finite part respectively. In the
92 following, we will denote with L_{ij} the operator L when $r \in \Gamma_i$ and $r' \in \Gamma_j$ with $L = D, S, N$ or D^* .

93 2.2. The Symmetric Formulation for the EEG Forward Problem

94 Several BEM formulations have been proposed to solve the EEG forward problem [18, 34]. Among them, the
95 symmetric formulation [34] is quite popular and known for providing high levels of accuracy [36]. In solving the
96 EEG forward problem, an efficient strategy is to build the unknown potential V starting from two functions, a function
97 u harmonic in R^3 and a function v that takes into account the source term. The starting point of the symmetric
98 formulation is to build u_i in each domain such that $u_i = V - v_i/\sigma_i$ in Ω_i and $u = -v_i/\sigma_i$ in $R^3 \setminus \Omega_i$, with v_i the solution
99 of (1) in an unbounded medium: $v_i(r) = \int_{\Omega_i} f_i(r')G(r, r')dr'$. In this fashion, u_i is harmonic in $R^3 \setminus \partial\bar{\Omega}_i = \Gamma_{i-1} \cup \Gamma_i$.
100 Using the boundary conditions (2a) and (2b) as well as the representation theorem [39], two integral equations for the
101 potential and its derivative can be obtained [34]. They read:

$$\sigma_{i+1}^{-1}(v_{i+1})_{\Gamma_i} - \sigma_i^{-1}(v_i)_{\Gamma_i} = D_{i,i-1}V_{i-1} - 2D_{ii}V_i + D_{i,i+1}V_{i+1} - \sigma_i^{-1}S_{i,i-1}p_{i-1} + (\sigma_i^{-1} + \sigma_{i+1}^{-1})S_{ii}p_i - \sigma_{i+1}^{-1}S_{i,i+1}p_{i+1} \quad (5)$$

$$(\partial_n v_{i+1})_{\Gamma_i} - (\partial_n v_i)_{\Gamma_i} = \sigma_i N_{i,i-1}V_{i-1} - (\sigma_i + \sigma_{i+1})N_{ii}V_i + \sigma_{i+1}N_{i,i+1}V_{i+1} - D_{i,i-1}^*p_{i-1} + 2D_{ii}^*p_i - D_{i,i+1}^*p_{i+1} \quad (6)$$

103 with V_i the potential on the surface Γ_i and $p_i = \sigma_i [\mathbf{n} \cdot \nabla V]_i$. Equations (5) and (6) are obtained by applying the
104 boundary conditions on the surface Γ_i . In a nested domain, it only involves the computation of the operators with
105 functions defined in the surrounding surfaces Γ_{i-1} , Γ_i and Γ_{i+1} . To clarify the ideas, equations (5) and (6) have been
106 rewritten in matrix form in equation (7). For a more detailed explanation on the symmetric formulation, the reader is

and (6) are tested with P_0 and P_1 basis functions respectively. The operator matrices arising are then given by

$$[\mathbf{D}_{ij}]_{kl} = \int_{t_k} D_{ij}(P_{1l}) P_{0k}(r) dr \quad (8a)$$

$$[\mathbf{S}_{ij}]_{kl} = \int_{t_k} S_{ij}(P_{0l}) P_{0k}(r) dr \quad (8b)$$

$$[\mathbf{N}_{ij}]_{kl} = \int_{\mu_{P_{1i}}} N_{ij}(P_{1l}) P_{1k}(r) dr \quad (8c)$$

$$[\mathbf{D}^*_{ij}]_{kl} = \int_{\mu_{P_{1i}}} D^*_{ij}(P_{0l}) P_{1k}(r) dr. \quad (8d)$$

117 Consequently, the system to be solved reads $\mathbf{Z}\mathbf{x} = \mathbf{b}$ with \mathbf{Z} given by

$\mathbf{Z} =$

$$\left[\begin{array}{ccccccc} (\sigma_1 + \sigma_2) \mathbf{N}_{11} & -2\mathbf{D}^*_{11} & -\sigma_2 \mathbf{N}_{12} & \mathbf{D}^*_{12} & & & \\ -2\mathbf{D}_{11} & (\sigma_1^{-1} + \sigma_2^{-1}) \mathbf{S}_{11} & \mathbf{D}_{12} & -\sigma_2^{-1} \mathbf{S}_{12} & & & \\ -\sigma_2 \mathbf{N}_{21} & \mathbf{D}^*_{21} & (\sigma_2 + \sigma_3) \mathbf{N}_{22} & -2\mathbf{D}^*_{22} & \cdots & & \\ \mathbf{D}_{21} & -\sigma_2^{-1} \mathbf{S}_{21} & -2\mathbf{D}_{22} & (\sigma_2^{-1} + \sigma_3^{-1}) \mathbf{S}_{22} & \cdots & & \\ & & \vdots & \vdots & \ddots & & \\ & & & & & (\sigma_{N-1} + \sigma_N) \mathbf{N}_{N-1,N-1} & 2\mathbf{D}^*_{N-1,N-1} & -\sigma_N \mathbf{N}_{N-1,N} \\ & & & & & -2\mathbf{D}_{N-1,N-1} & (\sigma_{N-1}^{-1} + \sigma_N^{-1}) \mathbf{S}_{N-1,N-1} & \mathbf{D}_{N-1,N} \\ & & & & & -\sigma_N \mathbf{N}_{N,N-1} & \mathbf{D}_{N,N-1} & \sigma_N \mathbf{N}_{N,N} \end{array} \right] \quad (9)$$

where

$$[\mathbf{x}]_{2l-1} = a_l \quad (10a)$$

$$[\mathbf{x}]_{2l} = b_l \quad (10b)$$

and where

$$[\mathbf{b}]_{2k} = \int_{\mu_k} (\sigma_{k+1}^{-1} v_{k+1} - \sigma_k^{-1} v_k) P_{0k} dr \quad (11a)$$

$$[\mathbf{b}]_{2k-1} = \int_{t_k} (\partial_n v_{k+1} - \partial_n v_k) P_{1k} dr. \quad (11b)$$

118 In the following, the operator associated with the matrix \mathbf{Z} (obtained by replacing in (9) \mathbf{D} , \mathbf{S} , \mathbf{N} , and \mathbf{D}^* with D ,
119 S , N , and D^*) will be denoted by Z .

120 3. A Calderon Preconditioner for the Symmetric Formulation

121 The high accuracy of the symmetric BEM formulation [34] has made of it a very popular tool for solving the EEG
122 forward problem. However its system matrix suffers from ill-conditioning that can lead to the non-convergence of the

123 employed iterative solver used to compute the solution [45], which has to be used for high refined meshes where the
 124 direct methods are impractical. Indeed, the operator S is compact [39]. This means that its spectrum will accumulate
 125 at zero when the mesh is refined and it will therefore have a condition number increasing inversely proportional to the
 126 average mesh length h . Moreover, the hypersingular operator N is an unbounded operator [39]. This implies that its
 127 condition number will also grow with $1/h$. Since these operators are the diagonal blocks of the matrix \mathbf{Z} in (9) and the
 128 off-diagonal blocks of the matrix are smoothers [62], it follows that the overall conditioning of \mathbf{Z} will increase when
 129 the mesh discretization will increase ($h \rightarrow 0$).

By leveraging on the Calderon identities, it is possible to build a preconditioner for the system matrix \mathbf{Z} . The rationale behind our strategy can be understood by considering the continuous operators first. The Calderon identities that for our approach read [63]

$$S_{ii}N_{ii} = \frac{1}{4}I - D_{ii}^2 \quad (12a)$$

$$N_{ii}S_{ii} = \frac{1}{4}I - D_{ii}^{*2} \quad (12b)$$

$$D_{ii}S_{ii} = D_{ii}^*S_{ii} \quad (12c)$$

$$D_{ii}^*N_{ii} = N_{ii}D_{ii}^* \quad (12d)$$

130 where the symbol I stands for the identity operator. The spectral analysis of (12a) and (12b) shows that the operators
 131 $S_{ii}N_{ii}$ and $N_{ii}S_{ii}$ are well conditioned. Indeed, given that D and D^* are compact operators, then D_{ii}^2 and D_{ii}^{*2} are also
 132 compact operators as a product of two compact operators. Then the spectrum of D_{ii}^2 and D_{ii}^{*2} is bounded above and
 133 accumulates at zero. However, the presence of the identity operator in (12a) and (12b) guarantees that the spectrum of
 134 the operators $S_{ii}N_{ii}$ and $N_{ii}S_{ii}$ will be bounded from below by $1/4$. In other words, $S_{ii}N_{ii}$ and $N_{ii}S_{ii}$ are second kind
 135 operators whose spectrum accumulates at $1/4$. This property can be exploited to build a left preconditioner for the
 136 symmetric operator Z .

137 **Before introducing the Calderon preconditioner, we present a regularization matrix for the coefficients of the sym-**
 138 **metric operator, which is necessary for the stability of the matrix condition number with respect to the conductivity**
 139 **ratio and for the correct behaviour of the subsequent mesh refinement preconditioner. Indeed, the symmetric operator**
 140 **is unstable when there is a high conductivity contrast between two adjacent domains Ω_i and Ω_j , due to the con-**
 141 **ductivity factors in the diagonal blocks of the matrix.** These conductivity factors are given by $a_{2l-1,2l-1} = \sigma_l + \sigma_{l+1}$
 142 and $a_{2l,2l} = \sigma_l^{-1} + \sigma_{l+1}^{-1}$. As a consequence, in the case of high conductivity contrast between two adjacent domains,
 143 i.e. asymptotically, when $\frac{\sigma_i}{\sigma_j} \rightarrow \infty$, that is $(\sigma_i + \sigma_j) \rightarrow \infty$ when $\sigma_i \rightarrow \infty$ or $(\sigma_i^{-1} + \sigma_j^{-1}) \rightarrow \infty$ when $\sigma_j \rightarrow 0$, the
 144 condition number of the system matrix will grow as a function of the **conductivity ratio** $CR_{ij} = \frac{\max(\sigma_i, \sigma_j)}{\min(\sigma_i, \sigma_j)}$. Because
 145 of high conductivity contrast between the brain and the skull [40, 41], this undesirable situation is likely to appear
 146 when solving the EEG forward problem. To solve this problem, we will rescale with respect to the conductivity the

The terms denoted with K_{ij} , contain linear combinations of the compact operators $D_{ij}D_{jk}$, $S_{ij}N_{jk}$, $S_{ij}D_{jk}^*$, $D_{ij}S_{jk}$, $D_{ij}^*N_{jk}$, $N_{ij}D_{jk}$, $N_{ij}S_{jk}$ and $D_{ij}^*D_{jk}^*$. They read

$$K_{2n-1,2m-1} = \sum_{i=n-1}^{n+1} \chi_m(i)(c_{2n-1,2i-1}c_{2i-1,2m-1}S_{ni}N_{im} + (-1)^{\delta_{ni}}\chi_i c_{2n-1,2i}c_{2i,2m-1}D_{ni}D_{im}) \quad (20a)$$

$$- c_{2n-1,2m-1}S_{nm}N_{nm}\delta_{nm}$$

$$K_{2n,2m} = \sum_{i=n-1}^{n+1} \chi_m(i)((-1)^{\delta_{ni}}c_{2n,2i-1}c_{2i-1,2m}D_{ni}^*D_{im}^* + \chi_i c_{2n,2i}c_{2i,2m}N_{ni}S_{im}) \quad (20b)$$

$$- c_{2n,2m}N_{nm}S_{nm}\delta_{nm}$$

$$K_{2n-1,2m} = \sum_{i=n-1}^{n+1} \chi_m(i)(c_{2n-1,2i-1}c_{2i-1,2m}S_{ni}D_{im}^* + (-1)^{\delta_{ni}}\chi_i c_{2n-1,2i}c_{2i,2m}D_{ni}S_{im}) \quad (20c)$$

$$K_{2n,2m-1} = \sum_{i=n-1}^{n+1} \chi_m(i)((-1)^{\delta_{ni}}c_{2n,2i-1}c_{2i-1,2m-1}D_{ni}^*N_{im} + \chi_i c_{2n,2i}c_{2i,2m-1}N_{ni}D_{im}) \quad (20d)$$

164 where the symbols $\chi_i, \chi_m(i)$ are given by

$$\chi_m(i) = \begin{cases} 1 & \text{if } |i - m| < 2 \\ 0 & \text{otherwise} \end{cases}, \quad (21)$$

165

$$\chi_i = \begin{cases} 1 & \text{if } i < N \\ 0 & \text{otherwise} \end{cases}, \quad (22)$$

166 and where δ_{nm} is the Kronecker's delta

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}. \quad (23)$$

167 As shown previously, the terms $S_{ii}N_{ii}$ and $N_{ii}S_{ii}$ are well conditioned second kind operator matrices, as a conse-
 168 quence the terms $c_{ii}S_{ii}N_{ii}$ $c_{ii}N_{ii}S_{ii}$ will also be well conditioned with respect to the mesh parameter h since $1 < c_{ii} < 2$.
 169 In order to show the compactness of the operators K_{ij} , we analyze each operator product in them. First, in the diag-
 170 onal blocks $K_{2n-1,2n-1}$ and $K_{2n,2n}$ we have the following products: $S_{ni}N_{in}$, $N_{ni}S_{in}$, $D_{ni}^*D_{in}^*$, $D_{ni}D_{in}$. The operators D
 171 and D^* are compact, therefore the products $D_{ni}D_{in}$ and $D_{ni}^*D_{in}^*$ are compact as well [63]. The products $S_{ni}N_{in}$, $N_{ni}S_{in}$
 172 are present when $i \neq n$, then the operators N_{in} and N_{ni} have a regular kernel due to the lack of singularity. Hence,
 173 the products with the compact operators S_{ni} and S_{in} are also compact operators [39, 64, 65]. Overall, in the diagonal
 174 block we have the sum of compact operators. In the off diagonal blocks $K_{2n-1,2m-1}$, $K_{2n,2m}$, when $n \neq m$, we have the
 175 products $S_{ni}N_{im}$, $D_{ni}D_{im}$, $D_{ni}^*D_{im}^*$, $N_{ni}S_{im}$. As stated before, the product of the D^* and D are compact. Moreover, in the
 176 products $S_{ni}N_{im}$ and $N_{ni}S_{im}$ at least one of the operators has a regular kernel, since $n \neq m$. Hence, these products yield
 177 compact operators [64, 65]. Finally, for the off diagonal blocks $K_{2n-1,2m}$, $K_{2n,2m-1}$ we have the following products:
 178 $S_{ni}D_{im}^*$, $D_{ni}S_{im}$, $D_{ni}^*N_{im}$, $N_{ni}D_{im}$. The products $S_{ni}D_{im}^*$ and $D_{ni}S_{im}$ are clearly compact, since all the operators involved
 179 are compact. The only not-compact terms are present in the products $D_{ni}^*N_{im}$ and $N_{ni}D_{im}$ when $n = i = m$. However,

180 recalling the Calderon identity (12d) and the properties (17), we have

$$-c_{2n,2n-1}c_{2n-1,2n-1}D_{nn}^*N_{nn} + c_{2n,2n}c_{2n,2n-1}N_{nn}D_{nn}^* = 0 \quad (24)$$

181 which is true for the continues operators. When the operators are discretized, this cancellation is not exact, however
182 the residue tends to zero.

Then, we can write

$$C_q Z_q = A + B$$

with

$$\begin{cases} [A]_{2l-1,2l-1} & = c_{2l-1,2l-1}^2 S_{ll} N_{ll} \\ [A]_{2l,2l} & = c_{2l,2l}^2 N_{ll} S_{ll} \\ [A]_{ij} & = 0 \text{ if } i \neq j \end{cases}$$

183 and B such that $[B]_{ij} = K_{ij}$, $C_q Z_q$ can be seen as the sum of the well conditioned matrix A and a compact perturbation
184 B (as the operators K_{ij} are compact operators). We can therefore expect the operator $C_q Z_q$ to be well conditioned with
185 respect to the mesh parameter.

186 For now on, we will refer to the product

$$Z_c = C_q Z_q, \quad (25)$$

187 as the Calderon-Symmetric operator, which is well conditioned for both mesh refinement and conductivity ratio be-
188 tween adjacent surfaces.

189 4. Discretization of the Calderon Preconditioner and Solution of the Preconditioned Symmetric Formulation

190 In order to solve the preconditioned symmetric integral equation, the proposed multiplicative preconditioner C_q
191 has to be discretized. This discretization should be carried out with care. In fact, the preconditioned operator in (25)
192 will contain operator products which will not directly translate into matrix products in the general case. A suitable
193 choice of basis functions should be made to guarantee that this could instead be the case here. To fix the ideas, we
194 could consider the discretization of the operator product $N_{11} S_{11}$ appearing in the top-left block of Z_c . The matrix \mathbf{S}_{11} is
195 obtained by using test and trial functions in P_0 while the matrix \mathbf{N}_{11} is obtained by using test and trial functions in P_1 .
196 Yet the number of vertices N_t and the number of cells that defines the dimensions of the space P_1 and P_0 are different.
197 As a consequence, the blocks \mathbf{N}_{11} and \mathbf{S}_{11} do not have compatible shapes and cannot be multiplied. Furthermore,
198 the basis functions used for discretizing N_{11} and S_{11} must satisfy appropriate inf-sup conditions with respect to the
199 duality pairing $\langle v, w \rangle : H^{1/2}(\Gamma_1) \times H^{-1/2}(\Gamma_1) \rightarrow \mathbb{R}$ (the reader should refer to [66] for further technical details on
200 this topic). This condition enables to get a stable condition number for the Gram matrices, necessarily present in
201 the discretized system as they orthonormalize the two chosen basis and testing functions sets. To properly take care
202 of this fact, we propose to discretize the preconditioner C_q on the dual mesh M_Ω^* of the standard mesh M_Ω and to

203 leverage on the dual basis functions introduced in [67] on such a mesh. In the dual mesh, each vertex corresponds to
 204 a cell of the standard mesh and vice-versa. This means that in M_Ω^* we can build a discrete space in $H^{1/2}(\Gamma)$ which has
 205 the same dimension as the discrete space associated with $H^{-1/2}(\Gamma)$ in M_Ω and vice-versa. Moreover, the dual basis
 206 functions introduced by [67] abide by the inf-sup conditions required to obtain stable discrete products [66]. As a
 207 consequence, the discretization of the preconditioner operator C_q by using these basis functions enables to perform
 208 the matrix multiplication associated with the operator multiplication $C_q Z_q$ in such a way that the spectral bounds
 209 holding for the continuous operator products will translate in well-conditioned matrix products.

210 The dual mesh M_Ω^* can be obtained by barycentric refinement of the standard mesh M_Ω by dividing the triangles t_k
 211 into six smaller triangles t_{bk} whose edges are built by tracing the medians of the standard triangles t_k [67]. The cells c_k
 212 of M_Ω^* are defined as the set of triangle t_{bk} sharing a common vertex v_k of M_Ω . The vertices of M_Ω^* are the barycenters
 213 b_k of the triangles t_k in M_Ω . The reader should refer to Fig. 2 for an example of such a refinement.

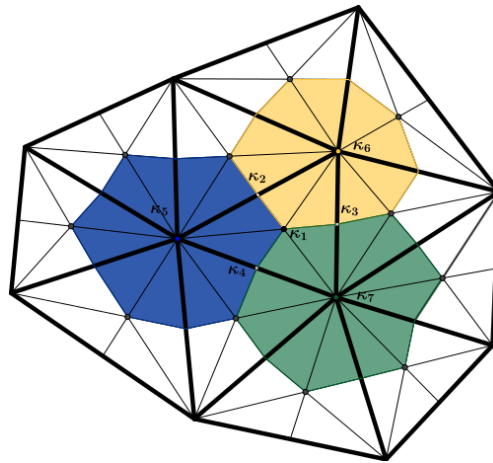


Figure 2. Standard Mesh in bold lines, its barycentric refinement in thin lines. Three cells of the dual mesh are evidenced. The coefficients κ are the coefficients used in the linear combination of primal P_1 functions to build the dual \tilde{P}_1 functions: $\kappa_1 = 1$, $\kappa_i = 1/2$ if $i \in 2, 3, 4$ and $\kappa_i = 1/t$ if $i \in 5, 6, 7$, with t the number of triangles in M_Ω sharing the corresponding node.

214 In M_Ω^* , we define the dual piecewise linear functions $\tilde{P}_1 = span\{\tilde{P}_{1_k}\}_{k=1}^{N_t}$ obtained with a linear combination of
 215 P_1 functions built on the barycentric refined mesh [67]. A dual piecewise linear function is shown Fig. 3d. The
 216 coefficients of the linear combination are shown Fig. 2. The support of \tilde{P}_{1_i} is denoted $\mu_{\tilde{P}_{1_i}}$. The dual piecewise
 217 constant functions in \tilde{P}_0 , denoted \tilde{P}_{0k} , are the constant functions equal to $1/A_{c_k}$ on the cell c_k , whose area is A_{c_k} , of
 218 M_Ω^* and equal to zero elsewhere. A dual piecewise constant function is shown in Fig. 3c. **An extended explanation of**
 219 **the dual mesh and dual basis functions is given by [67].**

operator and the test functions of the other operator. Hence, the computation of the Gram matrix does not require the evaluation of any operator. Additionally, it is almost diagonal, therefore the computational cost is very low. This matrix is obtained as

$$[\mathbf{G}_{2i-1}]_{kl} = \int_{\mu_{P_{1k}}} (\tilde{P}_{0l}) P_{1k}(r) dr \quad (28a)$$

$$[\mathbf{G}_{2i}]_{kl} = \int_{t_k} (\tilde{P}_{1l}) P_{0k}(r) dr. \quad (28b)$$

223 Finally, the discretization of the Calderon symmetric operator is given by

$$\mathbf{Z}_c = \tilde{\mathbf{C}}_q \mathbf{G}^{-1} \mathbf{Z}_q \quad (29)$$

224 where $\mathbf{Z}_q = \mathbf{QZ}_q\mathbf{Q}$. The solution of the preconditioned symmetric formulation is then obtained by solving the
225 following system $\mathbf{Z}_c \mathbf{y} = \mathbf{C}_q \mathbf{G}^{-1} \mathbf{Qb}$ and \mathbf{x} is obtained with $\mathbf{x} = \mathbf{Qy}$.

226 Summarizing, the Calderon preconditioning strategy is multiplicative in nature. Its aim is to build a precondition-
227 ing operator spectrally equivalent to the inverse of the original operator. Thus, once this operator is built, multiplying
228 the ill-conditioned operator with it yields an operator spectrally equivalent to an identity. The preconditioning opera-
229 tor is built on a dual mesh in order to allow matrix multiplication and stability. Moreover, regularization matrices are
230 added in order to get a condition number independent of the conductivity ratio. In a nutshell, the steps are:

- 231 1. Compute the standard symmetric system matrix \mathbf{Z} ;
- 232 2. Compute the Calderon preconditioning matrix $\tilde{\mathbf{C}}_q$ on the dual mesh;
- 233 3. Compute the Gram matrices linking the dual and standard discretization, known as Gram matrices \mathbf{G} ;
- 234 4. Normalize the operator \mathbf{Z} with the regularization matrices \mathbf{Q} ;
- 235 5. Perform the multiplication $\mathbf{Z}_c = \tilde{\mathbf{C}}_q \mathbf{G}^{-1} \mathbf{Z}_q$;
- 236 6. The right hand side \mathbf{b} must be modified accordingly : compute $\mathbf{b}_c = \mathbf{Q}\tilde{\mathbf{C}}_q\mathbf{G}^{-1}\mathbf{Qb}$;
- 237 7. Solve the system $\mathbf{Z}_c \mathbf{y} = \mathbf{b}_c$;
- 238 8. Get the solution using $\mathbf{x} = \mathbf{Qy}$.

239 5. Numerical Results

240 The new Calderon regularized symmetric formulation proposed in this work has been first tested on the canonical
241 scenario of three homogeneous and concentric spheres of radii 0.8, 0.9, and 1 in normalized units respectively. **Such**
242 **model is shown in Fig. 4a.** Indeed, in the case of homogeneous nested spheres, an analytical solution is available as a
243 reference [68, 69], this solution will be denoted with V_{ref} . In these simulations, a single dipole source is placed in (0,
244 0, 0.5) with a dipole moment of (0, 0, 1) and the **normalized** conductivity of the layers is chosen to be 1, 0.0125 and 1
245 starting from the inner domain, **which are the conductivity values used in the standard symmetric formulation in [34].**
246 As a complement to these results, and to validate the new formulation on a real case scenario, the new formulation
247 has been tested also on a realistic head model obtained from MRI data , **which is presented in Fig. 4b.**

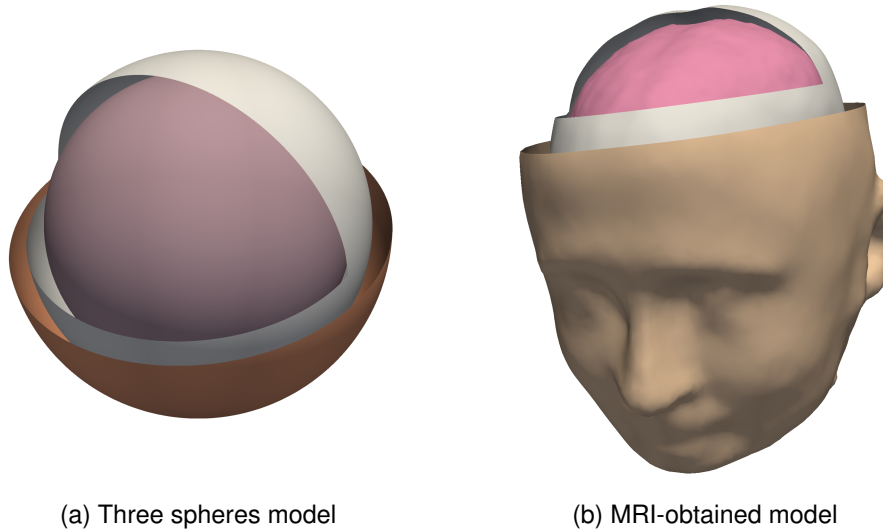


Figure 4. Head models used for testing the Calderon-Symmetric formulation

248 *5.1. Assessments on accuracy and condition number*

249 The first test conducted aimed at verifying that applying the proposed preconditioner to the symmetric formulation
 250 does not modify its accuracy. The assessment parameter is the relative error computed as $RE = \frac{\|V_{num} - V_{ref}\|}{\|V_{ref}\|}$ where V_{num}
 251 refers to the numerical solution. In Fig. 5a and Fig. 5b it is shown that the Calderon preconditioned symmetric
 252 formulation and the non preconditioned symmetric formulation provide exactly the same accuracy for different mesh
 253 refinement levels and **different conductivity ratios respectively**. This means that the proposed preconditioner does
 254 not alter the accuracy of the initial formulation. **These** figures also confirms the higher level of accuracy that the
 255 symmetric formulation can reach with respect to two others existing BEM formulations, namely the adjoint double
 256 layer formulation and the double layer formulation.

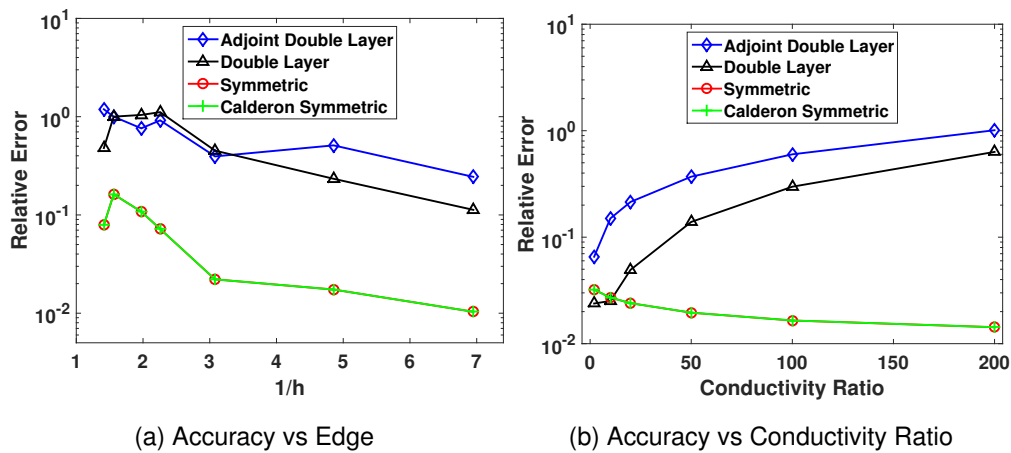


Figure 5. Accuracy of different boundary integral formulations of the EEG forward problem

257 In the second test we compared the condition numbers of the preconditioned and the non preconditioned symmetric
 258 formulation system matrices . **The variation of the condition number with respect to the mesh refinement is shown**
 259 **in Fig. 6a.** We see that the condition number of the symmetric formulation grows rapidly with the mesh refinement
 260 parameter $1/h$ while the condition number of the proposed formulation stays constant as expected by the theory.
 261 **In Fig. 6b is presented the condition number when the conductivity ratio increases.** Here, we can see that the new
 262 **formulation has a condition number independent of the conductivity contrast.**

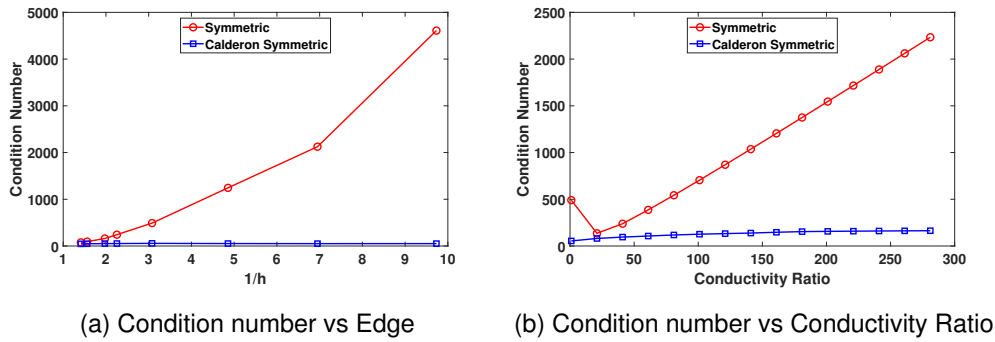


Figure 6. Condition number of the Symmetric Formulation and Calderon-Symmetric formulation.

263 The third test aimed at showing the efficiency in terms of number of iterations of the proposed preconditioner.
 264 For different spectral indices $1/h$, we present in Fig. 7a the number of iterations needed for a Conjugate Gradient
 265 Square (CGS) solver to reach a relative residual error of 10^{-6} . In this figure, it is clear that only with the proposed
 266 Calderon preconditioner this number of iterations stays constant with the mesh refinement. **The number of iterations**
 267 **for increasing conductivity ratio is shown in Fig. 7b.** In this case, all the preconditioners lead to a constant number of
 268 **iterations. Still, the best performance is given by the Calderon preconditioner.**

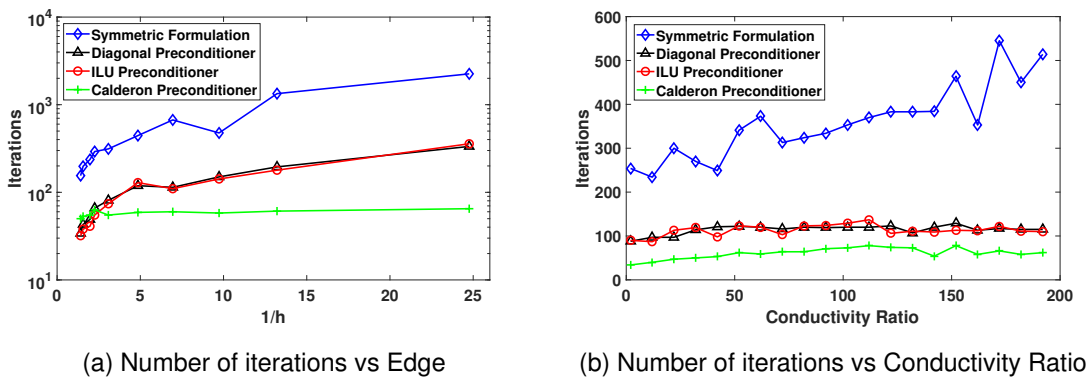


Figure 7. Number of iterations for an accuracy of $1e-6$ for different preconditioners.

269 5.2. Assessments on time

270 Since for small numbers of unknowns direct solvers can be used to solve the obtained system of equations, a
 271 preconditioner is only useful when the number of unknowns becomes too high for direct inversion, that is when it is
 272 necessary to use iterative solvers. Indeed, if N denotes the number of unknowns, then the complexity of a direct solver
 273 is $O(N^3)$ while the complexity of an iterative solver is $O(kN^2)$, where k is the number of iterations needed to reach
 274 a desired accuracy. However, for high number of unknowns, given that BEM matrices are dense, the time needed
 275 to compute the system matrix is also increasing. To deal with this issue and to show the benefits of the proposed
 276 Calderon preconditioner in a high number of unknowns context we coupled it with an Adaptive Cross Approximation
 277 (ACA) algorithm [70], that provides a compressed version of the system matrix.

278 The time necessary to compute the solution for different numbers of unknown is shown Fig.8. In this test, we
 279 compare the performance of direct and iterative solutions with or without the proposed Calderon preconditioner. It is
 280 clear that direct inversion (DI) has, as expected, a time complexity that increases rapidly with N . This prevents the use
 281 of this solver for very detailed models. The iterative solver used is the CGS algorithm. We see that without using any
 282 preconditioner the time necessary to obtain the solution with this iterative solver is also increasing with the number
 283 of unknowns even if this choice of solver is faster than DI. This is due to the fact the number of iterations increases
 284 with the mesh refinement parameter (due to the ill-conditioning of the matrices). However, employing the proposed
 285 Calderon preconditioner solves this issue.

286 The time for computing the dense and compressed Calderon-Symmetric operator is presented in Fig. 9. It can be seen
 287 that the use of the ACA yields in a linear time complexity

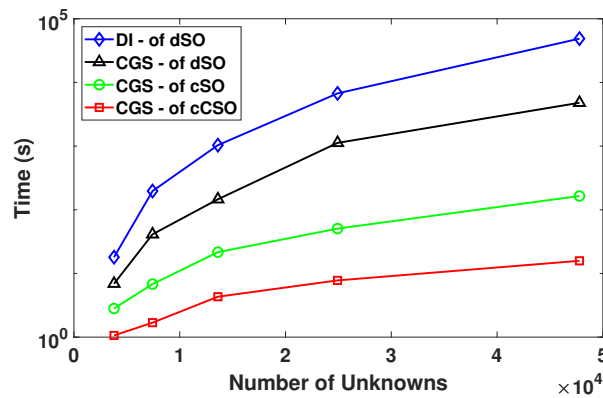


Figure 8. Time for solving one equation with respect to the mesh refinement. DI - Direct Inversion (DI), CGS - Conjugate Gradient Square, dSO - dense Symmetric Operator, cSO - compressed Symmetric Operator, cCSO - compressed Calderon-Symmetric Operator

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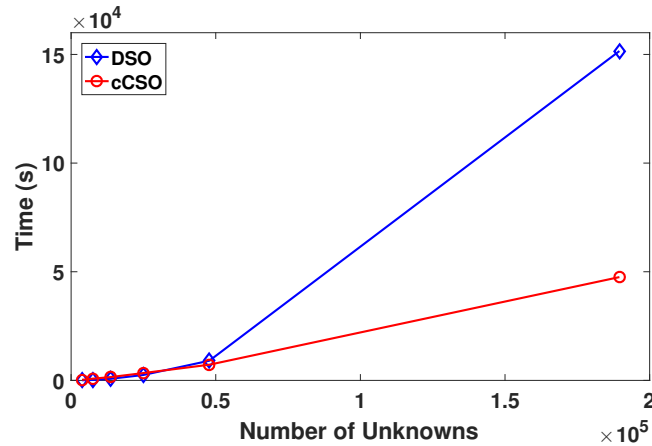


Figure 9. Time for computing the different Operators with respect to the mesh refinement. DSO - Dense Symmetric Operator, cCSO - compressed Calderon-Symmetric Operator

289 5.3. Assessments on a MRI-obtained head model

290 Finally, we seek to assess the performance of the proposed preconditioner in a realistic scenario that consists in
 291 computing a leadfield matrix using a head model obtained from MRI data. This matrix that provides the propagation
 292 model between known brain electric current sources and electrodes situated at the surface of the head of a patient is
 293 a key element in distributed inverse solution. For this purpose, we constructed a three layer mesh using [35]. These
 294 layers model the brain, the skull, and the scalp. They contain 11850, 11616 and 22948 triangular cells respectively.
 295 The potential generated on the scalp by a single dipole situated in the brain is presented Fig. 10a. This figure also
 296 shows the position of the 21 electrodes for which we computed the leadfield matrix. The error when the Calderon-
 297 Symmetric formulation is solved with the ACA compared with the original Symmetric formulation solved with direct
 298 inversion is displayed Fig. 10b. It can be seen that the error is never greater than 0.05%. To fill-in the leadfield matrix,
 299 we placed 1500 unitary dipole source in the brain layer, each having an orientation orthogonal to the brain surface.
 300 Using reciprocity [71], the forward model is then solved at each electrode position. We compared in Table 2, for four
 301 different cases, the time needed to compute the operator, the time needed to solve the forward system once, and the
 302 time needed to compute the full leadfield matrix. Hence, the total time needed to get the leadfield matrix is given
 303 by the sum of the computation for obtaining the operator and the leadfield matrix. It can be seen that even if the
 304 time necessary to compute the compressed Calderon-Symmetric operator is greater than the compressed Symmetric
 305 operator, the fast convergence of the new method allows to compute the complete leadfield matrix in 2.56 hours, that is
 306 almost 10 times faster than without the proposed preconditioner. This compensates largely the computation overload
 307 in computing the preconditioning operator.

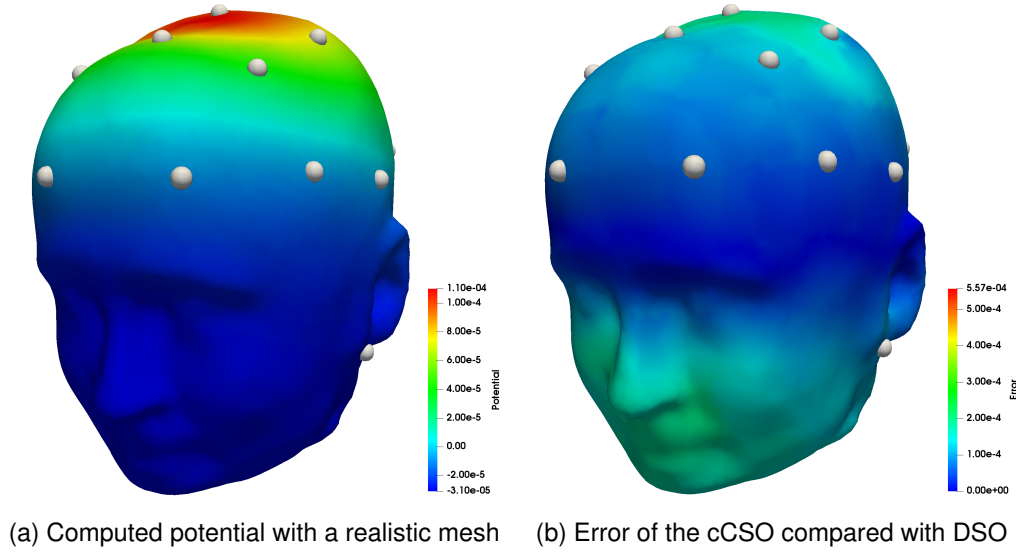


Figure 10. MRI-obtained head model

Method	Memory (GB)	Operator Time (s)	Solution Time (s)	Lead Field Time (h)
DI Dense Symmetric	16.234	10845.67	54609.99	18.19
CGS Dense Symmetric	16.234	10845.67	7294.61	45.56
CGS Compressed Symmetric	1.254	1436.63	2322.89	13.95
CGS Compressed Calderon-Symmetric	2.542	7888.86	62.40	2.56

Table 2. Memory and computation time information for computing a lead field matrix using the reciprocity method

308 **6. Conclusion**

309 In this work, leveraging on Calderon identities, we have proposed a Calderon preconditioned symmetric formu-
 310 lation for the EEG forward problem. When compared to the standard symmetric equation, the proposed formulation
 311 has the advantage of showing constant condition numbers both as a function of the mesh refinement and of the con-
 312 ductivity contrast. Employing this preconditioner does not degrade the accuracy of the symmetric formulation. This
 313 means that, for a given relative accuracy of the solution, the proposed formulation converges substantially faster than
 314 the standard one. Moreover, the integration of the proposed approach into existing symmetric formulation implemen-
 315 tations is achieved only at the cost of computing, on a barycentric refined mesh, the preconditioning operators with
 316 already existing tools. Numerical results have substantiated the theoretical claims and have shown the practical impact
 317 of the newly proposed scheme.

318

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 323 project 321).

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