Some Results and Applications of Geometric Counting Processes

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# Some results and applications of geometric counting processes

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**Abstract** Among Mixed Poisson processes, counting processes having geometrically distributed increments can be obtained when the mixing random intensity is exponentially distributed. Dealing with shock models and compound counting models whose shocks and claims occur according to such counting processes, we provide various comparison results and aging properties concerning total claim amounts and random lifetimes. Furthermore, the main characteristic distributions and properties of these processes are recalled and proved through a direct approach, as an alternative to those available in the literature. We also provide closed-form expressions for the first-crossing-time problem through monotone nonincreasing boundaries, and numerical estimates of first-crossing-time densities through other suitable boundaries. Finally, we present several applications in seismology, software reliability and other fields.

**Keywords** Counting processes · Multivariate Geometric distribution · First-crossing time · Shock models · Stochastic Orders · Aging

**Mathematics Subject Classification (2000)** 60J27 · 60K15 · 60K20

This paper is dedicated to the cherished memory of Moshe Shaked, to whom we are very grateful for much inspiration and advice on our studies in stochastic orderings and stochastic processes.

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## 1 Motivations and preliminaries

The relevance of the Poisson process in modeling random occurrences of events in time and space in the applied sciences is well-known. In fact, it is widely used for a good compromise between realistic representation of the phenomena and mathematical tractability of the model. The Poisson process also satisfies several properties and has a manageable structure that can be adapted to more general counting models. However, it is also well known that certain random occurrences of events in time cannot be properly described by Poisson processes. Indeed, in many cases the assumption of independence between increments is not realistic at all, as well as the assumption of memoryless property for the time intervals between occurrences. In particular, the assumption of finite expectations of these random intervals should be rejected in many disciplines, like, e.g., in software reliability or in earth sciences (climatology, hydrology, etc). For this reasons, several alternatives to Poisson processes may be taken into account, even if typically certain suitable processes are less mathematically tractable.

Among others, a possible alternative to Poisson processes is represented by a Mixed Poisson process, i.e. the process $\mathbf{N} = \{N(t), t \in \mathbb{R}_0^+\}$ whose marginal distributions can be expressed as

$$\mathbb{P}[N(t) = k] = \int_0^\infty \mathbb{P}[N^{(\alpha)}(t) = k] \, \mathrm{d}U(\alpha), \qquad t \in \mathbb{R}_0^+, \tag{1}$$

where $N^{(\alpha)}(t)$, $t \in \mathbb{R}_0^+$, is a Poisson process with intensity $\alpha$, and where $U(\cdot)$ is a distribution with support contained in $\mathbb{R}^+$ (cf. Chapter 4 of Grandell, 1997, or Chapter 8.5 of Rolski et al., 1999). Note that here, and throughout the paper, $\mathbb{R}^+$ denotes the set of strictly positive real numbers and $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$, while $\mathbb{N} = \{0, 1, \ldots\}$ and $\mathbb{N}^+ = \{1, 2, \ldots\}$.

Specifically, if $U(\cdot)$ is a gamma distribution then the resulting process $\mathbf{N}$ is termed *binomial counting process*, or *Pascal process*, or *Pólya-Lundberg process*.

From now on we deal with the special case when $U(\cdot) = U_\lambda(\cdot)$ is an exponential distribution with mean $\lambda \in \mathbb{R}^+$, for which the process $\mathbf{N}$ will be said a *Geometric counting process* with intensity $\lambda$, according to the terminology used by Cha and Finkelstein (2013), who studied dependence properties of its increments in the general case of non constant intensities. Note that this terminology should not be confused with the notion of geometric process discussed, for instance, in Lam (2007) and Finkelstein (2010).

The following characterization of the Geometric counting process is easy to prove (see, e.g., Rolski et al., 1999).

*Property 1* For fixed $\lambda \in \mathbb{R}^+$, the Geometric counting process with constant intensity $\lambda$ satisfies the following properties:

1. $N(0) = 0$;
2. $\mathbb{P}[N(t+s) - N(t) = k] = \dfrac{1}{1+\lambda s} \left( \dfrac{\lambda s}{1+\lambda s} \right)^k =: p_k(s), \quad \forall\, s, t \in \mathbb{R}_0^+, k \in \mathbb{N}.$

| | $N(t)$ Geometric counting process | $N^{(\lambda)}(t)$ Poisson process |
|---|---|---|
| $\mathbb{E}[N(t)]$ | $\lambda t$ | $\lambda t$ |
| $\mathrm{Var}[N(t)]$ | $\lambda t(1+\lambda t)$ | $\lambda t$ |
| $\mathrm{Cov}[N(s), N(t)]$ | $\lambda s(1+\lambda t)$ | $\lambda s$ |
| $\mathrm{Cov}\big[N(s), \big(N(t)-N(s)\big)\big]$ | $\lambda^2 s(t-s)$ | $0$ |
| $r[N(s), N(t)]$ | $\sqrt{\dfrac{s(1+\lambda t)}{t(1+\lambda s)}}$ | $\sqrt{\dfrac{s}{t}}$ |

**Table 1** Results for the Geometric counting process and for the Poisson process, both with parameter $\lambda \in \mathbb{R}^+$, for $t, s \in \mathbb{R}^+$, with $s < t$.

Since the marginal probability distribution of $\mathbf{N}$ is expressed as a mixture of the distribution of the Poisson process $N^{(\alpha)}(t)$ with exponential mixing distribution, the results listed in Table 1 can be easily verified (where $r[\cdot,\cdot]$ denotes the correlation coefficient). For suitable comparison, Table 1 also shows the analogous results for the Poisson process $N^{(\lambda)}(t)$. We remark that the process $\mathbf{N}$ is overdispersed, and that its increments are not independent but positively correlated. Moreover, for fixed $\lambda \in \mathbb{R}^+$, the following asymptotic result holds (cf. Proposition 4.2 of Grandell, 1997):

$$\frac{N(t)}{\lambda t} \xrightarrow{d} X \qquad \text{as } t \to \infty,$$

where $X$ is exponentially distributed with mean 1, whereas for the Poisson process one has $\frac{N^{(\lambda)}(t)}{\lambda t} \xrightarrow{p} 1$ as $t \to \infty$.

Other useful properties of $\mathbf{N}$ will be recalled in the next section. Some of them are similar to those satisfied by the Poisson process, with suitable mathematical tractability. Moreover, the inter-times of $\mathbf{N}$ are distributed according to modified Pareto distributions, thus such process is appropriate to be applied in those fields where random occurrences between events have infinite expectations, and are not independent.

The purpose of this paper is oriented toward several lines. First, we aim to provide a brief survey of this particular family of processes, describing the related distributions and main properties, and by using a direct approach based on the assumption of geometric distribution for the increments. We present proofs of these properties which are alternative to those already available in the literature. The second aim is to provide a simulation procedure for the Geometric counting process. It will be used to obtain estimates of some instances of the first-passage-time densities for the process under investigation, whereas we provide the exact results in the presence of monotone nonincreasing boundaries. The third aim is to study further characteristics, including conditions for aging properties and stochastic comparisons of shock models where shocks occur according to the Geometric processes. Finally, we purpose to provide examples of applications of such processes in seismology, software reliability, and other applied fields.

The paper is organized as follows. In Section 2 we provide the joint distributions of the increments of the Geometric counting process, and discuss the relevant properties of its marginal and conditional distributions, as well as the joint distributions of arrivals and of inter-times. In Section 3 we recall simulation procedures for the Geometric counting process. Section 4 is devoted to analyze the first-passage-time problem of such process through two types of boundaries: in the case of monotone nonincreasing boundaries we provide closed-form expressions for the relevant functions, whereas in the remaining case we estimate the first-passage-time densities via a simulation-based approach. Further characteristics, comparison results and aging properties for compound Geometric processes and shock models based on such processes are discussed in Sections 5 and 6. Finally, examples of applicative fields where Geometric counting processes can be used are discussed in Section 7, also with examples of applications of the results provided in Section 6.

## 2 Background on useful distributions

Aiming to develop an approach in which the joint laws of the increments of the process $\mathbf{N}$ follow a multivariate geometric distribution, for ease of reference we recall here the definition of multivariate geometric distributions as stated in Sreehari and Vasudeva (2012).

**Definition 1** Let $m \in \mathbb{N}^+$. Given the set of parameters $\{p_1, \ldots, p_m\}$ satisfying $p_i \in \mathbb{R}^+$, $i = 1, \ldots, m$ and $\sum_{i=1}^m p_i < 1$, the integer-valued random vector $(N_1, \ldots, N_m)$ is said to be distributed according to a *Multivariate Geometric distribution* (MG distribution) with parameters $p_1, \ldots, p_m$, and we will write $(N_1, \ldots, N_m) \sim MG(p_1, \ldots, p_m)$, if, for all $k_1, \ldots, k_m \in \mathbb{N}$,

$$\mathbb{P}[(N_1, \ldots, N_m) = (k_1, \ldots, k_m)] = \binom{\sum_{i=1}^m k_i}{k_1, \ldots, k_m} \prod_{i=1}^m p_i^{k_i} \left(1 - \sum_{i=1}^m p_i\right).$$

In the case $m = 1$ we will write $N \sim G(p)$ if $N$ has geometric distribution such that $\mathbb{P}[N = k] = p(1-p)^k$ for all $k \in \mathbb{N}$.

Hereafter the following notation will be used for the sets of multidimensional vectors of increasing times:

$\mathcal{T}_{m+1} = \{(t_0, t_1, \ldots, t_m) \in \mathbb{R}^{m+1} : 0 \le t_0 < t_1 < \ldots < t_m\}$,
$\mathcal{T}^0_{m+1} = \{(t_0, t_1, \ldots, t_m) \in \mathbb{R}^{m+1} : 0 = t_0 < t_1 < \ldots < t_m\}$,
$\mathcal{T}^+_{m+1} = \{(t_0, t_1, \ldots, t_m) \in \mathbb{R}^{m+1} : 0 < t_0 < t_1 < \ldots < t_m\}$.

### 2.1 Joint distribution of the increments

Let $\mathbf{N} = \{N(t), t \ge 0\}$ be a Geometric counting process defined as mentioned in the previous section. For every interval of time $(t, t+s]$ the increments of process $\mathbf{N}$ should be geometrically distributed with parameter $\frac{1}{1+\lambda s}$. Hence, one can immediately observe that, fixed $m \in \mathbb{N}^+$ and given $(t_0, \ldots, t_m) \in \mathcal{T}_{m+1}$, it should be

$$N(t_i) - N(t_{i-1}) \sim G\left(\frac{1}{1+\lambda(t_i - t_{i-1})}\right) \quad \forall\, i = 1, 2, \ldots, m,$$

and

$$N(t_m) - N(t_0) = \sum_{i=1}^{m} \big(N(t_i) - N(t_{i-1})\big) \ \sim G\left(\frac{1}{1+\lambda(t_m - t_0)}\right).$$

In other words, it is required for the $m$-dimensional random vector

$$\big(N(t_1) - N(t_0), N(t_2) - N(t_1), \ldots, N(t_m) - N(t_{m-1})\big) \tag{2}$$

to have geometrically distributed margins. Moreover, the sum of two, or more, of its components should still be geometrically distributed, with parameter given by the sum of the corresponding parameters. Consequently, vector (2) should have the MG distribution recalled in Definition 1, thus

$$\big(N(t_1) - N(t_0), N(t_2) - N(t_1), \ldots, N(t_m) - N(t_{m-1})\big) \sim MG(p_1, p_2, \ldots, p_m)$$

where, with easy computations, one has that the parameters $p_1, p_2, \ldots, p_m$ are defined as

$$p_i = \frac{\lambda(t_i - t_{i-1})}{1+\lambda(t_m - t_0)}, \quad i = 1, 2, \ldots, m,$$

with

$$1 - \sum_{i=1}^{m} p_i = \frac{1}{1+\lambda(t_m - t_0)}.$$

Hence, the following property for processes considered in Property 1 is immediately proved.

**Proposition 1** *Given a Geometric counting process* $\mathbf{N}$ *with intensity* $\lambda \in \mathbb{R}^+$, *the joint distribution of its increments is given by*

$$\begin{aligned} p_{\mathbf{k}}(\mathbf{t}) &:= \mathbb{P}\big[\big(N(t_1) - N(t_0), N(t_2) - N(t_1), \ldots, N(t_m) - N(t_{m-1})\big) = \mathbf{k}\big] \\ &= \binom{\sum_{i=1}^{m} k_i}{k_1, k_2, \ldots, k_m} \frac{\prod_{i=1}^{m} \big(\lambda(t_i - t_{i-1})\big)^{k_i}}{[1+\lambda(t_m - t_0)]^{1+\sum_{i=1}^{m} k_i}} \end{aligned} \tag{3}$$

*for all* $\mathbf{k} = (k_1, k_2, \ldots, k_m) \in \mathbb{N}^m$ *and* $\mathbf{t} = (t_0, t_1, \ldots, t_m) \in \mathcal{T}_{m+1}$.

*Remark 1* It is not hard to see that the distribution of increments for spaced intervals of $\mathbf{N}$ is identical to that of contiguous intervals. Indeed, for instance making use of (3), for $(t_0,t_1,t_2,t_3)\in\mathcal{T}_4$ and $k_1,k_3\in\mathbb{N}$ we have

$$
\begin{aligned}
&\mathbb{P}[N(t_1)-N(t_0)=k_1, N(t_3)-N(t_2)=k_3]\\
&=\sum_{k_2=0}^{+\infty} p_{(k_1,k_2,k_3)}(t_0,t_1,t_2,t_3)\\
&=\sum_{k_2=0}^{+\infty}\binom{k_1+k_2+k_3}{k_1,k_2,k_3}\frac{(\lambda(t_1-t_0))^{k_1}(\lambda(t_2-t_1))^{k_2}(\lambda(t_3-t_2))^{k_3}}{[1+\lambda(t_3-t_0)]^{1+k_1+k_2+k_3}}\\
&=\binom{k_1+k_3}{k_1}\frac{(\lambda(t_1-t_0))^{k_1}(\lambda(t_3-t_2))^{k_3}}{[1+\lambda(t_3-t_0)]^{1+k_1+k_3}}\sum_{k_2=0}^{+\infty}\binom{k_1+k_2+k_3}{k_2}\frac{(\lambda(t_2-t_1))^{k_2}}{[1+\lambda(t_3-t_0)]^{k_2}}\\
&=\binom{k_1+k_3}{k_1}\frac{(\lambda(t_1-t_0))^{k_1}(\lambda(t_3-t_2))^{k_3}}{[1+\lambda((t_3-t_2)+(t_1-t_0))]^{1+k_1+k_3}},
\end{aligned}
$$

where use of the following binomial formula has been made:

$$
\sum_{r=0}^{+\infty}\binom{r+k}{r}x^r=\left(\frac{1}{1-x}\right)^{k+1}, \qquad |x|<1;\;\; k\in\mathbb{N}.
$$

The following general formula for the distribution of the increments of $\mathbf{N}$ for spaced intervals can be proved or by reasoning as above, or by making use of Eq. (3) in Sreehari and Vasudeva (2012). For it, let $(t_0,t_1,\ldots,t_{2m+1})\in\mathcal{T}_{2m+2}$ and $(k_1,\ldots,k_m)\in\mathbb{N}^m$. Then:

$$
\begin{aligned}
&\mathbb{P}[N(t_{2i+1})-N(t_{2i})=k_i,\; i=1,2,\ldots,m]\\
&\qquad=\binom{\sum_{i=1}^m k_i}{k_1,k_2,\ldots,k_m}\frac{\prod_{i=1}^m[\lambda(t_{2i+1}-t_{2i})]^{k_i}}{\left[1+\lambda[\sum_{i=1}^m(t_{2i}-t_{2i-1})]\right]^{1+\sum_{i=1}^m k_i}}.
\end{aligned}
\tag{4}
$$

For the joint distribution of the process at different times, from (3) it is easy to verify that given a Geometric counting process with intensity $\lambda\in\mathbb{R}^+$, for all $(t_0,t_1,\ldots,t_m)\in\mathcal{T}_{m+1}$ and all integers $0\le k_0\le k_1\le\ldots\le k_m$ we have

$$
\begin{aligned}
&\mathbb{P}\big[N(t_0)=k_0, N(t_1)=k_1,\ldots,N(t_m)=k_m\big]\\
&\qquad=\binom{k_m}{k_0,k_1-k_0,\ldots,k_m-k_{m-1}}\frac{\prod_{i=1}^m\big(\lambda(t_i-t_{i-1})\big)^{k_i-k_{i-1}}}{[1+\lambda t_m]^{1+k_m}}.
\end{aligned}
$$

For instance, as immediate consequence of the above expression one obtains the results for $N(t)$ shown in Table 1.

## 2.2 Distribution of arrivals and inter-times

Let $T_i$, $i\in\mathbb{N}^+$, denote the arrival times of a Geometric counting process $\mathbf{N}$, and let $X_i=T_i-T_{i-1}$, $i\in\mathbb{N}^+$, be the inter-times, with $T_0=0$. Explicit expressions for the joint density of the arrival times and of the corresponding inter-times are described hereafter. (See also Rolski et al., 1999, where some of the following expressions are provided.)

From Property 1 it is immediate to observe that the univariate distribution of $T_i$, $i \in \mathbb{N}^+$, is given by

$$F_{T_i}(t) := \mathbb{P}[T_i \leq t] = \mathbb{P}[N(t) \geq i] = \left(\frac{\lambda t}{1+\lambda t}\right)^i, \qquad t \in \mathbb{R}_0^+, \tag{5}$$

with probability density function

$$f_{T_i}(t) = i\left(\frac{\lambda t}{1+\lambda t}\right)^{i-1} \frac{\lambda}{(1+\lambda t)^2}, \qquad t \in \mathbb{R}_0^+. \tag{6}$$

Concerning the vector $\mathbf{T_m} = (T_1, T_2, \ldots, T_m)$, one can immediately obtain that, for $(t_1, t_2, \ldots, t_m) \in \mathcal{T}_m$, it holds

$$\begin{aligned}\overline{F}_{\mathbf{T_m}}(t_1, t_2, \ldots, t_m) &= \mathbb{P}[T_1 > t_1, T_2 > t_2, \ldots, T_m > t_m] \\ &= \mathbb{P}[N(t_1) = 0, N(t_2) \leq 1, \ldots, N(t_m) \leq m-1] \\ &= \sum_{(k_1,k_2,\ldots,k_m)\in\mathcal{A}} p_{\mathbf{k}}(0, t_1, t_2, \ldots, t_m), \end{aligned} \tag{7}$$

where probabilities $p_{\mathbf{k}}(\mathbf{t})$ are defined in (3), while $\mathcal{A}$ is the set

$$\mathcal{A} = \Big\{(k_1, k_2, \ldots, k_m) \in \mathbb{N}^m : \sum_{i=1}^{r} k_i \leq r-1, \ \forall\, r = 1, 2, \ldots, m\Big\}. \tag{8}$$

**Proposition 2** *For all* $\mathbf{t} = (t_1, \ldots, t_m) \in \mathbb{R}^m$ *such that* $0 < t_1 < \ldots < t_m$ *we have*

$$f_{\mathbf{T_m}}(\mathbf{t}) = \frac{m!\ \lambda^m}{[1+\lambda t_m]^{m+1}} \equiv \frac{m!}{(t_m)^m}\, p_m(t_m), \tag{9}$$

*whereas* $f_{\mathbf{T_m}}(\mathbf{t}) = 0$ *otherwise.*

We remark that the joint density of $\mathbf{T_m}$ has been provided in Albrecht (2006) and references therein. A different approach finalized to compute the density (9) is proposed in Appendix A, and involves the survival function (7).

Recalling that, for $x_1, \ldots, x_m \in \mathbb{R}_0^+$,

$$f_{\mathbf{X_m}}(x_1, x_2, \ldots, x_m) = f_{\mathbf{T_m}}(x_1, x_1 + x_2, \ldots, x_1 + x_2 + \ldots + x_m),$$

from (9) one immediately obtains the joint density of the vector $\mathbf{X_m} = (X_1, X_2, \ldots, X_m)$ of the inter-times of the counting process $\mathbf{N}$ as given in McFadden (1965), for $x_1, \ldots, x_m \in \mathbb{R}_0^+$:

$$f_{\mathbf{X_m}}(x_1, x_2, \ldots, x_m) = \frac{m!\ \lambda^m}{[1+\lambda\sum_{i=1}^m x_i]^{m+1}} \equiv \frac{m!}{(t_m)^m}\, p_m\Big(\sum_{i=1}^{n} x_i\Big). \tag{10}$$

Note that the function $p_m(\cdot)$ used in the last terms of (9) and (10) corresponds to the geometric probability mass introduced in point 2 of Property 1.

Making use of (10) we get the marginal density for all inter-times $X_i$,

$$f_{X_i}(x) = \frac{\lambda}{(1+\lambda x)^2}, \qquad x \in \mathbb{R}_0^+, \quad i \in \mathbb{N}. \tag{11}$$

We recall that a random variable $Y$ is said to have a Pareto (Type I) distribution with parameters $\alpha, \beta \in \mathbb{R}^+$, shortly $Y \sim Pareto(\alpha, \beta)$, if it has density

$$f_Y(y) = \frac{\beta\alpha^\beta}{y^{\beta+1}}, \qquad y \in (\alpha, \infty).$$

Hence, from density (11) it is not hard to see that $X_i$ has a modified Pareto distribution, in the sense that

$$X_i =_{\mathrm{st}} \frac{Y-1}{\lambda} \tag{12}$$

where $Y \sim Pareto(1, 1)$, and $=_{\mathrm{st}}$ means equality in law.

It is worth pointing out that, due to (11), the inter-times of the Geometric counting process $\mathbf{N}$ have non-finite expectations.

Counting processes with inter-times having Pareto distributions or, more generally, non-finite expectations, have been applied in a variety of fields of engineering and environmental sciences. For example, applications may be found in geophysics (see Benson et al., 2007), in climatology (Lavergnat and Golé, 1998), in network modeling (see, e.g., Cai and Eun, 2009, or Gordon, 1995), in modeling for internet traffic (see Clegg et al., 2010, and references therein). Thus, the Geometric counting process can be proposed as a valuable alternative to the Poisson process in disciplines where exponential distribution has been observed to be not appropriate to describe time between occurrences of random phenomena (see, e.g., Paxson and Floyd, 1995, were critics to the exponential model are raised in the field of network models). See Pradhan and Kundu (2016) for discrimination problems and Bayesian model selection criterion for the Geometric and the Poisson distribution. We also recall Kozubowski and Podgórski, K (2009), where can be found other references on Negative Binomial processes and a survey on techniques for simulation and estimation of their parameters.

### 2.3 Conditional distributions

Since the process $\mathbf{N}$ has non-independent increments, one can be interested in the relationships between each inter-time and the history of the process up to the last arrival, i.e., in the distribution of $X_m$ conditional on $X_1, X_2, \ldots, X_{m-1}$, for any $m = 2, 3, \ldots$. Obviously, the corresponding conditional density can be immediately obtained from Eq. (10) as follows:

$$f_{X_m|X_1,X_2,\ldots,X_{m-1}}(x_m|x_1, x_2, \ldots, x_{m-1}) = \frac{m\lambda(1+\lambda\sum_{i=1}^{m-1} x_i)^m}{(1+\lambda\sum_{i=1}^{m} x_i)^{m+1}}, \tag{13}$$

for $x_1, x_2, \ldots, x_m \in \mathbb{R}_0^+$. It is interesting to note that the distribution of $X_m$ conditional on $X_1, X_2, \ldots, X_{m-1}$ actually depends on the sum of the past inter-times. In fact, Mixed Poisson processes satisfy the Markov property (see, e.g., Grandell, 1997). Moreover, the instantaneous jump rate of $\mathbf{N}$ depends on time and on the number of occurred jumps, being $\mathbb{P}[N(t+h)-N(t) \mid N(t)=k]/h \to (k+1)/(\lambda+t)$ as $h \to 0^+$. Hence, recalling that $T_{m-1} = X_1 + X_2 + \ldots + X_{m-1}$, from (13) one has the conditional density:

$$f_{X_m|T_{m-1}}(x|t) = \frac{m\lambda(1+\lambda t)^m}{[1+\lambda(t+x)]^{m+1}}, \qquad x, t \in \mathbb{R}_0^+. \tag{14}$$

The corresponding conditional survival function, for $m \in \mathbb{N}^+$, is:

$$\overline{F}_{X_m|T_{m-1}}(x|t) = \left(\frac{1+\lambda t}{1+\lambda(t+x)}\right)^m, \qquad x, t \in \mathbb{R}_0^+. \tag{15}$$

From (14) and (15) we obtain the failure rate function

$$h_{X_m|T_{m-1}}(x|t) = \frac{f_{X_m|T_{m-1}}(x|t)}{\overline{F}_{X_m|T_{m-1}}(x|t)} = \frac{m\lambda}{1+\lambda(t+x)}, \qquad x, t \in \mathbb{R}_0^+, \tag{16}$$

which represents the intensity that the $m$-th inter-time of $\mathbf{N}$ has duration close to $x$ given that it is larger than $x$, and given that the $(m-1)$-th arrival occured at time $t$, for $m \in \mathbb{N}^+$.

We recall that the stochastic intensity of $\mathbf{N}$ is provided by (see, for intance, Aven and Jensen, 2013)

$$\lambda_t = \lim_{h \to 0^+} \mathbb{P}[N(t+h) - N(t) = 1 | \mathcal{F}_{t^-}], \qquad t \in \mathbb{R}_0^+,$$

where $\mathcal{F}_{t^-}$ represents the history of the process prior to time $t$. Clearly, $\lambda_t$ can be interpreted as the (conditional) expected number of increments per unit of time at time $t$ given the available information at that time. Since $\lambda_t$ can be viewed as the failure rate of $[X_{N(t^-)+1} = t - T_{N(t^-)} | T_{N(t^-)}]$, from (16) we immediately get the stochastic intensity

$$\lambda_t = \frac{[N(t^-)+1]\lambda}{1+\lambda t}, \qquad t \in \mathbb{R}_0^+, \tag{17}$$

which shows how the previous history affects the occurrence of events. An analogue expression has been obtained in Cha (2014) for Pólya processes whose mixing variable has Gamma distribution.

Another interesting result for Geometric counting processes, corresponding to a similar result for Poisson processes, is the following expression for the conditional distribution of the process $\mathbf{N}$. Indeed, making use of Eq. (3), for $0 < s < t$ and $k = 0, 1, \ldots, n$ one has (cf. Theorem 6.1 of Grandell, 1997)

$$\mathbb{P}[N(s) = k | N(t) = n] = \frac{p_{(k,n-k)}(0, s, t)}{p_{(n)}(0, t)} = \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k},$$

i.e., the process at time $s < t$, given $N(t) = n$, has binomial distribution with parameters $s/t$ and $n$.

Because of the lack of independence among increments, a different expression, with respect to the case of Poisson processes, is obtained when $s > t$. In fact, recalling (3), for the Geometric counting process $\mathbf{N}$ we have, for $k, n \in \mathbb{N}$ and $0 < t < s$,

$$\mathbb{P}[N(s) - N(t) = n | N(t) = k] = \binom{n+k}{k} \left(\frac{1+\lambda t}{1+\lambda s}\right)^{k+1} \left(\frac{\lambda(s-t)}{1+\lambda s}\right)^{n}, \quad (18)$$

which is a negative binomial distribution. Hence, the conditional mean is

$$\mathbb{E}[N(s) - N(t) | N(t) = k] = \frac{\lambda(s-t)}{1+\lambda t}\,(k+1), \qquad \text{for } k \in \mathbb{N},\ \ 0 < t < s.$$

Further generalizations of these formulas, dealing with joint distributions of the process at different times, may be given by means of Eq. (4) and Theorem 2.5 in Sreehari and Vasudeva (2012). For example, applying Eq. (4) in Sreehari and Vasudeva (2012) one can obtain, for $k, n \in \mathbb{N}$, and $s, t \in \mathbb{R}^+$, $s < t$,

$$\mathbb{P}[N(s) = n | N(t) - N(s) = k] = \binom{n+k}{k} \left(\frac{1+\lambda(t-s)}{1+\lambda t}\right)^{k+1} \left(\frac{\lambda s}{1+\lambda t}\right)^{n}.$$

## 3 Simulation of Geometric counting process

In this section we discuss a simulation procedure for the Geometric counting process.

From (14), and recalling (12), one has that the inter-times of $\mathbf{N}$ conditioned on last arrivals can be represented in terms of modified Pareto distributions as

$$[X_m | T_{m-1} = t] =_{\mathrm{st}} \frac{Y_m - (1+\lambda t)}{\lambda}, \qquad \text{with } \ Y_m \sim \mathit{Pareto}(1+\lambda t, m), \quad (19)$$

for all $m \in \mathbb{N}^+$ and $t \in \mathbb{R}_0^+$. This representation can be applied to provide a first tool for simulations of the process, sampling from random variables having Pareto distributions whose parameters are defined from previous sampling. The corresponding simulation procedure is based on the fact that a random variable $Y \sim \mathit{Pareto}(\alpha, \beta)$ is generated by $Y = F_Y^{-1}(U)$, where $U$ is uniformly distributed in $(0, 1)$, and where

$$F_Y^{-1}(u) = \alpha\,(1-u)^{-1/\beta}, \qquad 0 < u < 1 \quad (20)$$

is the quantile function of $Y$. Hence, a simulation procedure for the arrival times $T_1, T_2, \ldots, T_n$ of process $\mathbf{N}$ can be specified as follows.

**Simulation procedure for $n$ arrival times**

```
1. input(λ, n)
2. T_0 = 0
3. for m = 1 to n
4.     begin
5.     U = rand(0,1)                      ⟦simulate an uniform variate in (0,1)⟧
6.     Y = (1 + λ · T_{m−1}) · U^{−1/m}   ⟦simulate Y ∼ Pareto(1 + λT_{m−1}, m)⟧
7.     X = ((Y − 1)/λ) − T_{m−1}          ⟦simulate [X_m|T_{m−1}]⟧
8.     T_m = T_{m−1} + X
9.     end
10. output(T_1, T_2, ..., T_n)
```

Note that steps 6 and 7 of the simulation procedure are founded on Eqs. (20) and (19), respectively. Simulated sample-paths of $\mathbf{N}$, obtained by means of the above sketched procedure, are provided in Fig. 1 for different values of $\lambda$.

As for the case of Poisson counting processes, is it possible to provide a simple algorithm able to simulate Geometric counting processes by conditioning on the number of arrivals up to a fixed time $t > 0$. In fact, by making use of the joint density of the arrivals in (42), and by conditioning with respect to $N(t) = n$, it is easy to verify that for every fixed $t \in \mathbb{R}^+$ it holds

$$f_{T_1|N(t)=1}(u) = \frac{1}{t}, \qquad 0 \le u \le t,$$

or, more generally (see Theorem 6.3 of Grandell, 1997),

$$f_{(T_1,\ldots,T_n)|N(t)=n}(u_1,\ldots,u_n) = \frac{\int_t^\infty f_{(T_1,\ldots,T_n,T_{n+1})}(u_1,\ldots,u_n,v)dv}{\mathbb{P}[N(t)=n]} = \frac{n!}{t^n}$$

for $0 < u_1 < u_2 < \ldots < u_n < t$, whereas $f_{(T_1,\ldots,T_n)|N(t)=n}(u_1,\ldots,u_n) = 0$ otherwise. It means that, as for Poisson processes, the conditional distribution of the first $n$ arrivals of $\mathbf{N}$, given that $N(t) = n$, is the same of the order statistics from an $n$-sized sample of independent uniformly distributed random variables having support $[0,t]$. Thus, simulations of Geometric processes can be similarly provided sampling from a geometrically distributed variable at first, and then sampling from a set of uniformly distributed random variables.

An application of the simulation procedure will be provided hereafter.

## 4 First-crossing-time problems

### 4.1 Geometric process

In this section we analyze some first-crossing-time problems for the Geometric counting processes. Let us consider a continuous function $t \mapsto \beta_k(t)$, where

**Fig. 1** Simulated sample-paths of $N(t)$ stopped at the 21-th arrival time, for $\lambda = 0.5$, 1, 2, 4 (from bottom to top).

$\beta_k(t) \geq 0$ for all $t \in \mathbb{R}_0^+$, and $\beta_k(0) = k$, for a fixed $k \in \mathbb{N}^+$. We define the first-crossing time of $\mathbf{N}$ through the boundary $\beta_k(t)$ as

$$T_{\beta_k} = \inf\{t > 0 : N(t) \geq \beta_k(t)\}. \tag{21}$$

Similarly as in Proposition 7.1 of Di Crescenzo et al. (2015) we have the following result for the survival function of (21).

**Proposition 3** *If $\beta_k(t)$ is monotone nonincreasing in $t$, then the first-crossing of $\mathbf{N}$ through $\beta_k(t)$ is certain, and for all $t \geq 0$*

$$\mathbb{P}[T_{\beta_k} > t] = \sum_{j=0}^{\lfloor \beta_k(t)^- \rfloor} p_j(t) = 1 - \left( \frac{\lambda t}{1 + \lambda t} \right)^{\lfloor \beta_k(t)^- \rfloor + 1}, \tag{22}$$

*where $\lfloor x^- \rfloor$ denotes the largest integer smaller than $x$.*

As an immediate consequence of Proposition 3 we obtain the closed-form results concerning the first-crossing time of $\mathbf{N}$ through a constant boundary. Indeed, if $\beta_k(t) = k \in \mathbb{N}^+$, from (22) we get, for all $t \geq 0$,

$$\mathbb{P}[T_k > t] = 1 - \left( \frac{\lambda t}{1 + \lambda t} \right)^k, \qquad \gamma_k(t) = \frac{k \lambda^k t^{k-1}}{(1 + \lambda t)^{k+1}}, \tag{23}$$

where $\gamma_k(t) = -d\mathbb{P}[T_k > t]/dt$ is the first-crossing-time density. It is worth pointing out that $T_k =_{\rm st} \max\{X_1, \dots, X_k\}$, where the $X_i$'s are i.i.d. random variables having Lomax distribution, i.e. $\mathbb{P}[X_1 \leq t] = \frac{\lambda t}{1+\lambda t}$, $t \geq 0$. Moreover, note that $T_k$ possesses a power-law distribution, in the sense that $\mathbb{P}[T_k > t] \sim L(t)\, t^{-1}$, where $L(t)$ is a slowly varying function, i.e. $\lim_{t\to\infty} L(r\,t)/L(t) = 1$ for any $r > 0$, so that the moments of $T_k$ are infinite. Fig. 2 shows some plots of the functions given in (23).

Another example concerns the linear decreasing boundary $\beta_k(t) = k - t$. Some instances of the corresponding first-crossing-time survival function are provided in Fig. 3, obtained by means of Eq. (22).

Further instances of interest arise when the boundary $\beta_k(t)$ does not satisfy the assumption of Proposition 3. In this case we estimate the first-crossing-time

**Fig. 2** First-crossing-time survival functions given in (23), for constant boundary $\beta_k(t) = k$, with (a) $k = 5$ and (b) $k = 10$, for $\lambda = 1, 2, 3, 5, 10$ (from top to bottom). The corresponding densities are given respectively in (c) and (d), from bottom to top near the origin.

density via histograms obtained by means of extensive simulations performed by use of MATHEMATICA®, resorting to the procedure exploited in Section 3. As example, we first consider the case of increasing boundary $\beta_k(t) = \log(t+1)+2$. (Here and in the remainder of the paper, 'log' means natural logarithm.) In this case the histograms exhibit changes of shapes for $t = \exp(k-2)-1$, with $k = 3, 4, \ldots$, i.e., when the boundary takes integer values (see Fig. 4). Another example deals with the periodic boundary $\beta_k(t) = \log(t+1)+2$, where the shape of the histograms reflects the periodicity of the boundary (cf. Fig. 5).

In all cases the obtained functions possess long tails, this being in agreement with the nature of the inter-times of the Geometric counting process.

For brevity, we limit ourselves to mention that the first-passage time of $\mathbf{N}$ through a linear increasing boundary can be studied by means of a renewal-based iterative procedure, similarly as shown in Section 7.1 of Di Crescenzo et al. (2015) for the iterated Poisson process.

**Fig. 3** First-crossing-time survival functions (22) for the linear boundary $\beta_k(t) = k - t$, with (a) $k = 5$ and (b) $k = 10$, for $\lambda = 1, 2, 3, 5, 10$ (from top to bottom).

**Fig. 4** Histogram estimating the first-crossing time density through the boundary $\beta_k(t) = \log(t+1) + 2$, for (a) $\lambda = 1$ and (b) $\lambda = 2$, obtained by $10^5$ simulated sample paths of $\mathbf{N}$. The sample mean and sample deviation standard are (a) $\overline{x} = 156.9$, $s = 16\,383.7$ and (b) $\overline{x} = 51.6$, $s = 3\,037.7$.

**Fig. 5** As Fig. 4, for the boundary $\beta_k(t) = 2\sin(\pi t/5) + 7$. The sample mean and sample deviation standard are (a) $\overline{x} = 60.3$, $s = 1\,686.6$ and (b) $\overline{x} = 31.1$, $s = 952.3$.

## 4.2 Compound Geometric process

First-crossing-time problems are of interest also for suitable extensions such as the compound Geometric counting process, defined as

$$Z(t) = \sum_{n=1}^{N(t)} W_n, \qquad t \in \mathbb{R}_0^+,$$

where $N(t)$ is the Geometric counting process with intensity $\lambda$, and where $\{W_n\}_{n\in\mathbb{N}^+}$ is assumed to be a sequence of independent absolutely continuous random variables with support $\mathbb{R}^+$. Consider the first-crossing time of $Z(t)$ through a constant level $k$, namely

$$T_k^Z = \inf\{t > 0 : Z(t) \geq k\}, \qquad k \in \mathbb{N}^+.$$

Since $Z(t)$ has increasing trajectories, and noting that the distribution $Z(t)$ has an atom at 0 and an absolutely continuous component over $\mathbb{R}^+$, for $k \in \mathbb{N}^+$ we have

$$\mathbb{P}[T_k^Z > t] = \int_{[0,k)} \mathrm{d}\mathbb{P}[Z(t) \in \mathrm{d}x] = \sum_{n=0}^{\infty} p_n(t) F_W^{(n)}(k), \qquad t \in \mathbb{R}_0^+, \tag{24}$$

where $p_n(t)$ is the distribution of $N(t) \sim G((1+\lambda t)^{-1})$, with $F_W^{(n)}(k) = \mathbb{P}[W_1 + \ldots + W_n \leq k]$, for $n \in \mathbb{N}^+$, and $F_W^{(0)}(k) = 1$. Hereafter we obtain closed form expressions of the first-crossing-time survival function (24).

*Example 1* Let $W_n$ be exponentially distributed with hazard rate $\nu_n$, $n \in \mathbb{N}^+$.

(a) If $\nu_n = 1$, $n \in \mathbb{N}^+$, then $F_W^{(n)}$ is an Erlang cumulative distribution function, i.e. $F_W^{(n)}(k) = 1 - \mathrm{e}^{-k} \sum_{i=0}^{n-1} k^i/i!$. Hence, from (24) we obtain

$$\mathbb{P}[T_k^Z > t] = 1 - \frac{\lambda t}{1+\lambda t} \exp\left\{-\frac{k}{1+\lambda t}\right\}, \qquad t \in \mathbb{R}_0^+.$$

(b) If $\nu_n = n$, $n \in \mathbb{N}^+$, then $F_W^{(n)}$ follows a generalized exponential distribution, i.e. $F_W^{(n)}(k) = (1 - \mathrm{e}^{-k})^n$. In this case, due to (24) one has

$$\mathbb{P}[T_k^Z > t] = \frac{\mathrm{e}^k}{\mathrm{e}^k + \lambda t}, \qquad t \in \mathbb{R}_0^+.$$

Some plots of the survival function of $T_k^Z$ are shown in Figure 6. In both cases the crossing occurs a.s., and $T_k^Z$ possesses an heavy-tailed distribution, with $\mathbb{E}[T_k^Z] = +\infty$. Clearly, in case (b) the survival function exhibits a heavier tail since the summands $W_n$ are stochastically smaller and smaller as $n$ grows.

## 5 Further Properties

In this section we point out further properties of Geometric counting processes, which are of general interest in applied fields like reliability or actuarial theory. Some of them will be applied in the sequel. In the following, given a function $g(\cdot)$ defined on $\mathbb{N}$, we set $\Delta g(n) := g(n+1) - g(n)$ for $n \in \mathbb{N}$.

The first property deals with independent Geometric counting processes with different intensities.

**Fig. 6** The survival function of the first-crossing times analyzed in the two cases of Example 1, for $\lambda = 0.5,\ 1,\ 2,\ 5$ (from top to bottom), with $k = 5$.

**Proposition 4** *Let $\{g(n);\ n \in \mathbb{N}\}$ be any sequence of real numbers. Let $N_\lambda(t)$ and $N_\mu(t)$ be two independent Geometric counting processes with intensities $\lambda$ and $\mu$, respectively. Then, if $\lambda, \mu \in \mathbb{R}^+$, with $\lambda < \mu$, then for any fixed $t \in \mathbb{R}^+$ the following property holds:*

$$\mathbb{E}[g(N_\mu(t))] - \mathbb{E}[g(N_\lambda(t))] = \mathbb{E}[\Delta g(Z_{N_\lambda(t),N_\mu(t)})]\,(\mu - \lambda)t, \tag{25}$$

*where $Z_{N_\lambda(t),N_\mu(t)}$ has the following probability distribution, for all $n \in \mathbb{N}$,*

$$\begin{aligned}\mathbb{P}[Z_{N_\lambda(t),N_\mu(t)} = n] &:= \frac{\mathbb{P}[N_\mu(t) > n] - \mathbb{P}[N_\lambda(t) > n]}{(\mu - \lambda)t} \\ &= \frac{1}{(\mu - \lambda)t}\left[\left(\frac{\mu t}{1 + \mu t}\right)^{n+1} - \left(\frac{\lambda t}{1 + \lambda t}\right)^{n+1}\right].\end{aligned} \tag{26}$$

*Proof* We recall a result given in Section 7 of Di Crescenzo (1999). Let $X$ and $Y$ be non-negative integer-valued random variables satisfying $\mathbb{P}[X \geq n] \leq \mathbb{P}[Y \geq n]$ for all $n \in \mathbb{N}$ and $\mathbb{E}(Y) < \infty$, and let $Z = Z_{X,Y}$ be a non-negative integer-valued random variable having probability mass function

$$\mathbb{P}[Z = n] = \frac{\mathbb{P}[Y > n] - \mathbb{P}[X > n]}{\mathbb{E}(Y) - \mathbb{E}(X)}, \qquad n \in \mathbb{N}.$$

If $\mathbb{E}[g(X)]$ and $\mathbb{E}[g(Y)]$ are finite, then $\mathbb{E}[\Delta g(Z)]$ is finite and

$$\mathbb{E}[g(Y)] - \mathbb{E}[g(X)] = \mathbb{E}[\Delta g(Z)]\,[\mathbb{E}(Y) - \mathbb{E}(X)].$$

Using the above result for the Geometric counting processes, with $X = N_\lambda(t)$, $Y = N_\mu(t)$ and $Z = Z_{N_\lambda(t),N_\mu(t)}$, $t \in \mathbb{R}^+$, the thesis thus follows.

From (26) is not hard to see that $Z_{N_\lambda(t),N_\mu(t)}$ has the same distribution of $N_\lambda(t) + N_\mu(t)$.

A result similar to Proposition 4 can be proved by considering two independent Geometric counting processes, with the same intensities, evaluated at different times.

We point out that the property stated in Proposition 4 does not hold for the Poisson process.

Let us now deal with the notions of stochastic monotonicity and stochastic convexity, whose definitions are recalled here (see Shaked and Shanthikumar, 1988, or Ch. 8 of Shaked and Shanthikumar, 2007, for further details, examples and applications of these notions).

**Definition 2** Let $\{X(\theta), \theta \in \Theta\}$ be a family of random variables, where $\Theta$ is an ordered set. The family is said to be

– stochastically increasing, denoted $\{X(\theta), \theta \in \Theta\} \in \mathrm{SI}$, if $\mathbb{E}[\phi(X(\theta))]$ is increasing in $\theta$ for all increasing functions $\phi$;
– stochastically convex, denoted $\{X(\theta), \theta \in \Theta\} \in \mathrm{SCX}$, if $\mathbb{E}[\phi(X(\theta))]$ is convex in $\theta$ for all convex functions $\phi$;
– stochastically increasing and convex, denoted $\{X(\theta), \theta \in \Theta\} \in \mathrm{SICX}$, if $\{X(\theta), \theta \in \Theta\} \in \mathrm{SI}$ and $\mathbb{E}[\phi(X(\theta))]$ is increasing convex in $\theta$ for all increasing convex functions $\phi$.

Hereafter we show that Geometric counting processes satisfy the above recalled notions. To this purpose, here and in the following we denote by $N_\lambda(t)$ the process at fixed time $t$, when it is appropriate to emphasize the dependence on parameter $\lambda \in \mathbb{R}^+$.

**Proposition 5** *For any fixed $t \in \mathbb{R}^+$ the Geometric counting process $N_\lambda(t)$ satisfies the following properties:*
*(a)* $\{N_\lambda(t), \lambda \in \mathbb{R}^+\} \in \mathrm{SI}$*;*
*(b)* $\{N_\lambda(t), \lambda \in \mathbb{R}^+\} \in \mathrm{SCX}$*;*
*(c)* $\{N_\lambda(t), \lambda \in \mathbb{R}^+\} \in \mathrm{SICX}$.

*Proof* We recall that for any fixed $t \in \mathbb{R}^+$ one has $N_\lambda(t) \sim G\left(\frac{1}{1+\lambda t}\right)$, this easily implying that $N_\lambda(t)$ is increasing in $\lambda$ in the usual stochastic order (see Definition 3(ii) below), so that statement (a) holds.

The proof of (b) can be obtained from Proposition 4, by assuming that the function $g(\cdot)$ in (25) is convex, and thus $\Delta g(\cdot)$ is increasing. It thus follows that the mean $\mathbb{E}[\Delta g(Z_{N_\lambda(t),N_\mu(t)})]$ is increasing in $\lambda$ and $\mu$. Hence, due to (25) we have that $\mathbb{E}[g(N_\lambda(t))]$ is convex in $\lambda \in \mathbb{R}^+$ for all convex functions $g$ and for any fixed $t \in \mathbb{R}^+$. This shows that $\{N_\lambda(t), \lambda \in \mathbb{R}^+\}$ is stochastically convex in $\lambda \in \mathbb{R}^+$ for any fixed $t \in \mathbb{R}^+$.

From point 2 of Property 1, for any fixed $t \in \mathbb{R}^+$ and for all $k \in \mathbb{N}$ one has $\sum_{\ell=k}^{\infty} \mathbb{P}[N_\lambda(t) \geq \ell] = (\lambda t)^k (1+\lambda t)^{1-k}$, this being an increasing convex function in $\lambda \in \mathbb{R}^+$. The proof of the statement (c) thus follows from Theorem 8.A.10(a) of Shaked and Shanthikumar (2007).

Various applications of Proposition 5 will be given in Section 6.

The next property of Geometric counting processes concerns the total positivity of its distribution and of its integral over $(0, t)$. Recall that a nonnegative measurable function $h(x,y)$ is said to be Totally Positive of order 2 (shortly, $TP_2$) in $(x,y) \in \mathcal{X} \times \mathcal{Y}$, with $\mathcal{X} \subseteq \mathbb{R}$ and $\mathcal{Y} \subseteq \mathbb{R}$, if

$$\begin{vmatrix} h(x_1,y_1) & h(x_1,y_2) \\ h(x_2,y_1) & h(x_2,y_2) \end{vmatrix} \geq 0 \qquad \text{for every } x_1 \leq x_2 \ \text{ and } y_1 \leq y_2.$$

See, e.g., Joag-Dev et al. (1995) for further details. For the sake of simplicity, we remind here the *basic composition property* for $TP_2$ functions (Karlin, 1968), which asserts that the bivariate measurable function

$$h(x,y) = \int_{\Theta} \phi(x,\theta)\psi(\theta,y)\mathrm{d}\theta$$

satisfies $TP_2$ if both $\phi$ and $\psi$ are $TP_2$ (in $(x,\theta)$ and $(\theta,y)$, respectively).

The following result will be used in the proof of Theorem 5 to obtain various stochastic comparisons.

**Proposition 6** *For a Geometric counting process* $\mathbf{N}$*, both functions* $p_k(t) = \mathbb{P}[N(t)=k]$ *and* $\int_0^t p_k(s)\,\mathrm{d}s = \int_0^t \mathbb{P}[N(s)=k]\,\mathrm{d}s$ *are* $TP_2$ *in* $(t,k) \in \mathbb{R}_0^+ \times \mathbb{N}^+$.

*Proof* Let $0 \leq s \leq t$ and $0 \leq k_1 \leq k_2$. It is easy to see that

$$\begin{vmatrix} p_{k_1}(s) & p_{k_2}(s) \\ p_{k_1}(t) & p_{k_2}(t) \end{vmatrix} = \Big(\frac{\lambda t}{1+\lambda t}\Big)^{k_1}\Big(\frac{\lambda s}{1+\lambda s}\Big)^{k_1}\Big[\Big(\frac{\lambda t}{1+\lambda t}\Big)^{k_2-k_1} - \Big(\frac{\lambda s}{1+\lambda s}\Big)^{k_2-k_1}\Big] \geq 0$$

where the inequality is due to monotonicity of $x/(1+x)$ in $x \geq 0$. Hence, $p_k(t)$ is $TP_2$ in $(t,k)$. Concerning the second assertion, it follows from the basic composition property of $TP_2$ functions, just observing that

$$\int_0^t p_k(s)\,\mathrm{d}s = \int_0^\infty \mathbf{1}_{[0,t]}(s) p_k(s)\,\mathrm{d}s,$$

where $\mathbf{1}_{[0,t]}(s) = 1$ if $s \in [0,t]$ and it vanishes otherwise, so that it is $TP_2$ in $(s,t)$.

*Remark 2* From point 2 of Property 1, with a direct calculation it can be verified that for any $k \in \mathbb{N}$ one has:

$$\int_0^t p_k(s)\,\mathrm{d}s = t\,\frac{(\lambda t)^k}{k+1}\ {}_2F_1(k+1,k+1;k+2;-\lambda t), \qquad t \in \mathbb{R}_0^+,$$

where

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n}\,\frac{z^n}{n!}$$

is the Gauss hypergeometric Function, and $(a)_n$ is the Pochhammer symbol.

## 6 Comparison results and aging properties

In many applied probability contexts, such as reliability or actuarial theory, counting processes are often considered to model occurrences of shocks or claims. In this section, we provide some applications in these fields of the properties listed in previous sections, dealing with comparisons of random quantities and lifetimes modeled through Geometric counting processes. To this aim, hereafter we recall various useful definitions of aging properties, stochastic orders and related notions (see, e.g. Shaked and Shanthikumar, 2007). Note that prime ($'$) means derivative, and the terms decreasing and increasing are used in non-strict sense.

**Definition 3** (i) Let $X$ be an absolutely continuous random variable with support $\mathbb{R}^+$, having differentiable probability density function $f(x)$, cumulative distribution function $F(x) = \mathbb{P}(X \le x)$, survival function $\overline{F}(x) = 1-F(x)$, and failure rate function $h_X(x) = f(x)/\overline{F}(x)$. We say that $X$ is

- increasing (decreasing) likelihood ratio, in short ILR (DLR), if $f(x)$ is log-concave (log-convex) or, equivalently, if $f'(x)/f(x)$ is decreasing (increasing) in $x \in \mathbb{R}^+$;
- increasing (decreasing) failure rate, in short IFR (DFR), if $\overline{F}(x)$ is log-concave (log-convex) or, equivalently, if $h_X(x)$ is increasing (decreasing) in $x \in \mathbb{R}^+$;
- increasing (decreasing) failure rate in average, in short IFRA (DFRA), if $-\frac{1}{x}\log \overline{F}(x)$ is increasing (decreasing) in $x \in \mathbb{R}^+$;
- new (worst) better than used, in short NBU (NWU), if $\overline{F}(x+t) \le (\ge)$ $\overline{F}(x)\overline{F}(t)$ for all $x, t \in \mathbb{R}^+$.

(ii) Moreover, if $Y$ is an absolutely continuous random variable with support $\mathbb{R}^+$, having differentiable probability density function $g(x)$, cumulative distribution function $G(x)$, survival function $\overline{G}(x)$, hazard rate function $h_Y(x) = g(x)/\overline{G}(x)$, and reversed hazard rate function $r_Y(x) = g(x)/G(x)$, then we say that $X$ is smaller than $Y$

- in the likelihood ratio order, denoted by $X \le_{\mathrm{lr}} Y$, if $f(x)g(y) \ge f(y)g(x)$ for all $x < y$, with $x, y \in \mathbb{R}^+$;
- in the hazard rate order, denoted by $X \le_{\mathrm{hr}} Y$, if $\overline{G}(x)/\overline{F}(x)$ is increasing in $x \in \mathbb{R}^+$, or, equivalently, if $h_X(x) \ge h_Y(x)$ for all $x \in \mathbb{R}^+$;
- in the reversed hazard rate order, denoted by $X \le_{\mathrm{rh}} Y$, if $G(x)/F(x)$ is increasing in $x \in \mathbb{R}^+$, or, equivalently, if $r_X(x) \le r_Y(x)$ for all $x \in \mathbb{R}^+$;
- in the usual stochastic order, denoted by $X \le_{\mathrm{st}} Y$, if $\overline{F}(x) \le \overline{G}(x)\ \forall\, x \in \mathbb{R}^+$ or, equivalently, if $\mathbb{E}[\phi(X)] \ge \mathbb{E}[\phi(Y)]$ for all increasing functions $\phi$ for which the expectations exist;
- in the increasing convex order, denoted by $X \le_{\mathrm{icx}} Y$, if $\int_x^\infty \overline{F}(y)\mathrm{d}y \le \int_x^\infty \overline{G}(y)\mathrm{d}y\ \forall\, x \in \mathbb{R}^+$ or, equivalently, if $\mathbb{E}[\phi(X)] \ge \mathbb{E}[\phi(Y)]$ for all increasing and convex functions $\phi$ for which the expectations exist;
- in the increasing concave order, denoted by $X \le_{\mathrm{icx}} Y$, if $\int_0^x \overline{F}(y)\mathrm{d}y \le \int_0^x \overline{G}(y)\mathrm{d}y\ \forall\, x \in \mathbb{R}^+$ or, equivalently, if $\mathbb{E}[\phi(X)] \ge \mathbb{E}[\phi(Y)]$ for all increasing and concave functions $\phi$ for which the expectations exist;

– in the mean inactivity time order, denoted by $X \leq_{\rm mit} Y$, if $\int_0^x F(y){\rm d}y / \int_0^x G(y){\rm d}y$ is increasing in $x \in \mathbb{R}^+$, or, equivalently, if $\mathbb{E}[x - X|X \leq x] \geq \mathbb{E}[x - Y|Y \leq x]$ for all $x \in \mathbb{R}^+$.

(iii) Let $(X_1, X_2)$ and $(Y_1, Y_2)$ be random vectors with joint distribution functions $F$ and $G$, respectively, and suppose that $F$ and $G$ have the same univariate marginals. If $F(x_1, x_2) \leq G(x_1, x_2)$ for all $x_1, x_2 \in \mathbb{R}$, then we say that $(X_1, X_2)$ is smaller than $(Y_1, Y_2)$ in the PQD (positive quadrant dependent) order, denoted by $(X_1, X_2) \leq_{\rm PQD} (Y_1, Y_2)$.

The notions in point (i) of Definition 3 are listed from the stronger to the weaker. Moreover, for the stochastic orders given in (ii) similar definition can be provided in the case of integer-valued variables $X$ and $Y$. We also recall that among these orders the following implications hold:

$$X \leq_{\rm lr} Y \quad \Rightarrow \quad X \leq_{\rm hr} Y \quad \Rightarrow \quad X \leq_{\rm st} Y \quad \Rightarrow \quad X \leq_{\rm icx} Y,$$

$$X \leq_{\rm lr} Y \quad \Rightarrow \quad X \leq_{\rm rh} Y \quad \Rightarrow \quad X \leq_{\rm icv} Y \quad \Rightarrow \quad X \leq_{\rm mit} Y.$$

Also note that the likelihood ratio order is one of the strongest stochastic orders considered in the literature to compare non negative random variables (see, e.g., Shaked and Shanthikumar, 2007, for details, properties and applications of stochastic orders in general).

The first application of the results stated previously and pointed out in this section follow from (14) and (15), and deals with comparisons among inter-times of the Geometric counting process. Recalling the failure rate (16) and the stochastic intensity (17) we can easily see that $[X_m|T_{m-1} = t]$ is increasing in $t$ and is decreasing in $m$ according to the hazard rate order. However, hereafter we shall prove that such monotonicity properties hold even for the stronger likelihood ratio order.

**Proposition 7** *For all $m = 2, 3, \ldots$ the $m$-th inter-time of process* $\mathbf{N}$ *is increasing in the last arrival according to the likelihood ratio order, i.e.,*

$$[X_m|T_{m-1} = t_1] \leq_{\rm lr} [X_m|T_{m-1} = t_2] \qquad \textit{for} \;\; 0 \leq t_1 \leq t_2.$$

*Proof* From (14) we have that the ratio $f_{X_m|T_{m-1}}(x|t_1)/f_{X_m|T_{m-1}}(x|t_2)$ is decreasing in $x \in \mathbb{R}_0^+$, for all $t_1 \leq t_2$. The proof thus follows from the definition of the likelihood ratio order.

We remark that, since the likelihood ratio order implies the usual stochastic order, as a corollary of Proposition 7 one can obtain Theorem 1 of Cha and Finkelstein (2013).

Let us now consider two further comparison results involving the likelihood ratio order. The proofs are similar to that of Proposition 7, thus are omitted.

**Proposition 8** *The conditional inter-times $X_m$ of process* $\mathbf{N}$ *are decreasing in $m$ according to the likelihood ratio order, i.e., for fixed $t \in \mathbb{R}_0^+$,*

$$[X_m|T_{m-1} = t] \geq_{\rm lr} [X_{m+1}|T_m = t] \qquad \textit{for} \;\; m = 2, 3, \ldots. \tag{27}$$

It should be pointed out that Proposition 8 strengthens the well known fact that Pólya-Lundberg processes satisfy the positive contagion property, i.e., that the stochastic inequality (27) holds for the usual stochastic order $\geq_{\rm st}$ (see, e.g., Rolski et al., 1999).

In the following proposition we write $X_m^{(\lambda)}$ to emphasize the dependence on parameter $\lambda$.

**Proposition 9** *For all $m = 2, 3, \ldots$ the $m$-th inter-time of process $\mathbf{N}$ is decreasing in $\lambda \in \mathbb{R}^+$ according to the likelihood ratio order, i.e., for fixed $t \in \mathbb{R}_0^+$,*

$$[X_m^{(\lambda_1)} | T_{m-1} = t] \geq_{\rm lr} [X_m^{(\lambda_2)} | T_{m-1} = t] \qquad \text{for} \ \ 0 < \lambda_1 \leq \lambda_2.$$

*Remark 3* We remark that the density of arrival times $T_i$ in (6) is $\mathrm{TP}_2$ in $(i, t) \in \mathbb{N}^+ \times \mathbb{R}_0^+$. The proof is similar to that of Proposition 6. Moreover, for $i = 1$ such density is DLR, i.e., it satisfies negative aging, whereas for $i \geq 2$ it is not ILR neither DLR, having reversed bathtube failure rate. Furthermore, for all $i \in \mathbb{N}^+$ the density $f_{T_i}(t)$ is DRFR, as one can easily verify.

Hereafter, we provide some applications of Proposition 5 of interest in insurance contexts, where total claim amounts are considered, or in reliability, where cumulative damage shock models are used to describe accumulated wear along time. To this aim, given a family of random variables $\{W_j, j \in \mathbb{N}^+\}$, for any fixed $t \in \mathbb{R}^+$ let us consider the compound sum

$$S_\lambda(t) = \sum_{j=1}^{N_\lambda(t)} W_j, \qquad \lambda \in \mathbb{R}^+. \tag{28}$$

Moreover, for fixed $t \in \mathbb{R}^+$ we denote by $S_\Lambda(t)$ the mixture of $S_\lambda(t)$ with respect to a given random variable $\Lambda$ taking values in $\mathbb{R}^+$, so that $\mathbb{P}[S_\Lambda(t) \in B] = \int_{\mathbb{R}^+} \mathbb{P}[S_\lambda(t) \in B] \, \mathrm{d}F_\Lambda(\lambda)$ for any Borel set $B$. The first immediate application of Proposition 5 provides simple comparison criteria among random sums defined as in (28) when the corresponding mixing random parameters are stochastically ordered.

**Theorem 1** *Let $\{W_j, j \in \mathbb{N}^+\}$ be a family of i.i.d. random variables that are independent from $N_\lambda(t)$, for any fixed $t \in \mathbb{R}^+$ and $\lambda \in \mathbb{R}^+$. Given two random variables $\Lambda_1$ and $\Lambda_2$ both taking values in $\mathbb{R}^+$, one has:*

$$\Lambda_1 \leq_{\rm st} [\leq_{\rm cx}, \leq_{\rm icx}] \ \Lambda_2 \qquad \Rightarrow \qquad S_{\Lambda_1}(t) \leq_{\rm st} [\leq_{\rm cx}, \leq_{\rm icx}] \ S_{\Lambda_2}(t) \quad \forall \ t \in \mathbb{R}^+.$$

*Proof* From Proposition 5 and use of Theorems 1.A.6, 3.A.21 and 4.A.18 in Shaked and Shanthikumar (2007) one has $N_{\Lambda_1}(t) \leq_{\rm st} [\leq_{\rm cx}, \leq_{\rm icx}] \ N_{\Lambda_2}(t)$. The proof of the assertion immediately follows recalling the closure property under random sums for these stochastic orders (see Theorems 1.A.4, 3.A.13 and 4.A.9 in Shaked and Shanthikumar, 2007, respectively).

An example of application of Theorem 1 is provided in Section 7.4 below.

The previous statement can be generalized by introducing dependence among the counting process and the summands. In this case, given now a family of random variables $\{W_{\lambda,j}, j \in \mathbb{N}^+\}$, for any fixed $t \in \mathbb{R}^+$ let us define the compound sum

$$Z_\lambda(t) = \sum_{j=1}^{N_\lambda(t)} W_{\lambda,j}, \qquad \lambda \in \mathbb{R}^+. \tag{29}$$

Moreover, for fixed $t \in \mathbb{R}^+$ we denote by $Z_\Lambda(t)$ the mixture of $Z_\lambda(t)$ with respect to a given random variable $\Lambda$ taking values in $\mathbb{R}^+$. Note that for the random sum $Z_\Lambda(t)$, defined as in (29), it is not assumed independence between number of summands and summands, because of the common random parameter $\Lambda$. Thus these sums can be used to model, for example, total claim amounts where both number of claims and their values can depend on a common random environment.

**Theorem 2** *For all $\lambda \in \mathbb{R}^+$, let $\{W_{\lambda,j}, j \in \mathbb{N}^+\}$ be a family of i.i.d. random variables that are independent from $N_\lambda(t)$, for any fixed $t \in \mathbb{R}^+$. Moreover, assume that for all $j \in \mathbb{N}^+$ the family $\{W_{\lambda,j}, \lambda \in \mathbb{R}^+\}$ is increasing in $\lambda \in \mathbb{R}^+$ in the usual stochastic order. Given two random variables $\Lambda_1$ and $\Lambda_2$ both taking values in $\mathbb{R}^+$, one has:*

$$\Lambda_1 \leq_{\mathrm{st}} \Lambda_2 \qquad \Rightarrow \qquad Z_{\Lambda_1}(t) \leq_{\mathrm{st}} Z_{\Lambda_2}(t) \quad \forall\, t \in \mathbb{R}^+.$$

*Proof* Due to Proposition 5 we have that $N_\lambda(t)$ is increasing in $\lambda$ in the usual stochastic order, for any fixed $t \in \mathbb{R}^+$. Moreover, by assumptions $\{W_{\lambda,j}, \lambda \in \mathbb{R}^+\}$ is increasing in $\lambda \in \mathbb{R}^+$ in the usual stochastic order for all $j \in \mathbb{N}^+$. Hence, recalling (29), Theorem 1.A.4 of Shaked and Shanthikumar (2007) implies that, for any fixed $t \in \mathbb{R}^+$, $Z_\lambda(t)$ is increasing in $\lambda \in \mathbb{R}^+$ in the usual stochastic order, too. The thesis then follows from Theorem 1.A.6 of Shaked and Shanthikumar (2007).

To provide a result similar to Theorem 2 concerning comparisons in the increasing convex, let us now recall the statement of Corollary 3.3 of Fernández-Ponce et al. (2008) for the case of one-dimensional form of parameters.

**Lemma 1** *Consider the family of random variables defined by the compound sum $Z_\lambda = \sum_{j=1}^{N_\lambda} W_{\lambda,j}$, $\lambda \in \mathbb{R}^+$, and let the following assumptions hold:*
*i) for all $\lambda \in \mathbb{R}^+$, the sequence $\{W_{\lambda,j}, j \in \mathbb{N}\}$ is formed by independent random variables, that are independent from the random variable $N_\lambda$;*
*ii) $\{W_{\lambda,j}, \lambda \in \mathbb{R}^+\} \in \mathrm{SICX}$, for all $j \in \mathbb{N}^+$;*
*iii) $\{N_\lambda, \lambda \in \mathbb{R}^+\} \in \mathrm{SICX}$;*
*iv) the sequence $\{W_{\lambda,j}, j \in \mathbb{N}\}$ is increasing in the usual stochastic order, i.e. $W_{\lambda,j} \leq W_{\lambda,k}$ for all $j, k \in \mathbb{N}$ with $j \leq k$, for any $\lambda \in \mathbb{R}^+$.*

*Then, given two random variables $\Lambda_1$ and $\Lambda_2$ both taking values in $\mathbb{R}^+$, one has:*

$$\Lambda_1 \leq_{\mathrm{icx}} \Lambda_2 \qquad \Rightarrow \qquad Z_{\Lambda_1} \leq_{\mathrm{icx}} Z_{\Lambda_2},$$

*where $Z_{\Lambda_i}$ denotes the mixture of $Z_\lambda$ with respect to $\Lambda_i$, for $i = 1, 2$.*

By applying Lemma 1, the following statement easily follows from Proposition 5. We remark that various examples of families of random variables possessing the SICX property are provided in Chapter 8 of Shaked and Shanthikumar (2007).

**Theorem 3** *Under the same assumptions of Theorem 2, let* $\{W_{\lambda,j}, \lambda \in \mathbb{R}^+\} \in$ SICX *for all* $j \in \mathbb{N}^+$*. Then, given two random variables* $\Lambda_1$ *and* $\Lambda_2$ *both taking values in* $\mathbb{R}^+$*, one has:*

$$\Lambda_1 \leq_{\text{icx}} \Lambda_2 \qquad \Rightarrow \qquad Z_{\Lambda_1}(t) \leq_{\text{icx}} Z_{\Lambda_2}(t) \quad \forall\, t \in \mathbb{R}^+.$$

*Proof* Under the given assumptions, we have that the hypothesis i), ii) and iv) of Lemma 1 are satisfied. Moreover, due to point (c) of Proposition 5 we have that $\{N_\lambda(t), \lambda \in \mathbb{R}^+\} \in$ SICX for any fixed $t \in \mathbb{R}^+$, and thus the assumption iii) of Lemma 1 holds. Hence, the thesis follows from Lemma 1.

The comparison results shown in Theorems 2 and 3 are concerning compound sums of the form (29), where both the random summands and the random number of terms depend on the same parameter $\lambda \in \mathbb{R}^+$. Hereafter we deal with the case in which the parameters of the considered variates are different, but not independent. Specifically, for any fixed $t \in \mathbb{R}^+$, let us now consider the compound sum

$$Z_{(\lambda,\theta)}(t) = \sum_{j=1}^{N_\lambda(t)} W_{\theta,j}, \qquad \lambda, \theta \in \mathbb{R}^+,$$

where, for all $\theta \in \mathbb{R}^+$, $\{W_{\theta,j}, j \in \mathbb{N}^+\}$ is a family of random variables that does not depend on $\lambda$. The following statement shows that, whenever the parameters $\lambda$ and $\theta$ describing the environmental conditions are random, then the compound sum $Z$ increases in the increasing convex order as the positive dependence among the two parameters increases in terms of the PQD order.

**Theorem 4** *For all* $j \in \mathbb{N}^+$*, let* $\{W_{\theta,j}, \theta \in \mathbb{R}^+\} \in$ SI*. Then, given two bivariate random vectors* $(\Lambda_1, \Theta_1)$ *and* $(\Lambda_2, \Theta_2)$*, both taking values in* $(\mathbb{R}^+)^2$*, one has:*

$$(\Lambda_1, \Theta_1) \leq_{\text{PQD}} (\Lambda_2, \Theta_2) \qquad \Rightarrow \qquad Z_{(\Lambda_1,\Theta_1)}(t) \leq_{\text{icx}} Z_{(\Lambda_2,\Theta_2)}(t) \quad \forall\, t \in \mathbb{R}^+,$$

*where, for fixed* $t \in \mathbb{R}^+$ *we denote by* $Z_{(\Lambda_i,\Theta_i)}(t)$ *the mixture of* $Z_{(\lambda,\theta)}(t)$ *with respect to* $(\Lambda_i, \Theta_i)$*, for* $i = 1, 2$*.*

*Proof* We recall that, due to point (a) of Proposition 5, $\{N_\lambda(t), \lambda \in \mathbb{R}^+\} \in$ SI for any fixed $t \in \mathbb{R}^+$. Hence, the proof follows from Theorem 2.1 of Belzunce *et al.* (2006).

An example of application of Theorem 4 will be provided in Section 7.4 below.

Comparison results similar to those described above and based on Proposition 5 can be proved for processes of interest in other applicative fields, such as in population dynamics. Consider, for example, a continuous time branching process $\{S_\lambda(t), t \in \mathbb{R}^+\}$ defined through the composition between a discrete time Galton-Watson branching process $\{B(n), n \in \mathbb{N}\}$, describing the number of individuals in a population along generations, and a Geometric counting process $\{N_\lambda(t), t \in \mathbb{R}^+\}$ describing the sequence of random times where new generations occur. Let $D$ denote an integer-valued random variable having the same distribution as the number of offsprings of an ancestor. It has been proved (see Theorem 8.B.17 in Shaked and Shanthikumar, 2007) that $\{B(n), n \in \mathbb{N}\}$ satisfies the SICX property in $n$ if $B(0) \geq 1$ a.s., $D \geq 1$ a.s. and $\mathbb{P}[D > 1] > 0$. Moreover, under assumption of independence, the composition of two SICX parametrized families mantains the SICX property (see Theorem 8.A.17 in Shaked and Shanthikumar, 2007, for details). Thus, from Proposition 5 it follows that $\{S_\lambda(t) = B(N_\lambda(t)), t \in \mathbb{R}^+, \lambda \in \mathbb{R}^+\}$ is SICX in $\lambda$ for every fixed value of $t$. Applying Theorem 4.A.18 in Shaked and Shanthikumar (2007), as a corollary of this property one has

$$\Lambda_1 \leq_{\rm icx} \Lambda_2 \qquad \Rightarrow \qquad S_{\Lambda_1}(t) \leq_{\rm icx} S_{\Lambda_2}(t) \quad \text{for all} \quad t \in \mathbb{R}^+,$$

i.e., the population at any fixed time $t \geq 0$ increases in increasing convex order as the random parameter of the Geometric process increases in the increasing convex order.

In the next results, we consider and stochastically compare two shock models with underlying the geometric counting process $\mathbf{N}$, such that the survival function of the random failure time $S_i$ is given by

$$\overline{F}_i(t) = \mathbb{P}[S_i > t] = \sum_{k=0}^{\infty} \overline{P}_{i,k}\, p_k(t), \qquad t \in \mathbb{R}_0^+, \;\; i = 1, 2, \tag{30}$$

where $\overline{P}_{i,k} = \mathbb{P}[M_i > k]$, $k \in \mathbb{N}$, with $M_i$ being the number of shocks causing the failure of the $i$th system, $i = 1, 2$, and where $p_k(t) = \mathbb{P}[N(t) = k]$ is the geometric distribution given in point 2 of Property 1. Specifically, if $\leq_*$ denotes any stochastic order, we show that for the stochastic orders listed in Definition 3 one has that if $M_1 \leq_* M_2$ then $S_1 \leq_* S_2$. (Note that conditions such that $M_1 \leq_* M_2$ are provided in Esary et al., 1973, and Pellerey, 1993). It should be pointed out that similar results have been proved for shock models governed by the Poisson process (see, e.g., Singh and Jain, 1989).

To this aim, first observe that the survival function of $S_i$ has the mixture representation

$$\overline{F}_i(t) = \int_0^\infty \overline{F}_i^{(\alpha)}(t)\, \mathrm{d}U_\lambda(\alpha), \qquad t \in \mathbb{R}_0^+, \;\; i = 1, 2, \tag{31}$$

where $U_\lambda(\cdot)$ is an exponential distribution with mean $\lambda \in \mathbb{R}^+$ and where

$$\overline{F}_i^{(\alpha)}(t) = \mathbb{P}[S_i^{(\alpha)} > t] = \sum_{k=0}^{\infty} \overline{P}_{i,k}\, \mathbb{P}[N^{(\alpha)}(t) = k], \qquad t \in \mathbb{R}_0^+, \;\; i = 1, 2, \tag{32}$$

where $\mathbf{N}^{(\alpha)} = \{N^{(\alpha)}(t), t \in \mathbb{R}_0^+\}$, for $\alpha \in \mathbb{R}^+$, is a Poisson process with intensity $\alpha$.

From (31) one immediately has that $M_1 \leq_* M_2$ implies $S_1 \leq_* S_2$ for all stochastic orders which are closed under mixture. For instance, the above implication holds for the usual stochastic order $\leq_{\rm st}$ and for the increasing concave order $\leq_{\rm icv}$, which are closed under mixture (cf. Shaked and Shanthikumar, 2007). Incidentally, also the increasing convex order $\leq_{\rm icx}$ is closed under mixture, but in this case the result cannot be applied because its definition involves integrals of $\mathbb{P}[N^{(\alpha)}(t) = k]$, which are divergent.

Let us now focus on stochastic orders that are not closed under mixture.

**Theorem 5** *Let $S_1$ and $S_2$ be the lifetimes of two components described as in (30). If $M_1 \leq_{\rm lr}$ $[\leq_{\rm hr}, \leq_{\rm rhr}, \leq_{\rm mit}]$ $M_2$, then $S_1 \leq_{\rm lr}$ $[\leq_{\rm hr}, \leq_{\rm rhr}, \leq_{\rm mit}]$ $S_2$.*

*Proof* First we see the proof of the statement for the case of hazard rate order $\leq_{\rm hr}$. Recall that $M_1 \leq_{\rm hr} M_2$ if the ratio $\overline{P}_{1,k}/\overline{P}_{2,k}$ is decreasing in $k \in \mathbb{N}$, i.e. if $\overline{P}_{i,k}$ is $TP_2$ in $(i,k) \in \{1,2\} \times \mathbb{N}$. Since the term $p_k(t)$ is $TP_2$ in $(k,t)$, by Proposition 6, then the $TP_2$ property of $\overline{F}_i(t)$ in $(i,t)$ follows from the basic composition property. Thus $\overline{F}_1(t)/\overline{F}_2(t)$ is decreasing in $t$, i.e., $S_1 \leq_{\rm hr} S_2$.

The proof is similar for the reversed hazard order $\leq_{\rm rh}$ and the likelihood ratio order $\leq_{\rm lr}$. For the last one, just observe that the density of $S_i$ is

$$f_i(t) = \sum_{k=1}^{\infty} p_{i,k}\, f_{T_k}(t), \qquad t \in \mathbb{R}_0^+,\ \ i = 1,2,$$

and that $f_{T_k}(t)$ is $TP_2$ in $(k,t)$.

For what concerns the mean inactivity time order $\leq_{\rm mit}$, recall that $M_1 \geq_{\rm mit} M_2$ if the ratio $\sum_{j=0}^{k} P_{1,j}/\sum_{j=0}^{k} P_{2,j}$ is decreasing in $k \in \mathbb{N}$, i.e., if $\sum_{j=0}^{k} P_{i,j}$ is $TP_2$ in $(i,k) \in \{1,2\} \times \mathbb{N}$. Then observe that

$$\int_0^t F_i(s)\mathrm{d}s = \sum_{k=0}^{\infty} P_{i,k} \int_0^t p_k(s)\mathrm{d}s, \qquad t \in \mathbb{R}_0^+,\ \ i = 1,2,$$

(where the switch between integral and sum is due to finite value of the integral). Recalling that $\int_0^t p_k(s)\mathrm{d}s$ is $TP_2$ in $(t,s)$ by Proposition 6, and observing that it is also decreasing in $k$, one can prove that $\int_0^t F_i(s)\mathrm{d}s$ is $TP_2$ in $(i,t)$ by applying Theorem 2.1 in Joag-Dev et al. (1995) (see also the comment in the first lines of pag 117 in the quoted reference). The thesis then follows.

An example of application of Theorem 5 is given in Section 7.3.

One may wonder if similar results can be proved considering two shock models governed by different counting processes, and identically distributed numbers $M_i$ of shocks causing the failure of components. Indeed, this is not satisfied. First, consider the case of two Poisson processes, with intensities $\alpha_1$ and $\alpha_2$. Recalling (32), the probability density of $S_i^{(\alpha_i)}$ can be expressed as a mixture of Erlang densities:

$$f_i^{(\alpha_i)}(t) = \sum_{k=1}^{\infty} p_{i,k}\, f_{T_k}^{(\alpha_i)}(t), \qquad t \in \mathbb{R}_0^+,\ \ i = 1,2, \tag{33}$$

**Fig. 7** Ratio of densities (a) in (33), for $\alpha_1 = 1$ and $\alpha_2 = 2$, with $p_{i,k}$ given in (34), and (b) in (35), for $\lambda_1 = 1$ and $\lambda_2 = 2$, with $p_{i,k}$ given in (34).

where $T_k^{(\alpha_i)} \sim Erlang(\alpha_i, k)$ are the arrival times of the Poisson processes, and $p_{i,k} = \mathbb{P}[M_i = k]$, $k \in \mathbb{N}^+$, for $i = 1, 2$.

**Counterexample 1** *For the shock models governed by Poisson processes, assume that $M_1$ and $M_2$ are identically distributed, with probability distribution*

$$p_{i,k} = 0.1 \cdot \mathbf{1}_{\{k=1\}} + 0.9 \cdot \mathbf{1}_{\{k=2\}}, \qquad k \in \mathbb{N}^+, \;\; i = 1, 2. \tag{34}$$

*Hence, it is not hard to show that if $\alpha_1 < \alpha_2$ then $S_1^{(\alpha_1)}$ and $S_2^{(\alpha_2)}$ are not ordered in the likelihood ratio ordering. Indeed, for instance, (33) yields that the ratio of densities $f_1^{(1)}(t)/f_2^{(2)}(t)$ is not monotone in $t \in \mathbb{R}^+$, as can be seen in Figure 7(a), for instance.*

The same result also holds for the shock models governed by the Geometric counting process, as shown hereafter.

**Counterexample 2** *For the shock models governed by Geometric counting processes, if $M_1$ and $M_2$ are identically distributed, with probability distribution (34), and if $\lambda_1 < \lambda_2$, then the random failure times having survival functions (30) are not ordered in the likelihood ratio ordering. Indeed, for instance, for $\lambda_1 = 1$ and $\lambda_2 = 2$ the ratio of densities $f_1(t)/f_2(t)$ is not monotone in $t \in \mathbb{R}^+$, see Figure 7(b), where the density of $S_i$ is*

$$f_i(t) = \sum_{k=1}^{\infty} p_{i,k}\, f_{T_i}(t), \qquad t \in \mathbb{R}_0^+, \;\; i = 1, 2, \tag{35}$$

*with $p_{i,k} = \mathbb{P}[M_i = k]$, and $f_{T_i}(t)$ given in (6).*

Hereafter, we provide aging properties for the lifetimes of components described as in (30). The proof of this statement follows from (31) and the closure under mixture of negative aging notions (cf. Barlow and Proschan, 1981).

**Theorem 6** *Let $S$ be the lifetime of a component described as in (30). If $M$ is DLR [DFR, DFRA, NWU], then $S$ is DLR [DFR, DFRA, NWU].*

| Year | Month | Day | Location | Latitude | Longitude |
|---|---|---|---|---|---|
| 1903 | 5 | 13 | Italy: Palermo | 38.100 | 13.400 |
| 1903 | 7 | 28 | Italy: Northern | 44.300 | 10.000 |
| 1905 | 8 | 26 | Italy: Central | 42.100 | 13.900 |
| 1905 | 9 | 8 | Italy: Monteleone,Tropea | 39.000 | 16.000 |
| 1906 | 9 | 17 | Italy: Sicily | 38.000 | 13.700 |
| 1907 | 2 | 21 | Italy: Sicily | 38.000 | 13.700 |
| 1907 | 10 | 23 | Italy: Ferruzzano | 38.133 | 16.017 |
| 1908 | 12 | 28 | Italy: Messina, Sicily, Calabria | 38.170 | 15.580 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |

**Table 2** First lines of the database describing the occurrences of significant earthquakes registered in Italy in the years from 1900 to 1999 by the National Geophysical Data Center.

*Remark 4* The negative aging of components whose lifetime is described as in (30) is not surprising. In fact, in the particular case where the mixing variable $M$ is geometrically distributed, with parameter $p \in (0,1)$, and thus it satisfies the non aging property, one has that the corresponding survival function and density function result to be, respectively,

$$\overline{F}_S(t) = \frac{1}{1+p\lambda t} \qquad \text{and} \qquad f_S(t) = \frac{p\lambda}{(1+p\lambda t)^2}, \qquad t \in \mathbb{R}_0^+. \tag{36}$$

Note that this distribution is DLR, thus having negative aging property.

## 7 Applications

As mentioned in Section 2.2, the Geometric counting processes can be applied in disciplines where inter-times between occurrences of events have infinite expectations. In this section we provide two examples of observed occurrences where such process seems to fit the available data in a satisfactory manner, and thus it can be appropriately used to model the failures along time. The first example deals with the occurrences of earthquakes in Italy, while the second one deals with failures of an electronic switching system. Note that many other examples of datasets can be found in the literature, where the Geometric process fits the observed data better than the Poisson process.

### 7.1 An application in seismology

The raw data considered here consist in the occurrences of significant earthquakes registered in Italy in the years from 1900 to 1999 by the National Geophysical Data Center of the U.S. Department of Commerce, available at *http://www.ngdc.noaa.gov/ngdc.html*. The total number of relevant registered earthquakes in the database is 113, the first lines being shown in Table 2.

| outcomes | obs. freq. | Geom. | Poisson |
|---|---|---|---|
| 0 | 43 | 49.9 | 32.3 |
| 1 | 29 | 24.9 | 36.5 |
| 2 | 14 | 13.2 | 20.6 |
| 3 | 7 | 7.0 | 7.8 |
| $\geq 4$ | 7 | 7.9 | 2.8 |

| outcomes | obs. freq. | Geom. | Poisson |
|---|---|---|---|
| 0 | 13 | 15.3 | 5.2 |
| 1 | 8 | 10.6 | 11.8 |
| 2 | 12 | 7.4 | 13.3 |
| 3 | 6 | 5.1 | 10.0 |
| 4 | 5 | 3.5 | 5.7 |
| $\geq 5$ | 6 | 8.0 | 4.0 |

**Table 3** Observed and theoretical values of the outcomes used for the $\chi^2$ tests of goodness of fit in the occurrences of earthquakes in Italy, with respect both to Geometric and Poisson distributions. In the left (right) side are considered intervals of one year (two years).

Starting from the given dataset, it is possible to count the frequency of earthquakes in each year, obtaining a sample of 100 observations from a counting random variable, and then to provide an estimate of the corresponding distribution. Assuming a geometric distribution for this variable, the estimated value for its parameter is $p = 0.4695$, which corresponds to $\lambda = 1.13$ for the parameter of the Geometric counting process defined in Section 1 (using year as time unit). A goodness of fit $\chi^2$ test for the geometric distribution gives a $p$-value greater than 0.76, thus not allowing for rejection of the geometric distribution hypothesis, while the same test for the Poisson distribution gives a $p$-value smaller than 0.004, this leading to rejection of the Poisson distribution hypothesis (see the left side of Table 3, for details).

A similar analysis can be performed considering time intervals of length two years, thus considering a number of 50 observations, and counting the number of earthquakes in each time interval. Assuming a geometric distribution for this variable, one obtains the estimate $p = 0.3067$ for the parameter $p$, which corresponds, to the same parameter $\lambda = 1.13$ for the Geometric counting process, recalling that now the width of the time interval is 2 years. The $\chi^2$ test for the geometric distribution gives now a $p$-value greater than 0.27, thus not allowing for rejection of the geometric distribution hypothesis, while the same test for the Poisson distribution gives again a $p$-value smaller than 0.004, this leading again to rejection of the Poisson distribution hypothesis (see Table 3 for details). Thus, the Geometric counting process seems to be more appropriate than the Poisson process to describe the occurrences of significant earthquakes in Italy in the considered time period.

The estimate of the parameter $\lambda$ of the Geometric counting process can be used to infer on further occurrences of earthquakes in the same region. For example, considering the 5-year time interval from year 2000 to year 2004, on the ground of the 113 earthquakes occurred from 1900 to 1999, by applying Eq. (18) one can compute the probability of having a fixed number of earthquakes in Italy, which is reported in Fig. 8. The corresponding expected value is 5.65. Thus, the true value of 6 registered earthquakes occurred from 2000 to 2004 is well estimated by the Geometric counting process.

**Fig. 8** Estimated probabilities $\mathbb{P}[N(2004) - N(1999) = k|N(1999) = 113]$, i.e., of the numbers of earthquakes in Italy in years from 2000 to 2004 conditioned on the registered occurrences up to year 1999.

## 7.2 An application in software reliability

In the past decades the Poisson process has been heavily criticized for software and electronic reliability modeling (see, e.g., Paxson and Floyd, 1995, and references therein). The example provided here shows that the Geometric counting process can be considered as a valid alternative to the Poisson one. The raw data considered in this example are in the list of data sets available at *http://www.cse.cuhk.edu.hk/ lyu/book/reliability/data.html* and analyzed in Lyu (1996). Specifically, we consider here the data set SS1, describing frequencies of failures of the Brazilian Electronic Switching System TROPICO R-1500 in a time interval of 81 time units (of 10 days each), 30 of them during system validation phase, 12 during field trials, and the remaining 39 during system operation. This set of data has been extensively considered in Chapters 10 and 11 of Lyu (1996), as a case study for software reliability growth modeling , as well as in Bastos Martini et al. (1990) or Kanoun et al. (1991) to introduce new techniques for software reliability evaluation.

Removing from the dataset the observations of failures referring to the first 42 time units of system validation and field trials, thus using a set of 39 observations referring to periods of system operation, one can consider 10-day unit-time intervals. By counting the failure frequencies in these units, we obtain a sample of 39 observations from a discrete random variable counting the occurrences of failures. Assuming geometric distribution for this variable, the estimate of its parameter is $p = 0.2635$, which corresponds to a parameter $\lambda = 2.7949$ for the Geometric counting process defined in Section 1 (using 10 days as unit of time). A goodness of fit $\chi^2$ test for the geometric distribution gives a $p$-value greater than 0.79, thus not allowing for rejection of the geometric distribution hypothesis, while the similar testing for the Poisson distribution gives a $p$-value smaller than 0.0001, this strongly leading to rejection of the Poisson distribution hypothesis. See the left side of Table 4 for details.

As a second step, one can consider time intervals of two units (from unit 44 to unit 81), with 19 observations. By counting the number of failures in

| outcomes | obs. freq. | Geom. | Poisson |
|---|---|---|---|
| 0 | 10 | 10.3 | 2.4 |
| 1 | 9 | 7.6 | 6.7 |
| $2-3$ | 7 | 9.7 | 18.0 |
| $4-5$ | 6 | 5.3 | 9.4 |
| $\geq 6$ | 7 | 6.2 | 2.5 |

| outcomes | obs. freq. | Geom. | Poisson |
|---|---|---|---|
| $0-2$ | 8 | 7.5 | 1.6 |
| $3-5$ | 3 | 4.5 | 8.3 |
| $6-8$ | 2 | 2.8 | 7.0 |
| $9-11$ | 4 | 1.7 | 1.8 |
| $\geq 12$ | 2 | 1.7 | 0.2 |

**Table 4** Observed and theoretical values of the outcomes used for the $\chi^2$ tests of goodness of fit in the occurrences of failures of Switching System TROPICO R-1500, with respect both to Geometric and Poisson distributions. In the left (right) side we consider intervals of ten (twenty) days.

each interval, we obtain a new sample from a discrete random variable which is assumed geometric distributed. Its parameter is estimated by $p = 0.1532$, which corresponds to $\lambda = 2.7632$ for the Geometric counting process defined in Section 1 (using 10 days as time unit, and recalling that now the time interval is of 2 units). The $\chi^2$ test for the geometric distribution gives now a $p$-value greater than 0.24, thus not allowing for rejection of the geometric distribution hypothesis. On the contrary, by testing the Poisson distribution the $p$-value is smaller than 0.0001, this strongly leading to rejection of the Poisson distribution hypothesis (see Table 4 for details).

It is worth noting that the hypothesis of Geometric distribution can not be rejected by considering time intervals of different length, this confirming the goodness of fit of this model in representing the occurrences of failures along time. It is remarkable to mention that, as observed also in Lyu (1996), the inter-times between failures statistically increase as the failures occur (that is, the reliability increases along time). This fact is actually coherent with the kind of dependence described in (19), which justify that inter-times stochastically increases as the time evolves, and as the number of observed failures increases.

### 7.3 An application to item's reliability

Consider an item, or a system, that performs a task and is subject to failures caused by shocks arriving according to a Geometric process $\mathbf{N}$ with known intensity $\lambda$. Also, assume that not all shocks are fatal for the item, and denote by $M$ the number of shocks causing the item's failure. Let $q_i = \mathbb{P}[M = i|M \geq i]$, $i \in \mathbb{N}^+$, be the probability that the $i$-th shock causes the failure of the item, given that it survived the previous $i-1$ shocks. Denoting by $S$ the item's failure time, according to (30) its survival function is given by

$$\mathbb{P}[S > t] = \sum_{k=0}^{\infty} \overline{Q}_k \, p_k(t), \qquad t \in \mathbb{R}_0^+, \tag{37}$$

where $\overline{Q}_k = \mathbb{P}[M > k] = \prod_{i=1}^{k}(1 - q_i)$ and $p_k(t) = \mathbb{P}[N(t) = k]$. In concrete applications it is reasonable to assume monotonicity of the sequence $\{q_i, i \in \mathbb{N}^+\}$. For instance, if the failure risk is increasing with the occurrence of the shocks, then we have $q_i \leq q_{i+1}$ for every $i \in \mathbb{N}^+$. Under this assumption, Theorem 5 provides a lower bound to the survival function (37) as follows. Let $q = q_1$, and denote by $S^*$ the lifetime of another item subject to the same shocks of the previous, but whose failure is caused by a geometric random number of shocks, say $M^* \sim G(q)$. Hence, as shown in the first of (36), the survival function of the second item is

$$\mathbb{P}[S^* > t] = \frac{1}{1 + q\lambda t}, \qquad t \in \mathbb{R}_0^+.$$

Since by the increasingness of $\{q_i, i \in \mathbb{N}^+\}$ one has $M^* \leq_{\mathrm{hr}} M$, from Theorem 5 it follows $S^* \leq_{\mathrm{hr}} S$. For instance, an immediate consequence is

$$\mathbb{P}[S > t + s | S > s] \geq \mathbb{P}[S^* > t + s | S^* > s] = \frac{1 + p\lambda s}{1 + p\lambda(t + s)} \qquad \forall t, s \in \mathbb{R}_0^+.$$

### 7.4 An application in insurance

Consider an insurance company that activates a new policy to cover customers for certain risks that occur in accordance to a Geometric counting process $\mathbf{N}$, whose intensity $\lambda \in \mathcal{L} \subseteq \mathbb{R}_0^+$ depends on the policyholder. Assume that the claims $W_j$ are independent and identically distributed. Let $\Lambda$ denote a random variable taking values in $\mathcal{L}$, and distributed as the parameters $\lambda$ of the potential customers interested in the policy. Thus, for a randomly chosen policyholder, the accumulated claim amount along time is described by a stochastic process $\mathbf{S}_\Lambda = \{S_\Lambda(t), t \in \mathbb{R}_0^+\}$ defined as in Theorem 1. Hence, $\mathbf{S}_\Lambda$ is the mixture of processes

$$S_\lambda(t) = \sum_{j=1}^{N_\lambda(t)} W_j, \qquad t \in \mathbb{R}_0^+, \ \ \lambda \in \mathcal{L}$$

with respect to $\Lambda$. Assume that an estimate $\widehat{\lambda}$ of the mean of $\Lambda$ is available. Thus, since $\widehat{\lambda} \leq_{\mathrm{cx}} \Lambda$ by Theorem 3.A.24 in Shaked and Shanthikumar (2007), from Theorem 1 one gets the stochastic bound

$$\sum_{j=1}^{N_{\widehat{\lambda}}(t)} W_j \leq_{\mathrm{cx}} S_\Lambda(t).$$

This relation yields useful inequalities concerning, for example, the variance or other convex measures of risk, such as stop-loss measures. In the case of the variance one obtains immediately

$$\mathrm{Var}[S_\Lambda(t)] \geq \mathrm{Var}\left[\sum_{j=1}^{N_{\widehat{\lambda}}(t)} W_j\right] = \widehat{\lambda} t \mathbb{E}[W_j^2], \qquad t \in \mathbb{R}_0^+.$$

Let us now assume that the claims $W_j$ depend on the policyholder, where such a dependence is described by a parameter $\theta \in \mathcal{T} \subseteq \mathbb{R}$, so that the sequence of claims is $\{W_{j,\theta}; j \in \mathbb{N}^+\}$. Thus, a pair $(\lambda, \theta)$ of parameters is associated to any policyholder. Let $(\Lambda, \Theta)$ be a random vector whose distribution characterizes the potential customers interested in the policy. Assume that the claims $W_{i,\theta}$ are stochastically increasing in $\theta$, and assume that a positive dependence exists between individual's parameters $\lambda$ and $\theta$, i.e. the vector $(\Lambda, \Theta)$ satisfies $(\Lambda^\perp, \Theta^\perp) \leq_{\mathrm{PQD}} (\Lambda, \Theta)$, where $(\Lambda^\perp, \Theta^\perp)$ is the version of $(\Lambda, \Theta)$ with independent component. Hence, one can apply Theorem 4 obtaining $S_{(\Lambda^\perp,\Theta^\perp)}(t) \leq_{\mathrm{icx}} S_{(\Lambda,\Theta)}(t)$ for all $t \in \mathbb{R}_0^+$.

If $\widehat{\theta}$ is an estimate of the mean of $\Theta$, then due to the previous stochastic inequalities for all $t \in \mathbb{R}_0^+$ we have

$$S_{(\Lambda,\Theta)}(t) \geq_{\mathrm{icx}} \sum_{j=1}^{N_{\Lambda^\perp}(t)} W_{j,\Theta^\perp} \geq_{\mathrm{cx}} \sum_{j=1}^{N_{\widehat{\lambda}}(t)} W_{j,\Theta^\perp} \geq_{\mathrm{cx}} \sum_{j=1}^{N_{\widehat{\lambda}}(t)} W_{j,\widehat{\theta}} = S_{(\widehat{\lambda},\widehat{\theta})}(t).$$

The last stochastic inequality is justified by Theorems 3.A.24 and 3.A.12(d) of Shaked and Shanthikumar (2007). Thus, for example one obtains that the stop-loss coverage $\mathbb{E}\left[|S_{(\Lambda,\Theta)} - a|^+\right]$ of a randomly chosen policyholder is always bounded from above by $\mathbb{E}\left[|S_{(\widehat{\lambda},\widehat{\theta})}(t) - a|^+\right]$, which can be easily provided for any $a \in \mathbb{R}_0^+$.

## 8 Concluding remarks

Further generalizations of homogeneous and nonhomogeneous Poisson processes based on alternative expressions of the stochastic intensity of the process, both for the univariate and the multivariate cases, have been recently introduced in the literature. For example, a time dependent stochastic intensity is considered in Cha and Finkelstein (2013), while a multivariate version of Pólya processes recently has been introduced and studied in Cha and Giorgio (2016). Also, shock models based on these generalizations (with random delays for shocks' effects) have been considered in Cha and Finkelstein (2018). Possible future developments can be oriented to results similar to those presented in this paper for such general counting processes and shock models, with meaningful potential applications in engineering and actuarial sciences.

## Appendix. Proof of Proposition 2.2

Fix $\mathbf{t} = (t_1, t_2, \ldots, t_m) \in \mathcal{T}_m^+$. From (7) we have

$$f_{\mathbf{T_m}}(\mathbf{t}) = (-1)^m \sum_{(k_1,k_2,\ldots,k_m)\in\mathcal{A}} \frac{\partial^m}{\partial t_1 \partial t_2 \cdots \partial t_m} p_{(k_1,k_2,\ldots,k_m)}(0, t_1, t_2, \ldots, t_m). \tag{38}$$

Recall now that, for $\mathbf{k} = (k_1, k_2, \ldots, k_m) \in \mathcal{A}$,

$$p_{\mathbf{k}}(0, t_1, t_2, \ldots, t_m) = \binom{\sum_{i=1}^m k_i}{k_1, k_2, \ldots, k_m} \frac{\lambda^{\sum k_i}}{[1+\lambda t_m]^{1+\sum k_i}} \left[t_1^{k_1}(t_2 - t_1)^{k_2} \cdots (t_m - t_{m-1})^{k_m}\right],$$

and observe that it holds

$$\frac{\partial^{m-1}}{\partial t_1 \partial t_2 \cdots \partial t_{m-1}} \left[ t_1^{k_1} (t_2 - t_1)^{k_2} \cdots (t_m - t_{m-1})^{k_m} \right] \neq 0 \tag{39}$$

if and only if the term $t_1 t_2 \cdots t_m$ appears in expansion of the product $t_1^{k_1}(t_2 - t_1)^{k_2} \cdots (t_m - t_{m-1})^{k_m}$. With a straightforward computation of such product, and considering the constraints in (8), it is easy to see that the condition

$$\frac{\partial^m}{\partial t_1 \partial t_2 \cdots \partial t_m} \, p_{\mathbf{k}}(0, t_1, t_2, \ldots, t_m) \neq 0 \tag{40}$$

is fulfilled if and only if $k_1 + k_2 + \ldots + k_m = m - 1$. Recalling that it should be $k_1 = 0$, and again considering the constrains (8), we have that (40) holds if, and only if, $k_1 = 0$ and $k_i = 1$ for all $i = 2, 3, \ldots, m$. In this case we have

$$\begin{aligned}
&\frac{\partial^{m-1}}{\partial t_1 \partial t_2 \cdots \partial t_{m-1}} \, p_{(0,1,1,\ldots,1)}(0, t_1, t_2, \ldots, t_m) \\
&\quad = \frac{\partial^{m-1}}{\partial t_1 \partial t_2 \ldots \partial t_{m-1}} \binom{m-1}{0, 1, \ldots, 1} \lambda^{m-1} \frac{\prod_{i=2}^m (t_i - t_{i-1})}{[1 + \lambda t_m]^m} \\
&\quad = (m-1)! \, \lambda^{m-1} \, \frac{\partial^m}{\partial t_1 \partial t_2 \ldots \partial t_m} \, \frac{\prod_{i=2}^m (t_i - t_{i-1})}{[1 + \lambda t_m]^m} \\
&\quad = \frac{(m-1)! \, \lambda^{m-1}}{[1 + \lambda t_m]^m}.
\end{aligned} \tag{41}$$

In conclusion, from (38), (39) and (41), for $0 < t_1 < t_2 < \ldots < t_m$ we have

$$\begin{aligned}
f_{\mathbf{T_m}}(\mathbf{t}) &= (-1)^m \frac{\partial^m}{\partial t_1 \partial t_2 \cdots \partial t_m} \, p_{(0,1,\ldots,1)}(0, t_1, t_2, \ldots, t_m) \\
&= (-1)^m \frac{\partial}{\partial t_m} \left[ \frac{\partial^{m-1}}{\partial t_1 \partial t_2 \cdots \partial t_{m-1}} \, p_{(0,1,\ldots,1)}(0, t_1, t_2, \ldots, t_m) \right] \\
&= (-1)^m \frac{\partial}{\partial t_m} \, \frac{(m-1)! \, \lambda^{m-1}}{[1 + \lambda t_m]^m} \\
&= \frac{m! \, \lambda^m}{[1 + \lambda t_m]^{m+1}},
\end{aligned} \tag{42}$$

while the density is zero whenever $\mathbf{t} \notin \mathcal{T}_m^+$. Finally, this gives (9).

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