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An adaptive hp -DG-FE Method for Elliptic Problems. Convergence and Optimality in the 1D Case

Dedicated to the memory of Professor Ben-yu Guo

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Abstract

We propose and analyze an hp -adaptive DG-FEM algorithm, termed **hp-ADFEM**, and a realization of it in one space dimension which is convergent, instance optimal, and h - and p -robust. The procedure consists of iterating two routines: one hinges on Binev's algorithm for the adaptive hp -approximation of a given function, and finds a near-best hp -approximation of the current discrete solution and data to a desired accuracy; the other one improves the discrete solution to a finer but comparable accuracy, by iteratively applying Dörfler marking and h -refinement.

1 Introduction

The design and analysis of adaptive hp -type finite element methods for elliptic problems is significantly more challenging than for h -type methods. Indeed, as demonstrated e.g. by some examples given in [6, Sect.1], one should include in the adaptive procedure the possibility of stepping back from an early choice between h -refinement and p -enrichment: while the true structure of the solution reveals itself along the iterations, one should be able to re-distribute the allocated degrees of freedom between h - and p -resolution. The existence of (rather) pathological situations has not prevented the development of practical hp -adaptive algorithms that work (see e.g. [9] and the references therein), but in most cases these procedures are not supported by a sound mathematical theory, which assesses the optimality, and even the convergence, of the method (unless a-priori assumptions on the structure of the solution are made).

The crucial issue is an approximation problem: how can we build an hp -finite element space of minimal dimension in which a given function can be approximated with a prescribed accuracy? A constructive answer to this question has been given by P. Binev in the past few years (see [5]), who designed a greedy hp -algorithm, which is incremental with respect to the dimension and has instance optimality properties (see Sect. 2.3).

With a good answer to such an approximation problem, one may think of recursively applying the hp -adaptive algorithm to a sequence of Galerkin discrete solutions of the elliptic problem, built in a way to get closer and closer to the exact solution. This idea

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has been implemented in [6], where a general framework for adaptive hp -discretizations has been devised, and an adaptive algorithm termed **hp-AFEM** has been proposed, which guarantees convergence and instance optimality of the sequence of generated Galerkin solutions. The algorithm is both h - and p -optimal in one space dimension, whereas in higher dimensions p -robustness is lost, partly due to the need of going from the non-conforming meshes produced by Binev’s algorithm to the conforming ones needed in a continuous Galerkin method, and partly due to the use of a residual-based error estimator (the latter obstruction may be removed by resorting to equilibrated flux estimators, as done in [7]).

Since Binev’s algorithm produces non-conforming meshes and discontinuous approximations, it is quite natural to associate to it a Discontinuous, rather than a Continuous, Galerkin discretization of the elliptic problem. The purpose of this paper is to take a step forward in this direction. In particular, hereafter we propose an hp -adaptive DG-FEM algorithm, termed **hp-ADFEM**, and a realization of it in one space dimension which is convergent, instance optimal, and h - and p -robust. No restriction on the relative size of neighboring elements, nor on the polynomial degrees used on them, is required. In building a discrete solution that matches a prescribed accuracy, we extend to the hp -case the approach developed in [4] for h -type DG methods, using in the analysis several results on hp -type a posteriori error estimators (see e.g. [8] and the references therein). The multi-dimensional case is currently under investigation [1]; while our general convergence theorem holds in any dimension, proving p -robustness seems to require a grading property in the distribution of polynomial degrees over the partition, which is not guaranteed by the algorithm proposed in [5].

The paper is organized as follows. In Sect. 2, we introduce our general framework for the hp -approximation of a given function, and we present Binev algorithm. Sect. 3 describes the hp -DG discretizations that we consider, and collects some of their properties. Sect. 4 contains the general convergence and instance optimality result, based on the concatenation of Binev’s algorithm and a procedure to compute DG-solutions with polynomial data, matching a prescribed tolerance. Finally, in Sect. 5 we illustrate a possible realization of this procedure, which is based on the classical SOLVE \rightarrow ESTIMATE \rightarrow MARK \rightarrow REFINE paradigm.

The following notation will be used throughout the paper. By $A \lesssim B$ we will mean that A can be bounded by a multiple of B , independently of parameters which A and B may depend on. Likewise, $A \simeq B$ is defined as both $A \lesssim B$ and $B \lesssim A$.

C.C. wishes to remember the long-lasting friendship and mutual esteem with Professor Ben-yu Guo, a person of great humanity and a devoted scientist.

2 hp -partitions and hp -approximations

Let Ω be a bounded open interval of the real line. In view of the hp -adaptive discretization of a boundary-value problem therein, we introduce some notation concerning partitions in Ω and function spaces built on them.

2.1 Partitions of the domain

We assume that we are given an essentially disjoint initial partition \mathcal{K}_0 of $\bar{\Omega}$ into finitely many closed subintervals, which will be the initial geometric elements; the initial subdivision may depend upon the data of the problem at hand. Then, we apply subsequent dyadic subdivisions, by halving each element K that we encounter into two closed subintervals K' and K'' of equal size, the ‘children’ of K , such that $K = K' \cup K''$ and $|K' \cap K''| = 0$.

The set \mathfrak{K} of all these geometric elements forms an infinite binary ‘master tree’, having as its roots the elements of the initial partition of $\bar{\Omega}$. A subtree of the master tree is a finite subset of \mathfrak{K} that contains all roots and for each element in the subset both its parent and its sibling are in the subset. The leaves of a subtree form an essentially disjoint partition of $\bar{\Omega}$. The set of all such geometric partitions, or ‘ h -partitions’, will be denoted as \mathbb{K} . For $\mathcal{K}, \tilde{\mathcal{K}} \in \mathbb{K}$, we call $\tilde{\mathcal{K}}$ a refinement of \mathcal{K} , and denoted as $\mathcal{K} \leq \tilde{\mathcal{K}}$, when any $K \in \tilde{\mathcal{K}}$ is either in \mathcal{K} or has an ancestor in \mathcal{K} .

Starting from an h -partition $\mathcal{K} \in \mathbb{K}$, we obtain an hp -partition \mathcal{D} by associating an integer $p \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ to each element $K \in \mathcal{K}$. This integer will represent a polynomial degree, which will identify certain finite dimensional spaces of polynomial functions defined in K . A pair $D = (K_D, p_D) \in \mathfrak{K} \times \mathbb{N}_0$ formed by a geometric element K_D and an integer p_D will be termed an hp -element. Thus, a collection $\mathcal{D} = \{D = (K_D, p_D)\}$ of hp -elements is an hp -partition provided $\mathcal{K}(\mathcal{D}) := \{K_D : D \in \mathcal{D}\} \in \mathbb{K}$; the latter will be the associated h -partition. The collection of all hp -partitions is denoted as \mathbb{D} . Since $p+1$ is the dimension of the space $\mathbb{P}_p(K)$ of the univariate polynomials of degree $\leq p$ in K , we define the *dimension* of the hp -partition \mathcal{D} as the integer

$$\#\mathcal{D} := \sum_{D \in \mathcal{D}} (p_D + 1).$$

For $\mathcal{D}, \tilde{\mathcal{D}} \in \mathbb{D}$, we call $\tilde{\mathcal{D}}$ a refinement of \mathcal{D} , and write $\mathcal{D} \leq \tilde{\mathcal{D}}$, when both $\mathcal{K}(\mathcal{D}) \leq \mathcal{K}(\tilde{\mathcal{D}})$, and $d_{\tilde{D}} \geq d_D$, for any $D \in \mathcal{D}$, $\tilde{D} \in \tilde{\mathcal{D}}$ with K_D being either equal to $K_{\tilde{D}}$ or an ancestor of $K_{\tilde{D}}$.

2.2 Approximation spaces on hp -partitions

Let Z be a normed space of vector-valued functions $z : \Omega \rightarrow \mathbb{R}^m$ ($m \geq 1$), which is relevant for our application. For any geometric element $K \in \mathfrak{K}$, let Z_K be the space collecting the restrictions $z|_K$ to K of all functions $z \in Z$. Then, for any geometric partition $\mathcal{K} \in \mathbb{K}$, we define

$$Z_{\mathcal{K}} := \{z : \Omega \rightarrow \mathbb{R}^m : z|_K \in Z_K \forall K \in \mathcal{K}\} = \prod_{K \in \mathcal{K}} Z_K; \quad (1)$$

obviously, $Z \subseteq Z_{\mathcal{K}}$. In the sequel, we will work with functions that belong to $Z_{\mathcal{K}}$ for some partition $\mathcal{K} \in \mathbb{K}$; therefore, we set

$$\mathcal{Z} := \bigcup_{\mathcal{K} \in \mathbb{K}} Z_{\mathcal{K}}.$$

We assume that for any $K \in \mathfrak{K}$, the space Z_K contains all polynomial functions of any degree, and this set of functions is dense in Z_K . Then, for $p \in \mathbb{N}_0$ we assume we have chosen finite dimensional spaces $Z_{K,p} \subset Z_K$ of polynomial functions on K of degree related to p , satisfying $Z_{K,p} \subset Z_{K,p+1}$ and $Z_{K,p} \subset Z_{K',p} \times Z_{K'',p}$ (K' and K'' being the children of K). For any $D = (K_D, p_D) \in \mathfrak{K} \times \mathbb{N}_0$, we set $Z_D := Z_{K_D, p_D}$. Then, given an hp -partition \mathcal{D} , we define

$$Z_{\mathcal{D}} := \{z : \Omega \rightarrow \mathbb{R}^m : z|_{K_D} \in Z_D \forall D \in \mathcal{D}\} = \prod_{D \in \mathcal{D}} Z_D, \quad (2)$$

which obviously satisfies $Z_{\mathcal{D}} \subset Z_{\mathcal{K}(\mathcal{D})}$. We will use the notation $z_{\mathcal{D}}$ to indicate a function in $Z_{\mathcal{D}}$. Note that no interelement continuity is imposed in the definition of $Z_{\mathcal{D}}$. Also note that the dimension of $Z_{\mathcal{D}}$ is proportional to the cardinality $\#\mathcal{D}$.

For all $D \in \mathfrak{K} \times \mathbb{N}_0$, we assume a local projector $Q_D : \mathcal{Z} \rightarrow Z_D$, and a local error functional $e_D = e_D(z) \geq 0$, that, for any $z \in \mathcal{Z}$ gives a measure for some function of the

distance between $z|_{K_D}$ and its local approximation $z_D := Q_D(z)$. We assume that this error functional is non-increasing under both ‘ h -refinements’ and ‘ p -enrichments’, in the sense that

$$\begin{aligned} e_{D'} + e_{D''} &\leq e_D \text{ when } K_{D'}, K_{D''} \text{ are the children of } K_D, \text{ and } p_{D'} = p_{D''} = p_D; \\ e_{D'} &\leq e_D \text{ when } K_{D'} = K_D \text{ and } p_{D'} \geq p_D. \end{aligned} \quad (3)$$

Given any hp -partition $\mathcal{D} \in \mathbb{D}$, we define the global projector $Q_{\mathcal{D}} : \mathcal{Z} \rightarrow Z_{\mathcal{D}}$ as $Q_{\mathcal{D}}(z) := (z_D)_{D \in \mathcal{D}}$, and the global error functional

$$E_{\mathcal{D}}(z) := \sum_{D \in \mathcal{D}} e_D(z), \quad (4)$$

which is a measure for the distance between z and its projection $z_{\mathcal{D}} := Q_{\mathcal{D}}(z)$. Note that (3) is equivalent to

$$E_{\tilde{\mathcal{D}}}(z) \leq E_{\mathcal{D}}(z) \quad \forall \tilde{\mathcal{D}} \geq \mathcal{D}. \quad (5)$$

2.3 The instance optimal hp -approximation algorithm

Herafter, we present the greedy algorithm proposed by P. Binev [5] to produce a near-best adaptive hp -approximation of a function $z \in \mathcal{Z}$, based on the associated local error functionals $e_D = e_D(z)$ and global error functional $E_{\mathcal{D}} = E_{\mathcal{D}}(z)$ introduced above.

Denote by $R \geq 1$ the cardinality of the initial geometric partition K_0 . Using property (3), Binev’s algorithm builds a sequence of hp -partitions \mathcal{D}_N , $N \geq R$, satisfying $\#\mathcal{D}_N = N$; the construction is incremental, in that going from \mathcal{D}_N to \mathcal{D}_{N+1} one exploits the work already done to build \mathcal{D}_N . The main feature of the algorithm is its instance optimality, expressed as follows.

Theorem 2.1 ([5]). *For $n \geq R$ let*

$$\sigma_n := \inf_{\#\mathcal{D} \leq n} E_{\mathcal{D}}$$

be the smallest error achievable with an hp -partition of cardinality $\leq n$. Then, the hp -partitions \mathcal{D}_N produced by Binev’s algorithm yield error functionals $E_{\mathcal{D}_N}$ satisfying the bounds

$$E_{\mathcal{D}_N} \leq \frac{2N}{N - n + 1} \sigma_n \quad \forall n \leq N. \quad \square \quad (6)$$

Binev’s construction can be easily used to produce an instance optimal hp -partition for which the error functional is below a given threshold.

Corollary 2.1 ([6]). *Let $B > 1$ arbitrary. Given $\varepsilon > 0$, let $\mathcal{D} \in \mathbb{D}$ be the first partition in Binev’s sequence for which $E_{\mathcal{D}}^{\frac{1}{2}} \leq \varepsilon$. Then, setting $b = \frac{1}{2}(1 - \frac{1}{B}) < 1$, it holds*

$$\#\mathcal{D} \leq B \#\hat{\mathcal{D}}$$

for all partitions $\hat{\mathcal{D}} \in \mathbb{D}$ satisfying $E_{\hat{\mathcal{D}}}^{\frac{1}{2}} \leq b\varepsilon$. □

This result motivates the introduction of the following routine, which will constitute one of the two major building blocks of our proposed hp -adaptive algorithm.

- $[\mathcal{D}, z_{\mathcal{D}}] := \mathbf{hp}\text{-NEARBEST}(\varepsilon, z)$

The routine **hp-NEARBEST** takes as input $\varepsilon > 0$, and $z \in \mathcal{Z}$, and outputs $\mathcal{D} \in \mathbb{D}$ as well as $z_{\mathcal{D}} \in Z_{\mathcal{D}}$ such that $E_{\mathcal{D}}(z)^{\frac{1}{2}} \leq \varepsilon$ and, for some constants $0 < b < 1 < B$, $\#\mathcal{D} \leq B \#\widehat{\mathcal{D}}$ for any $\widehat{\mathcal{D}} \in \mathbb{D}$ with $E_{\widehat{\mathcal{D}}}(z)^{\frac{1}{2}} \leq b\varepsilon$.

The approximation $z_{\mathcal{D}}$ of the input z is just the element-wise projection given by the operator $Q_{\mathcal{D}}$ associated with the partition \mathcal{D} , i.e., we set

$$z_{\mathcal{D}} := Q_{\mathcal{D}}(z). \quad (7)$$

3 Discontinuous Galerkin *hp*-discretizations

We are interested in solving numerically the model boundary-value problem

$$Au = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega \quad (8)$$

with $Au := -(\nu u_x)_x + \xi u$, where $\nu \in L^\infty(\Omega)$ satisfies $\text{essinf}_\Omega \nu > 0$, $\xi \in L^2(\Omega)$ satisfies $\xi \geq 0$ a.e. in Ω , $f \in L^2(\Omega)$. We actually assume that ν, ξ are piecewise- H^1 functions, precisely that $\nu|_K, \xi|_K \in H^1(K)$ for each element K of the initial partition \mathcal{K}_0 introduced in Sect. 2.1; we will write $\nu, \xi \in H^1(\Omega; \mathcal{K}_0)$. It will be convenient to refer to a triple $g := (\nu, \xi, f)$ as to a “data” of our problem; we thus have $g \in G(\Omega) := (H^1(\Omega; \mathcal{K}_0))^2 \times L^2(\Omega)$. The solution $u \in H_0^1(\Omega)$ of Problem (8) for given data g will be indicated by $u(g)$.

The following notation will be useful in the design of a DG discretization of our problem. For any element $K \in \mathfrak{K}$, let $(v, w)_K$ denote the L^2 -inner product in K , with corresponding norm $\|v\|_K$. For any geometric partition $\mathcal{K} \in \mathbb{K}$, let us set

$$V_{\mathcal{K}} := \{v \in L^2(\Omega) : v|_K \in H^1(K) \quad \forall K \in \mathcal{K}\}. \quad (9)$$

For $v \in V_{\mathcal{K}}$, it will be convenient to denote by \tilde{v}_x the function in $L^2(\Omega)$ such that $(\tilde{v}_x)|_K = (v|_K)_x$ for all $K \in \mathcal{K}$; thus, $\|\tilde{v}_x\|_\Omega^2 = \sum_{K \in \mathcal{K}} \|(v|_K)_x\|_K^2$. Let us denote by $\mathcal{E}_{\mathcal{K}}$ the set of all endpoints of elements in \mathcal{K} , and let us define the jumps and averages of a piecewise smooth function ϕ on \mathcal{K} as follows: if $e \in \mathcal{E}_{\mathcal{K}}$ is shared by two contiguous elements K^- and K^+ , then we set

$$[[\phi]]_e := \phi|_{K^-}(e) - \phi|_{K^+}(e), \quad \{\{\phi\}\}_e := \frac{1}{2}(\phi|_{K^-}(e) + \phi|_{K^+}(e)),$$

whereas if e is the left/right boundary point of Ω , we set $[[\phi]]_e = +/ -\phi(e)$ and $\{\{\phi\}\}_e = \phi(e)$.

For any *hp*-partition $\mathcal{D} \in \mathbb{D}$ let us set

$$V_{\mathcal{D}} := \{v \in L^2(\Omega) : v|_{K_D} \in \mathbb{P}_{p_D}(K_D) \quad \forall D \in \mathcal{D}\} \subset V_{\mathcal{K}(\mathcal{D})}. \quad (10)$$

If $D \in \mathcal{D}$, let $h_D := |K_D|$ denote the size of the element K_D . If $e \in \mathcal{E}_{\mathcal{D}} := \mathcal{E}_{\mathcal{K}(\mathcal{D})}$, we define the weight

$$\sigma_{\mathcal{D}, e} := \max\left(\frac{p_{D^-}^2}{h_{D^-}}, \frac{p_{D^+}^2}{h_{D^+}}\right) \quad (11)$$

if $e \in K_{D^-} \cap K_{D^+}$, and $\sigma_{\mathcal{D}, e} := \frac{p_D^2}{h_D}$ if $e \in \partial\Omega \cap K_D$.

It is convenient to introduce the inner product $(\phi, \psi)_{\mathcal{E}_{\mathcal{D}}} := \sum_{e \in \mathcal{E}_{\mathcal{D}}} \phi_e \psi_e$ in $\mathbb{R}^{|\mathcal{E}_{\mathcal{D}}|}$ between two quantities $\phi = (\phi_e)$ and $\psi = (\psi_e)$ indexed in $\mathcal{E}_{\mathcal{D}}$. The corresponding norm will be denoted by $\|\phi\|_{\mathcal{E}_{\mathcal{D}}}$.

At this point, we are ready to introduce the symmetric bilinear form $a_{\mathcal{D}}$ defined on $V_{\mathcal{D}} \times V_{\mathcal{D}}$ as

$$a_{\mathcal{D}}(w, v) := (\nu \tilde{w}_x, \tilde{v}_x)_\Omega + (\xi w, v)_\Omega - (\{\{\nu w_x\}\}, [v])_{\mathcal{E}_{\mathcal{D}}} - (\{\{\nu v_x\}\}, [w])_{\mathcal{E}_{\mathcal{D}}} + \gamma (\sigma_{\mathcal{D}}[w], [v])_{\mathcal{E}_{\mathcal{D}}}, \quad (12)$$

where $\gamma > 0$ is a sufficiently large stabilization parameter, as well as the DG-norm defined on $V_{\mathcal{D}}$ as

$$\|v\|_{\mathcal{D}} := \left((\nu \tilde{v}_x, \tilde{v}_x)_{\Omega} + \gamma \|\sigma_{\mathcal{D}}^{1/2} \llbracket v \rrbracket\|_{\mathcal{E}_{\mathcal{D}}}^2 \right)^{\frac{1}{2}}. \quad (13)$$

It is well-known (see [2, 3]) that $a_{\mathcal{D}}$ is a continuous form with respect to the DG-norm, and it is coercive provided γ is chosen large enough, with coercivity and continuity constants independent of \mathcal{D} ; in the sequel, we will assume that this condition is satisfied.

Since $a_{\mathcal{D}}$ depends on the choice of coefficients ν and ξ , and since in the adaptive algorithm we will consider a sequence of DG discretizations with changing (piecewise polynomial) data, sometimes we will prefer the more precise notation $a_{\mathcal{D}}(w, v; \nu, \xi)$ to indicate the right-hand side of (12).

Problem 8 with data $g = (\nu, \xi, f) \in G(\Omega)$ is then discretized by the following Symmetric Interior Penalty Discontinuous-Galerkin method ([2]):

$$u_{\mathcal{D}} \in V_{\mathcal{D}} \quad : \quad a_{\mathcal{D}}(u_{\mathcal{D}}, v_{\mathcal{D}}; \nu, \xi) = (f, v_{\mathcal{D}})_{\Omega} \quad \forall v_{\mathcal{D}} \in V_{\mathcal{D}}. \quad (14)$$

We will write $u_{\mathcal{D}} = u_{\mathcal{D}}(g)$ when we want to stress the dependence of $u_{\mathcal{D}}$ upon the given data g .

3.1 Approximation spaces and error functionals

Hereafter, we specify the choice of approximation spaces and error functionals, introduced in a general setting in Sect. 2.2, that is tailored to the discretization problem of interest.

Since we will deal with approximations of a specific solution of Problem 8, and approximations of the corresponding data, our functions z will be of the form $z = (v, g) = (v, \nu, \xi, f)$. Then, a natural choice for the “base” space Z is $Z = H^1(\Omega) \times G(\Omega) = H^1(\Omega) \times (H^1(\Omega; \mathcal{K}_0))^2 \times L^2(\Omega)$. Note that for $\mathcal{K} \in \mathbb{K}$, the local spaces Z_K that form the global space $Z_{\mathcal{X}}$ according to (1) are given by $Z_K = (H^1(K))^3 \times L^2(K)$.

For any element $K \in \mathfrak{K}$ and integer $p \in \mathbb{N}_0$, we set

$$Z_{K,p} = V_{K,p} \times G_{K,p} \quad \text{with} \quad V_{K,p} = \mathbb{P}_p(K) \quad \text{and} \quad G_{K,p} = \mathbb{P}_{p+1}(K) \times \mathbb{P}_{p+1}(K) \times \mathbb{P}_{p-1}(K).$$

Then, for any $\mathcal{D} \in \mathbb{D}$, we define $Z_{\mathcal{D}}$ according to (2); it is easily seen that $Z_{\mathcal{D}} =: V_{\mathcal{D}} \times G_{\mathcal{D}}$, where $V_{\mathcal{D}}$ has been already introduced in (10). We will write $z_{\mathcal{D}} = (v_{\mathcal{D}}, g_{\mathcal{D}}) = (v_{\mathcal{D}}, \nu_{\mathcal{D}}, \xi_{\mathcal{D}}, f_{\mathcal{D}})$ for the generic element in $Z_{\mathcal{D}}$.

In order to define the projectors Q_D , let $\Pi_{K,p}^0 : L^2(K) \rightarrow \mathbb{P}_p(K)$ be the L^2 -orthogonal projector, and let $\Pi_{K,p}^1 : H^1(K) \rightarrow \mathbb{P}_p(K)$ be the projector such that

$$(\Pi_{K,p}^1 v)_x = \Pi_{K,p}^0 v_x \quad \text{and} \quad \int_K \Pi_{K,p}^1 v = \int_K v, \quad \forall v \in H^1(K).$$

The latter definition can be extended to functions v that are just piecewise- H^1 on K , by replacing v_x with \tilde{v}_x in the L^2 -projection. Then, for $z = (v, g) = (v, \nu, \xi, f) \in \mathcal{Z}$ and $D = (K_D, p_D)$ we set

$$Q_D(z) = (\Pi_{K_D, p_D}^1 v|_{K_D}, \Pi_{K_D, p_D+1}^1 \nu|_{K_D}, \Pi_{K_D, p_D+1}^1 \xi|_{K_D}, \Pi_{K_D, p_D-1}^0 f|_{K_D}).$$

The corresponding local error functional is defined as

$$e_D(z) := e_{1,D}(v) + \frac{1}{\kappa^2} \text{osc}_D^2(g) =: e_{1,D}(v) + \frac{1}{\kappa^2} (e_{1,D}(\nu) + e_{1,D}(\xi) + e_{0,D}(f)), \quad (15)$$

where for $\varphi = v, \nu, \xi$

$$e_{1,D}(\varphi) := \|(\mathbb{I} - \Pi_{K_D, p_D}^0)(\tilde{\varphi}_x)|_{K_D}\|_{K_D}^2, \quad e_{0,D}(f) := \frac{h_D}{p_D} \|(\mathbb{I} - \Pi_{K_D, p_D}^0)f|_{K_D}\|_{K_D}^2,$$

and $\kappa > 0$ is a (sufficiently small) penalization parameter to be chosen later on.

Finally, for a given hp -partition $\mathcal{D} \in \mathbb{D}$, the global projector $Q_{\mathcal{D}} : \mathcal{Z} \rightarrow Z_{\mathcal{D}}$ and the global error functional $E_{\mathcal{D}}(z) = E_{\mathcal{D}}(v, g)$ are defined as in Sect. 2.2 (see (4)).

We now establish some properties involving the functional $E_{\mathcal{D}}$, that will be useful in the sequel.

Property 3.1. *There exists a constant $C_0 > 0$ such that for any $z = (v, \nu, \xi, f) \in \mathcal{Z}$ and for any partition $\mathcal{D} \in \mathbb{D}$ one has*

$$\|\nu - \nu_{\mathcal{D}}\|_{L^\infty(\Omega)} + \|\xi - \xi_{\mathcal{D}}\|_{L^\infty(\Omega)} \leq C_0 \kappa E_{\mathcal{D}}(z)^{\frac{1}{2}},$$

where $z_{\mathcal{D}} = (v_{\mathcal{D}}, \nu_{\mathcal{D}}, \xi_{\mathcal{D}}, f_{\mathcal{D}}) = Q_{\mathcal{D}}(z)$.

Proof. For any $\mathcal{D} \in \mathbb{D}$ and any $D \in \mathcal{D}$, set $\psi := (\nu - \nu_{\mathcal{D}})|_{K_D}$. Since by construction ψ vanishes at a point in K_D , we have for any $x \in K_D$

$$|\psi(x)| \leq h_D^{1/2} \|\psi_x\|_{K_D} \leq |\Omega| e_{1,D}(\nu)^{1/2},$$

from which the bound for $\|\nu - \nu_{\mathcal{D}}\|_{L^\infty(\Omega)}$ easily follows. The coefficient ξ can be treated similarly. \square

At this point, let us fix once and for all the data of interest $g_\star = (\nu_\star, \xi_\star, f_\star) \in G(\Omega)$ for Problem (8), and let $u_\star := u(g_\star)$ be the corresponding solution.

Let us set $\nu_0 := \text{ess inf}_\Omega \nu_\star > 0$.

Assumption 3.1. *Let \mathcal{D}_0 denote the root partition \mathcal{X}_0 endowed with polynomials of degree 1 in each element. Setting $z_0 := (0, g_\star) \in \mathcal{Z}$, we assume that \mathcal{D}_0 is chosen to satisfy*

$$C_0 \kappa E_{\mathcal{D}_0}(z_0)^{\frac{1}{2}} \leq \frac{\nu_0}{\lambda}, \quad \text{where } \lambda := 2 + \frac{1}{\sqrt{2}} |\Omega|.$$

Recalling (5), this assumption together with Property 3.1 guarantees that for any $\mathcal{D} \in \mathbb{D}$ (which trivially satisfies $\mathcal{D} \geq \mathcal{D}_0$), Problem 8 with approximate data $\nu_{\mathcal{D}}$ and $\xi_{\mathcal{D}}$ is coercive in $H_0^1(\Omega)$, precisely one has

$$(\nu_{\mathcal{D}} v_x, v_x)_\Omega + (\xi_{\mathcal{D}} v, v)_\Omega \geq \frac{\nu_0}{2} \|v_x\|_\Omega^2 \quad \forall v \in H_0^1(\Omega). \quad (16)$$

This easily follows using the bound $\|v\|_\Omega \leq \frac{1}{2\sqrt{2}} |\Omega| \|v_x\|_\Omega$.

The following result is fundamental for establishing the convergence of our adaptive algorithm.

Proposition 3.1. *i) There exists a constant $C_\star > 0$ with the following property: for all $\mathcal{D} \in \mathbb{D}$ and all $z \in \mathcal{Z}$ of the form $z = (v, g_\star)$, let $z_{\mathcal{D}} = (v_{\mathcal{D}}, g_{\mathcal{D}}) := Q_{\mathcal{D}}(z)$, and let $u(g_{\mathcal{D}}) \in H_0^1(\Omega)$ be the solution of Problem 8 with data $g_{\mathcal{D}}$; then, it holds*

$$\|u_\star - u(g_{\mathcal{D}})\|_{H_0^1(\Omega)} \leq C_\star \kappa E_{\mathcal{D}}(z_0)^{\frac{1}{2}} \leq C_\star \kappa E_{\mathcal{D}}(z)^{\frac{1}{2}}, \quad (17)$$

where κ is the penalization parameter introduced in (15).

ii) For all $\mathcal{D} \in \mathbb{D}$, $v \in V_{\mathcal{X}(\mathcal{D})}$, $w \in H_0^1(\Omega)$ and $g \in G(\Omega)$, it holds

$$|E_{\mathcal{D}}(v, g)^{\frac{1}{2}} - E_{\mathcal{D}}(w, g)^{\frac{1}{2}}| \leq \|v - w\|_{\mathcal{D}}. \quad (18)$$

The proof follows step by step the proof of Proposition 3 in [6], to which we refer.

4 The adaptive algorithm *hp*-ADFEM

As anticipated in the Introduction, the algorithm we propose consists in alternating between a stage in which a new *hp*-partition is found, which is near-optimal for the current accuracy, and a stage in which this partition is further refined to guarantee a higher accuracy for the corresponding DG discrete solution; the data used in the latter stage to define the DG problem are approximations of the exact data, provided by the former stage.

The first stage will be accomplished by a call to the routine **hp-NEARBEST** introduced in Sect. 2.3. The second stage will be realized through a routine **DG-SOLVE** that we present now, postponing to Sect. 5 the detailed description of the underlying algorithm and the analysis of its properties. Essentially, starting from a given *hp*-partition and a corresponding data approximation, several DG problems are solved on subsequently refined partitions, whose generation is driven by an a posteriori error estimator, until a contraction property guarantees that the discretization error is brought below a prescribed threshold. In this stage, optimality is not an issue for the output partition, provided its cardinality remains comparable to that of the input partition.

- $[\bar{\mathcal{D}}, \bar{u}] := \mathbf{DG-SOLVE}(\varepsilon, \mathcal{D}, z_{\mathcal{D}})$

The routine **DG-SOLVE** takes as input $\varepsilon > 0$, $\mathcal{D} \in \mathbb{D}$, and $z_{\mathcal{D}} = (v_{\mathcal{D}}, g_{\mathcal{D}}) \in Z_{\mathcal{D}}$. It outputs $\bar{\mathcal{D}} \in \mathbb{D}$ with $\mathcal{D} \leq \bar{\mathcal{D}}$ and $\bar{u} := u_{\bar{\mathcal{D}}}(g_{\mathcal{D}}) \in V_{\bar{\mathcal{D}}}$ such that $\|u(g_{\mathcal{D}}) - \bar{u}\|_{\bar{\mathcal{D}}} \leq \varepsilon$.

We recall that $u_{\bar{\mathcal{D}}}(g_{\mathcal{D}})$ denotes the solution of the following DG problem (see (14)): for $g_{\mathcal{D}} = (\nu_{\mathcal{D}}, \xi_{\mathcal{D}}, f_{\mathcal{D}}) \in G_{\mathcal{D}}$,

$$u_{\bar{\mathcal{D}}} \in V_{\bar{\mathcal{D}}} \quad : \quad a_{\bar{\mathcal{D}}}(u_{\bar{\mathcal{D}}}, v_{\bar{\mathcal{D}}}; \nu_{\mathcal{D}}, \xi_{\mathcal{D}}) = (f_{\mathcal{D}}, v_{\bar{\mathcal{D}}})_{\Omega} \quad \forall v_{\bar{\mathcal{D}}} \in V_{\bar{\mathcal{D}}}. \quad (19)$$

The input function $v_{\mathcal{D}} \in V_{\mathcal{D}}$ may be used in the algorithm to define the starting point of the adaptive iterations.

Assumption 4.1. *Let $b < 1 < B$ be the constants that appear in the statement of the instance optimality property for the routine **hp-NEARBEST**. We assume that the penalization parameter κ in (15) is chosen small enough, so that it holds*

$$C_{\star} \kappa < b.$$

We are ready to present our algorithm **hp-ADFEM**. Let us introduce the parameters and the input data.

Parameters: two real numbers $\eta \in (0, 1)$, $\omega > 0$ satisfying

$$C_{\star} \kappa < b(1 - \eta) \quad \text{and} \quad \omega \in \left(\frac{1}{b}, \frac{1 - \eta}{C_{\star} \kappa} \right).$$

(Note that such a choice of ω is equivalent to $b\omega - 1 > 0$ and $C_{\star} \kappa \omega + \eta < 1$, which are two quantities that will appear below.)

Input data: $g_{\star} \in G(\Omega)$, $\varepsilon_0 > 0$, and $\bar{u}_0 \in V_{\bar{\mathcal{D}}_0}$ for some $\bar{\mathcal{D}}_0 \in \mathbb{D}$ such that $\|u_{\star} - \bar{u}_0\|_{\bar{\mathcal{D}}_0} \leq \varepsilon_0$.

Algorithm hp-ADFEM($\varepsilon_0, \bar{u}_0, g_{\star}$)

```

for  $i = 1, 2, \dots$  do
   $[\mathcal{D}_i, (v_{\mathcal{D}_i}, g_{\mathcal{D}_i})] := \mathbf{hp-NEARBEST}(\omega \varepsilon_{i-1}, (\bar{u}_{i-1}, g_{\star}))$ 
   $[\bar{\mathcal{D}}_i, \bar{u}_i] := \mathbf{DG-SOLVE}(\eta \varepsilon_{i-1}, \mathcal{D}_i, (v_{\mathcal{D}_i}, g_{\mathcal{D}_i}))$ 
   $\varepsilon_i := (C_{\star} \kappa \omega + \eta) \varepsilon_{i-1}$ 
end do

```

Theorem 4.1. *Under Assumptions 3.1 and 4.1, the sequences (\bar{u}_i) , (\mathcal{D}_i) produced by **hp-ADFEM** satisfy the following properties:*

$$\|u_\star - \bar{u}_i\|_{\bar{\mathcal{D}}_i} \leq \varepsilon_i \quad \forall i \geq 0, \quad \mathbb{E}_{\mathcal{D}_i}(u_\star, g_\star)^{\frac{1}{2}} \leq (\omega + 1)\varepsilon_{i-1} \quad \forall i \geq 1, \quad (20)$$

and

$$\#\mathcal{D}_i \leq B\#\mathcal{D} \quad \text{for any } \mathcal{D} \in \mathbb{D} \text{ with } \mathbb{E}_{\mathcal{D}}(u_\star, g_\star)^{\frac{1}{2}} \leq (b\omega - 1)\varepsilon_{i-1}. \quad (21)$$

Proof. The bound $\|u_\star - \bar{u}_0\|_{\bar{\mathcal{D}}_0} \leq \varepsilon_0$ is valid by assumption. For $i \geq 1$, the tolerances used for **hp-NEARBEST** and **DG-SOLVE**, together with (17) show that

$$\begin{aligned} \|u_\star - \bar{u}_i\|_{\bar{\mathcal{D}}_i} &\leq \|u_\star - u(g_{\mathcal{D}_i})\|_{H_0^1(\Omega)} + \|u(g_{\mathcal{D}_i}) - \bar{u}_i\|_{\bar{\mathcal{D}}_i} \\ &\leq C_\star \kappa \mathbb{E}_{\mathcal{D}_i}(\bar{u}_{i-1}, g_\star)^{\frac{1}{2}} + \mu \varepsilon_{i-1} \leq (C_\star \kappa \omega + \mu) \varepsilon_{i-1} = \varepsilon_i. \end{aligned} \quad (22)$$

The first statement follows for all $i \geq 0$. Using this and (18) implies the second assertion

$$\mathbb{E}_{\mathcal{D}_i}(u_\star, g_\star)^{\frac{1}{2}} \leq \mathbb{E}_{\mathcal{D}_i}(\bar{u}_{i-1}, g_\star)^{\frac{1}{2}} + \|u_\star - \bar{u}_{i-1}\|_{\bar{\mathcal{D}}_{i-1}} \leq (\omega + 1)\varepsilon_{i-1} \quad \forall i \geq 1.$$

Finally, let $\mathcal{D} \in \mathbb{D}$ with $\mathbb{E}_{\mathcal{D}}(u_\star, g_\star)^{\frac{1}{2}} \leq (b\omega - 1)\varepsilon_{i-1}$. Then, again by (18), $\mathbb{E}_{\mathcal{D}}(\bar{u}_{i-1}, g_\star)^{\frac{1}{2}} \leq b\omega \varepsilon_{i-1}$ and so $\#\mathcal{D}_i \leq B\#\mathcal{D}$ because of the optimality property of **hp-NEARBEST**. \square

The main result of Theorem 4.1 can be summarized by saying that **hp-ADFEM** is *instance optimal* for reducing $\mathbb{E}_{\mathcal{D}}(u_\star, g_\star)$ over $\mathcal{D} \in \mathbb{D}$.

5 The routine DG-SOLVE

The purpose of this section is the description and analysis of a realization of the routine **DG-SOLVE**. It is based on an iterative procedure of the form SOLVE \rightarrow ESTIMATE \rightarrow MARK \rightarrow REFINES, in which ESTIMATE uses a residual-type estimator, whereas REFINES applies a dyadic splitting of each marked element while preserving the polynomial degree. The procedure satisfies a contraction property, which guarantees the reduction of a suitable ‘‘error’’ by a fixed amount at each iteration. Our construction is strongly inspired by [4], whose arguments are hereafter extended to cover the *hp*-case.

In the sequel, the input partition \mathcal{D} will be denoted by \mathcal{D}_{in} , whereas the symbol \mathcal{D} will be used to denote any refinement of \mathcal{D}_{in} generated by the procedure. Similarly, the input function will be denoted by $z_{\text{in}} = (v_{\text{in}}, g_{\text{in}})$. To avoid cumbersome notation, we will actually write $g_{\text{in}} =: g = (\nu, \xi, f)$, but we will recall that g is a piecewise polynomial approximation on the input partition \mathcal{D}_{in} of the given data $g_\star = (\nu_\star, \xi_\star, f_\star) \in G(\Omega)$. Coherently, the exact solution of Problem (8) with input data g will be denoted by $u = u(g)$, whereas for any *hp*-partition $\mathcal{D} \leq \mathcal{D}_{\text{in}}$, $u_{\mathcal{D}} = u_{\mathcal{D}}(g)$ will be the solution of the corresponding DG Problem (14).

For the analysis of the procedure, following [3], we extend the definition of the DG form $a_{\mathcal{D}}$ given in (12) on $V_{\mathcal{D}} \times V_{\mathcal{D}}$ to the infinite dimensional space $V_{\mathcal{K}(\mathcal{D})} \times V_{\mathcal{K}(\mathcal{D})}$ (recall (9)). To this end, we introduce the lifting operator $L_{\mathcal{D}} : V_{\mathcal{K}(\mathcal{D})} \rightarrow V_{\mathcal{D}}$ such that for all $w \in V_{\mathcal{K}(\mathcal{D})}$

$$L_{\mathcal{D}} w \in V_{\mathcal{D}} : (\nu v, L_{\mathcal{D}} w)_{\Omega} = (\{\nu v\}, [w])_{\mathcal{E}_{\mathcal{D}}} \quad \forall v \in V_{\mathcal{D}}. \quad (23)$$

Then, on $V_{\mathcal{K}(\mathcal{D})} \times V_{\mathcal{K}(\mathcal{D})}$ we define the bilinear form

$$\begin{aligned} a_{\mathcal{D}}(w, v) &:= (\nu \tilde{w}_x, \tilde{v}_x)_{\Omega} + (\xi w, v)_{\Omega} \\ &\quad - (\nu \tilde{w}_x, L_{\mathcal{D}} v)_{\mathcal{E}_{\mathcal{D}}} - (\nu \tilde{v}_x, L_{\mathcal{D}} w)_{\mathcal{E}_{\mathcal{D}}} + \gamma (\sigma_{\mathcal{D}} [w], [v])_{\mathcal{E}_{\mathcal{D}}}, \end{aligned} \quad (24)$$

which is readily seen to coincide with (12) on $V_{\mathcal{D}} \times V_{\mathcal{D}}$.

The lifting operator satisfies the following stability bound.

Property 5.1. *There exists a constant $C_1 > 0$ independent of \mathcal{D} such that*

$$\|L_{\mathcal{D}}w\|_{\Omega} \leq C_1 \|\sigma_{\mathcal{D}}^{1/2}[[w]]\|_{\varepsilon_{\mathcal{D}}} \quad \forall w \in V_{\mathcal{K}(\mathcal{D})}. \quad (25)$$

Proof. If K is any interval of length h and e is one of its endpoints, the inverse inequality $|\phi(e)| \lesssim \frac{p}{h^{1/2}} \|\phi\|_K$ holds for any $\phi \in \mathbb{P}_p(K)$. Then, the result easily follows by choosing $v = L_{\mathcal{D}}w$ in (23). \square

Using (25), one proves the existence of a constant $\gamma_0 > 0$ independent of \mathcal{D} such that for any $\gamma \geq \gamma_0$ the bilinear form $a_{\mathcal{D}}$ is continuous and coercive in $V_{\mathcal{K}(\mathcal{D})}$ with respect to the DG norm $\|v\|_{\mathcal{D}}$, uniformly in \mathcal{D} . For future references, let us denote by $0 < \alpha_* \leq \alpha^*$ the coercivity and continuity constants. Since $a_{\mathcal{D}}$ is symmetric, it defines an inner product in $V_{\mathcal{K}(\mathcal{D})}$; the corresponding norm will be denoted by $\|v\|_{a,\mathcal{D}}$ and is uniformly equivalent to the DG norm $\|v\|_{\mathcal{D}}$ introduced in (13).

It is well-known that while the DG-solution $u_{\mathcal{D}} \in V_{\mathcal{D}}$ satisfies the variational equations

$$a_{\mathcal{D}}(u_{\mathcal{D}}, v_{\mathcal{D}}) = (f, v_{\mathcal{D}})_{\Omega} \quad \forall v_{\mathcal{D}} \in V_{\mathcal{D}}, \quad (26)$$

the exact solution $u \in H_0^1(\Omega)$ need not satisfy $a_{\mathcal{D}}(u, v) = (f, v)_{\Omega}$ for all $v \in V_{\mathcal{K}(\mathcal{D})}$ (inconsistency of the DG formulation). However, we do have the partial consistency property

$$a_{\mathcal{D}}(u, v) = (f, v)_{\Omega} \quad \forall v \in H_0^1(\Omega). \quad (27)$$

This motivates the introduction of the conforming subspace $V_{\mathcal{D}}^c := V_{\mathcal{D}} \cap H_0^1(\Omega)$. Then, by subtraction of (26) from (27), we obtain the *partial orthogonality* property

$$a_{\mathcal{D}}(u - u_{\mathcal{D}}, v_{\mathcal{D}}) = 0 \quad \forall v_{\mathcal{D}} \in V_{\mathcal{D}}^c. \quad (28)$$

It is useful for the sequel to introduce the orthogonal decomposition

$$V_{\mathcal{D}} = V_{\mathcal{D}}^c \oplus V_{\mathcal{D}}^{\perp}, \quad (29)$$

where $V_{\mathcal{D}}^{\perp}$ is the orthogonal complement of $V_{\mathcal{D}}^c$ with respect to the inner product $a_{\mathcal{D}}(w, v)$. Any $v_{\mathcal{D}} \in V_{\mathcal{D}}$ will be split according to (29) as $v_{\mathcal{D}} = v_{\mathcal{D}}^c + v_{\mathcal{D}}^{\perp}$.

Property 5.2. *There exists a constant $C_2 > 0$ independent of \mathcal{D} for which the following bound on the DG discretization error holds:*

$$\|u - u_{\mathcal{D}}\|_{\mathcal{D}} \leq C_2 \left(\inf_{w_{\mathcal{D}} \in V_{\mathcal{D}}^c} \|u - w_{\mathcal{D}}\|_{H_0^1(\Omega)} + \|u_{\mathcal{D}}^{\perp}\|_{\mathcal{D}} \right).$$

Proof. For any $w_{\mathcal{D}} \in V_{\mathcal{D}}^c$, using (28) we have

$$\begin{aligned} a_{\mathcal{D}}(u_{\mathcal{D}} - w_{\mathcal{D}}, u_{\mathcal{D}} - w_{\mathcal{D}}) &= a_{\mathcal{D}}(u_{\mathcal{D}} - w_{\mathcal{D}}, u_{\mathcal{D}}^c - w_{\mathcal{D}}) + a_{\mathcal{D}}(u_{\mathcal{D}} - w_{\mathcal{D}}, u_{\mathcal{D}}^{\perp}) \\ &= a_{\mathcal{D}}(u - w_{\mathcal{D}}, u_{\mathcal{D}}^c - w_{\mathcal{D}}) + a_{\mathcal{D}}(u_{\mathcal{D}}^{\perp}, u_{\mathcal{D}} - w_{\mathcal{D}}) \\ &= a_{\mathcal{D}}(u - w_{\mathcal{D}}, u_{\mathcal{D}} - w_{\mathcal{D}}) - a_{\mathcal{D}}(u_{\mathcal{D}}^{\perp}, u - u_{\mathcal{D}}), \end{aligned}$$

whence, by the coercivity and continuity of the form $a_{\mathcal{D}}$,

$$\|u_{\mathcal{D}} - w_{\mathcal{D}}\|_{\mathcal{D}}^2 \lesssim \|u - w_{\mathcal{D}}\|_{\mathcal{D}} \|u_{\mathcal{D}} - w_{\mathcal{D}}\|_{\mathcal{D}} + \|u_{\mathcal{D}}^{\perp}\|_{\mathcal{D}} \|u - u_{\mathcal{D}}\|_{\mathcal{D}}.$$

We conclude by the triangle inequality. \square

We also introduce an approximation operator $\mathbb{I}_{\mathcal{D}} : V_{\mathcal{K}(\mathcal{D})} \rightarrow V_{\mathcal{D}}^c$ that will be useful in the sequel. For any $D \in \mathcal{D}$, set $K_D =: [e_l, e_r]$ and let $\mathcal{P}_D : H^1(K_D) \rightarrow \mathbb{P}_{p_D}(K_D)$ be defined as follows:

$$(\mathcal{P}_D v)(x) := v(e_l) + \int_{e_l}^x (\Pi_{K_D, p_D-1}^0 v_x)(s) ds$$

(recall that Π^0 means L^2 -orthogonal projection). Furthermore, consider the Legendre Gauss-Lobatto grid in K_D containing $p_D + 1$ nodes, and let ψ_{D,e_l} and ψ_{D,e_r} denote the Lagrange basis functions of degree p_D on this grid, associated with the boundary nodes. Then, we define $(\mathbb{I}_D v)|_{K_D} := \mathcal{J}_D v|_{K_D}$, where

$$\mathcal{J}_D v := \mathcal{P}_D v - \tau_{e_l} \llbracket v \rrbracket_{e_l} \psi_{D,e_l} + \tau_{e_r} \llbracket v \rrbracket_{e_r} \psi_{D,e_r} \quad (30)$$

with $\tau_e = 1$ if $e \in \partial\Omega$, $\tau_e = \frac{1}{2}$ otherwise. Checking that $\mathbb{I}_D v \in V_D^c$ is straightforward.

Property 5.3. *The following error estimates hold for any $v \in H_0^1(\Omega)$:*

$$\|(v - \mathbb{I}_D v) \omega_D^{-1/2}\|_{K_D} \leq \frac{1}{(p_D(p_D + 1))^{1/2}} \|v_x\|_{K_D}, \quad \|(\mathbb{I}_D v)_x\|_{K_D} \leq \|v_x\|_{K_D}, \quad (31)$$

where ω_D is the quadratic bubble function in K_D , defined as $\omega_D(x) = (x - e_l)(e_r - x)$.

The following error estimates hold for any $v \in V_D$:

$$\|v - \mathbb{I}_D v\|_{K_D} \lesssim \frac{h_D^{1/2}}{p_D} (\llbracket v \rrbracket_{e_l} + \llbracket v \rrbracket_{e_r}), \quad \|(v - \mathbb{I}_D v)_x\|_{K_D} \lesssim \frac{p_D}{h_D^{1/2}} (\llbracket v \rrbracket_{e_l} + \llbracket v \rrbracket_{e_r}). \quad (32)$$

The latter inequality implies the bound

$$\|\tilde{v}_x - (\mathbb{I}_D v)_x\|_{\Omega} \lesssim \|\sigma_D^{1/2} \llbracket v \rrbracket\|_{\varepsilon_D} \quad \forall v \in V_D. \quad (33)$$

Proof. The first inequality in (31) can be found in [10], whereas the second one is just the stability of the orthogonal projection. The inequalities (32) easily follow from the bounds $\|\psi_{D,e}\|_{K_D} \simeq \frac{h_D^{1/2}}{p_D}$ and $\|(\psi_{D,e})_x\|_{K_D} \lesssim \frac{p_D^2}{h_D} \|\psi_{D,e}\|_{K_D}$. \square

Corollary 5.1. *There exists a constant $C_3 > 0$ independent of \mathcal{D} such that for any $v = v^c \oplus v^\perp \in V_D = V_D^c \oplus V_D^\perp$ one has*

$$\|v^\perp\|_{\mathcal{D}} \leq C_3 \gamma^{1/2} \|\sigma_D^{1/2} \llbracket v \rrbracket\|_{\varepsilon_D}$$

Proof. One has

$$\|v^\perp\|_{\mathcal{D}} \simeq \|v^\perp\|_{a,\mathcal{D}} = \inf_{w \in V_D^c} \|v - w\|_{a,\mathcal{D}} \simeq \inf_{w \in V_D^c} \|v - w\|_{\mathcal{D}} \leq \|v - \mathbb{I}_D v\|_{\mathcal{D}},$$

then one concludes by (33). \square

5.1 The residual estimator

Given any $v \in V_D$ and any $D \in \mathcal{D}$, let us define the local residual

$$\text{res}_D(v) := (f - Av)|_{K_D};$$

for any $e \in \partial K_D$, let us define the jump of the flux at e

$$J_e(v) = \llbracket \nu v_x \rrbracket_e.$$

Then, the (squared) local error estimator is defined as follows

$$\eta_D^2(v) := \frac{1}{p_D(p_D + 1)} \|\text{res}_D(v) \omega_D^{1/2}\|_{K_D}^2 + \sum_{e \in \partial K_D} \sigma_{D,e}^{-1} J_e^2(v),$$

where ω_D denotes the quadratic bubble function introduced in Property 5.3 above. The (squared) global error estimator is

$$\eta_{\mathcal{D}}^2(v) := \sum_{D \in \mathcal{D}} \eta_D^2(v),$$

whereas its restriction to a subset $\mathcal{D}' \subseteq \mathcal{D}$ of elements will be denoted by

$$\eta_{\mathcal{D}}^2(v; \mathcal{D}') := \sum_{D \in \mathcal{D}'} \eta_D^2(v).$$

We show that $\eta_{\mathcal{D}}(u_{\mathcal{D}})$ is a reliable estimator for our DG problem in two steps.

Proposition 5.1. *There exists a constant $C_4 > 0$ independent of \mathcal{D} such that*

$$a_{\mathcal{D}}(u - u_{\mathcal{D}}, u - u_{\mathcal{D}}) \leq C_4 \left(\eta_{\mathcal{D}}^2(u_{\mathcal{D}}) + \gamma \|\sigma_{\mathcal{D}}^{1/2} \llbracket u_{\mathcal{D}} \rrbracket\|_{\varepsilon_{\mathcal{D}}}^2 \right).$$

Proof. We adapt the proof of [4], Lemma 3.1, to our hp setting. Let us split the DG solution as $u_{\mathcal{D}} = u_{\mathcal{D}}^c + u_{\mathcal{D}}^{\perp}$ and let us set $e := u - u_{\mathcal{D}}$ and $w := u - u_{\mathcal{D}}^c \in H_0^1(\Omega)$, so that $e = w - u_{\mathcal{D}}^{\perp}$. Then, recalling (27) and (28),

$$\begin{aligned} a_{\mathcal{D}}(e, e) &= a_{\mathcal{D}}(e, w) - a_{\mathcal{D}}(e, u_{\mathcal{D}}^{\perp}) = a_{\mathcal{D}}(e, w - \mathbb{I}_{\mathcal{D}}w) - a_{\mathcal{D}}(e, u_{\mathcal{D}}^{\perp}) \\ &= (f, w - \mathbb{I}_{\mathcal{D}}w)_{\Omega} - a_{\mathcal{D}}(u_{\mathcal{D}}, w - \mathbb{I}_{\mathcal{D}}w) - a_{\mathcal{D}}(e, u_{\mathcal{D}}^{\perp}), \end{aligned}$$

Integrating back by parts, we get

$$a_{\mathcal{D}}(u_{\mathcal{D}}, w - \mathbb{I}_{\mathcal{D}}w) = \sum_{D \in \mathcal{D}} (Au_{\mathcal{D}}, w - \mathbb{I}_{\mathcal{D}}w)_{K_D} + (L_{\mathcal{D}}u_{\mathcal{D}}, \nu(w - \mathbb{I}_{\mathcal{D}}w)_x)_{\Omega},$$

whence

$$a_{\mathcal{D}}(e, w) = \sum_{D \in \mathcal{D}} (\text{res}_D(u_{\mathcal{D}}), w - \mathbb{I}_{\mathcal{D}}w)_{K_D} + (L_{\mathcal{D}}u_{\mathcal{D}}, \nu(w - \mathbb{I}_{\mathcal{D}}w)_x)_{\Omega}.$$

Writing $(\text{res}_D(u_{\mathcal{D}}), w - \mathbb{I}_{\mathcal{D}}w)_{K_D} = (\text{res}_D(u_{\mathcal{D}})\omega_D^{1/2}, (w - \mathbb{I}_{\mathcal{D}}w)\omega_D^{-1/2})_{K_D}$ and using (31) as well as (25), we obtain

$$a_{\mathcal{D}}(e, w) \leq (\eta_{\mathcal{D}}(u_{\mathcal{D}}) + C_1 \|\sigma_{\mathcal{D}}^{1/2} \llbracket u_{\mathcal{D}} \rrbracket\|_{\varepsilon_{\mathcal{D}}}) \|w_x\|_{\Omega},$$

where the last norm can be bounded using the coercivity of the form $a_{\mathcal{D}}$:

$$\|w_x\|_{\Omega} = \|w\|_{\mathcal{D}} \leq \|e\|_{\mathcal{D}} + \|u_{\mathcal{D}}^{\perp}\|_{\mathcal{D}} \leq \alpha_*^{1/2} a_{\mathcal{D}}(e, e)^{1/2} + \|u_{\mathcal{D}}^{\perp}\|_{\mathcal{D}}.$$

By Young's inequality, we obtain for a suitable constant $C > 0$

$$a_{\mathcal{D}}(e, w) \leq \frac{1}{4} a_{\mathcal{D}}(e, e) + C \left(\eta_{\mathcal{D}}^2(u_{\mathcal{D}}) + \|u_{\mathcal{D}}^{\perp}\|_{\mathcal{D}}^2 + \|\sigma_{\mathcal{D}}^{1/2} \llbracket u_{\mathcal{D}} \rrbracket\|_{\varepsilon_{\mathcal{D}}}^2 \right).$$

It remains to bound the term $a_{\mathcal{D}}(e, u_{\mathcal{D}}^{\perp})$, which is easily done using the continuity of $a_{\mathcal{D}}$:

$$a_{\mathcal{D}}(e, u_{\mathcal{D}}^{\perp}) \leq a_{\mathcal{D}}(e, e)^{1/2} a_{\mathcal{D}}(u_{\mathcal{D}}^{\perp}, u_{\mathcal{D}}^{\perp})^{1/2} \leq a_{\mathcal{D}}(e, e)^{1/2} (\alpha^*)^{1/2} \|u_{\mathcal{D}}^{\perp}\|_{\mathcal{D}} \leq \frac{1}{4} a_{\mathcal{D}}(e, e) + \alpha^* \|u_{\mathcal{D}}^{\perp}\|_{\mathcal{D}}^2.$$

We obtain the desired result by invoking Corollary 5.1. \square

Proposition 5.2. *There exists a constant $C_5 > 0$ independent of \mathcal{D} such that for any γ large enough, say $\gamma \geq \gamma_1 \geq \gamma_0$, one has*

$$\gamma \|\sigma_{\mathcal{D}}^{1/2} \llbracket u_{\mathcal{D}} \rrbracket\|_{\mathcal{E}_{\mathcal{D}}} \leq C_5 \eta_{\mathcal{D}}(u_{\mathcal{D}}).$$

Proof. Here, we adapt the proof of [4], Lemma 3.3, to our hp setting. By the coercivity of the form $a_{\mathcal{D}}$ applied to $u_{\mathcal{D}} - \mathbb{I}_{\mathcal{D}}u_{\mathcal{D}}$, we have

$$\gamma \|\sigma_{\mathcal{D}}^{1/2} \llbracket u_{\mathcal{D}} \rrbracket\|_{\mathcal{E}_{\mathcal{D}}}^2 \leq \alpha_*^{-1} a_{\mathcal{D}}(u_{\mathcal{D}} - \mathbb{I}_{\mathcal{D}}u_{\mathcal{D}}, u_{\mathcal{D}} - \mathbb{I}_{\mathcal{D}}u_{\mathcal{D}}) \quad (34)$$

since $\llbracket \mathbb{I}_{\mathcal{D}}u_{\mathcal{D}} \rrbracket = 0$. For simplicity, let us set $w := u_{\mathcal{D}} - \mathbb{I}_{\mathcal{D}}u_{\mathcal{D}}$ and $v := \mathbb{I}_{\mathcal{D}}u_{\mathcal{D}} \in H_0^1(\Omega)$. Then,

$$a_{\mathcal{D}}(w, w) = (f, w)_{\Omega} - a_{\mathcal{D}}(v, w)$$

and, using $L_{\mathcal{D}}v = 0$ several times, we have

$$\begin{aligned} a_{\mathcal{D}}(v, w) &= (\nu v_x, \tilde{w}_x)_{\Omega} + (\xi v, w)_{\Omega} - (L_{\mathcal{D}}u_{\mathcal{D}}, \nu v_x)_{\Omega} \\ &= (\nu \tilde{u}_{\mathcal{D},x}, \tilde{w}_x)_{\Omega} + (\xi u_{\mathcal{D}}, w)_{\Omega} - \|\nu^{1/2} \tilde{w}_x\|_{\Omega}^2 - \|\xi^{1/2} w\|_{\Omega}^2 - (L_{\mathcal{D}}u_{\mathcal{D}}, \nu v_x)_{\Omega}. \end{aligned}$$

Using in this identity

$$(\nu v_x, \tilde{w}_x)_{\Omega} = - \sum_{D \in \mathcal{D}} ((\nu u_{\mathcal{D},x})_x, w)_{K_D} + (\llbracket \nu u_{\mathcal{D},x} \rrbracket, \{\!\{w\}\!\})_{\mathcal{E}_{\mathcal{D}}} + (\llbracket w \rrbracket, \{\!\{ \nu u_{\mathcal{D},x} \}\!\})_{\mathcal{E}_{\mathcal{D}}}$$

and observing that $(\llbracket w \rrbracket, \{\!\{ \nu u_{\mathcal{D},x} \}\!\})_{\mathcal{E}_{\mathcal{D}}} = (L_{\mathcal{D}}w, \nu \tilde{u}_{\mathcal{D},x})_{\Omega}$, we obtain

$$\begin{aligned} a_{\mathcal{D}}(w, w) &= \sum_{D \in \mathcal{D}} (\text{res}_D(u_{\mathcal{D}}), w)_{K_D} + (J_{\mathcal{D}}(u_{\mathcal{D}}), \{\!\{w\}\!\})_{\mathcal{E}_{\mathcal{D}}} \\ &\quad + \|\nu^{1/2} \tilde{w}_x\|_{\Omega}^2 + \|\xi^{1/2} w\|_{\Omega}^2 + (L_{\mathcal{D}}u_{\mathcal{D}}, \nu \tilde{w}_x)_{\Omega}. \end{aligned} \quad (35)$$

By (32) we have

$$\|w\|_{K_D} \leq \sum_{e \in \partial K_D} \frac{h_D^{1/2}}{p_D} |\llbracket u_{\mathcal{D}} \rrbracket_e| = \sum_{e \in \partial K_D} \frac{h_D^{1/2}}{p_D} \sigma_{\mathcal{D},e}^{-1/2} \sigma_{\mathcal{D},e}^{1/2} |\llbracket u_{\mathcal{D}} \rrbracket_e| \leq \frac{h_D}{p_D^2} \sum_{e \in \partial K_D} \sigma_{\mathcal{D},e}^{1/2} |\llbracket u_{\mathcal{D}} \rrbracket_e|,$$

whence

$$\begin{aligned} (\text{res}_D(u_{\mathcal{D}}), w)_{K_D} &\leq \frac{h_D}{p_D^2} \|\text{res}_D(u_{\mathcal{D}})\|_{K_D} \sum_{e \in \partial K_D} \sigma_{\mathcal{D},e}^{1/2} |\llbracket u_{\mathcal{D}} \rrbracket_e| \\ &\lesssim \frac{1}{p_D} \|\text{res}_D(u_{\mathcal{D}}) \omega_D^{1/2}\|_{K_D} \sum_{e \in \partial K_D} \sigma_{\mathcal{D},e}^{1/2} |\llbracket u_{\mathcal{D}} \rrbracket_e| \leq \eta_D(u_{\mathcal{D}}) \sum_{e \in \partial K_D} \sigma_{\mathcal{D},e}^{1/2} |\llbracket u_{\mathcal{D}} \rrbracket_e|, \end{aligned}$$

where we have used the inverse inequality $\|\phi\|_{K_D} \lesssim \frac{p_D}{h_D} \|\phi \omega_D^{1/2}\|_{K_D}$ which holds for all polynomials of degree $\simeq p_D$, since $\text{res}_D(u_{\mathcal{D}})$ is such a polynomial. Thus, we obtain

$$\sum_{D \in \mathcal{D}} (\text{res}_D(u_{\mathcal{D}}), w)_{K_D} \lesssim \eta_{\mathcal{D}}(u_{\mathcal{D}}) \|\sigma_{\mathcal{D}}^{1/2} \llbracket u_{\mathcal{D}} \rrbracket\|_{\mathcal{E}_{\mathcal{D}}}.$$

Concerning the second term on the right-hand side of (35), we observe that by construction of $\mathbb{I}_{\mathcal{D}}u_{\mathcal{D}}$, one has $w(e) = \frac{1}{2} \llbracket u_{\mathcal{D}} \rrbracket_e$ at any internal inter-element point e , whereas $w(e) = 0$ at the boundary points of Ω . Thus,

$$\begin{aligned} (J_{\mathcal{D}}(u_{\mathcal{D}}), \{\!\{w\}\!\})_{\mathcal{E}_{\mathcal{D}}} &\lesssim \sum_{e \in \mathcal{E}_{\mathcal{D}}} |J_e(u_{\mathcal{D}})| |\llbracket u_{\mathcal{D}} \rrbracket_e| = \sum_{e \in \mathcal{E}_{\mathcal{D}}} \sigma_{\mathcal{D},e}^{-1/2} |J_e(u_{\mathcal{D}})| \sigma_{\mathcal{D},e}^{1/2} |\llbracket u_{\mathcal{D}} \rrbracket_e| \\ &\leq \eta_{\mathcal{D}}(u_{\mathcal{D}}) \|\sigma_{\mathcal{D}}^{1/2} \llbracket u_{\mathcal{D}} \rrbracket\|_{\mathcal{E}_{\mathcal{D}}}. \end{aligned}$$

Finally, using (32) and (25), the three last terms on the right-hand side of (35) can be bounded by $C\|\sigma_{\mathcal{D}}^{1/2}[[u_{\mathcal{D}}]]\|_{\mathcal{E}_{\mathcal{D}}}^2$. Substituting all the previous bounds in (34), we obtain

$$\gamma\|\sigma_{\mathcal{D}}^{1/2}[[u_{\mathcal{D}}]]\|_{\mathcal{E}_{\mathcal{D}}}^2 \lesssim (\eta_{\mathcal{D}}(u_{\mathcal{D}})\|\sigma_{\mathcal{D}}^{1/2}[[u_{\mathcal{D}}]]\|_{\mathcal{E}_{\mathcal{D}}} + \|\sigma_{\mathcal{D}}^{1/2}[[u_{\mathcal{D}}]]\|_{\mathcal{E}_{\mathcal{D}}}^2),$$

where the constant implied by the symbol \lesssim is independent of γ . Therefore, choosing γ large enough, we get the desired result. \square

Corollary 5.2. *There exists a constant $C_6 > 0$ independent of \mathcal{D} such that for any $\gamma \geq \gamma_1$, one has*

$$a_{\mathcal{D}}(u - u_{\mathcal{D}}, u - u_{\mathcal{D}}) \leq C_6 \eta_{\mathcal{D}}^2(u_{\mathcal{D}}). \quad \square$$

5.2 The adaptive iterations

The routine **DG-SOLVE** iterates the mapping

$$(\mathcal{D}, u_{\mathcal{D}}, \eta_{\mathcal{D}}(u_{\mathcal{D}})) \rightarrow (\mathcal{D}_*, u_{\mathcal{D}_*}, \eta_{\mathcal{D}_*}(u_{\mathcal{D}_*})), \quad (36)$$

where \mathcal{D}_* is a refinement of \mathcal{D} obtained by first applying a Dörfler marking to the elements of \mathcal{D} based on the error estimator $\eta_{\mathcal{D}}(u_{\mathcal{D}})$, and then performing a dyadic subdivision to the marked elements and its neighbors.

To be precise, let $\vartheta \in (0, 1)$ be the Dörfler parameter. Let us order the local error estimators $\eta_D(u_{\mathcal{D}})$, $D \in \mathcal{D}$, by decreasing value, and let us choose a set $\mathcal{M} \subseteq \mathcal{D}$ of minimal cardinality for which

$$\eta_{\mathcal{D}}(u_{\mathcal{D}}; \mathcal{M}) \geq \vartheta \eta_{\mathcal{D}}(u_{\mathcal{D}}). \quad (37)$$

Let $\partial\mathcal{M} \subseteq \mathcal{D}$ denote the set of elements D that share an interface with an element in \mathcal{M} . Then, we replace each $D = (K_D, p_D) \in \mathcal{M} \cup \partial\mathcal{M}$ by the two elements $D' = (K'_D, p_D)$ and $D'' = (K''_D, p_D)$, where K'_D and K''_D are the two children of K_D . Thus, the new partition \mathcal{D}_* is defined by

$$\mathcal{D}_* = \{D', D'' : D \in \mathcal{M} \cup \partial\mathcal{M}\} \cup \{D : D \in \mathcal{D} \setminus (\mathcal{M} \cup \partial\mathcal{M})\}. \quad (38)$$

Our aim is to prove that a suitable combination of (squared) DG error and error estimator, i.e.,

$$\|u - u_{\mathcal{D}}\|_{a, \mathcal{D}}^2 + \beta \eta_{\mathcal{D}}^2(u_{\mathcal{D}})$$

for some $\beta > 0$, is reduced by a fixed rate $\varrho \in (0, 1)$ in performing the mapping (36). The proof, which extends [4] to our hp -setting, will be based on the following results.

Lemma 5.1. *There exists a constant $C_7 > 0$ independent of \mathcal{D} such that for any real $\lambda \in (0, 1)$, one has*

$$\eta_{\mathcal{D}_*}^2(u_{\mathcal{D}_*}) \leq (1 + \lambda) \left(1 - \frac{\vartheta^2}{2}\right) \eta_{\mathcal{D}}^2(u_{\mathcal{D}}) + \frac{C_7}{\lambda} \|u_{\mathcal{D}_*} - u_{\mathcal{D}}\|_{\mathcal{D}_*}^2.$$

Proof. We first establish a few results about the Lipschitz continuity of the local error estimators. Assume that $v, w \in V_{\mathcal{D}}$ and let $D \in \mathcal{D}$. By Minkowski's inequality,

$$|\eta_D(v) - \eta_D(w)| \leq \left(\frac{1}{p_D^2} \|(\text{res}_D(v) - \text{res}_D(w)) \omega_D^{1/2}\|_{K_D}^2 + \sum_{e \in \partial K_D} \sigma_{\mathcal{D}, e}^{-1} |J_e(v) - J_e(w)|^2 \right)^{1/2}.$$

One has

$$\begin{aligned}
\|(\text{res}_D(v) - \text{res}_D(w))\omega_D^{1/2}\|_{K_D} &\leq \|(\nu(v-w)_x)_x\omega_D^{1/2}\|_{K_D} + \|\xi(v-w)\omega_D^{1/2}\|_{K_D} \\
&\lesssim p_D\|\nu(v-w)_x\|_{K_D} + h_D\|\xi(v-w)\|_{K_D} \\
&\lesssim p_D\|(v-w)_x\|_{K_D} + h_D\|(v-w)\|_{K_D},
\end{aligned}$$

where we have used the inverse inequality $\|\phi_x\omega_D^{1/2}\|_{K_D} \lesssim p_D\|\phi\|_{K_D}$, which holds for all polynomial ϕ of degree $\simeq p_D$ in K_D , as well as the bound $\|\omega_D^{1/2}\|_{L^\infty(K_D)} \leq h_D$.

On the other hand, for each $e \in \partial K_D$, let us denote by D' the element in \mathcal{D} sharing the interface e with D . Then,

$$\begin{aligned}
|J_e(v) - J_e(w)| &\leq |\nu(v-w)_x|_{K_D}(e) + |\nu(v-w)_x|_{K_{D'}}(e) \\
&\lesssim |(v-w)_x|_{K_D}(e) + |(v-w)_x|_{K_{D'}}(e) \\
&\lesssim \frac{p_D}{h_D^{1/2}}\|(v-w)_x\|_{K_D} + \frac{p_{D'}}{h_{D'}^{1/2}}\|(v-w)_x\|_{K_{D'}} \\
&\leq \sigma_{D',e}^{1/2}(\|(v-w)_x\|_{K_D} + \|(v-w)_x\|_{K_{D'}}),
\end{aligned}$$

where we have used the inverse inequality $|\psi(e)| \lesssim \frac{p_{D'}}{h_{D'}^{1/2}}\|\psi\|_{K_{D'}}$, which holds for all polynomial ψ of degree $\simeq p_{D'}$ in $K_{D'}$. We conclude that

$$|\eta_D(v) - \eta_D(w)| \lesssim \mathcal{N}_D(v-w), \quad \text{with } \mathcal{N}_D^2(\phi) := \sum_{D'} \|\phi_x\|_{K_D}^2 + \frac{h_D^2}{p_D^2}\|\phi\|_{K_D}^2,$$

where summation is extended to all $D' \in \mathcal{D}$ such that $K_{D'} \cap K_D$ is nonempty; this implies

$$\eta_D^2(v) \leq (1+\lambda)\eta_D^2(w) + \frac{C}{\lambda}\mathcal{N}_D^2(v-w) \quad (39)$$

for a suitable constant $C > 0$ independent of D .

We now apply these bounds, with $v = u_{\mathcal{D}_*}$ and $w = u_{\mathcal{D}}$, to the partition (38) generated by the refinement procedure. If $D \in \mathcal{M}$, let D_m , $m = 1, 2$ be the two children in which D is split. We have $\omega_{D_m}(x) \leq \frac{1}{2}\omega_D(x)$ for all $x \in D_m$. By definition of refinement, we have $h_{D_m} = \frac{1}{2}h_D$ as well as $h_{D'_m} = \frac{1}{2}h_{D'}$ for any neighborhood $D' \in \mathcal{D}$ of D , which implies $\sigma_{D'_m,e}^{-1} \leq \frac{1}{2}\sigma_{D',e}^{-1}$ for any $e \in \partial K_{D_m}$. Hence, we immediately have $\sum_{m=1}^2 \eta_{D_m}^2(u_{\mathcal{D}}) \leq \frac{1}{2}\eta_D^2(u_{\mathcal{D}})$ and $\sum_{m=1}^2 \mathcal{N}_{D_m}(u_{\mathcal{D}_*} - u_{\mathcal{D}}) \leq \mathcal{N}_D(u_{\mathcal{D}_*} - u_{\mathcal{D}})$, whence

$$\sum_{m=1}^2 \eta_{D_m}^2(u_{\mathcal{D}_*}) \leq \frac{1}{2}(1+\lambda)\eta_D^2(u_{\mathcal{D}}) + \frac{C}{\lambda}\mathcal{N}_D^2(u_{\mathcal{D}_*} - u_{\mathcal{D}}).$$

If $D \in \partial\mathcal{M}$, we can only say that $\sigma_{D'_m,e}^{-1} \leq \sigma_{D',e}^{-1}$ for any $e \in \partial K_D$, whence

$$\sum_{m=1}^2 \eta_{D_m}^2(u_{\mathcal{D}_*}) \leq (1+\lambda)\eta_D^2(u_{\mathcal{D}}) + \frac{C}{\lambda}\mathcal{N}_D^2(u_{\mathcal{D}_*} - u_{\mathcal{D}}).$$

Finally, for any unsplit $D \in \mathcal{D} \setminus (\mathcal{M} \cup \partial\mathcal{M})$, we just have

$$\eta_D^2(u_{\mathcal{D}_*}) \leq (1+\lambda)\eta_D^2(u_{\mathcal{D}}) + \frac{C}{\lambda}\mathcal{N}_D^2(u_{\mathcal{D}_*} - u_{\mathcal{D}}).$$

Summing-up all contributions and using the marking condition, we obtain

$$\begin{aligned}
\eta_{\mathcal{D}_*}^2(u_{\mathcal{D}_*}) &\leq (1+\lambda) \left(\eta_{\mathcal{D}}^2(u_{\mathcal{D}}) - \frac{1}{2}\eta_{\mathcal{D}}^2(u_{\mathcal{D}}; \mathcal{M}) \right) + \frac{C}{\lambda} \sum_{D \in \mathcal{D}} \mathcal{N}_D^2(u_{\mathcal{D}_*} - u_{\mathcal{D}}) \\
&\leq (1+\lambda) \left(1 - \frac{\vartheta^2}{2} \right) \eta_{\mathcal{D}}^2(u_{\mathcal{D}}) + \frac{C}{\lambda} \sum_{D \in \mathcal{D}} \mathcal{N}_D^2(u_{\mathcal{D}_*} - u_{\mathcal{D}}).
\end{aligned}$$

It remains to prove that $\sum_{D \in \mathcal{D}} \mathcal{N}_D^2(u_{\mathcal{D}_*} - u_{\mathcal{D}}) \lesssim \|u_{\mathcal{D}_*} - u_{\mathcal{D}}\|_{\mathcal{D}_*}^2$. Setting now $w := u_{\mathcal{D}_*} - u_{\mathcal{D}}$, we have

$$\sum_{D \in \mathcal{D}} \mathcal{N}_D^2(w) = \|\tilde{w}_x\|_{\Omega}^2 + \sum_{D \in \mathcal{D}} \frac{h_D^2}{p_D^2} \|w\|_{K_D}^2.$$

Writing, for a.e. $x \in \Omega$,

$$w(x) = \sum_{e \in \mathcal{E}_{\mathcal{D}_*}, e < x} \llbracket w \rrbracket_e + \int_{\min \Omega}^x \tilde{w}_x(s) ds = \sum_{e \in \mathcal{E}_{\mathcal{D}_*}, e < x} \sigma_{\mathcal{D}_*, e}^{-1/2} \sigma_{\mathcal{D}_*, e}^{-1/2} \llbracket w \rrbracket_e + \int_{\min \Omega}^x \tilde{w}_x(s) ds,$$

we have

$$w^2(x) \lesssim \left(\sum_{e \in \mathcal{E}_{\mathcal{D}_*}} \sigma_{\mathcal{D}_*, e}^{-1} \right) \sum_{e \in \mathcal{E}_{\mathcal{D}_*}} \sigma_{\mathcal{D}_*, e} \llbracket w \rrbracket_e^2 + |\Omega| \|\tilde{w}_x\|_{\Omega}^2.$$

Since $\sum_{e \in \mathcal{E}_{\mathcal{D}_*}} \sigma_{\mathcal{D}_*, e}^{-1} \leq |\Omega|$, we easily obtain the desired bound. \square

Lemma 5.2. *There exists a constant $C_8 > 0$ independent of \mathcal{D} such that for any real $\delta \in (0, 1)$ and any $\gamma \geq \gamma_1$, one has*

$$\|u - u_{\mathcal{D}_*}\|_{a, \mathcal{D}_*}^2 \leq (1 + \delta) \|u - u_{\mathcal{D}}\|_{a, \mathcal{D}}^2 - \frac{\alpha_*}{2} \|u_{\mathcal{D}_*} - u_{\mathcal{D}}\|_{\mathcal{D}_*}^2 + \frac{C_8}{\delta \gamma} (\eta_{\mathcal{D}_*}^2(u_{\mathcal{D}_*}) + \eta_{\mathcal{D}}^2(u_{\mathcal{D}})).$$

Proof. Let us set $w_* := u - u_{\mathcal{D}_*}$, $w := u - u_{\mathcal{D}}$, $d := u_{\mathcal{D}_*} - u_{\mathcal{D}}$, $d^c := u_{\mathcal{D}_*}^c - u_{\mathcal{D}}^c$ and $d^\perp := u_{\mathcal{D}_*}^\perp - u_{\mathcal{D}}^\perp$. Observing that $a_{\mathcal{D}_*}(w_*, d^c) = 0$ by the partial orthogonality property (28), one easily gets

$$\|w_*\|_{a, \mathcal{D}_*}^2 = a_{\mathcal{D}_*}(w_*, w_*) = a_{\mathcal{D}_*}(w_* + d^c, w_* + d^c) - a_{\mathcal{D}_*}(d^c, d^c).$$

Using $u_{\mathcal{D}} = u_{\mathcal{D}}^c + u_{\mathcal{D}}^\perp$ and $u_{\mathcal{D}_*} = u_{\mathcal{D}_*}^c + u_{\mathcal{D}_*}^\perp$, one has $w_* + d^c = w - d^\perp$, whence

$$\begin{aligned} a_{\mathcal{D}_*}(w_* + d^c, w_* + d^c) &= a_{\mathcal{D}_*}(w, w) - 2a_{\mathcal{D}_*}(w, d^\perp) + a_{\mathcal{D}_*}(d^\perp, d^\perp) \\ &\leq \|w\|_{a, \mathcal{D}_*}^2 + 2(\alpha^*)^{1/2} \|w\|_{a, \mathcal{D}_*} \|d^\perp\|_{\mathcal{D}_*} + \alpha^* \|d^\perp\|_{\mathcal{D}_*}^2, \end{aligned}$$

where we have used the uniform continuity of the form $a_{\mathcal{D}_*}$ with respect to the DG-norm. Using the uniform coercivity and the triangle inequality, we get

$$a_{\mathcal{D}_*}(d^c, d^c) \geq \alpha_* \|d^c\|_{\mathcal{D}_*}^2 \geq \alpha_* \left(\frac{1}{2} \|d\|_{\mathcal{D}_*}^2 - \|d^\perp\|_{\mathcal{D}_*}^2 \right).$$

Collecting these inequalities and using Young's inequality, we obtain

$$\|w_*\|_{a, \mathcal{D}_*}^2 \leq (1 + \delta) \|w\|_{a, \mathcal{D}_*}^2 - \frac{\alpha_*}{2} \|d\|_{\mathcal{D}_*}^2 + \frac{C}{\delta} \|d^\perp\|_{\mathcal{D}_*}^2. \quad (40)$$

At this point, we observe that $\|u_{\mathcal{D}}^\perp\|_{\mathcal{D}_*}^2 \leq 2\|u_{\mathcal{D}}^\perp\|_{\mathcal{D}}^2$. Indeed, $\|u_{\mathcal{D}}^\perp\|_{\mathcal{D}_*}^2 = \|(u_{\mathcal{D}}^\perp)^\sim\|_{\Omega}^2 + \gamma \sum_{e \in \mathcal{E}_{\mathcal{D}_*}} \sigma_{\mathcal{D}_*, e} \llbracket u_{\mathcal{D}}^\perp \rrbracket_e^2$, but the jumps of $u_{\mathcal{D}}^\perp$ occur only at the interfaces $e \in \mathcal{E}_{\mathcal{D}}$, and $\sigma_{\mathcal{D}_*, e} \leq 2\sigma_{\mathcal{D}, e}$ by definition of the refinement strategy. Thus, using Corollary 5.1, we get

$$\|d^\perp\|_{\mathcal{D}_*}^2 \lesssim \|u_{\mathcal{D}_*}^\perp\|_{\mathcal{D}_*}^2 + \|u_{\mathcal{D}}^\perp\|_{\mathcal{D}}^2 \lesssim \gamma \|\sigma_{\mathcal{D}_*}^{1/2} \llbracket u_{\mathcal{D}_*} \rrbracket\|_{\mathcal{E}_{\mathcal{D}_*}}^2 + \gamma \|\sigma_{\mathcal{D}}^{1/2} \llbracket u_{\mathcal{D}} \rrbracket\|_{\mathcal{E}_{\mathcal{D}}}^2. \quad (41)$$

It remains to replace $\|w\|_{a, \mathcal{D}_*}^2$ by $\|w\|_{a, \mathcal{D}}^2$. To this end, let us write

$$a_{\mathcal{D}_*}(w, w) = a_{\mathcal{D}}(w, w) + 2(L_{\mathcal{D}_*} w, \nu \tilde{w}_x)_{\Omega} - 2(L_{\mathcal{D}} w, \nu \tilde{w}_x)_{\Omega} - \gamma \|\sigma_{\mathcal{D}}^{1/2} \llbracket w \rrbracket\|_{\mathcal{E}_{\mathcal{D}}}^2 + \gamma \|\sigma_{\mathcal{D}_*}^{1/2} \llbracket w \rrbracket\|_{\mathcal{E}_{\mathcal{D}_*}}^2.$$

Using Property 5.1 and the coercivity of the form $a_{\mathcal{D}}$, one gets

$$(L_{\mathcal{D}^*} w, \nu \tilde{w}_x)_{\Omega} \lesssim \|\sigma_{\mathcal{D}^*}^{1/2} \llbracket w \rrbracket\|_{\varepsilon_{\mathcal{D}^*}} a_{\mathcal{D}}(w, w)^{1/2} \lesssim \|\sigma_{\mathcal{D}}^{1/2} \llbracket u_{\mathcal{D}} \rrbracket\|_{\varepsilon_{\mathcal{D}}} a_{\mathcal{D}}(w, w)^{1/2}.$$

A similar bound holds for $(L_{\mathcal{D}} w, \nu \tilde{w}_x)_{\Omega}$. Therefore, using once more Young's inequality, we arrive at

$$\|w\|_{a, \mathcal{D}^*}^2 \leq (1 + \delta) \|w\|_{a, \mathcal{D}}^2 + \frac{C}{\delta} \gamma \|\sigma_{\mathcal{D}}^{1/2} \llbracket u_{\mathcal{D}} \rrbracket\|_{\varepsilon_{\mathcal{D}}}^2. \quad (42)$$

Replacing (41)-(42) into (40), we obtain

$$\begin{aligned} \|u - u_{\mathcal{D}^*}\|_{a, \mathcal{D}^*}^2 &\leq (1 + \delta)^2 \|u - u_{\mathcal{D}}\|_{a, \mathcal{D}}^2 - \frac{\alpha_*}{2} \|u_{\mathcal{D}^*} - u_{\mathcal{D}}\|_{\mathcal{D}^*}^2 \\ &\quad + \frac{C}{\delta} \gamma \left(\|\sigma_{\mathcal{D}^*}^{1/2} \llbracket u_{\mathcal{D}^*} \rrbracket\|_{\varepsilon_{\mathcal{D}^*}}^2 + \|\sigma_{\mathcal{D}}^{1/2} \llbracket u_{\mathcal{D}} \rrbracket\|_{\varepsilon_{\mathcal{D}}}^2 \right). \end{aligned}$$

The desired result follows from Proposition 5.2, after replacing δ by $\delta/3$. \square

We are ready to establish the main result of this section.

Theorem 5.1. *Consider the mapping (36) defined above. There exist constants $\beta > 0$ and $\varrho \in (0, 1)$, independent of \mathcal{D} , such that, choosing $\gamma > 0$ large enough in the definition (24), one has*

$$\|u - u_{\mathcal{D}^*}\|_{a, \mathcal{D}^*}^2 + \beta \eta_{\mathcal{D}^*}^2(u_{\mathcal{D}^*}) \leq \varrho \left(\|u - u_{\mathcal{D}}\|_{a, \mathcal{D}}^2 + \beta \eta_{\mathcal{D}}^2(u_{\mathcal{D}}) \right).$$

Proof. Let us simplify our notation by setting $E_*^2 := \|u - u_{\mathcal{D}^*}\|_{a, \mathcal{D}^*}^2$, $E^2 := \|u - u_{\mathcal{D}}\|_{a, \mathcal{D}}^2$, $e_*^2 := \|u_{\mathcal{D}^*} - u_{\mathcal{D}}\|_{\mathcal{D}^*}^2$ and $\eta_*^2 := \eta_{\mathcal{D}^*}^2(u_{\mathcal{D}^*})$, $\eta^2 := \eta_{\mathcal{D}}^2(u_{\mathcal{D}})$. Then, the inequalities of Lemmas 5.2-5.1 read as follows:

$$\begin{aligned} E_*^2 &\leq (1 + \delta) E^2 - \frac{\alpha_*}{2} e_*^2 + \frac{C_8}{\delta \gamma} (\eta_*^2 + \eta^2) \\ \eta_*^2 &\leq (1 + \lambda) \left(1 - \frac{\vartheta^2}{2}\right) \eta^2 + \frac{C_7}{\lambda} e_*^2. \end{aligned}$$

Thus, for any real $\beta > 0$,

$$\begin{aligned} E_*^2 + \beta \eta_*^2 &\leq (1 + \delta) E^2 - \frac{\alpha_*}{2} e_*^2 + \left(\beta + \frac{C_8}{\delta \gamma} \right) \eta_*^2 + \frac{C_8}{\delta \gamma} \eta^2 \\ &\leq (1 + \delta) E^2 - \frac{\alpha_*}{2} e_*^2 + \left(\beta + \frac{C_8}{\delta \gamma} \right) \left((1 + \lambda) \left(1 - \frac{\vartheta^2}{2}\right) \eta^2 + \frac{C_7}{\lambda} e_*^2 \right) + \frac{C_8}{\delta \gamma} \eta^2. \end{aligned}$$

Writing $1 - \frac{\vartheta^2}{2} = \left(1 - \frac{\vartheta^2}{4}\right) - \frac{\vartheta^2}{4}$ and using $E^2 \leq C_6 \eta^2$ from Corollary (5.2), we easily obtain for $\gamma \geq \gamma_1$

$$\begin{aligned} E_*^2 + \beta \eta_*^2 &\leq \left[(1 + \delta) - \left(\beta + \frac{C_8}{\delta \gamma} \right) \frac{1 + \lambda \vartheta^2}{C_6} \frac{1}{4} \right] E^2 + \left[\left(\beta + \frac{C_8}{\delta \gamma} \right) \frac{C_7}{\lambda} - \frac{\alpha_*}{2} \right] e_*^2 \\ &\quad + \left[(1 + \lambda) \left(1 - \frac{\vartheta^2}{4}\right) + \frac{C_8}{\beta \delta \gamma} \left(1 + (1 + \lambda) \left(1 - \frac{\vartheta^2}{4}\right)\right) \right] \beta \eta^2 \\ &=: \varrho_1 E^2 + \varrho_2 e_*^2 + \varrho_3 \beta \eta^2. \end{aligned}$$

At this point, we first choose λ sufficiently small to have $(1 + \lambda) \left(1 - \frac{\vartheta^2}{4}\right) < 1$. Next, we choose δ sufficiently small to have $\varrho_1 < 1$ for $\gamma = \gamma_1$, hence for any $\gamma \geq \gamma_1$. Then, the parameter $\beta > 0$ is determined by imposing $\varrho_2 = 0$, which is possible provided γ is large enough, say $\gamma \geq \gamma_2 \geq \gamma_1$. Finally, for γ even larger, say $\gamma \geq \gamma_3 \geq \gamma_2$, the second addend in ϱ_3 can be made so small that $\varrho_3 < 1$. In conclusion, the desired result holds for all $\gamma \geq \gamma_3$ with $\varrho := \max(\varrho_1, \varrho_3)$. \square

Corollary 5.3. Denote by $\{(\mathcal{D}_k, u_{\mathcal{D}_k}, \eta_{\mathcal{D}_k}(u_{\mathcal{D}_k})) : k \geq 0\}$ the sequence produced by iterating the mapping (36) from the input partition $\mathcal{D}_0 := \mathcal{D}_{in}$. Then,

$$\|u - u_{\mathcal{D}_k}\|_{\mathcal{D}_k}^2 \leq \alpha_*^{-1} \varrho^k (\|u - u_{\mathcal{D}_0}\|_{a, \mathcal{D}_0}^2 + \beta \eta_{\mathcal{D}_0}^2(u_{\mathcal{D}_0})). \quad \square$$

The latter result guarantees that the target accuracy $\|u - u_{\mathcal{D}_k}\|_{\mathcal{D}_k}^2 \leq \varepsilon^2$ of **DG-SOLVE** can be matched provided the iterations are stopped at a sufficiently large k . In particular, if there exists a constant $C_9 > 0$ such that

$$\|u - u_{\mathcal{D}_0}\|_{a, \mathcal{D}_0}^2 + \beta \eta_{\mathcal{D}_0}^2(u_{\mathcal{D}_0}) \leq C_9 \varepsilon^2, \quad (43)$$

then the number K of iterations in **DG-SOLVE** is bounded independently of ε . In this case, since the mapping (36) at most doubles the cardinality of the partition, i.e., $|\mathcal{D}_*| \leq 2|\mathcal{D}|$, we conclude that the cardinality of the output partition $\mathcal{D}_{out} := \mathcal{D}_K$ is uniformly bounded by the cardinality of the input partition \mathcal{D}_{in} , precisely

$$|\mathcal{D}_{out}| \leq 2^K |\mathcal{D}_{in}|.$$

Remark 5.1. (*Arithmetic complexity*) According to [5], if $N := \#\mathcal{D}$ denotes the cardinality of the current *hp*-partition, the arithmetic complexity of **hp-NEARBEST** is $O(N^2)$ (or $O(N \log N)$ in some specific situations). On the other hand, **DG-SOLVE** performs a bounded numbers of solutions of DG problems, which can be achieved in linear complexity. \square

5.3 Initialization

Let us discuss a possible strategy to fulfill (43). Recall that we enter **DG-SOLVE** at iteration i of **hp-ADFEM** with input partition \mathcal{D}_i and data $g_{\mathcal{D}_i}$. This means that, with the notation of **hp-ADFEM**, condition (43) reads

$$\|u(g_{\mathcal{D}_i}) - u_{\mathcal{D}_i}\|_{a, \mathcal{D}_i}^2 + \beta \eta_{\mathcal{D}_i}^2(u_{\mathcal{D}_i}) \leq C_9 \varepsilon_i^2. \quad (44)$$

The first term on the left-hand side can be bounded from above by using the uniform continuity of the form $a_{\mathcal{D}_i}$ and the bounds given in Property 5.2, Corollary 5.1 and Proposition 5.2. This yields

$$\|u(g_{\mathcal{D}_i}) - u_{\mathcal{D}_i}\|_{a, \mathcal{D}_i}^2 + \beta \eta_{\mathcal{D}_i}^2(u_{\mathcal{D}_i}) \leq C_{10} \inf_{w_{\mathcal{D}_i} \in V_{\mathcal{D}_i}^c} \|u(g_{\mathcal{D}_i}) - w_{\mathcal{D}_i}\|_{H_0^1(\Omega)}^2 + C_{11} \eta_{\mathcal{D}_i}^2(u_{\mathcal{D}_i})$$

for constants $C_{10}, C_{11} > 0$ independent of \mathcal{D}_i . We now show that the infimum on the right-hand side can be bounded by a multiple of ε_i^2 .

Property 5.4. *There exists a constant $C_{12} > 0$ independent of \mathcal{D}_i such that*

$$\inf_{w_{\mathcal{D}_i} \in V_{\mathcal{D}_i}^c} \|u(g_{\mathcal{D}_i}) - w_{\mathcal{D}_i}\|_{H_0^1(\Omega)} \leq C_{12} \varepsilon_i$$

Proof. For simplicity, set again $u := u(g_{\mathcal{D}_i})$. Then, for any $w_{\mathcal{D}_i} \in V_{\mathcal{D}_i}^c$, let us write $u - w_{\mathcal{D}_i} = (u - u_\star) + (u_\star - \bar{u}_{i-1}) + (\bar{u}_{i-1} - w_{\mathcal{D}_i})$. Using (17), we get

$$\|u - u_\star\|_{H_0^1(\Omega)} = \|u(g_\star) - u(g_{\mathcal{D}_i})\|_{H_0^1(\Omega)} \leq C_\star \kappa E_{\mathcal{D}_i}(\bar{u}_{i-1}, g_\star)^{\frac{1}{2}} \leq C_\star \kappa \omega \varepsilon_{i-1}. \quad (45)$$

On the other hand, recalling (20), we have

$$\|u_\star - \bar{u}_{i-1}\|_{\bar{\mathcal{D}}_{i-1}} \leq \varepsilon_{i-1}. \quad (46)$$

Let us define $w_{\mathcal{D}_i}$ as follows. Set $\psi := (\bar{u}_{i-1})_x^\sim \in L^2(\Omega)$ and let $q \in L^2(\Omega)$ be the piecewise polynomial function such that $q|_{K_D} = \Pi_{K_D, p_D-1}^0 \psi|_{K_D}$ for all $D \in \mathcal{D}_i$. Notice that, recalling the definition (15), we have

$$\|\psi - q\|_\Omega^2 = \sum_{D \in \mathcal{D}_i} \|\psi - q\|_{K_D}^2 \leq \sum_{D \in \mathcal{D}_i} e_D(\bar{u}_{i-1}, g_\star) = E_{\mathcal{D}_i}(\bar{u}_{i-1}, g_\star) \leq \omega^2 \varepsilon_{i-1}^2.$$

On the other hand, it holds

$$\int_\Omega q = \int_\Omega \psi = \sum_{D \in \mathcal{D}_{i-1}} \int_{K_D} \bar{u}_{i-1, x} = - \sum_{e \in \mathcal{E}_{\mathcal{D}_{i-1}}} \llbracket \bar{u}_{i-1} \rrbracket_e = - \sum_{e \in \mathcal{E}_{\mathcal{D}_{i-1}}} \sigma_{\mathcal{D}_{i-1}, e}^{-1/2} \sigma_{\mathcal{D}_{i-1}, e}^{1/2} \llbracket \bar{u}_{i-1} \rrbracket_e,$$

whence

$$\left(\int_\Omega q \right)^2 \leq \left(\sum_{e \in \mathcal{E}_{\mathcal{D}_{i-1}}} \sigma_{\mathcal{D}_{i-1}, e}^{-1} \right) \|\sigma_{\mathcal{D}_{i-1}}^{1/2} \llbracket \bar{u}_{i-1} \rrbracket\|_{\mathcal{E}_{\mathcal{D}_{i-1}}}^2 \leq |\Omega| \|\sigma_{\mathcal{D}_{i-1}}^{1/2} \llbracket \bar{u}_{i-1} \rrbracket\|_{\mathcal{E}_{\mathcal{D}_{i-1}}}^2 \leq \frac{|\Omega|}{\gamma} \varepsilon_{i-1}^2$$

by (46). Therefore, if we set

$$w_{\mathcal{D}_i}(x) = \int_{x_0}^x q(s) ds - (x - x_0) \int_\Omega q$$

where $x_0 = \min \Omega$, we realize $w_{\mathcal{D}_i} \in V_{\mathcal{D}_i}^c$ and $\|(\bar{u}_{i-1})_x^\sim - w_{\mathcal{D}_i, x}\|_\Omega \leq C \varepsilon_{i-1}$. This concludes the proof, since $\varepsilon_{i-1} \simeq \varepsilon_i$. \square

By Property 5.4, we get the bound

$$\|u(g_{\mathcal{D}_i}) - u_{\mathcal{D}_i}\|_{a, \mathcal{D}_i}^2 + \beta \eta_{\mathcal{D}_i}^2(u_{\mathcal{D}_i}) \leq C_{13} \varepsilon_i^2 + C_{11} \eta_{\mathcal{D}_i}^2(u_{\mathcal{D}_i}).$$

At this point, we may proceed as follows. Assume that we have chosen, once and for all, an absolute constant $\hat{C} > 0$. We check the validity of

$$\eta_{\mathcal{D}_i}^2(u_{\mathcal{D}_i}) \leq \hat{C} \varepsilon_i^2.$$

- In the affirmative case, $u_{\mathcal{D}_i}$ does satisfy condition (44), and we can start the iterations of **DG-SOLVE**.
- In the negative case, we discard $u_{\mathcal{D}_i}$ and compute $\hat{u}_{\mathcal{D}_i}^c \in V_{\mathcal{D}_i}^c$, the (continuous) Galerkin approximation of $u(g_{\mathcal{D}_i})$ on the partition \mathcal{D}_i . For such an approximation, it is known that the residual estimator is both reliable and efficient; hence, resorting once more to Property 5.4,

$$\eta_{\mathcal{D}_i}(\hat{u}_{\mathcal{D}_i}^c) \simeq \|u(g_{\mathcal{D}_i}) - \hat{u}_{\mathcal{D}_i}^c\|_a \simeq \|u(g_{\mathcal{D}_i}) - \hat{u}_{\mathcal{D}_i}^c\|_{H_0^1(\Omega)} \leq C_{12} \varepsilon_i.$$

Therefore, condition (44) is satisfied with $u_{\mathcal{D}_i}$ replaced by $\hat{u}_{\mathcal{D}_i}^c$, and we start the iterations of **DG-SOLVE** from this approximation.

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