Structural Properties of a Class of Linear Hybrid Systems and Output Feedback Stabilization

Structural Properties of a Class of Linear Hybrid Systems and Output Feedback Stabilization

Corrado Possieri and Andrew R. Teel

***Abstract*—In this paper, we deal with the problem of output feedback stabilization for a class of linear hybrid systems. This problem is addressed by characterizing the structural properties of such a class of systems. Namely, reachability, controllability, stabilizability, observability, constructibility and detectability are framed in terms of algebraic and geometric conditions on the data of the system. Two canonical forms, recalling the classical Kalman decompositions with respect to reachability and observability, are given. By taking advantage of this characterization, duality between control and observation structural properties is established and necessary and sufficient conditions for the existence of a linear time–invariant output feedback compensator are stated. Compared with previous results, no assumption is needed on the plant about minimum phaseness, relative degree or squareness.**

I. Introduction

Hybrid dynamical systems provide a comprehensive framework to characterize processes evolving according to continuous–time dynamics (*flow*) and discrete–time dynamics (*jump*) (see [1], [2] and references therein). Many tools for the analysis and control of such a class of systems have been developed [3]–[9]. For instance, in [10], necessary and sufficient conditions for reachability and observability of linear switched system have been stated assuming nonsingularity of impulsive gain matrices, while, in [11], the same structural properties have been characterized by removing the latter assumption. In this work, we focus the attention on a class of widely studied linear hybrid systems (see [12]–[19] and references therein) where the clock variable satisfies a constant dwell–time and is available for feedback. This allows us to focus our attention on a linear setting and to extend many of the classical results for non–hybrid linear systems. As a matter of fact, even if the results given in this paper are formalized in the modern hybrid formalism, they are strongly related to the scientific research carried out in the 1980's, in the context of multi–rate sampled–data systems, generalized holders and periodic systems (we refer the interested reader to [20]–[22] for multirate sampled–data systems, to [23] for generalized holders, and to [24]–[27] for periodic systems).

One of the main reasons of interest in output feedback stabilization for such a class of hybrid systems is in completing a key aspect on necessary conditions for hybrid output regulation [28]. Such a problem has been also proven relevant in many applications involving hybrid systems, as, for instance, billiard systems, juggling and walking robots [29]–[32]. In this scenario, the objective is to drive the output of the hybrid plant to zero, despite the presence of a hybrid disturbance input that is generated by an external exosystem with known dynamics and unknown initial condition. In [33], [34], compensator structures are proposed to generate desired steady–state solutions solving this problem. In [35], [36], it is shown that a time–invariant compensator is able to generate the steady state, but stabilization is achieved by exploiting a time–varying compensator. This paper shows that, under the mere stabilizability and detectability hypotheses, a linear dynamic time–invariant output feedback stabilizer actually exists. To achieve such an objective, the structural properties of this class of systems are framed in terms of the hybrid system data. Namely, we propose conditions, wholly similar to the Popov-Belevitch-Hautus (PBH) tests [37], [38], that guarantee reachability, controllability, and stabilizability of the hybrid system. Similar conditions are given also for observation objectives, leading to the characterization of observability, constructibility, and detectability. Two standard forms, mimicking the classical Kalman decomposition [39] for non–hybrid linear systems, are proposed. Taking advantage of this characterization, a linear dynamic time–invariant output feedback compensator for this class of hybrid systems is given. Hence, the employment of the output feedback stabilizer given in this paper with the steady–state generator designed in [35] allows to solve the output regulation problem proposed in [33]. Moreover, as shown in Example 3, the output feedback stabilizer proposed in this paper can be also employed to achieve stability of a class of mechanical systems subject to periodic jumps (see also [40]).

The remainder of the paper is organized as follows: In Section II the considered class of hybrid systems is introduced and some preliminary results are stated. In Section III, a comprehensive characterization of structural properties of these hybrid systems is given. Namely, in Section III-A, reachability, controllability, stabilizability, and a new structural property, called strong reachability, are characterized and a control standard form is proposed. In Section III-B, observability, constructibility, and detectability are framed in terms of the data of the hybrid system, and an observation standard form is given. In Section IV, a duality theorem, relating "control" structural properties of a given system with "observation" properties of a dual hybrid linear systems, is stated. In Section V, necessary and sufficient conditions for the existence of a stabilizing linear dynamic time–invariant output feedback are given and a compensator structure is proposed. In Section VI the extension of the results of this paper to arbitrary initial conditions for the timer variable is analyzed. Conclusions and future work are discussed in Section VII.

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II. Notation and Preliminaries

Let $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{C}$ denote the set of real, integer and complex numbers, respectively. Define $\mathbb{C}_g := \{s \in \mathbb{C} : |s| < 1\}$. Letting M be a square matrix, $\Lambda(M)$ denotes the spectrum of M. Letting $S \in \mathbb{R}^{n\times\ell}$, define $\mathrm{Im}(S) := \{x \in \mathbb{R}^n : \exists z \in \mathbb{R}^\ell$ such that $x = Sz\}$ and $\mathrm{Ker}(S) := \{z \in \mathbb{R}^\ell : Sz = 0\}$. Let $\mathcal{X} \neq \{0\}$ be a subspace of $\mathbb{R}^n$; the *dimension of* $\mathcal{X}$, denoted $\dim(\mathcal{X})$, is the number of elements in any basis of $\mathcal{X}$.

Consider the hybrid system governed by the flow dynamics

$$\begin{aligned} \dot{\tau} &= 1, && \text{(1a)} \\ \dot{x} &= Ax + Bu_F, && \text{(1b)} \end{aligned}$$

whether $(\tau, x) \in [0, \tau_M] \times \mathbb{R}^n$, and subject to jumps according to the rules

$$\begin{aligned} \tau^+ &= 0, && \text{(1c)} \\ x^+ &= Ex + Fu_J, && \text{(1d)} \end{aligned}$$

whether $(\tau, x) \in \{\tau_M\} \times \mathbb{R}^n$, with state $x(t,k) \in \mathbb{R}^n$, flow input $u_F(t,k) \in \mathbb{R}^{m_1}$, jump input $u_J(k) \in \mathbb{R}^{m_2}$, initial conditions $x(0,0) = x_0$, $x_0 \in \mathbb{R}^n$, and $\tau(0,0) = 0$ (in the subsequent Section VI, extensions of the results of this paper to $\tau(0,0) = \tau_0$, $\tau_0 \in [0, \tau_M]$ are discussed). In the previous equations, τ_M is a positive known constant that imposes a fixed dwell–time constraint between two consecutive jumps. Hence, each solution (usually called *hybrid arc*) to system (1) is defined on the *hybrid time domain*

$$\mathcal{T} := \{(t,k), t \in [k\tau_M, (k+1)\tau_M], k \in \mathbb{Z}_{\geqslant 0}\},$$

which is then *a priori* fixed. Solutions to system (1) are *hybrid arcs*, *i.e.*, locally absolutely continuous functions mapping $(t,k) \in \mathcal{T}$ in the indicated set. For compactness, given $(t,k) \in \mathcal{T}$, the shortcut $t_k := k\tau_M$ will be used.

Let $\varphi(t,k,x_0,u_F,u_J)$ be the solution to system (1) at hybrid time $(t,k) \in \mathcal{T}$, with initial condition x_0 and inputs $u_F(\cdot,\cdot)$, $u_J(\cdot)$. With some abuse of notation, we say that the system (1) is *Linear Time–Invariant* (briefly, *LTI*), to underline the fact that, given $x_{0,1}, x_{0,2} \in \mathbb{R}^n$, $u_{F,1}(\cdot,\cdot)$, $u_{F,2}(\cdot,\cdot)$, $u_{J,1}(\cdot)$, and $u_{J,2}(\cdot)$, $\varphi(t,k,\alpha x_{0,1} + \beta x_{0,2}, \alpha u_{F_1} + \beta u_{F,2}, \alpha u_{J,1} + \beta u_{J,2}) = \alpha\varphi(t,k,x_{0,1},u_{F,1},u_{J,1}) + \beta\varphi(t,k,x_{0,2},u_{F,2},u_{J,2})$, for each $\alpha, \beta \in \mathbb{R}$, and that, for each $x_0 \in \mathbb{R}^n$, $u_F(\cdot,\cdot)$, $u_J(\cdot)$, $\varphi(t,k,x_0,u_F,u_J)$ equals the solution to system (1) at hybrid time $(t+h\tau_M, k+h) \in \mathcal{T}$, with initial condition $x(t_h,h) = x_0$ and inputs $\tilde{u}_F(t,k) = u_F(t-h\tau_M, k-h)$, $\tilde{u}_J(k) = u_J(k-h)$, for each $(t,k) \in \mathcal{T} \cap ([t_h,\infty) \times \{h, h+1, \cdots\})$, $h \in \mathbb{Z}_{\geqslant 0}$. System (1) is *stable* if for each $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that $|x_0| < \delta_\varepsilon \implies |\varphi(t,k,x_0,0,0)| \leqslant \varepsilon, \forall (t,k) \in \mathcal{T}$. System (1) is *attractive* if $\lim_{t+k\to\infty} |\varphi(t,k,x_0,0,0)| = 0$. System (1) is *asymptotically stable* if is stable and attractive. Let $Ee^{A\tau_M}$ be the *monodromy matrix* of system (1). As discussed in [35], [36], system (1) is asymptotically stable if and only if all the eigenvalues of the monodromy matrix $Ee^{A\tau_M}$ (or, equivalently, $e^{A\tau_M}E$, because $\Lambda(Ee^{A\tau_M}) = \Lambda(e^{A\tau_M}E)$ even if, in general, $Ee^{A\tau_M} \neq e^{A\tau_M}E$ [41, Ex. 7.1.19]) lie in the open unit circle in the complex plane $\mathbb{C}_g$.

III. Structural properties of hybrid systems

Let $y_F(t,k) \in \mathbb{R}^{q_1}$ and $y_J(k) \in \mathbb{R}^{q_2}$ be the measurable outputs of system (1), defined as

$$\begin{aligned} y_F(t,k) &:= C_F x(t,k), && \text{(2a)} \\ y_J(k) &:= C_J x(t_k, k-1). && \text{(2b)} \end{aligned}$$

Note that, as a matter of convenience, we separate the continuous–time input u_F (resp., output y_F) from the discrete–time input u_F (resp., output y_J), but, by [42], there is no conceptual difficulty in keeping them together in a single vector $u = [\ u_F' \quad u_J'\ ]'$ (resp., $y = [\ y_F' \quad y_J'\ ]'$), by simply redefining the matrices in (1), (2).

Mimicking the definitions for non–hybrid linear systems, system (1), (2) is said to be

- *stabilizable* if, for any initial condition $x_0 \in \mathbb{R}^n$, there exist inputs $u_F(\cdot,\cdot)$ and $u_J(\cdot)$ such that $\lim_{t+k\to\infty} \varphi(t,k,x_0,u_F,u_J) = 0$;
- *controllable* if, for any initial condition $x_0 \in \mathbb{R}^n$, there exist inputs $u_F(\cdot,\cdot)$, $u_J(\cdot)$, and a finite hybrid time $(\theta,\kappa) \in \mathcal{T}$ such that $\varphi(\theta,\kappa,x_0,u_F,u_J) = 0$;
- *reachable* if, for each $x \in \mathbb{R}^n$, there exist inputs $u_F(\cdot,\cdot)$, $u_J(\cdot)$, and a finite hybrid time $(\theta,\kappa) \in \mathcal{T}$ such that $\varphi(\theta,\kappa,0,u_F,u_J) = x$;
- *detectable* if, for any initial condition $x_0 \in \mathbb{R}^n$, by using only measurements of the input functions $u_F(\cdot,\cdot)$, $u_J(\cdot)$ and of the outputs $y_F(\cdot,\cdot)$, $y_J(\cdot)$, it is possible to determine an estimate $\hat{x}(t,k)$ of $x(t,k) = \varphi(t,k,x_0,u_F,u_J)$ that is such that $\lim_{t+k\to\infty} \hat{x}(t,k) - x(t,k) = 0$;
- *constructible* if, for any initial condition $x_0 \in \mathbb{R}^n$, there exists a hybrid time $(\theta,\kappa) \in \mathcal{T}$ such that, by using only measurements of the input functions $u_F(\cdot,\cdot)$, $u_J(\cdot)$ and of the outputs $y_F(\cdot,\cdot)$, $y_J(\cdot,)$ up to time (θ,κ), it is possible to determine $\varphi(\theta,\kappa,x_0,u_F,u_J)$.
- *observable* if, for any initial condition $x_0 \in \mathbb{R}^n$, there exists a hybrid time $(\theta,\kappa) \in \mathcal{T}$ such that, by using only measurements of the input functions $u_F(\cdot,\cdot)$, $u_J(\cdot)$ and of the outputs $y_F(\cdot,\cdot)$, $y_J(\cdot,)$ up to time (θ,κ), it is possible to determine x_0.

The goal of this section is to frame the structural properties defined above in terms of algebraic and geometric conditions on the data of the hybrid system. Namely, we present simple tests, wholly similar to the Popov-Belevitch-Hautus (PBH) tests for non–hybrid linear systems, characterizing the structural properties of such a class of systems. It is worth pointing out that, by the definitions given above, if system (1), (2) is controllable it is also stabilizable, while if it is observable it is also constructible and if it is constructible, it is also detectable. On the other hand, as shown in the subsequent Example 2, reachability does not imply controllability and stabilizability for this class of hybrid systems. Therefore, in order to reestablish these classical implications, a new structural property, called strong reachability, is defined in the subsequent Definition 1.

A. Reachability, controllability and stabilizability

Define the *reachable set*

$$\mathcal{X}_r := \{x \in \mathbb{R}^n : \exists u_F(\cdot,\cdot), u_J(\cdot) \text{ s.t. } \varphi(\theta,\kappa,0,u_F,u_J) = x, \text{for some } (\theta,\kappa) \in \mathcal{T}\}.$$

The following theorem characterizes the set $\mathcal{X}_r$ in terms of the data (A, B, E, F, τ_M) of the hybrid system (1).

Theorem 1. *The set $\mathcal{X}_r$ is given by*

$$\mathcal{X}_r = \bigcup_{t\in[0,\tau_M]} \left(\mathrm{Im}(e^{At} R_{Ee^{A\tau_M},\bar{F}}) + \mathrm{Im}(\delta_t\, R_{A,B})\right),$$

where $R_{Ee^{A\tau_M},\bar{F}} = [\ \bar{F} \quad Ee^{A\tau_M}\bar{F} \quad \cdots \quad (Ee^{A\tau_M})^{n-1}\bar{F}\ ]$, $\bar{F} = [\ F \quad ER_{A,B}\ ]$, $R_{A,B} = [\ B \quad \cdots \quad A^{n-1}B\ ]$, and $\delta_t = 0$ if $t = 0$, or $\delta_t = 1$, if $t \in (0, \tau_M]$.

Proof. Define the *reachable set with fixed final time* $(t, k) \in \mathcal{T}$

$$\mathcal{X}_r^{(t,k)} := \{x \in \mathbb{R}, \exists u_F(\cdot,\cdot), u_J(\cdot) \text{ s.t. } \varphi(t,k,0,u_F,u_J) = x\}.$$

Clearly, the reachable set $\mathcal{X}_r$ is given by $\bigcup_{(t,k)\in\mathcal{T}} \mathcal{X}_r^{(t,k)}$ [43]. By classical results about reachability of continuous–time dynamical systems [44], $\mathcal{X}_r^{(t,0)} = \mathrm{Im}(R_{A,B})$, for all $t \in (0, \tau_M]$. Therefore, for each $x \in \mathcal{X}_r^{(\tau_M,0)}$, there exists v such that $x = R_{A,B}v$. Consider the set $\mathcal{X}_r^{(\tau_M,1)}$. For each $x \in \mathcal{X}_r^{(\tau_M,1)}$, there exist vectors u_1, v_1 such that $x = Fu_1 + ER_{A,B}v_1$. Consider now the set $\mathcal{X}_r^{(t,1)}$, for all $t \in (\tau_M, 2\tau_M]$. Since system (1) is LTI, for each $x \in \mathcal{X}_r^{(t,1)}$, there exist vectors u_1, v_1, v_2 such that $x = e^{A(t-\tau_M)}(Fu_1+ER_{A,B}v_1)+R_{A,B}v_2$. As a matter of fact, since system (1) is LTI, $\varphi(t, 1, 0, u_{F,1} + u_{F,2}, u_{J,1}) = \varphi(t, 1, 0, \bar{u}_{F,1}, u_{J,1}) + \varphi(t, 1, 0, \bar{u}_{F,2}, 0)$, where

$$\bar{u}_{F,1}(t,k) = \begin{cases} u_{F,1}(t,k) + u_{F,2}(t,k), & \text{if } t \leqslant \tau_M, \\ 0, & \text{otherwise,} \end{cases}$$
$$\bar{u}_{F,2}(t,k) = \begin{cases} 0, & \text{if } t \leqslant \tau_M, \\ u_{F,1}(t,k) + u_{F,2}(t,k), & \text{otherwise.} \end{cases}$$

Hence, for each $x_1 \in \mathbb{R}^n$ such that there exist $\bar{u}_{F,1}(\cdot,\cdot)$, $u_{J,1}(\cdot)$ such that $x_1 = \varphi(t, 1, 0, \bar{u}_{F,1}, u_{J,1})$, there exist vectors u_1, v_1 such that $x_1 = e^{A(t-\tau_M)}(Fu_1 + ER_{A,B}v_1)$, while, for each $x_2 \in \mathbb{R}^n$ such that there exists $\bar{u}_{F,2}(\cdot,\cdot)$ such that $x_2 = \varphi(t, 1, 0, \bar{u}_{F,2}, 0)$, there exists a vector v_2 such that $x_2 = R_{A,B}v_2$, for all $t \in (\tau_M, 2\tau_M]$. By iterating such a procedure, one obtains that, for each $x \in \mathcal{X}_r^{(t,k)}$, $t \in (t_k, t_{k+1}]$, $k \in \mathbb{Z}_{\geqslant 0}$, there exist vectors $u_1, \ldots, u_k, v_1, \ldots, v_{k+1}$ such that

$$x = e^{A(t-t_k)} \sum_{h=1}^{k} (Ee^{A\tau_M})^{k-h} \bar{F} \begin{bmatrix} u_h \\ v_h \end{bmatrix} + R_{A,B}v_{k+1}, \quad (3)$$

while, for each $\bar{x} \in \mathcal{X}_r^{(t_k,k)}$, $k \in \mathbb{Z}_{\geqslant 0}$, there exist vectors $u_1, \ldots, u_k, v_1, \ldots, v_k$ such that

$$\bar{x} = \sum_{h=1}^{k} (Ee^{A\tau_M})^{k-h} \bar{F} \begin{bmatrix} u_h \\ v_h \end{bmatrix}.$$

Hence, by considering that, by the Cayley-Hamilton theorem [44], there exist $a_0, \ldots, a_{n-1} \in \mathbb{R}$ such that $(Ee^{A\tau_M})^n = \sum_{i=0}^{n-1} a_i (Ee^{A\tau_M})^i$, one has that $x \in \mathcal{X}_r$ if and only if (at least) one of the following two conditions holds:

- $\exists t \in (0, \tau_M]$ and vectors $w_1, \ldots, w_n, v_{n+1}$, such that $x = e^{At} \sum_{h=1}^{n} (Ee^{A\tau_M})^{n-h} \bar{F} w_h + R_{A,B}v_{n+1}$.
- $\exists w_1, \ldots, w_n$ such that $x = \sum_{h=1}^{n} (Ee^{A\tau_M})^{k-h} \bar{F} w_h$.

Therefore, $\mathcal{X}_r = \bigcup_{(t,k)\in\mathcal{T}} \mathcal{X}_r^{(t,k)} = \mathrm{Im}(R_{Ee^{A\tau_M},\bar{F}}) \cup (\bigcup_{t\in(0,\tau_M]} \mathrm{Im}([\ e^{At}R_{Ee^{A\tau_M},\bar{F}} \quad R_{A,B}\ ]))$. Thus, the fact that $\mathrm{Im}(R_{Ee^{A\tau_M},\bar{F}}) = \mathrm{Im}([\ e^{At}R_{Ee^{A\tau_M},\bar{F}} \quad \delta_t\, R_{A,B}\ ]\big|_{t=0})$ concludes the proof. □

Corollary 1. *The reachable set with fixed time $\mathcal{X}_r^{(t,k)}$, $(t,k) \in \mathcal{T}$, $t \geqslant t_n$, $k \geqslant n$, is a subspace of $\mathbb{R}^n$ and is given by*

$$\mathcal{X}_r^{(t,k)} = \begin{cases} \mathrm{Im}([\ e^{A(t-t_k)}R_{Ee^{A\tau_M},\bar{F}} \quad R_{A,B}\ ]), & \text{if } t \neq t_k, \\ \mathrm{Im}(R_{Ee^{A\tau_M},\bar{F}}), & \text{if } t = t_k. \end{cases}$$

Note that, even if the reachable set in fixed time $\mathcal{X}_r^{(t,k)}$ is a subspace of the state space $\mathbb{R}^n$ of system (1) for all $(t,k) \in \mathcal{T}$, the set $\mathcal{X}_r$ needs not be a subspace of $\mathbb{R}^n$ (see also [43]).

Example 1. Consider system (1) with data $n = 3$, $\tau_M = 1$,

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$
$$E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

For such a system, $\mathrm{Im}([\ e^{At}R_{Ee^{A\tau_M},\bar{F}} \quad R_{A,B}\ ]) = \mathrm{Im}([\ 1 \quad t \quad t^2/2\ ]')$. Therefore, the points $x_a = [\ 1 \quad 0 \quad 0\ ]'$ and $x_b = [\ 1 \quad 1 \quad 1/2\ ]'$ are in $\mathcal{X}_r$, but $x_a + x_b \notin \mathcal{X}_r$, whence $\mathcal{X}_r$ is not a subspace of $\mathbb{R}^n$. The reachable set $\mathcal{X}_r$ for such a system is depicted in Figure 1. △

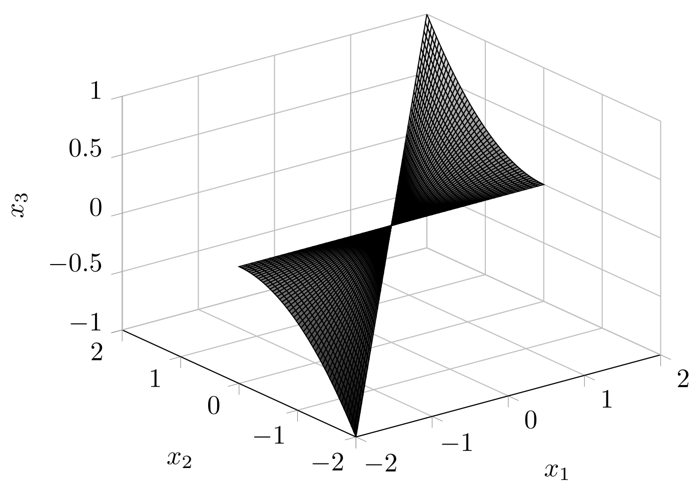

Fig. 1. Set $\mathcal{X}_r$ for the hybrid system given in Example 1.

The following technical lemma characterizes the reachable set of LTI continuous–time non–hybrid systems.

Lemma 1. *Let $x \in \mathbb{R}^n$. There exists a vector v such that $x = R_{A,B}v$ if and only if, for each $\theta \in \mathbb{R}$, there exists a vector v_θ such that $e^{A\theta}x = R_{A,B}v_\theta$.*

Proof. Trivially, if, for each $\theta \in \mathbb{R}$, there exists v_θ such that $e^{A\theta}x = R_{A,B}v_\theta$, then there exists v such that $x = R_{A,B}v$.

Consider the LTI continuous–time system $\dot{x} = Ax + Bu$, and let $\phi_1(t, x_0, u)$ be the solution to such a system with input $u(\cdot)$ and initial condition x_0. If there exists a vector v such that $x = R_{A,B}v$, then there exists an input $u(\cdot)$ and a time $t \in \mathbb{R}$ such that $\phi_1(t, 0, u) = x$ (i.e., x is in the reachable set of $\dot{x} = Ax + Bu$) [44]. Hence, $\tilde{x}_\theta = \phi_1(\theta, x, 0)$ is reachable, i.e., there exists v_θ such that $\tilde{x}_\theta = e^{A\theta}x = R_{A,B}v_\theta$, $\forall \theta \geqslant 0$.

Consider now the LTI continuous–time system $\dot{x} = -Ax + Bu$, and let $\phi_2(t, x_0, u)$ be the solution to such a system with input $u(\cdot)$ and initial condition x_0. Define the matrix $T = \mathrm{diag}(1, -1, 1, -1, \dots)$. Note that, if there exists a vector v such that $x = R_{A,B}v$, then there exists a vector $\bar{v} = Tv$ such that $x = R_{-A,B}\bar{v}$, where $R_{-A,B} = [\ B \quad -AB \quad A^2B \quad \cdots \quad (-A)^{n-1}B\ ]^\top$. Thus, if there exists a vector $\bar{v} \in \mathbb{R}^n$ such that $x = R_{-A,B}\bar{v}$, then there exists an input $u(\cdot)$ and a time $t \in \mathbb{R}$ such that $\phi_2(t, 0, u) = x$ (i.e., x is in the reachable set of $\dot{x} = -Ax + Bu$). Hence, there exists a vector $\bar{v}_\theta$ such that $e^{-A\theta}x = R_{-A,B}\bar{v}_\theta$, for all $\theta \geqslant 0$. Therefore, the vector $v_\theta = T\bar{v}_\theta$ is such that $e^{A\theta}x = R_{-A,B}\bar{v}_\theta = R_{A,B}v_\theta$, for all $\theta \leqslant 0$. ☐

By taking advantage of Lemma 1, the following proposition gives a sufficient condition for system (1) to be reachable.

Proposition 1. If $\mathrm{rank}([\ R_{Ee^{A\tau_M},\bar{F}} \quad R_{A,B}\ ]) = n$, then $\mathcal{X}_r^{(t,k)} = \mathbb{R}^n$, for all $t > t_n$, $t \neq t_k$, $k \geqslant n$.

Proof. If $\mathrm{rank}([\ R_{Ee^{A\tau_M},\bar{F}} \quad R_{A,B}\ ]) = n$, then $\mathrm{rank}(e^{At}[\ R_{Ee^{A\tau_M},\bar{F}} \quad R_{A,B}\ ]) = n$, for all $t \in \mathbb{R}$. As a matter of fact, by Lemma 1, for any vector v of suitable dimensions and any $t \in \mathbb{R}$, there exists a vector $\bar{v}$ of suitable dimensions such that $e^{At}R_{A,B}v = R_{A,B}\bar{v}$. Therefore, one has that $\mathrm{rank}([\ e^{At}R_{Ee^{A\tau_M},\bar{F}} \quad R_{A,B}\ ]) = \mathrm{rank}([\ e^{At}R_{Ee^{A\tau_M},\bar{F}} \quad e^{At}R_{A,B}\ ])$. Thus, since e^{At} is a nonsingular matrix for every $t \in \mathbb{R}$, by Corollary 1, $\mathcal{X}_r^{(t,k)} = \mathbb{R}^n$, for all $t > t_n$, $t \neq t_k$, $k \geqslant n$. ☐

Note that, if there exists a hybrid time $(t, k) \in \mathcal{T}$ such that $\mathcal{X}_r^{(t,k)} = \mathbb{R}^n$, then system (1) is reachable. However, such a condition is only sufficient to guarantee reachability of system (1) (see the subsequent Example 2).

Consider now the *controllable set*

$$\mathcal{X}_c := \{x \in \mathbb{R}^n : \exists u_F(\cdot,\cdot), u_J(\cdot) \text{ s.t. } \varphi(\theta, \kappa, x, u_F, u_J) = 0, \text{for some } (\theta, \kappa) \in \mathcal{T}\}.$$

The following proposition states that, differently from $\mathcal{X}_r$, the set $\mathcal{X}_c$ is a subspace of the state space $\mathbb{R}^n$.

Proposition 2. The set $\mathcal{X}_c$ is a subspace of $\mathbb{R}^n$.

Proof. Let $x_a, x_b \in \mathcal{X}_c$. Hence, there exist $(t_a, k_a), (t_b, k_b) \in \mathcal{T}$, $u_{F,a}(\cdot,\cdot)$, $u_{J,a}(\cdot)$, $u_{F,b}(\cdot,\cdot)$, and $u_{J,b}(\cdot)$ such that $\varphi(t_a, k_a, x_a, u_{F,a}, u_{J,a}) = 0$ and $\varphi(t_b, k_b, x_b, u_{F,b}, u_{J,b}) = 0$. Assume, without loss of generality, that $t_a \geqslant t_b$ and $k_a \geqslant k_b$. Note that system (1) is causal, i.e., that the solution $\varphi(t_b, k_b, x_b, u_{F,b}, u_{J,b})$ depends only on the input $u_{F,b}(t, k)$, for all times $(t, k) \in \mathcal{T}$ such that $t < t_b$ and $k < k_b$, and on $u_{J,b}(k)$, for all $k \in \mathbb{Z}_{\geqslant 0}$, $k \leqslant k_b$. Hence, we can assume, without loss of generality, that $u_{F,b}(t, k) = 0$, for all $(t, k) \in \mathcal{T}$ such that $t \geqslant t_b$ and $k \geqslant k_b$ and that $u_{J,b}(k) = 0$, for all $k \in \mathbb{Z}_{\geqslant 0}$, $k > k_b$. Thus by the linearity of system (1), one has that $\varphi(t_a, k_a, \alpha x_a + \beta x_b, \alpha u_{F,a} + \beta u_{F,b}, \alpha u_{J,a} + \beta u_{J,b}) = \alpha\varphi(t_a, k_a, x_a, u_{F,a}, u_{J,a}) + \beta\varphi(t_b, k_b, x_b, u_{F,b}, u_{J,b}) = 0$, for any $\alpha, \beta \in \mathbb{R}$. Hence, if $x_a, x_b \in \mathcal{X}_c$, then $\alpha x_a + \beta x_b \in \mathcal{X}_c$, for any $\alpha, \beta \in \mathbb{R}$, i.e., $\mathcal{X}_c$ is a subspace of $\mathbb{R}^n$. ☐

The following lemma states that if $x \in \mathcal{X}_c$, then there exist inputs $u_F(\cdot,\cdot)$ and $u_J(\cdot)$ that drive the system to zero after (at most) n jumps of the state of system (1).

Lemma 2. *If $x \in \mathcal{X}_c$, then there exist $u_F(\cdot,\cdot)$ and $u_J(\cdot)$ such that $\varphi(t, n, x, u_F, u_j) = 0$, for some $t \in [t_n, t_{n+1}]$.*

Proof. Define the controllable set in k steps, $k \in \mathbb{Z}_{\geqslant 0}$,

$$\mathcal{X}_c^k := \{x \in \mathbb{R}^n, \exists u_F(\cdot,\cdot), u_J(\cdot) \text{ s.t. } \varphi(\tau, k, x, u_F, u_J) = 0, \text{for some } \tau \in [t_k, t_{k+1}]\}.$$

By the same reasonings given in the proof of Proposition 2, $\mathcal{X}_c^k$ is a subspace of $\mathbb{R}^n$ for any $k \in \mathbb{Z}_{\geqslant 0}$. The subspaces $\mathcal{X}_c^k$ are such that $\mathcal{X}_c^\kappa \subset \mathcal{X}_c^{\kappa+1}$, $\forall \kappa \in \mathbb{Z}_{\geqslant 0}$. As a matter of fact, if $x \in \mathcal{X}_c^\kappa$, then there exist $u_F(\cdot,\cdot)$ and $u_J(\cdot)$ such that $\varphi(\theta, \kappa, x, u_F, u_J) = 0$, for some $\theta \in [t_\kappa, t_{\kappa+1}]$. Thus, inputs

$$\bar{u}_F(t,k) = \begin{cases} u_F(t,k), & \text{if } t \leqslant \theta, \\ 0, & \text{otherwise}, \end{cases}$$
$$\bar{u}_J(k) = \begin{cases} u_J(k), & \forall k \in \{1, \dots, \kappa\}, \\ 0, & \text{otherwise}, \end{cases}$$

are such that $\varphi(\theta, \kappa+1, x, u_F, u_J) = 0$, $\forall \theta \in [t_{\kappa+1}, t_{\kappa+2}]$, i.e., $x \in \mathcal{X}_c^{\kappa+1}$. Thus, the sets $\mathcal{X}_c^\kappa$ are such that

$$\dim(\mathcal{X}_c^0) \leqslant \dim(\mathcal{X}_c^1) \leqslant \cdots \leqslant \dim(\mathcal{X}_c^n) \leqslant \dim(\mathcal{X}_c^{n+1}). \quad (4)$$

Additionally, if $\mathcal{X}_c^\kappa = \mathcal{X}_c^{\kappa+1}$, for some $\kappa \in \mathbb{Z}_{\geqslant 0}$, then $\mathcal{X}_c^\kappa = \mathcal{X}_c^{\kappa+j}$, $j \in \mathbb{Z}_{\geqslant 0}$. Indeed, assume that $\mathcal{X}_c^\kappa = \mathcal{X}_c^{\kappa+1}$, and consider a point $x \in \mathcal{X}_c^{\kappa+2}$. Since $x \in \mathcal{X}_c^{\kappa+2}$, there exist functions $\hat{u}_F(\cdot,\cdot)$ and $\hat{u}_J(\cdot)$ such that $\tilde{x} = \varphi(\tau_M, 1, x, \hat{u}_F, \hat{u}_J) \in \mathcal{X}_c^{\kappa+1} = \mathcal{X}_c^\kappa$. Thus, there exist functions $\tilde{u}_F(\cdot,\cdot)$ and $\tilde{u}_J(\cdot)$ such that $\varphi(\theta, \kappa, \tilde{x}, \tilde{u}_F, \tilde{u}_J) = 0$, for some $\theta \in [t_\kappa, t_{\kappa+1}]$. Hence, since system (1) is LTI, one has that the functions

$$\check{u}_F(t,k) = \begin{cases} \hat{u}_F(t,k), & \text{if } (t,k) \in [0, \tau_M] \times \{0\}, \\ \tilde{u}_F(t,k), & \text{otherwise}, \end{cases}$$
$$\check{u}_J(k) = \begin{cases} \hat{u}_J(k), & k = 1, \\ \tilde{u}_J(k), & \text{otherwise}, \end{cases}$$

are such that $\varphi(\theta, \kappa+1, x, \check{u}_F, \check{u}_J) = 0$, for some $\theta \in [t_{\kappa+1}, t_{\kappa+2}]$, i.e., $x \in \mathcal{X}_c^{\kappa+1}$. Therefore, by iterating such a procedure, one has that if $\mathcal{X}_c^\kappa = \mathcal{X}_c^{\kappa+1}$, for some $\kappa \in \mathbb{Z}_{\geqslant 0}$, then $\mathcal{X}_c^\kappa = \mathcal{X}_c^{\kappa+j}$, $j \in \mathbb{Z}_{\geqslant 0}$. Hence, by considering that $\dim \mathcal{X}_c^\kappa \leqslant n$, for any $\kappa \in \mathbb{Z}_{\geqslant 0}$, $\kappa \geqslant 1$, one has that, in (4), at most the first n inequality signs hold, whence $\mathcal{X}_c^n = \mathcal{X}_c^{n+1}$, and thus $\mathcal{X}_c^{n+j} = \mathcal{X}_c^n$, $j \in \mathbb{Z}_{\geqslant 0}$. ☐

The following theorem gives geometric conditions that characterizes the subspace $\mathcal{X}_c$ in terms of the data (A, B, E, F, τ_M) of the hybrid system (1).

Theorem 2. *The set $\mathcal{X}_c$ is the subspace of $\mathbb{R}^n$ given by*

$$\{x \in \mathbb{R}^n : (Ee^{A\tau_M})^n x \in \mathrm{Im}([\ R_{e^{A\tau_M},\bar{F}} \quad R_{A,B}\ ])\}. \quad (5)$$

Proof. By Lemma 2, if $x \in \mathcal{X}_c$, then there exist $u_F(\cdot,\cdot)$ and $u_J(\cdot)$ such that $\varphi(t, n, x, u_F, u_j) = 0$, for some $t \in$

$[t_n, t_{n+1}]$. By Corollary 1, for a fixed $t \geqslant t_n$, $k \geqslant n$, the set of all the x such that $\exists u_F(\cdot,\cdot), u_J(\cdot)$ such that $\varphi(t,k,0,u_F,u_J) = x$ is $\mathrm{Im}([\ e^{A(t-t_k)}R_{Ee^{A\tau_M},\bar F} \quad R_{A,B}\ ])$, if $t \neq t_k$, or $\mathrm{Im}(R_{Ee^{A\tau_M}})$, if $t = t_k$, $k \in \mathbb{Z}_{\geqslant 0}$. Hence, since $\varphi(t,k,x,u_F,u_J) = \varphi(t,k,0,u_F,u_J) + \varphi(t,k,x,0,0)$ and $\varphi(t,k,x,0,0) = e^{A(t-t_K)}(Ee^{A\tau_M})^k x$, $x \in \mathcal{X}_c$ if and only if there exist vectors w, v and $t \in [0,\tau_M]$ such that

$$e^{At}(Ee^{A\tau_M})^n x - e^{At}R_{Ee^{A\tau_M},\bar F}w - \delta_t R_{A,B}v = 0.$$

By considering that the matrix e^{At} is invertible for any $t \in \mathbb{R}$ and that, by Lemma 1, for any $t \in \mathbb{R}$ and any vector v, there exists a vector $\bar v$ such that $e^{-At}R_{A,B}v = R_{A,B}\bar v$, one has that $x \in \mathcal{X}_c$ if and only if there exist vectors $w, \bar v$ such that

$$(Ee^{A\tau_M})^n x - R_{Ee^{A\tau_M},\bar F}w - R_{A,B}\bar v = 0.$$

Thus $x \in \mathcal{X}_c$ if and only if $(Ee^{A\tau_M})^n x \in \mathrm{Im}([\ R_{e^{A\tau_M},\bar F} \quad R_{A,B}\ ])$. $\square$

Corollary 2. *System* (1) *is controllable if and only if*

$$\mathrm{rank}([\ (Ee^{A\tau_M})^n \quad R_{Ee^{A\tau_M},\bar F} \quad R_{A,B}\ ]) = \mathrm{rank}([\ R_{Ee^{A\tau_M},\bar F} \quad R_{A,B}\ ]). \quad (6)$$

A direct consequence of Corollary 2 is that if E is nonsingular, then system (1) is controllable if and only if $\mathrm{rank}([\ R_{Ee^{A\tau_M},\bar F} \quad R_{A,B}\ ]) = n$. On the other hand, if E is singular, then the condition $\mathrm{rank}([\ R_{Ee^{A\tau_M},\bar F} \quad R_{A,B}\ ]) = n$ still guarantees that system (1) is controllable. Note that, differently from classical non–hybrid linear systems, even if system (1) is reachable, it may not be controllable, as shown in the following example.

Example 2. Consider system (1) with data $n = 2$, $\tau_M = \pi$,

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

For such a system, $\mathrm{Im}([\ e^{At}R_{Ee^{A\tau_M},\bar F} \quad R_{A,B}\ ]) = \mathrm{Im}([\ \cos(t) \quad \sin(t)\ ]')$, whence the set of the reachable state $\mathcal{X}_r = \bigcup_{t\in[0,\pi]} \mathrm{Im}([\ \cos(t) \quad \sin(t)\ ]') = \mathbb{R}^2$. However, there does not exist a hybrid time $(t,k) \in \mathcal{T}$ such that $\mathcal{X}_r^{(t,k)} = \mathbb{R}^2$. As a matter of fact, for each $t \in [0,\tau_M]$, one has that $\mathrm{rank}([\ e^{At}R_{Ee^{A\tau_M},\bar F} \quad R_{A,B}\ ]) < n$. Additionally, since $\mathrm{rank}([\ (Ee^{A\tau_M})^n \quad R_{Ee^{A\tau_M},\bar F} \quad R_{A,B}\ ]) = 2$, while $\mathrm{rank}([\ R_{Ee^{A\tau_M},\bar F} \quad R_{A,B}\ ]) = 1$, system (1) is not controllable, even if it is reachable. $\triangle$

Define the subspace $\mathcal{N}$ of $\mathbb{R}^n$,

$$\mathcal{N} := \mathrm{Im}([\ R_{Ee^{A\tau_M},\bar F} \quad R_{A,B}\ ]).$$

Lemma 3. *The subspace* $\mathcal{N}$ *is* $Ee^{A\tau_M}$*–invariant.*

Proof. Let $x \in \mathcal{N}$ and let $\bar x = Ee^{A\tau_M}x$. By considering that $\bar F = [\ F \quad ER_{A,B}\ ]$ and $R_{Ee^{A\tau_M},\bar F} = [\ \bar F \ \cdots \ (Ee^{A\tau_M})^{n-1}\bar F\ ]$, there exist vectors $v_1,\ldots,v_n,w_1,\ldots,w_n,w_{n+1}$ of suitable dimensions such that $x = \sum_{h=1}^n (Ee^{A\tau_M})^{n-h}Fv_h + \sum_{h=1}^n (Ee^{A\tau_M})^{n-h}ER_{A,B}w_h + R_{A,B}w_{n+1}$. Hence, the vectors $v_1,\ldots,v_n,w_1,\ldots,w_n,w_{n+1}$ are such that $\bar x = \sum_{h=1}^n (Ee^{A\tau_M})^{n-h+1}Fv_h + \sum_{h=1}^n (Ee^{A\tau_M})^{n-h+1}ER_{A,B}w_h + Ee^{A\tau_M}R_{A,B}w_{n+1}$. By the Cayley–Hamilton theorem, there exist $a_0,\ldots,a_{n-1} \in \mathbb{R}$ such that $(Ee^{A\tau_M})^n = \sum_{h=0}^{n-1} a_h (Ee^{A\tau_M})^h$, while, by Lemma 1, there exists $\bar w_{n+1}$ such that $e^{A\tau_M}R_{A,B}w_{n+1} = R_{A,B}\bar w_{n+1}$, for all w_{n+1}. By these reasonings, there exist vectors $\bar v_1,\ldots,\bar v_n,\bar w_1,\ldots,\bar w_n$ of suitable dimensions such that $\bar x = \sum_{h=1}^n (Ee^{A\tau_M})^{n-h}F\bar v_h + \sum_{h=1}^n (Ee^{A\tau_M})^{n-h}ER_{A,B}\bar w_h$. Hence, $\bar x = Ee^{A\tau_M} \in \mathcal{N}$, i.e., $\mathcal{N}$ is $Ee^{A\tau_M}$–invariant. $\square$

Since the subspace $\mathcal{N}$ is $Ee^{A\tau_M}$–invariant, classical results about Kalman decomposition for non–hybrid linear systems can be mimicked. Consider the following proposition.

Proposition 3. Assume that $\mathrm{rank}([\ R_{Ee^{A\tau_M},\bar F} \quad R_{A,B}\ ]) = \nu < n$. Let $\nu_c = \dim(\mathrm{Im}(R_{A,B})) \leqslant \nu$. There exists a matrix $T \in \mathbb{R}^{n\times n}$ such that

$$\hat M := TEe^{A\tau_M}T^{-1} = \begin{bmatrix} \hat M_{r_1,r_1} & \hat M_{r_1,r_2} & \hat M_{r_1,u} \\ \hat M_{r_2,r_1} & \hat M_{r_2,r_2} & \hat M_{r_2,u} \\ 0 & 0 & \hat M_{u,u} \end{bmatrix}, \quad (7a)$$

$$\tilde F := T\bar F = \begin{bmatrix} \tilde F_{r_1} \\ \tilde F_{r_2} \\ 0 \end{bmatrix}, \quad \hat F := TF = \begin{bmatrix} \hat F_{r_1} \\ \hat F_{r_2} \\ 0 \end{bmatrix}, \quad (7b)$$

$$\hat A := TAT^{-1} = \begin{bmatrix} \hat A_{r,r} & \hat A_{r,u_1} & \hat A_{r,u_2} \\ 0 & \hat A_{u_1,u_1} & \hat A_{u_1,u_2} \\ 0 & \hat A_{u_2,u_1} & \hat A_{u_2,u_2} \end{bmatrix}, \quad (7c)$$

$$\hat B := TB = \begin{bmatrix} \hat B_r \\ 0 \\ 0 \end{bmatrix}, \quad (7d)$$

where $\hat M_{r_1,r_1} \in \mathbb{R}^{\nu_c\times\nu_c}$, $\hat M_{r_1,r_2} \in \mathbb{R}^{\nu_c\times\nu-\nu_c}$, $\hat M_{r_1,u} \in \mathbb{R}^{\nu_c\times n-\nu}$, $\hat M_{r_2,r_1} \in \mathbb{R}^{\nu-\nu_c\times\nu_c}$, $\hat M_{r_2,r_2} \in \mathbb{R}^{\nu-\nu_c\times\nu-\nu_c}$, $\hat M_{r_2,u} \in \mathbb{R}^{\nu-\nu_c\times n-\nu}$, $\hat M_{u,u} \in \mathbb{R}^{n-\nu\times n-\nu}$, $\hat F_{r,1} \in \mathbb{R}^{\nu_c\times m_2}$, $\hat F_{r,2} \in \mathbb{R}^{\nu-\nu_c\times m_2}$, and $\hat B_r \in \mathbb{R}^{\nu_c\times m_1}$. Additionally, by letting $M_{r,r} = \begin{bmatrix} \hat M_{r_1,r_1} & \hat M_{r_1,r_2} \\ \hat M_{r_2,r_1} & \hat M_{r_2,r_2} \end{bmatrix}$ and $\tilde F_r = \begin{bmatrix} \tilde F_{r,1} \\ \tilde F_{r,2} \end{bmatrix}$, one has

$$\mathrm{rank}\left(\begin{bmatrix} R_{\hat M_{r,r},\tilde F_r} & \hat R_{\hat A,\hat B} \end{bmatrix}\right) = \nu, \quad (8)$$

with $\hat R_{\hat A,\hat B} = [\ R'_{\hat A_{r,r},\hat B_r} \quad 0'\ ]'$, $R_{\hat M_{r,r},\tilde F_r} = [\ \tilde F_r \ \cdots \ (\hat M_{r,r})^{n-1}\tilde F_r\ ]$ and $R_{\hat A_{r,r},\hat B_r} = [\ \hat B_r \ \cdots \ (\hat A_{r,r})^{n-1}\hat B_r\ ]$.

Proof. Let $e_1,\ldots,e_{\nu_c}$ be a basis of the subspace $\mathrm{Im}(R_{A,B})$, and let $e_{\nu_c+1},\ldots,e_\nu$ be such that $e_1,\ldots,e_\nu$ is a basis of the subspace $\mathcal{N}$. Thus, let $e_{\nu+1},\ldots,e_n$ be such that $e_1,\ldots,e_n$ is a basis of $\mathbb{R}^n$. Note that the vectors $e_1,\ldots,e_n$ are all chosen linearly independent. Let $T = [\ e_1 \ \cdots \ e_n\ ]^{-1}$. Hence, consider the vector $\hat x = Tx$, $\hat x = [\ \hat x_r' \quad \hat x_u'\ ]'$, where $\hat x_r \in \mathbb{R}^\nu$, $\hat x_u \in \mathbb{R}^{n-\nu}$. Note that, by construction, a vector $\hat x \in \mathbb{R}^n$ is in $\mathcal{N}$ if and only if the corresponding sub–vector $\hat x_u = 0$. Consider the matrix

$$\hat M := TEe^{A\tau_M}T^{-1} = \begin{bmatrix} \hat M_{r,r} & \hat M_{r,u} \\ \hat M_{u,r} & \hat M_{u,u} \end{bmatrix},$$

with $\hat{M}_{r,r} \in \mathbb{R}^{\nu\times\nu}$, $\hat{M}_{r,u} \in \mathbb{R}^{\nu\times n-\nu}$, $\hat{M}_{u,r} \in \mathbb{R}^{n-\nu\times\nu}$, and $\hat{M}_{u,u} \in \mathbb{R}^{n-\nu\times n-\nu}$. One has that

$$\hat{M}\hat{x} = \begin{bmatrix} \hat{M}_{r,r}\hat{x}_r + \hat{M}_{r,u}\hat{x}_u \\ \hat{M}_{u,r}\hat{x}_r + \hat{M}_{u,u}\hat{x}_u \end{bmatrix}.$$

By Lemma 3, the subspace $\mathcal{N}$ is $Ee^{A\tau_M}$–invariant. Hence, $\hat{M}_{u,r}\hat{x}_r = 0$, for all $\hat{x}_r$, i.e., $\hat{M}_{u,r} = 0$. Moreover, since $\mathrm{Im}(\tilde{F}) \in \mathcal{N}$, letting $\tilde{F} := T\bar{F} = [\ \tilde{F}_r' \quad \tilde{F}_u'\ ]'$, one has that $\tilde{F}_u = 0$. Thus, by considering that $e_1, \ldots, e_{\nu_c}$ is a basis of $R_{A,B}$, the matrix T is such that (7c)–(7d) hold. Additionally, by considering that the rank of a matrix is invariant with respect to a change of basis, (8) holds. □

Proposition 3 provides a control standard form to represent the dynamics of the hybrid system (1). By taking advantage of this standard form, the following corollary provides an algebraic condition wholly similar to the classical PBH (Popov-Belevitch-Hautus) test to verify controllability of system (1).

Corollary 3. *System* (1) *is controllable if and only if*

$$\mathrm{rank}([\ Ee^{A\tau_M} - sI \quad F \quad R_{A,B}\ ]) = n, \tag{9}$$

$\forall s \in \Lambda(Ee^{A\tau_M})$, $s \neq 0$.

Proof. By Corollary 2, system (1) is controllable if and only if (6) holds. Therefore, in order to prove the statement of this corollary, it suffices to prove that (6) and (9) are equivalent. Assume that (6) holds but (9) does not. Hence, there exists a vector $v \neq 0$ such that $v'Ee^{A\tau_M} = \lambda v'$, with $\lambda \neq 0$ and such that $v'F = 0$, $v'R_{A,B} = 0$. Thus, $v'[\ (Ee^{A\tau_M})^n \quad R_{Ee^{A\tau_M},\bar{F}} \quad R_{A,B}\ ] = [\ v'\lambda^n \quad 0 \quad 0\ ]$ and $v'[\ R_{Ee^{A\tau_M},\bar{F}} \quad R_{A,B}\ ] = 0$. As a matter of fact, by Lemma 1, for each x such that $x = R_{A,B}w$ there exists $\bar{w}$ such that $x = e^{A\tau_M}R_{A,B}\bar{w}$. Hence, $v'ER_{A,B}w = v'Ee^{A\tau_M}R_{A,B}\bar{w} = \lambda v'R_{A,B}\bar{w} = 0$, for each $w \in \mathbb{R}^{n\,m_1}$, and thus $v'ER_{A,B} = 0$. Therefore, since $\lambda \neq 0$, (6) does not hold, leading to a contradiction.

On the other hand, assume that (9) holds, but (6) does not. If (6) does not hold, then $\mathrm{rank}([\ R_{Ee^{A\tau_M},\bar{F}} \quad R_{A,B}\ ]) < n$. Thus, by Proposition 3, there exists a matrix T such that (7) hold. It can be easily checked that $\mathrm{rank}([\ Ee^{A\tau_M} - sI \quad F \quad R_{A,B}\ ]) = \mathrm{rank}([\ \hat{M} - sI \quad \hat{F} \quad R_{\hat{A},\hat{B}}\ ])$. Note that,

$$R_{\hat{A},\hat{B}} = \begin{bmatrix} \hat{B}_r & \hat{A}_{r,r}\hat{B}_r & \cdots & (\hat{A}_{r,r})^{n-1}\hat{B}_r \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Hence, let $\lambda_u \in \Lambda(\hat{M}_{u,u})$, and let $v_u \neq 0$ be such that $v_u'\hat{M}_{u,u} = \lambda_u v_u'$. One has that

$$[\ 0' \quad v_u'\ ][\ \hat{M} - \lambda_u I \quad \hat{F} \quad R_{\hat{A},\hat{B}}\ ] = v_u'[\ 0 \quad \hat{M}_{uu} - \lambda_u I \quad 0 \quad 0\ ] = 0,$$

leading to a contradiction, because if $\lambda_u \in \Lambda(\hat{M}_{u,u})$, then $\lambda_u \in \Lambda(\hat{M}) = \Lambda(Ee^{A\tau_M})$. □

The condition given in (9) is usually known in classical non–hybrid control theory as PBH test. As shown in Example 2, such a condition may not be satisfied by a reachable hybrid system. To reestablish classical implications for non–hybrid linear systems (as, for instance, "reachability implies controllability"), consider the following structural property.

Definition 1. System (1) is *strongly reachable* if, for each $x \in \mathbb{R}^n$, there exist a finite $\kappa \in \mathbb{Z}_{\geqslant 0}$ such that, for all $t \in (t_\kappa, t_{\kappa+1})$, there exist inputs $u_F(\cdot,\cdot)$ and $u_J(\cdot)$ such that $\varphi(t, \kappa, 0, u_F, u_J) = x$.

It is worth noticing that a non–hybrid linear system is reachable if and only if it is strongly reachable. As a matter of fact, if $A = 0$ and $B = 0$, (i.e., if the system is purely discrete), $\varphi(t, \kappa, 0, u_F, u_J) = \varphi(t_\kappa, \kappa, 0, u_F, u_J)$, for all $t \in (t_\kappa, t_{\kappa+1})$, and hence the system is strongly reachable if and only if it is reachable. On the other hand, if $E = I$ and $F = 0$ (i.e., the system is purely continuous), for each $x \in \mathbb{R}^n$, there exist $\bar{t} \in \mathbb{R}$ and $u_F(\cdot,\cdot)$ such that $x = \varphi(\bar{t}, 0, 0, u_F, 0)$ if and only if for each $t \in \mathbb{R}_{>0}$ there exists an input $\tilde{u}_F(\cdot)$ such that $x = \varphi(t, 0, 0, \tilde{u}_F, 0)$. Moreover, by definition, if the system (1) is strongly reachable it is also reachable. In the following theorem, a PBH test wholly similar to (9) is stated for strong reachability.

Theorem 3. *The hybrid system* (1) *is strongly reachable if and only if*

$$\mathrm{rank}([\ R_{Ee^{A\tau_M},\bar{F}} \quad R_{A,B}\ ]) = n, \tag{10}$$

or, equivalently, $\forall s \in \Lambda(Ee^{A\tau_M})$,

$$\mathrm{rank}([\ Ee^{A\tau_M} - sI \quad F \quad R_{A,B}\ ]) = n. \tag{11}$$

Proof. By Corollary 1, for each $\kappa \geqslant n$, $t \in (t_\kappa, t_{\kappa+1})$, there exist $u_F(\cdot,\cdot)$, $u_J(\cdot)$ such that $x = \varphi(t, \kappa, 0, u_F, u_J)$, if and only if $x \in \mathrm{Im}([\ e^{A(t-t_\kappa)}R_{Ee^{A\tau_M},\bar{F}} \quad R_{A,B}\ ])$. By Proposition 1, if (10) holds, then $\mathrm{Im}([\ e^{A(t-t_n)}R_{Ee^{A\tau_M},\bar{F}} \quad R_{A,B}\ ]) = \mathbb{R}^n$ for all $t \in (t_n, t_{n+1})$ and hence, for each $x \in \mathbb{R}^n$ and $t \in (t_n, t_{n+1})$, there exist inputs $u_F(\cdot,\cdot)$, $u_J(\cdot)$ such that $x = \varphi(t, n, 0, u_F, u_J)$, i.e., system (1) is strongly reachable.

Assume now that the system is strongly reachable, but that (10) does not hold. By Theorem 1 and (3), $\mathcal{X}_r^{(\tau+t_k,k)} \subset \mathcal{X}_r^{(\tau+t_{k+1},k+1)}$, for all $\tau \in (0, \tau_M)$, $k \in \mathbb{Z}_{\geqslant 0}$, and $\mathcal{X}_r^{(\tau+t_n,n)} = \mathcal{X}_r^{(\tau+t_{n+h},k+h)}$, for all $\tau \in (0, \tau_M)$, $h \in \mathbb{Z}_{\geqslant 0}$. Therefore, if the system (1) is strongly reachable (i.e., $\mathcal{X}_r^{(t,k)} = \mathbb{R}^n$ for some $k \in \mathbb{Z}_{\geqslant 0}$ and for all $t \in (t_k, t_{k+1})$), for all $x \in \mathbb{R}^n$ and $t \in (t_n, t_{n+1})$, there exist inputs $u_F(\cdot,\cdot)$ and $u_J(\cdot)$ such that $\varphi(t, n, 0, u_F, u_J) = x$. Hence, for each $x \in \mathbb{R}^n$ and $t \in (t_n, t_{n+1})$, there exists a vector w such that $x = [\ e^{A(t-t_n)}R_{Ee^{A\tau_M},\bar{F}} \quad R_{A,B}\ ]w$. Thus, by Lemma 1, there exists a vector $\bar{w}$ such that $x = e^{A(t-t_n)}[\ R_{Ee^{A\tau_M},\bar{F}} \quad R_{A,B}\ ]\bar{w}$. By considering that the matrices $e^{A(t-t_n)}$ and $e^{-A(t-t_n)}$ are nonsingular for each $t \in (t_n, t_{n+1})$ and that $\mathrm{Im}(e^{-A(t-t_n)}) = \mathbb{R}^n$, this is in contradiction with $[\ R_{Ee^{A\tau_M},\bar{F}} \quad R_{A,B}\ ]$ being rank deficient. The equivalence of (10) and (11) follows by the same arguments given in the proof of Corollary 3. □

It is worth pointing out that if the system (1) is strongly reachable, and hence (11) holds, then, by Corollary 3, the system (1) is also controllable (and hence stabilizable). Therefore, thanks to Definition 1, we reestablished the classical implications "strong reachability implies controllability" and "strong reachability implies stabilizability". Furthermore, by considering that, by Corollary 3, the system is controllable if and only if (6) holds and that, if E is nonsingular, (6) holds if and only if (10) holds, then, if E is nonsingular, the system (1) is controllable if and only if it is strongly reachable. Thus, strong reachability reestablish, in the hybrid framework of this paper, the equivalence stated in [10, Thm. 7] for switched linear systems. Moreover, in the subsequent Section IV, we show that strong reachability is the "dual property" of observability.

The following proposition extends the results given in [45, Cor. 1], by stating necessary and sufficient conditions for assigning the eigenvalues different from zero of the closed loop system to an arbitrary autoconjugate set of n complex values with a time–invariant dynamic linear state feedback.

Proposition 4. There exist matrices K_F and K_J such that

$$\Xi := \Lambda(e^{A\tau_M}E + e^{A\tau_M}FK_J + R_{A,B}K_F) \tag{12}$$

is an arbitrary autoconjugate set of n complex values if and only if the system (1) is strongly reachable. Additionally, by letting $\bar{K}_F$ be such that $R_{A,B}K_F = G(\tau_M)\bar{K}_F$, where $G(\tau_M) := \int_0^{\tau_M} e^{A(\tau_M - t)}BB'e^{A'(\tau_M - t)}dt$ is the *reachability Gramian* of system (1) during flow, the *time–invariant dynamic state feedback* with flow dynamics

$$\begin{aligned} \dot{\tau} &= 1, && (13a)\\ \dot{\xi} &= -A'\xi, && (13b)\end{aligned}$$

whether $(\tau, x) \in [0, \tau_M] \times \mathbb{R}^n$, jump dynamics

$$\begin{aligned} \tau^+ &= 0, && (13c)\\ \xi^+ &= e^{A'\tau_M}\bar{K}_F x, && (13d)\end{aligned}$$

whether $(\tau, x) \in \{\tau_M\} \times \mathbb{R}^n$, output

$$\begin{aligned} u_F &= B'\xi, && (13e)\\ u_J &= K_J x, && (13f)\end{aligned}$$

and initial conditions $\tau(0,0) = 0$, $\xi(0,0) = \xi_0$, $\xi_0 \in \mathbb{R}^n$, is such that the resulting closed loop monodromy matrix has spectrum $\Xi \cup \{0\}$.

Proof. By Theorem 3, the system (1) is strongly reachable if and only if (11) (or, equivalently, (10)) holds. By classical results about discrete–time dynamical systems [46], if (11) holds, then here exist matrices K_F and K_J such that the set Ξ given in (12) is an arbitrary autoconjugate set of n complex values (see also [45, Cor. 2]).

Assume now that the set Ξ can be assigned arbitrarily, but that (10) does not hold. Then, $\mathrm{rank}([\ R_{Ee^{A\tau_M},\bar{F}} \quad R_{A,B}\ ]) = \nu < n$. Let $\nu_c = \dim(\mathrm{Im}(R_{A,B}))$ and let T be the matrix given in Proposition 3. Consider the matrix $\hat{M}$ given in (7a). Let $\hat{N} = e^{\hat{A}\tau_M} = Te^{A\tau_M}T^{-1}$. Since $\hat{A}$ satisfies (7c), $\hat{N}$ is such that there exist matrices $\hat{N}_1 \in \mathbb{R}^{\nu_c\times\nu_c}$, $\hat{N}_2 \in \mathbb{R}^{\nu_c\times\nu-\nu_c}$, $\hat{N}_3 \in \mathbb{R}^{\nu_c\times n-\nu}$, $\hat{N}_4 \in \mathbb{R}^{\nu-\nu_c\times\nu-\nu_c}$, $\hat{N}_5 \in \mathbb{R}^{\nu-\nu_c\times n-\nu}$ $\hat{N}_6 \in \mathbb{R}^{n-\nu\times\nu-\nu_c}$, and $\hat{N}_7 \in \mathbb{R}^{n-\nu\times n-\nu}$ such that

$$\hat{N} = \begin{bmatrix} \hat{N}_1 & \hat{N}_2 & \hat{N}_3 \\ 0 & \hat{N}_4 & \hat{N}_5 \\ 0 & \hat{N}_6 & \hat{N}_7 \end{bmatrix}. \tag{14}$$

By considering that $\det(\hat{N}) = \det(\hat{N}_1)\det\left(\left[\begin{smallmatrix} \hat{N}_4 & \hat{N}_5 \\ \hat{N}_6 & \hat{N}_7\end{smallmatrix}\right]\right) \neq 0$, one has that $\det(\hat{N}_1) \neq 0$, i.e., $\hat{N}_1$ is a nonsingular matrix. Consider now the matrix

$$\hat{E} := TET^{-1} = \begin{bmatrix} \hat{E}_1 & \hat{E}_2 & \hat{E}_3 \\ \hat{E}_4 & \hat{E}_5 & \hat{E}_6 \\ \hat{E}_7 & \hat{E}_8 & \hat{E}_9 \end{bmatrix}, \tag{15}$$

where $\hat{E}_1 \in \mathbb{R}^{\nu_c\times\nu_c}$, $\hat{E}_2 \in \mathbb{R}^{\nu_c\times\nu-\nu_c}$, $\hat{E}_3 \in \mathbb{R}^{\nu_c\times n-\nu}$, $\hat{E}_4 \in \mathbb{R}^{\nu-\nu_c\times\nu_c}$, $\hat{E}_5 \in \mathbb{R}^{\nu-\nu_c\times\nu-\nu_c}$ $\hat{E}_6 \in \mathbb{R}^{\nu-\nu_c\times n-\nu}$ $\hat{E}_7 \in \mathbb{R}^{n-\nu\times\nu_c}$ $\hat{E}_8 \in \mathbb{R}^{n-\nu\times\nu-\nu_c}$, and $\hat{E}_9 \in \mathbb{R}^{n-\nu\times n-\nu}$. Consider the matrix, $\hat{M} = TEe^{A\tau_M}T^{-1} = \hat{E}\hat{N}$ given in (7a). Since $\hat{N}_1$ is nonsingular, one has that $\hat{E}_7 = 0$, $\hat{M}_{u,u} = \hat{E}_8\hat{N}_5 + \hat{E}_9\hat{N}_7$, and $\hat{E}_8\hat{N}_4 + \hat{E}_9\hat{N}_6 = 0$.

Let $\bar{K}_F \in \mathbb{R}^{m_1\times n}$ and $K_J \in \mathbb{R}^{m_2\times n}$ and let $\hat{K}_F := K_FT^{-1} = [\ \hat{K}_{F,r} \quad \hat{K}_{F,u}\ ]$, $\hat{K}_{F,r} \in \mathbb{R}^{m_1\times\nu_c}$, and $\hat{K}_J = K_JT^{-1} = [\ \hat{K}_{J,r} \quad \hat{K}_{J,u}\ ]$, $\hat{K}_{J,r} \in \mathbb{R}^{m_2\times\nu}$. By (7c)–(7d),

$$R_{\hat{A},\hat{B}}\hat{K}_F = \begin{bmatrix} R_{\hat{A}_{r,r},\hat{B}_r}\hat{K}_{F,r} & R_{\hat{A}_{r,r},\hat{B}_r}\hat{K}_{F,u} \\ 0 & 0 \end{bmatrix},$$

for any $\bar{K}_F \in \mathbb{R}^{m_1\times n}$. Consider now the matrix $(\hat{E} + \hat{F}\hat{K}_J)$. It can be easily checked that, for any $K_J \in \mathbb{R}^{m_2\times n}$,

$$\bar{E} := \hat{E} + \hat{F}\hat{K}_F = \begin{bmatrix} \bar{E}_1 & \bar{E}_2 & \bar{E}_3 \\ \bar{E}_4 & \bar{E}_5 & \bar{E}_6 \\ 0 & \hat{E}_8 & \hat{E}_9 \end{bmatrix},$$

where $\hat{E}_8$ and $\hat{E}_9$ are the ones given in (15). Consider now the matrix $\bar{M} := \bar{E}\hat{N} = (\hat{E} + \hat{F}\hat{K}_J)e^{\hat{A}\tau_M}$. Since $\hat{M}_{u,u} = \hat{E}_8\hat{N}_5 + \hat{E}_9\hat{N}_7$, and $\hat{E}_8\hat{N}_4 + \hat{E}_9\hat{N}_6 = 0$, one has that

$$\bar{M} = \begin{bmatrix} \bar{M}_1 & \bar{M}_2 & \bar{M}_3 \\ \bar{M}_4 & \bar{M}_5 & \bar{M}_6 \\ 0 & 0 & \hat{M}_{u,u} \end{bmatrix}. \tag{16}$$

Therefore, $\Lambda(\hat{M}_{u,u}) \subset \Lambda(Ee^{A\tau_M} + FK_Je^{A\tau_M} + R_{A,B}K_F)$, for every $K_F \in \mathbb{R}^{m_1\times n}$ and $K_J \in \mathbb{R}^{m_2\times n}$. Since $e^{A\tau_M}$ is invertible, it defines a change of basis. Therefore, $\Lambda(Ee^{A\tau_M} + BK_Je^{A\tau_M} + R_{A,B}K_F) = \Lambda(e^{A\tau_M}E + e^{A\tau_M}BK_J + e^{A\tau_M}R_{A,B}\tilde{K}_F)$, where $\tilde{K}_F = K_Fe^{-A\tau_M}$. Thus, since, by Lemma 1, $\mathrm{Im}(R_{A,B}) = \mathrm{Im}(e^{A\tau_M}R_{A,B})$ and the matrix $e^{-A\tau_M}$ is nonsingular, $\Lambda(\hat{M}_{u,u}) \subset \Lambda(e^{A\tau_M}E + e^{A\tau_M}BK_J + R_{A,B}K_F)$, for every $K_F \in \mathbb{R}^{m_1\times n}$ and $K_J \in \mathbb{R}^{m_2\times n}$, leading to a contradiction. The fact that the time–invariant dynamic state feedback (13) is such that the resulting closed loop monodromy matrix has spectrum $\Xi \cup \{0\}$ follows by [45, Prop. 1]. □

By (14), (15), and (16), it can be easily proved that, if there exists $\lambda \in \Lambda(\hat{M}_{u,u})$, $\lambda \notin \mathbb{C}_g$, then it is not possible to stabilize the system with a static time–invariant state feedback. The following theorem states that such a condition is indeed necessary for the stabilizability of system (1) and that it is, in fact, equivalent to [45, (7)].

Theorem 4. *System* (1) *is stabilizable if and only if* $\nexists\lambda \in \Lambda(\hat{M}_{u,u})$ *such that* $\lambda \notin \mathbb{C}_g$*, where* $\hat{M}_{u,u}$ *is the matrix given in* (7a)*, or, equivalently,* $\forall s \in \Lambda(Ee^{A\tau_M})$*,* $s \notin \mathbb{C}_g$*,*

$$\text{rank}([\ Ee^{A\tau_M} - sI \quad F \quad R_{A,B}\ ]) = n. \qquad (17)$$

Proof. By Propositions 3, 4, and (8), if $\nexists\lambda \in \Lambda(\hat{M}_{u,u})$ such that $\lambda \notin \mathbb{C}_g$, then there exists a dynamic time–invariant state feedback such that the closed loop monodromy matrix has spectrum contained in $\mathbb{C}_g$. Hence, then there exist $u_F(\cdot,\cdot)$ and $u_J(\cdot)$ such that $\lim_{t+k\to\infty} \varphi(t,k,x_0,u_F,u_J) = 0$, i.e., system (1) is stabilizable [36, Prop. 1].

Assume that system (1) is stabilizable and that there exists $\lambda \in \Lambda(\hat{M}_{u,u})$ such that $\lambda \notin \mathbb{C}_g$. Let $\hat{\varphi}(t,k,\hat{x},u_F,u_J)$ denote the solution to system (1) with initial condition $\hat{x}$ and inputs $u_F(\cdot,\cdot)$, $u_J(\cdot)$ in the coordinates $\hat{x} = [\ \hat{x}_r' \quad \hat{x}_u'\ ]' = Tx$, i.e., $\hat{\varphi}(t,k,\hat{x},u_F,u_J) := T\varphi(t,k,Tx,u_F,u_J)$. It can be easily checked that, for all $u_F(\cdot,\cdot)$, $u_J(\cdot)$, the dynamics of $\hat{x}_u(t_k,k)$, $k \in \mathbb{Z}_{\geqslant 0}$, are given by

$$\hat{x}_u(t_{k+1},k+1) = \hat{M}_{u,u}\hat{x}_u(t_k,k). \qquad (18)$$

Let $\gamma(k,w_u)$ be the solution to (18) with initial condition w_u. Hence, since $\lambda \notin \mathbb{C}_g$ and $|\hat{\varphi}(t_k,k,[\ \hat{x}_r' \quad \hat{x}_u'\ ]',u_F,u_J)| \geqslant |\gamma(k,\hat{x}_u)|$, for all $k \in \mathbb{Z}_{\geqslant 0}$, one has that there exists an initial condition w such that $\lim_{t+k\to\infty} \hat{\varphi}(t,k,w,u_F,u_J) \neq 0$, for all $u_F(\cdot,\cdot)$, $u_J(\cdot)$, leading to a contradiction.

We conclude the proof by showing that the set $\Theta = \{\lambda \in \Lambda(Ee^{A\tau_M}) : \text{rank}([\ Ee^{A\tau_M} - sI \quad F \quad R_{A,B}\ ]) \neq n\}$ equals $\Lambda(\hat{M}_{u,u})$. Assume that $\exists\lambda_r \in \Theta$ such that $\lambda_r \notin \Lambda(\hat{M}_{u,u})$. Then, $\lambda_r \in \Lambda(Ee^{A\tau_M}) \setminus \Lambda(\hat{M}_{u,u})$ is such that there exists $v = [\ v_r' \quad v_u'\ ]' \neq 0$ such that

$$\begin{bmatrix} v_r \\ v_u \end{bmatrix}' \begin{bmatrix} \hat{M}_{r,r} - \lambda_r I & \hat{M}_{r,u} & \hat{F}_r & \hat{R}_{\hat{A},\hat{B}} \\ 0 & \hat{M}_{u,u} - \lambda_r I & 0 & 0 \end{bmatrix} = 0,$$

where $\hat{R}_{\hat{A},\hat{B}} = [\ R'_{\hat{A}_{r,r},\hat{B}_r} \quad 0'\ ]'$. By considering that, by Proposition 3, $\text{rank}([\ R_{\hat{M}_{r,r},\tilde{F}_r} \quad \hat{R}_{\hat{A},\hat{B}}\ ]) = \nu$, and hence, by Corollary 3 and (11), $\text{rank}[\ \hat{M}_{r,r} - \lambda I \quad \hat{F}_r \quad \hat{R}_{\hat{A},\hat{B}}\ ] = \nu$, one has that $v_r = 0$, for all $\lambda \in \Lambda(\hat{M}_{r,r})$. Thus, there exists $v_u \neq 0$ such that $v_u'(\hat{M}_{u,u} - \lambda_r I) = 0$, leading to a contradiction. Therefore, $\Theta \subset \Lambda(\hat{M}_{u,u})$. Consider now $\lambda_u \in \Lambda(\hat{M}_{u,u})$, and let $v_u \neq 0$ be such that $v_u'\hat{M}_{u,u} = \lambda_u v_u'$. One has that

$$[\ 0' \quad v_u'\ ][\ \hat{M} - \lambda_u I \quad \hat{F} \quad R_{\hat{A},\hat{B}}\ ] = v_u'[\ 0 \quad \hat{M}_{uu} - \lambda_u I \quad 0 \quad 0\ ] = 0,$$

and hence $\Lambda(\hat{M}_{u,u}) \subset \Theta$. □

B. Observability, constructibility and detectability

Consider system (1), with the measurable outputs (2). By the linearity of such a system it can be easily checked that, for any initial condition x_0 and any control inputs $u_F(\cdot,\cdot)$, $u_J(\cdot)$,

$$\begin{aligned} y_F(t,k) &= C_F\varphi(t,k,x_0,0,0) + C_F\varphi(t,k,0,u_F,u_J), \\ y_J(k) &= C_J\varphi(t_k,k-1,x_0,0,0) + {} \\ &\quad + C_J\varphi(t_k,k-1,0,u_F,u_J). \end{aligned}$$

Note that, since the inputs $u_F(\cdot,\cdot)$ and $u_J(\cdot)$ are assumed to be known, it is always possible to compute the *forced response* $\varphi(t,k,0,u_F,u_J)$ of system (1) to such inputs. Therefore, system (1), (2) is observable, constructible, or detectable if and only if the hybrid system with flow dynamics

$$\dot{\tau} = 1, \qquad (19a)$$
$$\dot{x} = Ax, \qquad (19b)$$

whether $(\tau,x) \in [0,\tau_M] \times \mathbb{R}^n$, jump dynamics

$$\tau^+ = 0, \qquad (19c)$$
$$x^+ = Ex, \qquad (19d)$$

whether $(\tau,x) \in \{\tau_M\} \times \mathbb{R}^n$, outputs

$$y_F(t,k) = C_F x(t,k), \qquad (19e)$$
$$y_J(k) = C_J x(t_k,k), \qquad (19f)$$

and initial conditions $\tau(0,0) = 0$, $x(0,0) = x_0$, is observable, constructible, or detectable, respectively. Note that the *free response* $\varphi(t,k,x_0,0,0)$ of system (1) with initial condition $x_0 \in \mathbb{R}^n$ is given by $\varphi(t,k,x_0,0,0) = e^{A(t-t_k)}(Ee^{A\tau_M})^k x_0$. Hence, the outputs $y_F(t,k)$ and $y_J(k)$ defined in (19e) and (19f), respectively, are given by

$$y_F(t,k) = C_F e^{A(t-t_k)}(Ee^{A\tau_M})^k x_0, \qquad (20a)$$
$$y_J(k) = C_J e^{A\tau_M}(Ee^{A\tau_M})^{k-1} x_0. \qquad (20b)$$

Consider the *unobservable set*

$$\begin{aligned} \mathcal{X}_i := \{x \in \mathbb{R}^n : {} & C_F e^{A(t-t_k)}(Ee^{A\tau_M})^k x = 0 \text{ and} \\ & C_J e^{A\tau_M}(Ee^{A\tau_M})^{k-1} x = 0, \forall (t,k) \in \mathcal{T}\}. \end{aligned}$$

As for classical non–hybrid linear systems, if there exists $x \in \mathcal{X}_i$, $x \neq 0$, then system (19) is not observable, while if $\mathcal{X}_i = \{0\}$, then system (19) is observable.

Lemma 4. $\mathcal{X}_i$ *is an* $Ee^{A\tau_M}$*–invariant subspace of* $\mathbb{R}^n$*.*

Proof. Let x_a, $x_b \in \mathcal{X}_i$, i.e., $C_F e^{A(t-t_k)}(Ee^{A\tau_M})^k x_a = 0$, $C_J e^{A\tau_M}(Ee^{A\tau_M})^{k-1} x_a = 0$, $C_F e^{A(t-t_k)}(Ee^{A\tau_M})^k x_b = 0$, and $C_J e^{A\tau_M}(Ee^{A\tau_M})^{k-1} x_b = 0$, for all $(t,k) \in \mathcal{T}$. Hence, one has that $C_F e^{A(t-t_k)}(Ee^{A\tau_M})^k(\alpha x_a + \beta x_b) = 0$ and $C_J e^{A\tau_M}(Ee^{A\tau_M})^{k-1}(\alpha x_a + \beta x_b) = 0$, for any $\alpha,\beta \in \mathbb{R}$ and for all $(t,k) \in \mathcal{T}$. Thus, $\mathcal{X}_i$ is a subspace of $\mathbb{R}^n$.

Consider now $x \in \mathcal{X}_i$. One has that $C_F e^{A(t-t_k)}(Ee^{A\tau_M})^k x = 0$, $C_J e^{A\tau_M}(Ee^{A\tau_M})^{k-1} x = 0$, for all $(t,k) \in \mathcal{T}$. Let $\bar{x} = Ee^{A\tau_M}x$. One has that $C_F e^{A(t-t_k)}(Ee^{A\tau_M})^k \bar{x} = C_F e^{A(t-t_k)}(Ee^{A\tau_M})^{k+1} x = 0$ and $C_J e^{A\tau_M}(Ee^{A\tau_M})^{k-1}\bar{x} = C_J e^{A\tau_M}(Ee^{A\tau_M})^k x = 0$. Thus, if $x \in \mathcal{X}_i$, then $\bar{x} = Ee^{A\tau_M}x \in \mathcal{X}_i$, i.e., $\mathcal{X}_i$ is $Ee^{A\tau_M}$–invariant. □

Let O_{A,C_F} be the *observability matrix* of the continuous–time LTI system (19b)–(19e), $O_{A,C_F} := [\ C_F' \ \cdots \ (C_F A^{n-1})'\ ]'$, and let $C := [\ (C_J e^{A\tau_M})' \quad O'_{A,C_F}\ ]'$. The following theorem characterizes the set $\mathcal{X}_i$ in terms of the data (A,E,C_F,C_J,τ_M) of the hybrid system (19).

Theorem 5. *The set* $\mathcal{X}_i$ *is the subspace of* $\mathbb{R}^n$ *given by*

$$\mathcal{X}_i = \text{Ker}(O_{Ee^{A\tau_M},C}), \qquad (21)$$

with $O_{Ee^{A\tau_M},C} := [\ C' \ \ \cdots \ \ (C(Ee^{A\tau_M})^{n-1})'\ ]'$.

Proof. Define the matrix $O_{Ee^{A\tau_M},C_Je^{A\tau_M}} := [\ (C_Je^{A\tau_M})' \ \ \cdots \ \ (C_Je^{A\tau_M}(Ee^{A\tau_M})^{n-1})'\ ]'$. By classical results about non–hybrid linear systems, one has that $y_F(t,k) = 0$ for all $(t,k) \in \mathcal{T}$ if and only if $(Ee^{A\tau_M})^k x_0 \in \operatorname{Ker}(O_{A,C_F})$, for all $k \in \mathbb{Z}_{\geqslant 0}$, while $y_J(k) = 0$, for all $k \in \mathbb{Z}_{\geqslant 0}$, if and only if $x_0 \in \operatorname{Ker}(O_{Ee^{A\tau_M},C_Je^{A\tau_M}})$. By the Caylely–Hamilton theorem, there exist $a_0,\ldots,a_{n-1} \in \mathbb{R}$ such that $(Ee^{A\tau_M})^n = \sum_{h=0}^{n-1} a_h (Ee^{A\tau_M})^h$. Hence, $y_F(t,k) = 0$, for all $(t,k) \in \mathcal{T}$, if and only if $(Ee^{A\tau_M})^k x_0 \in \operatorname{Ker}(O_{A,C_F})$, for all $k \in \mathbb{Z}_{\geqslant 0}$, $k \leqslant n-1$. Hence, by considering that $(Ee^{A\tau_M})^k x_0 \in \operatorname{Ker}(O_{A,C_F})$ if and only if $x_0 \in \operatorname{Ker}(O_{A,C_F}(Ee^{A\tau_M})^k)$, one has that $x_0 \in \mathcal{X}_i$ if and only if $x_0 \in \operatorname{Ker}(O_{Ee^{A\tau_M},C_Je^{A\tau_M}}) \cap \operatorname{Ker}([\ O'_{A,C_F} \ \ \cdots \ \ (O_{A,C_F}(Ee^{A\tau_M})^{n-1})'\ ]')$, i.e. if and only if $x_0 \in \operatorname{Ker}(O_{Ee^{A\tau_M},C})$. $\square$

Corollary 4. *The system* (19) *is observable if and only if*

$$\operatorname{rank}(O_{Ee^{A\tau_M},C}) = n, \tag{22}$$

or, equivalently, $\forall s \in \Lambda(Ee^{A\tau_M})$,

$$\operatorname{rank}([\ (Ee^{A\tau_M})' - sI \quad (C_Je^{A\tau_M})' \quad O'_{A,C_F}\ ]') = n. \tag{23}$$

By the same reasonings given in Proposition 4, there exist matrices $L_F \in \mathbb{R}^{n\times q_1}$ and $L_J \in \mathbb{R}^{n\times q_2}$ such that

$$\Upsilon = \Lambda((E + L_JC_J)e^{A\tau_M} + L_FO_{A,C_F}). \tag{24}$$

is an arbitrary autoconjugate set of n complex values if and only if the system (19) is observable.

Define the *observability Gramian* of the system during flow

$$W(\tau_M) = \int_0^{\tau_M} e^{A't}C'_FC_Fe^{At}dt.$$

For each $L_F \in \mathbb{R}^{n\times q_2}$, there exists $\bar{L}_F \in \mathbb{R}^{n\times q_2}$ such that $L_FO_{A,C_F} = \bar{L}_FW(\tau_M)$, since, for each $t \neq 0$, $\operatorname{Ker}(W(t)) = \operatorname{Ker}(O_{A,C_F})$. Hence, consider the system with flow dynamics

$$\begin{aligned} \dot{\tau} &= 1, && (25a)\\ \dot{\check{x}} &= A\check{x}, && (25b)\\ \dot{\zeta} &= C'_F(C_F\check{x} - y_F) - A'\zeta, && (25c) \end{aligned}$$

whether $(\tau,\check{x},\zeta) \in [0,\tau_M]\times\mathbb{R}^n\times\mathbb{R}^n$, jump dynamics

$$\begin{aligned} \tau^+ &= 0, && (25d)\\ \check{x}^+ &= E\check{x} + L_J(C_J\check{x} - y_J) + \bar{L}_Fe^{A'\tau_M}\zeta, && (25e)\\ \zeta^+ &= 0, && (25f) \end{aligned}$$

whether $(\tau,\check{x},\zeta) \in \{\tau_M\}\times\mathbb{R}^n\times\mathbb{R}^n$, and initial conditions $\tau(0,0) = 0$, $\check{x}(0,0) = \check{x}_0$, $\check{x}_0 \in \mathbb{R}^n$, $\zeta(0,0) = \zeta_0$, $\zeta_0 \in \mathbb{R}^n$.

Let L_F be such that $L_FO_{A,C_F} = \bar{L}_FW(\tau_M)$. Let $\psi(t,k,x_0)$ be the solution to system (19) with initial condition x_0, and let $[\ \check{\psi}'(t,k,\check{x}_0,\zeta_0) \quad \mu'(t,k,\check{x}_0,\zeta_0)\ ]'$ be the solution to system (25), with initial condition $[\ \check{x}'_0 \quad \zeta'_0\ ]'$. The following proposition shows that if L_J, L_F are such that $\Upsilon \subset \mathbb{C}_g$, then $\lim_{t+k\to\infty} \check{\psi}(t,k,\check{x}_0,\zeta_0) - \psi(t,k,x_0) = 0$, for all $x_0,\check{x}_0,\zeta_0 \in \mathbb{R}^n$, i.e., system (25) is a state observer for system (19).

Proposition 5. Let Υ be the set given in (24), let L_J and L_F be matrices such that $\Upsilon \subset \mathbb{C}_g$, and let $\bar{L}_F$ be such that $L_FO_{A,C_F} = \bar{L}_FW(\tau_M)$. Then, $\lim_{t+k\to\infty} \check{\psi}(t,k,\check{x}_0,\zeta_0) - \psi(t,k,x_0) = 0$, for all $x_0,\check{x}_0,\zeta_0 \in \mathbb{R}^n$.

Proof. Define the estimation error $\tilde{x}(t,k) := \check{x}(t,k) - x(t,k)$. Since $y_F = C_Fx$ and $y_J = C_Jx$, the flow dynamics of $[\ \tilde{x}' \quad \zeta'\ ]'$ are given by

$$\begin{aligned} \dot{\tau} &= 1, && (26a)\\ \dot{\tilde{x}} &= A\tilde{x}, && (26b)\\ \dot{\zeta} &= C'_FC_F\tilde{x} - A'\zeta, && (26c) \end{aligned}$$

whether $(\tau,\tilde{x},\zeta) \in [0,\tau_M]\times\mathbb{R}^n\times\mathbb{R}^n$, and the jump dynamics of $[\ \tilde{x}' \quad \zeta'\ ]'$ are given by

$$\begin{aligned} \tau^+ &= 0, && (26d)\\ \tilde{x}^+ &= (E + L_JC_J)\tilde{x} + \bar{L}_Fe^{A'\tau_M}\zeta, && (26e)\\ \zeta^+ &= 0, && (26f) \end{aligned}$$

whether $(\tau,\tilde{x},\zeta) \in \{\tau_M\}\times\mathbb{R}^n\times\mathbb{R}^n$. Define $\chi = [\ \tilde{x}' \quad \zeta'\ ]'$,

$$H := \begin{bmatrix} A & 0 \\ C'_FC_F & -A' \end{bmatrix},\ J := \begin{bmatrix} E + L_JC_J & \bar{L}_Fe^{A'\tau_M} \\ 0 & 0 \end{bmatrix}.$$

Clearly, $\dot{\chi} = H\chi$ and $\chi^+ = J\chi$. By considering that

$$e^{H\tau_M} = \begin{bmatrix} e^{A\tau_M} & 0 \\ e^{-A'\tau_M}W(\tau_M) & e^{-A'\tau_M} \end{bmatrix},$$

one has that the monodromy matrix of system (26) is

$$Je^{H\tau_M} = \begin{bmatrix} (E + L_JC_J)e^{A\tau_M} + \bar{L}_FW(\tau_M) & \bar{L}_F \\ 0 & 0 \end{bmatrix}.$$

Hence, the matrix $Je^{H\tau_M}$ has spectrum $\Upsilon \cup \{0\}$. Thus, by [36, Prop. 1], $\lim_{t+k\to\infty} \tilde{x}(t,k) = 0$. $\square$

The state observer given in (25) employs the observability Gramian $W(\tau_M)$ to estimate the state of the system (1) from the outputs $y_F(\cdot,\cdot)$ and $y_J(\cdot)$ given in (19e) and (19f), respectively. It is worth noticing that Gramian–based observers have been used in the literature (see [36], [47]) to estimate the state of system (1) when just continuous–time outputs are available. Proposition 5 generalizes such results for hybrid systems having both continuous– and discrete–time outputs.

In the remainder part of this section, hybrid systems such that (22) (or, equivalently, (23)) does not hold are considered.

Proposition 6. Assume that $\dim(\operatorname{Ker}(O_{Ee^{A\tau_M},C})) = \nu > 0$. Let $\nu_c = \dim(\operatorname{Ker}(O_{A,C_F})) \geqslant \nu$. There exists a matrix $T \in \mathbb{R}^{n\times n}$ such that

$$\hat{M} := TEe^{A\tau_M}T^{-1} = \begin{bmatrix} \hat{M}_{i,i} & \hat{M}_{i,o_1} & \hat{M}_{i,o_2} \\ 0 & \hat{M}_{o_1,o_1} & \hat{M}_{o_1,o_2} \\ 0 & \hat{M}_{o_2,o_1} & \hat{M}_{o_2,o_2} \end{bmatrix}, \tag{27a}$$

$$\hat{C}_J := C_Je^{A\tau_M}T^{-1} = \begin{bmatrix} 0 & \hat{C}_{J,o_1} & \hat{C}_{J,o_2} \end{bmatrix}, \tag{27b}$$

$$\hat{A} := TAT^{-1} = \begin{bmatrix} \hat{A}_{i_1,i_1} & \hat{A}_{i_1,i_2} & \hat{A}_{i_1,o} \\ \hat{A}_{i_2,i_1} & \hat{A}_{i_2,i_2} & \hat{A}_{i_2,o} \\ 0 & 0 & \hat{A}_{o,o} \end{bmatrix}, \tag{27c}$$

$$\hat{C}_F := C_FT^{-1} = \begin{bmatrix} 0 & 0 & \hat{C}_{F,o} \end{bmatrix}, \tag{27d}$$

with $\hat{M}_{i,i} \in \mathbb{R}^{\nu\times\nu}$, $\hat{M}_{i,o_1} \in \mathbb{R}^{\nu\times\nu_c-\nu}$, $\hat{M}_{i,o_2} \in \mathbb{R}^{\nu\times n-\nu_c}$, $\hat{M}_{o_1,o_1} \in \mathbb{R}^{\nu_c-\nu\times\nu_c-\nu}$, $\hat{M}_{o_1,o_2} \in \mathbb{R}^{\nu_c-\nu\times n-\nu_c}$, $\hat{M}_{o_2,o_1} \in \mathbb{R}^{n-\nu_c\times\nu_c-\nu}$, $\hat{M}_{o_2,o_2} \in \mathbb{R}^{n-\nu_c\times n-\nu_c}$, $\hat{C}_{J,o_1} \in \mathbb{R}^{q_2\times\nu_c-\nu}$, $\hat{C}_{J,o_2} \in \mathbb{R}^{q_2\times\nu_c-\nu}$, and $\hat{C}_{F,o} \in \mathbb{R}^{q_1\times n-\nu_c}$. Additionally, by letting $\hat{M}_{o,o} = \begin{bmatrix} \hat{M}_{o_1,o_1} & \hat{M}_{o_1,o_2} \\ \hat{M}_{o_2,o_1} & \hat{M}_{o_2,o_2} \end{bmatrix}$ and $\hat{C} = \begin{bmatrix} \hat{C}_{J,o_1} & \hat{C}_{J,o_2} \\ 0 & O_{\hat{A}_{o,o},\hat{C}_{F,o}} \end{bmatrix}$, where $O_{\hat{A}_{o,o},\hat{C}_{F,o}} := [\ \hat{C}'_{F,o} \ \cdots \ (\hat{C}_{F,o}\hat{A}^{n-1}_{o,o})'\ ]'$, one has that

$$\dim(\mathrm{Ker}(O_{\hat{M}_{o,o},\hat{C}})) = 0, \tag{28}$$

where $O_{\hat{M}_{o,o},\hat{C}} := [\ \hat{C}' \ \cdots \ (\hat{C}\hat{M}^{n-1}_{o,o})'\ ]'$.

Proof. Let $e_1,\dots,e_\nu$ be a basis of the subspace $\mathrm{Ker}(O_{Ee^{A\tau_M},C})$ and let $e_{\nu+1},\dots,e_{\nu_c}$ be such that $e_1,\dots,e_{\nu_c}$ is a basis of $\mathrm{Ker}(O_{A,C_F})$. Thus, let $e_{\nu_c+1},\dots,e_n$ be such that $e_1,\dots,e_n$ is a basis of $\mathbb{R}^n$. Note that the vector $e_1,\dots,e_n$ are chosen linearly independent. Hence, let $T = [\ e_1 \ \cdots \ e_n\ ]^{-1}$. Define the vector $\hat{x} = Tx$, $\hat{x} = [\ \hat{x}'_i \ \ \hat{x}'_o\ ]$, where $\hat{x}_i \in \mathbb{R}^\nu$, $\hat{x}_o \in \mathbb{R}^{n-\nu}$. Note that, by construction, $\hat{x} \in \mathcal{X}_i$ if and only if $\hat{x}_o = 0$. Consider

$$\hat{M} := TEe^{A\tau_M}T^{-1} = \begin{bmatrix} \hat{M}_{i,i} & \hat{M}_{i,o} \\ \hat{M}_{o,i} & \hat{M}_{o,o} \end{bmatrix},$$

where $\hat{M}_{i,i} \in \mathbb{R}^{\nu\times\nu}$, $\hat{M}_{i,o} \in \mathbb{R}^{\nu\times n-\nu}$, $\hat{M}_{o,i} \in \mathbb{R}^{n-\nu\times\nu}$, and $\hat{M}_{o,o} \in \mathbb{R}^{n-\nu\times n-\nu}$. One has that

$$\hat{M}\hat{x} = \begin{bmatrix} \hat{M}_{i,i}\hat{x}_i + \hat{M}_{i,o}\hat{x}_o \\ \hat{M}_{o,i}\hat{x}_i + \hat{M}_{o,o}\hat{x}_o \end{bmatrix}.$$

By Lemma 4 the subspace $\mathcal{X}_i$ is $Ee^{A\tau_M}$–invariant, whence $\hat{M}_{o,i} = 0$. Since $C_Je^{A\tau_M}$ is in the orthogonal complement of $\mathcal{X}_i$, letting $\hat{C}_J = C_Je^{A\tau_M}T^{-1} = [\ \hat{C}_{J,i} \ \ \hat{C}_{j,o}\ ]$, one has that $\hat{C}_{J,i} = 0$. Additionally, by considering that $e_1,\dots,e_{\nu_c}$ is a basis of $\mathrm{Ker}(O_{A,C_F})$, by classical analysis, the matrix T is such that (27c)–(27d) hold. Moreover, since $\dim(\mathrm{Ker}(O_{Ee^{A\tau_M},C})) = \dim(\mathrm{Ker}(O_{\hat{M},\hat{C}}))$, (28) holds. □

Note that the results given in Proposition 6 with respect to observability of system (19) are similar to the results given in Proposition 3 with respect to strong reachability of system (1). Consider the following theorem.

Theorem 6. *System* (19) *is constructible if and only if*

$$\mathcal{X}_i \subset \mathrm{Ker}((Ee^{A\tau_M})^n). \tag{29}$$

Proof. Let $\psi(t,k,x_0)$ be the solution to system (19) at time $(t,k) \in \mathcal{T}$ with initial condition x_0, i.e., $\psi(0,0,x_0) = x_0$. By (20), one has that $C_F\psi(t,k,x_0) = C_F\psi(t,k,\bar{x}_0)$ and $C_J\psi(t_k,k,x_0) = C_J\psi(t_k,k,\bar{x}_0)$, for all $(t,k) \in \mathcal{T}$, $k \in \mathbb{Z}_{\geqslant 0}$, if and only if $\bar{x} := x_0 - \bar{x}_0 \in \mathcal{X}_i$. Therefore, if (29) holds, then $(Ee^{A\tau_M})^n\bar{x} = 0$, whence $\psi(t,k,x_0) = \psi(t,k,\bar{x}_0)$, for all $(t,k) \in \mathcal{T}$ with $t \geqslant t_n$ and $k \in \mathbb{Z}_{\geqslant 0}$, $k \geqslant n$. Thus, system (19) is constructible, because by using only measurements of the outputs $y_F(\cdot,\cdot)$, $y_J(\cdot,)$ up to time (t_n,n), it is possible to determine $\psi(t_n,n,x_0)$.

Assume now that system (19) is constructible and that (29) does not hold. Hence, there exists $\bar{x}_0 \in \mathcal{X}_i$ such that $(Ee^{A\tau_M})^n\bar{x}_0 \neq 0$. By classical results about discrete–time linear systems, if $(Ee^{A\tau_M})^n\bar{x}_0 \neq 0$, then $(Ee^{A\tau_M})^h\bar{x}_0 \neq 0$, for each $h \in \mathbb{Z}_{\geqslant 0}$, $h \geqslant n$. Hence, consider the solutions to system (19) $\psi(t,k,x_0)$ and $\psi(t,k,x_0+\bar{x}_0)$. Since $\bar{x}_0 \in \mathcal{X}_i$, one has that $C\psi(t,k,x_0) = C\psi(t,k,x_0+\bar{x}_0)$, but $\psi(t_h,h,x_0) \neq \psi(t_h,h,x_0+\bar{x}_0)$, $h \in \mathbb{Z}_{\geqslant 0}$, $h \geqslant n$, because $(Ee^{A\tau_M})^h\bar{x}_0 \neq 0$, i.e., system (19) is not constructible, leading to a contradiction. □

Note that (29) holds if and only if $\mathrm{Ker}(O_{Ee^{A\tau_M},C}) = \mathrm{Ker}([\ O'_{Ee^{A\tau_M},C} \ \ ((Ee^{A\tau_M})^n)'\ ]')$, or, equivalently,

$$\mathrm{rank}\left(\begin{bmatrix} Ee^{A\tau_M} \\ O_{Ee^{A\tau_M},C} \end{bmatrix}\right) = \mathrm{rank}(O_{Ee^{A\tau_M},C}). \tag{30}$$

By taking advantage of (30), in the following corollary, we provide a PBH test for constructibility of system (19).

Corollary 5. *The system* (19) *is constructible if and only if for all $s \in \Lambda(Ee^{A\tau_M})$, $s \neq 0$,*

$$\mathrm{rank}([\ (Ee^{A\tau_M})' - sI \ \ (C_Je^{A\tau_M})' \ \ O'_{A,C_F}\ ]') = n. \tag{31}$$

Proof. By Theorem 6, the hybrid system (19) is constructible if and only if (29) (or, equivalently, (30)) holds. Hence, in order to prove the statement of this corollary, it suffices to prove that (30) is equivalent to (31). Assume that (30) hold, but (31) does not. Hence, there exists $v \neq 0$ such that $Ee^{A\tau_M}v = \lambda v$, $Cv = 0$, with $\lambda \neq 0$. Thus, $O_{Ee^{A\tau_M},C}v = 0$, while $[\ (Ee^{A\tau_M})' \ \ O'_{Ee^{A\tau_M},C}\ ]'v = [\ \lambda^n v' \ \ 0\ ]'$, leading to a contradiction.

Assume now that (31) holds, but (30) does not. If (30) does not hold, then $\mathrm{rank}(O_{Ee^{A\tau_M},C}) < n$, whence, by Proposition 6, there exists a matrix T such that (27) holds. Note that $\mathrm{rank}([\ (Ee^{A\tau_M})' \ \ O'_{Ee^{A\tau_M},C}\ ]') = \mathrm{rank}([\ \hat{M}' \ \ O'_{\hat{M},\hat{C}}\ ]')$. Hence, let $\lambda_i \in \Lambda(\hat{M}_{i,i})$ and let v_i be such that $\hat{M}_{i,i}v_i = \lambda_i v_i$. One has that $[\ \hat{M}' - \lambda_i I \ \ \hat{C}'\ ]'[\ v'_i \ \ 0'\ ]' = [\ ((\hat{M}_{ii} - \lambda_i I)v_i)' \ \ 0'\ ]' = 0$, leading to a contradiction. □

The following two results characterize the detectability of system (19) in terms of the data (A,E,C_F,C_J,τ_M).

Theorem 7. *The system* (19) *is detectable if and only if $\nexists\lambda \in \Lambda(\hat{M}_{i,i})$ such that $\lambda \notin \mathbb{C}_g$, or, equivalently, for all $s \in \Lambda(Ee^{A\tau_M})$, $s \notin \mathbb{C}_g$,*

$$\mathrm{rank}([\ (Ee^{A\tau_M})' - sI \ \ (C_Je^{A\tau_M})' \ \ O'_{A,C_F}\ ]') = n. \tag{32}$$

Proof. If $\mathrm{rank}(O_{Ee^{A\tau_M},C}) = n$, then system (19) is detectable (indeed, observable) and there exists no $\lambda \in \Lambda(\hat{M}_{i,i})$ such that $\lambda \notin \mathbb{C}_g$. Assume now that $\mathrm{rank}(O_{Ee^{A\tau_M},C}) < n$. The set of all the s such that (32) does not hold is $\Lambda(\hat{M}_{i,i})$. Hence, $\nexists\lambda \in \Lambda(\hat{M}_{i,i})$ such that $\lambda \notin \mathbb{C}_g$ if and only if (32) holds for all $s \in \Lambda(Ee^{A\tau_M})$, $s \notin \mathbb{C}_g$.

Assume that (32) holds. Hence, there exist matrices $L_F \in \mathbb{R}^{n\times q_1}$ and $L_J \in \mathbb{R}^{n\times q_2}$ such that the set Υ in (24) is contained in $\mathbb{C}_g$. Thus, by Proposition 5, for any $x_0 \in \mathbb{R}^n$, by using only measurements of $y_F(\cdot,\cdot)$, $y_J(\cdot)$, it is possible to determine an estimate $\hat{x}(t,k)$ of $x(t,k)$ that is such that $\lim_{t+k\to\infty}\hat{x}(t,k) - x(t,k) = 0$, i.e., system (19) is detectable.

Assume now that there exists $\lambda \in \Lambda(\hat{M}_{i,i})$ such that $\lambda \notin \mathbb{C}_g$. Let $\hat{\psi}(t,k,\hat{x})$ denote the solution to system (19) with initial condition $\hat{x} = Tx$, i.e., $\hat{\psi}(t,k,\hat{x}) := T\psi(t,k,Tx)$.

Hence, there exists $w \in \mathbb{R}^n$ such that $\hat{C}_F\hat{\psi}(t,k,w) = \hat{C}_F\hat{\psi}(t,k,0) = 0$, $\hat{C}_J\hat{\psi}(t_k,k-1,w) = \hat{C}_J\hat{\psi}(t_k,k-1,0) = 0$, and $\lim_{t+k\to\infty}\hat{\psi}(t,k,w) \neq 0$. Hence, since $\hat{\psi}(t,k,w)$ is indistinguishable from 0 by using only measurements of the outputs $y_F(\cdot,\cdot)$ and $y_J(\cdot)$, it is not possible to determine an estimate $\hat{x}(t,k)$ of $x(t,k)$ that is such that $\lim_{t+k\to\infty}\hat{x}(t,k) - x(t,k) = 0$, i.e., system (19) is not detectable. □

IV. Duality

In this section, a duality theorem for the hybrid system (1), (2) is stated to characterize its "control" structural properties in terms of "observation" structural properties of a dual system. In order to achieve such a result, parity between "control" and "observation" properties has to be established. In fact, in Section III, four "control" and three "observation" structual properties have been defined and framed in terms of algebraic and geometrical conditions on the data of the hybrid system. As highlighted in Section III, the classical implications "strong reachability implies controllability", "controllability implies stabilizability", "observability implies constructibility", and "constructibility implies detectability" hold for the hybrid system (1), (2), while there is not direct implication between reachability and the other structural properties. By this reasoning, in this section, a duality principle is stated neglecting the latter structural property.

Define the *monodromy* discrete–time, LTI system

$$z(k) = \bar{A}z(k) + \bar{B}v(k), \tag{33a}$$
$$w(k) = \bar{C}z(k), \tag{33b}$$

where $\bar{A} := Ee^{A\tau_M}$, $\bar{B} := [\ F \quad R_{A,B}\ ]$, and $\bar{C} := [\ (C_Je^{A\tau_M})' \quad O'_{A,C_F}\ ]'$, where $R_{A,B} := [\ B \ \cdots \ A^{n-1}B\ ]$ and $O_{A,C_F} := [\ C'_F \ \cdots \ (C_FA^{n-1})'\ ]'$. The following proposition characterizes the structural properties of the hybrid system (1), (2) in terms of the ones of the monodromy system (33).

Proposition 7. The system (1), (2) is strongly reachable (resp., controllable, stabilizable, observable, constructible, detectable) if and only if the system (33) is reachable (resp., controllable, stabilizable, observable, constructible, detectable).

Proof. By classical results about discrete–time LTI systems [44], the monodromy system (33) is reachable if and only if $\mathrm{rank}[\ \bar{B} \ \cdots \ \bar{A}^{n-1}\bar{B}\ ] = n$, or, equivalently, $\mathrm{rank}[\ \bar{A}-sI \quad \bar{B}\ ] = n$, for all $s \in \Lambda(\bar{A})$. By considering that $[\ \bar{A}-sI \quad \bar{B}\ ] = [\ Ee^{A\tau_M}-sI \quad F \quad R_{A,B}\ ]$ and that the system (1) is strongly reachable if and only if (11) holds, the system (1), (2) is strongly reachable if and only if the system (33) is reachable . The necessary and sufficient conditions for controllability, stabilizablity, observability, constructibility, and detectability of the system (1), (2) given in Corollaries 3, 4, 5 and in Theorems 3, 4, 7, conclude the proof. □

By [48, Sec. 3.1.A.3], the monodromy system (33) is reachable (resp., controllable, stabilizable, observable, constructible, detectable) if and only if the *dual monodromy* system

$$z_D(k) = \bar{A}_Dz(k) + \bar{B}_Dv(k), \tag{34a}$$
$$w_D(k) = \bar{C}_Dz(k), \tag{34b}$$

where $\bar{A}_D := \bar{A}'$, $\bar{B}_D := \bar{C}'$, and $\bar{C}_D := \bar{B}'$, is observable (resp., constructible, detectable, reachable, controllable, stabilizable). Thus, consider the hybrid system with flow dynamics

$$\dot{\tau} = 1, \tag{35a}$$
$$\dot{x}_D = A_Dx_D + B_Du_{D,F}, \tag{35b}$$

whether $(\tau,x_D) \in [0,\tau_M]\times\mathbb{R}^n$, jump dynamics

$$\tau^+ = 0, \tag{35c}$$
$$x_D^+ = E_Dx_D + F_Du_{D,J}, \tag{35d}$$

whether $(\tau,x_D) \in \{\tau_M\}\times\mathbb{R}^n$, and measurable outputs

$$y_{D,F}(t,k) = C_{D,F}x_D(t,k), \tag{35e}$$
$$y_{D,J}(k) = C_{D,J}x_D(t_k,k-1). \tag{35f}$$

The following theorem is known for non–hybrid linear system as duality theorem.

Theorem 8. *The system* (1), (2) *is strongly reachable (resp., controllable, stabilizable, observable, constructible, detectable) if and only if the system* (35) *with data*

$$\begin{aligned} A_D &= A', & B_D &= C'_F,\\ E_D &= e^{A'\tau_M}E'e^{-A'\tau_M}, & F_D &= e^{A'\tau_M}C'_J,\\ C_{D,F} &= B', & C_{D,J} &= F'e^{-A'\tau_M}, \end{aligned}$$

is observable (resp., constructible, detectable, strongly reachable, controllable, stabilizable).

Proof. By Proposition 7, the system (35) is strongly reachable (resp., controllable, stabilizable, observable, constructible, detectable) if and only if the discrete–time system (34) with $\bar{A}_D = E_De^{A_D\tau_M}$, $\bar{B}_D = [\ F_D \quad R_{A_D,B_D}\ ]$, and $\bar{C}_D = [\ (C_{D,J}e^{A_D\tau_M})' \quad O'_{A_D,C_{D,F}}\ ]'$, where $R_{A_D,B_D} := [\ B_D \ \cdots \ A_D^{n-1}B_D\ ]$ and $O_{A_D,C_{D,F}} := [\ C'_{D,F} \ \cdots \ (C_{D,F}A_D^{n-1})'\ ]'$ is reachable (resp., controllable, stabilizable, observable, constructible, detectable). Therefore, if the data of the hybrid system (35) are the ones given above, one has that $\bar{A}_D := E_De^{A_D\tau_M} = e^{A'\tau_M}E'e^{-A'\tau_M}e^{A'\tau_M} = e^{A'\tau_M}E' = \bar{A}'$, $F_D = e^{A'\tau_M}C'_J$, $R_{A_D,B_D} = [\ C'_F \ \cdots \ (A')^{n-1}C'_F\ ] = O'_{A,C_F}$, $C_{D,J}e^{A_D\tau_M} = F'$, and $O_{A_D,C_{D,J}} = [\ B \ \cdots \ A^{n-1}B\ ]' = R'_{A,B}$. Hence, since $\bar{A}_D = \bar{A}'$, $\bar{B}_D = [\ F_D \quad R_{A_D,B_D}\ ] = [\ (C_Je^{A\tau_M})' \quad O'_{A,C_F}\ ] = \bar{C}'$, and $\bar{C}_D = [\ F \quad R_{A,B}\ ]' = \bar{B}'$, and the monodromy system (33) is reachable (resp., controllable, stabilizable, observable, constructible, detectable) if and only if the dual system (34) is observable (resp., constructible, detectable, reachable, controllable, stabilizable), then, by Proposition 7, the system (1), (2) is strongly reachable (resp., controllable, stabilizable, observable, constructible, detectable) if and only if the system (35) is observable (resp., constructible, detectable, strongly reachable, controllable, stabilizable). □

V. OUTPUT FEEDBACK STABILIZATION

Consider the *dynamic time–invariant output feedback* with flow dynamics

$$\begin{aligned} \dot{\tau} &= 1, && (36a)\\ \dot{\check{x}} &= A\check{x} + Bu_F, && (36b)\\ \dot{\zeta} &= C_F'(C_F\check{x} - y_F) - A'\zeta, && (36c)\\ \dot{\xi} &= -A'\xi, && (36d)\end{aligned}$$

whether $(\tau, \check{x}, \zeta, \xi) \in [0, \tau_M] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, jump dynamics

$$\begin{aligned} \tau^+ &= 0, && (36e)\\ \check{x}^+ &= E\check{x} + Fu_J + L_J(C_J\check{x} - y_J) + \bar{L}_F e^{A'\tau_M}\zeta, && (36f)\\ \zeta^+ &= 0, && (36g)\\ \xi^+ &= e^{A'\tau_M}\bar{K}_F\check{x}, && (36h)\end{aligned}$$

whether $(\tau, \check{x}, \zeta, \xi) \in \{\tau_M\} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, output

$$\begin{aligned} u_F &= B'\xi, && (36i)\\ u_J &= K_J\check{x}, && (36j)\end{aligned}$$

and initial conditions $\tau(0,0) = 0$, $\check{x}(0,0) = \check{x}_0$, $\check{x}_0 \in \mathbb{R}^n$, $\zeta(0,0) = \zeta_0$, $\zeta_0 \in \mathbb{R}^n$, $\xi(0,0) = \xi_0$, $\xi_0 \in \mathbb{R}^n$.

The following theorem is known for non–hybrid linear system as separation principle.

Theorem 9. *Let $G(\tau_M)$ and $W(\tau_M)$ be, respectively, the reachability and observability Gramian of system* (1) *during flow. Let matrices K_F and L_F be given, and let $\bar{K}_F$ and $\bar{L}_F$ be such that $R_{A,B}K_F = G(\tau_M)\bar{K}_F$ and $L_F O_{A,C_F} = \bar{L}_F W(\tau_M)$, respectively. The time invariant dynamic output feedback* (36) *is such that the closed loop monodromy matrix has spectrum $\Xi \cup \Upsilon \cup \{0\}$, where Ξ and Υ are the sets given in* (12) *and* (24)*, respectively.*

Proof. Let $\chi = [\ x \quad \xi' \quad \check{x}' \quad \zeta'\ ]'$ and let

$$A_\Sigma = \begin{bmatrix} A & BB' & 0 & 0\\ 0 & -A' & 0 & 0\\ 0 & BB' & A & 0\\ -C_F'C_F & 0 & C_F'C_F & -A' \end{bmatrix},$$

$$E_\Sigma = \begin{bmatrix} E & 0 & FK_J & 0\\ 0 & 0 & e^{A'\tau_M}\bar{K}_F & 0\\ -L_JC_J & 0 & E + FK_J + L_JC_J & \bar{L}_F e^{A'\tau_M}\\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly, the dynamics of the closed loop system are given by $\dot{\chi} = A_\Sigma\chi$, $\chi^+ = E_\Sigma\chi$. Define the matrices

$$T := \begin{bmatrix} I & 0 & 0 & 0\\ 0 & I & 0 & 0\\ -I & 0 & I & 0\\ 0 & 0 & 0 & I \end{bmatrix},$$

$$\hat{A}_\Sigma = \begin{bmatrix} A & BB' & 0 & 0\\ 0 & -A' & 0 & 0\\ 0 & 0 & A & 0\\ 0 & 0 & C_F'C_F & -A' \end{bmatrix},$$

$$\hat{E}_\Sigma = \begin{bmatrix} E + FK_J & 0 & FK_J & 0\\ e^{A'\tau_M}\bar{K}_F & 0 & e^{A'\tau_M}\bar{K}_F & 0\\ 0 & 0 & E + L_JC_J & \bar{L}_F e^{A'\tau_M}\\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $\hat{A}_\Sigma = TA_\Sigma T^{-1}$ and $\hat{E}_\Sigma = TE_\Sigma T^{-1}$. Note that the matrix $e^{\hat{A}_\Sigma\tau_M}$ equals

$$\begin{bmatrix} e^{A\tau_M} & G(\tau_M)e^{-A'\tau_M} & 0 & 0\\ 0 & e^{-A'\tau_M} & 0 & 0\\ 0 & 0 & e^{A\tau_M} & 0\\ 0 & 0 & e^{-A'\tau_M}W(\tau_M) & e^{-A'\tau_M} \end{bmatrix}.$$

Hence, by computing the closed loop monodromy matrix $\hat{E}_\Sigma e^{\hat{A}_\Sigma\tau_M}$, it can be easily checked that the spectrum of such a matrix is $\Xi \cup \Upsilon \cup \{0\}$. □

The main goals of this section are formalized in the following problem.

Problem 1. Let system (1) with outputs (2) be given. Find, if any, a linear dynamic time–invariant output feedback with state $\eta(t,k) \in \mathbb{R}^{n_K}$, flow dynamics

$$\begin{aligned} \dot{\tau} &= 1, && (37a)\\ \dot{\eta} &= A_K\eta + B_Ky_F, && (37b)\end{aligned}$$

whether $(\tau, \eta) \in [0, \tau_M] \times \mathbb{R}^{n_K}$, jump dynamics

$$\begin{aligned} \tau^+ &= 0, && (37c)\\ \eta^+ &= E_K\eta + F_Ky_J, && (37d)\end{aligned}$$

whether $(\tau, \eta) \in \{\tau_M\} \times \mathbb{R}^{n_K}$, output

$$\begin{aligned} u_F &= C_{K,F}\eta, && (37e)\\ u_J &= C_{K,J}\eta, && (37f)\end{aligned}$$

and initial condtions $\tau(0,0) = 0$, $\eta(0,0) = \eta_0$, $\eta_0 \in \mathbb{R}^{n_K}$, such that, letting M_Σ be the closed loop monodromy matrix,

(I) $\Lambda(M_\Sigma) \subset \mathbb{C}_g$;
(II) $\Lambda(M_\Sigma) \subset \{s \in \mathbb{C} : |s| < \varrho\}$, for a given $0 < \varrho < 1$.
(III) $\Lambda(M_\Sigma) = \{0\}$;

Note that, if one is able to find a solution to Problem 1.I, then the dynamic time–invariant output feedback (37) is such that the closed loop system is asymptotically stable [36, Prop. 1]. On the other hand, if one is able to find a solution to Problem 1 II, then, letting $\chi = [\ x' \quad \eta'\ ]'$, there exists a constant $c \in \mathbb{R}$, $c > 0$, such that, for any initial condition $\chi(0,0)$ of the closed loop dynamical system, $|\chi(t,k)| \leqslant c\varrho^k|\chi(0,0)|$. Finally, if one is able to find a solution to Problem 1.III, then the controller (37) is such that the state of the closed loop system is driven to 0 in finite time.

The following three propositions give conditions to guarantee the existence of a solution to Problem 1.

Proposition 8. There exists a solution to Problem 1.I if and only if system (1), (2) is stabilizable and detectable.

Proof. If system (1) is stabilizable, then, by Theorem 4, there exist matrices K_J and $\bar{K}_F$ such that the set Ξ given in (12) is contained in $\mathbb{C}_g$. On the other hand, if system (1) with outputs (2) is detectable, then, by Theorem 7, there exist matrices L_J and $\bar{L}_F$ such that the set Υ given in (24) is contained in $\mathbb{C}_g$. Hence, by Theorem 9, the dynamic time–invariant output feedback (36) is such that the monodromy matrx of the closed loop system has spectrum contained in $\mathbb{C}_g$.

Assume now that there exists a dynamic time–invariant output feedback (37) that solves Problem 1.I, but system (1) is not stabilizable. Hence, by Proposition 3, there exists a matrix T such that (7) holds. Additionally, since system (1) is not stabilizable, by Theorem 4, there exists $\lambda \in \Lambda(\hat{M}_{u,u})$ such that $\lambda \notin \mathbb{C}_g$. By letting $\hat{x} = Tx$, one has that the state of the closed loop system is $\hat{\chi} = [\ \hat{x}_r' \quad \hat{x}_u' \quad \eta'\ ]'$. Thus, there exists $w \in \mathbb{R}^\nu$ such that, by letting the initial condition of the closed loop system be $\hat{\chi}(0,0) = [\ 0 \quad w' \quad 0\ ]'$, $|\hat{x}_u(t,k)| \leqslant |\hat{\chi}(t,k)|$. By assumption, the closed loop system has all eigenvalues in $\mathbb{C}_g$, whence $\lim_{t+k\to\infty} \hat{\chi}(t,k) = 0$. However, since $\exists\lambda \in \Lambda(\hat{M}_{u,u})$ such that $\lambda \notin \mathbb{C}_g$, one has that $\lim_{k\to\infty} \hat{x}_u(t_k,k)$ is not equal to zero, leading to a contradiction. On the other hand, assume that there exists a dynamic time–invariant output feedback (37) that solves Problem 1.I, but system (1) with output (2) is not detectable. Hence, by Proposition 6, there exists a matrix T such that (27) holds. Moreover, since the system is not detectable, by Theorem 7, there exists $\lambda \in \Lambda(\hat{M}_{i,i})$ such that $\lambda \notin \mathbb{C}_g$. Thus, by a reasoning wholly similar to the one given for stabilizability, this leads to a contradiction. □

Proposition 9. There exists a solution to Problem 1.II if and only if system (1), (2) is such that, $\forall s \in \Lambda(Ee^{A\tau_M})$, $|s| \geqslant \varrho$,

$$\operatorname{rank}([\ Ee^{A\tau_M} - sI \quad F \quad R_{A,B}\ ]) = n, \tag{38a}$$

$$\operatorname{rank}([\ (Ee^{A\tau_M})' - sI \quad (C_J e^{A\tau_M})' \quad O'_{A,C_F}\ ]') = n. \tag{38b}$$

Proof. If system (1) is such that (38a) holds, then, by Proposition 4, the set Ξ given in (12) can be chosen so that $\Xi \subset \{s \in \mathbb{C} : |s| < \varrho\}$. On the other hand, if system (1) with outputs (2) is such that (38b) holds, then the set Υ given in (24) can be chosen so that $\Upsilon \subset \{s \in \mathbb{C} : |s| < \varrho\}$. Thus, given $\varrho > 0$, let $\bar{K}_F$, K_J, $\bar{L}_F$ and L_J be matrices such that $\Xi \cup \Upsilon \subset \{s \in \mathbb{C} : |s| < \varrho\}$. Hence, by Theorem 9, the dynamic time–invariant output feedback (36) is such that the monodromy matrx of the closed loop system has spectrum contained in $\{s \in \mathbb{C} : |s| < \varrho\}$.

Assume now that there exists a dynamic time–invariant output feedback (37) that solves Problem 1.II, but system (1) is not such that (38a) holds. Thus, by Theorem 4, there exists $\lambda \in \Lambda(\hat{M}_{u,u})$ such that $\lambda \notin \{s \in \mathbb{C} : |s| < \varrho\}$. By Proposition 3, there exists a matrix T such that (7) holds. By letting $\hat{x} = Tx$, one has that the state of the closed loop system is $\hat{\chi} = [\ \eta' \quad \hat{x}_r' \quad \hat{x}_u'\ ]'$. Let $\hat{C}_F = [\ \hat{C}_{F,r} \quad \hat{C}_{F,u_1} \quad \hat{C}_{F,u_2}\ ] = C_F T^{-1}$ and $\hat{C}_J = [\ \hat{C}_{J,r} \quad \hat{C}_{J,u_1} \quad \hat{C}_{J,u_2}\ ] = C_J T^{-1}$. By defining matrices

$$A_\Sigma := \begin{bmatrix} A_K & B_K\hat{C}_{F,r} & B_K\hat{C}_{F,u_1} & B_K\hat{C}_{F,u_2} \\ \hat{B}_r C_{K,F} & \hat{A}_{r,r} & \hat{A}_{r,u_1} & \hat{A}_{r,u_2} \\ 0 & 0 & \hat{A}_{u_1,u_1} & \hat{A}_{u_1,u_2} \\ 0 & 0 & \hat{A}_{u_2,u_1} & \hat{A}_{u_2,u_2} \end{bmatrix},$$

$$E_\Sigma := \begin{bmatrix} E_K & F_K\hat{C}_{J,r} & F_K\hat{C}_{J,u_1} & F_K\hat{C}_{J,u_2} \\ \hat{F}_{r_1} C_{K,J} & \hat{E}_1 & \hat{E}_2 & \hat{E}_3 \\ \hat{F}_{r_2} C_{K,J} & \hat{E}_4 & \hat{E}_5 & \hat{E}_6 \\ 0 & 0 & \hat{E}_7 & \hat{E}_8 \end{bmatrix},$$

one has that, by (15), the dynamics of the closed loop system are given by $\dot{\hat{\chi}} = A_\Sigma \hat{\chi}$, $\hat{\chi}^+ = E_\Sigma \hat{\chi}$. Clearly, the closed loop monodromy matrix is given by $M_\Sigma = E_\Sigma e^{A_\Sigma \tau_M}$. Hence, by (14), (15) and (16), one has that the eigenvalues of the matrix $\hat{M}_{u,u}$ are eigenvalues of M_Σ, i.e., $\Lambda(\hat{M}_{u,u}) \subset \Lambda(M_\Sigma)$. Hence, the controller (37) is not such that (II) holds, leading to a contradiction. On the other hand, assume that there exists a dynamic time–invariant output feedback (37) that solves Problem 1.II, but system (1) is not such that (38b) holds. Thus, by Theorem 7, there exists $\lambda \in \Lambda(\hat{M}_{i,i})$ such that $\lambda \notin \{s \in \mathbb{C} : |s| < \varrho\}$. By Proposition 6, there exists a matrix T such that (27) holds. By letting $\hat{x} = Tx$, one has that the state of the closed loop system is $\hat{\chi} = [\ \hat{x}_i' \quad \hat{x}_o' \quad \eta'\ ]'$. Let $\hat{B} = [\ \hat{B}_{i_1}' \quad \hat{B}_{i_2}' \quad \hat{B}_o'\ ]' = TB$ and $\hat{F} = [\ \hat{F}_{i_1}' \quad \hat{F}_{i_2}' \quad \hat{F}_o'\ ]' = TF$. By defining matrices

$$A_\Sigma := \begin{bmatrix} \hat{A}_{i_1,i_1} & \hat{A}_{i_1,i_2} & \hat{A}_{i_1,o} & B_{i_1}C_{F,K} \\ \hat{A}_{i_2,i_1} & \hat{A}_{i_2,i_2} & \hat{A}_{i_2,o} & B_{i_2}C_{F,K} \\ 0 & 0 & \hat{A}_{o,o} & B_o C_{F,K} \\ 0 & 0 & B_K\hat{C}_{F,o} & A_K \end{bmatrix},$$

$$E_\Sigma := \begin{bmatrix} E_1 & E_2 & E_3 & \hat{F}_{i_1}C_{J,K} \\ E_4 & E_5 & E_6 & \hat{F}_{i_2}C_{J,K} \\ 0 & E_7 & E_8 & \hat{F}_o C_{J,K} \\ 0 & \hat{C}_{J,o_1}F_K & \hat{C}_{J,o_2}F_K & E_K \end{bmatrix},$$

one has that the dynamics of the closed loop system are given by $\dot{\hat{\chi}} = A_\Sigma \hat{\chi}$, $\hat{\chi}^+ = E_\Sigma \hat{\chi}$. By a reasoning wholly similar to the one given for strong reachability, this is in contradiction with the existence of a controller (37) such that II holds. □

Proposition 10. There exists a solution to Problem 1.III if and only if system (1), (2) is controllable and constructible.

Proof. The proof of this proposition is wholly similar to the proof of Proposition 9, by replacing $\{s \in \mathbb{C} : |s| < \varrho\}$ with $\{0\}$, (38a) with (9), and (38b) with (31). □

By Propositions 8, 9, and 10, by choosing the matrices $\bar{K}_F$, K_J, $\bar{L}_F$, and L_J so that $\Xi \cup \Upsilon \subset \mathbb{C}_g$, $\Xi \cup \Upsilon \subset \{s \in \mathbb{C} : |s| < \varrho\}$, or $\Xi \cup \Upsilon = \{0\}$, one has that the closed loop system is asymptotically stable, converge to zero exponentially with decrease rate ϱ, or is driven to zero in finite time, respectively. It is worth pointing out that matrices $\bar{K}_F$, K_J, $\bar{L}_F$, and L_J can be computed by using any design technique that ensures either $\Xi \cup \Upsilon \subset \mathbb{C}_g$, $\Xi \cup \Upsilon \subset \{s \in \mathbb{C} : |s| < \varrho\}$, or $\Xi \cup \Upsilon = \{0\}$. For instance, one can use the separation principle of Theorem 9. In fact, one can compute disjointly matrices $\bar{K}_F$, K_J such that $\Xi \subset \mathbb{C}_g$ and matrices $\bar{L}_F$, L_J such that $\Upsilon \subset \mathbb{C}_g$. Thus, by Theorem 9, one has that the closed loop system has eigenvalues $\Xi \cup \Upsilon \cup \{0\} \subset \mathbb{C}_g$, i.e., the linear dynamic time–invariant output feedback (37) stabilizes the hybrid system. Note that if system (1), (2) is strongly reachable and observable, then there exists a solution to (I), (II), and (III) of Problem 1.

Example 3. The mechanical system analyzed in this example is used in [47] to illustrate some issues in regulation for the class of hybrid systems analyzed in this paper. Consider a disk of radius r, total mass m, and inertia $\mathcal{I}$, moving on an horizontal plane between two parallel walls, orthogonal to the plane of motion and infinitely massive. Let $l+2r$, $l > 0$, be the distance between the two walls, let (x_c, y_c) be the coordinates of the center of mass of the disk, and let α denote the angular position of the disk (Fig. 2).

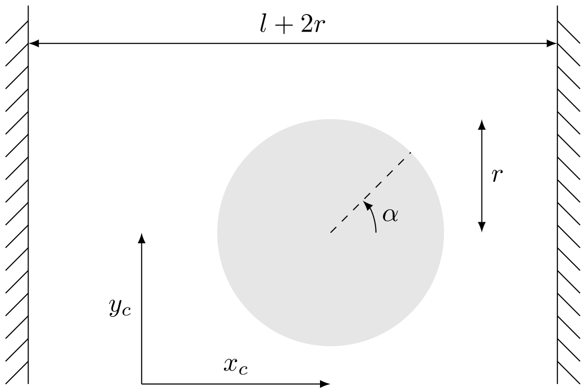

Fig. 2. A rotating disk bouncing between two walls.

Assume that all the impacts are elastic and occur with pre–impact conditions such that the infinitesimal interval in which the disk is in contact with the wall consists in a first interval of sliding followed by a second interval of rolling, i.e.,

$$|\dot{y}_c(t_k,k-1)+r\dot{\alpha}(t_k,k-1)|\leqslant 2\zeta\mu|\dot{x}_c(t_k,k-1)|, \quad (39)$$

where $\zeta=\frac{r^2m}{\mathcal{I}}$ and μ is the coefficient of kinetic friction characterizing the infinitesimal sliding phase [47]. Assuming, additionally, that $x_c(0)=0$, $|\dot{x}_c(t)|=|\dot{x}_c(0)|=v>0$, a hybrid state–space description of this mechanical system, with state $\chi=[\begin{array}{cccc} y_c & \dot{y}_c & \alpha & \dot{\alpha}\end{array}]'$ and input $u=[\begin{array}{cc} u_1 & u_2\end{array}]'$, is

$$\dot{\chi} = \begin{bmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 \end{bmatrix}\chi+\begin{bmatrix} 0 & 0\\ \frac{1}{M} & 0\\ 0 & 0\\ 0 & \frac{1}{\mathcal{I}} \end{bmatrix}u, \quad (40a)$$

$$\chi^+ = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1-\zeta^{-1} & 0 & -\zeta^{-1}r\\ 0 & 0 & 1 & 0\\ 0 & -r^{-1}(1-\zeta^{-1}) & 0 & \zeta^{-1} \end{bmatrix}\chi, \quad (40b)$$

and $\chi(0,0)=[\begin{array}{cccc} y_{c,0} & \dot{y}_{c,0} & \alpha_0 & \dot{\alpha}_0\end{array}]'$. Assume that the only measurable outputs of the system (40) are the pre–impact vertical and angular positions $y_c(t_k,k-1)$ and $\alpha(t_k,k-1)$, $k\in\mathbb{Z}_{>0}$, i.e.,

$$C_F=0,\quad C_J=\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}. \quad (40c)$$

By (11), (23), and (31), the system (40) is strongly reachable and controllable, but not observable. Thanks to the separation principle stated in Theorem 9, matrices $\bar{K}_F$, K_J, and $\bar{L}_F$, L_J such that $\Xi\cup\Upsilon\in\mathbb{C}_g$ can be computed disjointly. Namely, let $\tilde{A}=e^{A\tau_M}E$ and let $\tilde{B}=[\begin{array}{cc} e^{A\tau_M}F & R_{A,B}\end{array}]$. In order to compute matrices K_F and K_J such that the set Ξ given in (12) is a subset of $\mathbb{C}_g$, a possible approach is to solve the following equation (usually known as *Algebraic Riccati Equation* [49])

$$P=I+\tilde{A}'P\tilde{A}-\tilde{A}'P\tilde{B}(I+\tilde{B}'P\tilde{B})^{-1}\tilde{B}'P\tilde{A}.$$

By considering that the discrete–time linear system with data $(\tilde{A},\tilde{B},I)$ is stabilizable and detectable, the matrix

$$\tilde{K}=-(I+\tilde{B}'P\tilde{B})^{-1}\tilde{B}'P\tilde{A},$$

is such that $\Lambda(\tilde{A}+\tilde{B}\tilde{K})\subset\mathbb{C}_g$. Therefore, letting $[\begin{array}{cc} K_J' & K_F'\end{array}]'=\tilde{K}$, one has that the set Ξ given in (12) is a subset of $\mathbb{C}_g$ [49]. By exploiting the duality principle stated in Theorem 8, a wholly similar procedure can be carried out to compute L_F and L_J such that the set Υ given in (24) is a subset of $\mathbb{C}_g$. Hence, by Theorem 9, the time–invariant dynamic output feedback (36) is such that the eigenvalues of the closed loop system are in $\mathbb{C}_g$, and hence the closed loop system is asymptotically stable [36, Lem. 1].

A numerical simulation of the solution to closed loop system with the time–invariant dynamic output feedback (36) have been carried out assuming the following data: $m=0.22\text{Kg}$, $r=0.5\text{m}$, $v=1\text{m/s}$, and $l=1\text{m}$, $\chi(0,0)=[\begin{array}{cccc} 0.1\text{m} & -0.2\text{m/s} & -0.1\text{rad} & 0.2\text{rad/s}\end{array}]'$, and null initial conditions of the feedback controller. Figure 3 depicts the time history of the state $\chi(t,k)$, the estimation error $\chi(t,k)-\check{\chi}(t,k)$ and the applied control input $u(t,k)=[\begin{array}{cc} u_1(t,k) & u_2(t,k)\end{array}]'$. The admissible motion condition (39) is satisfied in such a simulations with $\mu\geqslant 0.025$. △

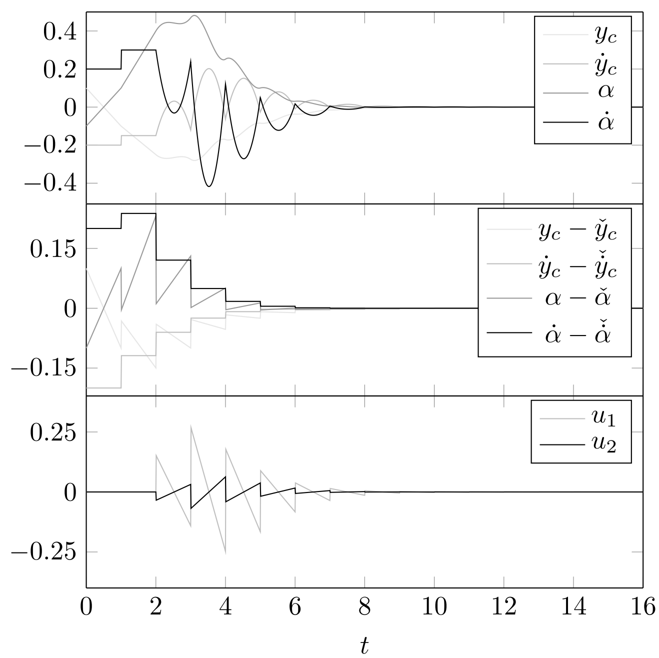

Fig. 3. Numerical simulations of the closed loop system.

Remark 1. The robustness of the proposed compensator relies on the same continuity arguments of non–hybrid linear systems. Namely, if matrices $\bar{K}_F$, K_J, $\bar{L}_F$, and L_J are such that $\Xi\cup\Upsilon\subset\mathbb{C}_g$, then small perturbations of the nominal parameters of the hybrid systems are such that the eigenvalues of the monodromy matrix of the closed loop system remains in $\mathbb{C}_g$. Namely, if the parameters of the hybrid system vary in a sufficiently small neighborhood of their nominal values, continuity implies preservation of asymptotic stability of the closed loop system.

VI. Arbitrary initial conditions for the timer τ

All throughout this paper, we have assumed that the initial condition of the timer τ governing the jumps of system (1) is

$\tau(0,0)=0$. In this section, we discuss the extension of the results given in this work to arbitrary initial conditions of the timer variable $\tau(0,0)=\tau_0$, $\tau_0\in[0,\tau_M]$.

Given $\tau_0\in[0,\tau_M]$, all the solutions to the hybrid system (1) are defined over the hybrid time domain

$$\mathcal{T}(\tau_0):=\{(t,k): t\in[\tilde{t}_k,\tilde{t}_{k+1}], k\in\mathbb{Z}_{\geqslant 0}\}, \tag{41a}$$

$$\tilde{t}_k:=\begin{cases} 0, & \text{if } k=0,\\ k\tau_M-\tau_0, & \text{if } k\in\mathbb{Z}_{>0}.\end{cases} \tag{41b}$$

Let $\phi(t,k,\tau_0,x_0,u_F,u_J)$ be the solution to system (1) at hybrid time $(t,k)\in\mathcal{T}(\tau_0)$, with initial conditions $\tau(0,0)=\tau_0$, $x(0,0)=x_0$, $x_0\in\mathbb{R}^n$, and inputs $u_F(\cdot,\cdot)$, $u_J(\cdot)$. By redefining the inputs $u_F(\cdot,\cdot)$ and $u_J(\cdot)$ so that the domain of such functions is $\mathcal{T}(\tau_0)$, it can be easily checked that the system (1) with initial condition $\tau(0,0)=0$ is reachable (resp., strongly reachable, controllable, stabilizable) if and only if the system (1) with initial condition $\tau(0,0)=\tau_0$, $\tau_0\in[0,\tau_M]$ is reachable (resp., strongly reachable, controllable, stabilizable).

More attention is needed when dealing with "observation" structural properties. In fact, consider the hybrid system (19) with initial condition $\tau(0,0)=\tau_M$ and $x(0,0)=x_0$. If the matrix E is singular and the pair (E,C_J) is not observable (in the classical sense), then there exists an initial condition $x_0\neq 0$ such that $y_F(t,k)=0$ and $y_J(k)=0$, for all $(t,k)\in\mathcal{T}(\tau_0)$, even if (23) holds. Therefore, it can be easily proved that, if the matrix E is singular, the hybrid system (19) is observable for any initial condition $\tau(0,0)=\tau_0$, $\tau_0\in[0,\tau_M]$, $x(0,0)=x_0$, if and only if the pair (E,C_J) is observable. On the other hand, if the matrix E is nonsingular, for any $\tau_0\in[0,\tau_M]$, the hybrid system (19) with $\tau(0,0)=\tau_0$ is observable if and only if the hybrid system (19) with $\tau_0=0$ is observable. As a matter of fact, if the latter condition holds, then there exists a hybrid time $(\theta,\kappa)\in\mathcal{T}(\tau_0)$ such that, by using only measurements of the outputs $y_F(t,k)$, $y_J(t,k)$ for all the times $\mathcal{T}(\tau_0)\cap[\tilde{t}_1,\theta]\times\{1,\kappa\}$, it is possible to determine $x(t_1,1)$. Hence, since E is nonsingular, the initial condition can be determined by letting $x_0=(Ee^{A\tilde{t}_1})^{-1}x(t_1,1)$.

VII. Conclusions and Future Work

In this work, we focus the attention on a class of linear hybrid systems where the clock variable satisfies a fixed dwell–time and is available for feedback. This allows us to focus the attention on a linear setting and to extend many of the classical results for non–hybrid linear systems. Namely, the main contributions of this paper are the following:

- provide simple tests to analyze structural properties of hybrid systems;
- provide two standard forms to represent the dynamics of the hybrid system;
- provide a duality theorem, relating structural properties of a given system with the structural properties of a dual linear hybrid system.
- provide necessary and sufficient conditions guaranteeing the existence of a linear dynamic time–invariant output feedback that stabilizes the system;
- propose a structure for such a linear dynamic time–invariant output feedback;
- provide a separation principle showing that the observer and the state feedback controller can be designed independently.

Robustness of the proposed compensator with respect to small variations of the parameters of the nominal hybrid system is discussed.

Future work will take advantage of this algebraic and geometric characterization of structural properties to solve challenging problems for this class of linear hybrid systems as, for instance, linear quadratic optimal control over finite and infinite horizon [50], characterization of the $\mathcal{L}_2$ gain properties, robust output regulation, and disturbance decoupling.

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