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Index and first Betti number of f-minimal hypersurfaces and self-shrinkers

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19 April 2024

# Index and first Betti number of $f$-minimal hypersurfaces and self-shrinkers

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**Abstract.** We study the Morse index of self-shrinkers for the mean curvature flow and, more generally, of $f$-minimal hypersurfaces in a weighted Euclidean space endowed with a convex weight. When the hypersurface is compact, we show that the index is bounded from below by an affine function of its first Betti number. When the first Betti number is large, this improves index estimates known in literature. In the complete non-compact case, the lower bound is in terms of the dimension of the space of weighted square summable $f$-harmonic 1-forms; in particular, in dimension 2, the procedure gives an index estimate in terms of the genus of the surface.

## 1. Introduction

### 1.1. Main definitions

It is well-known that an immersed hypersurface $\Sigma^m$ of a given Riemannian manifold $(M^{m+1}, g)$ is minimal (i.e. it has everywhere vanishing mean curvature) if and only if it is critical for the volume functional

$$\Sigma \mapsto \mathrm{vol}(\Sigma) = \int_\Sigma d\mu$$

where $d\mu$ is the Riemannian measure associated to the induced metric $g$ on $\Sigma$. More generally, given a smooth function $f$ on $M^{m+1}$, one can consider the so-called $f$-*volume*:

$$\mathrm{vol}_f(\Sigma) = \int_\Sigma e^{-f} \, d\mu.$$

It is then natural to study the immersions $x : \Sigma^m \to M^{m+1}$ which are critical for the $f$-volume. For the sake of exposition assume, for the moment, that $\Sigma$ is

compact and orientable. Given $u \in C^{\infty}(\Sigma)$ and a unit normal vector field $N$ on $\Sigma$, consider the associated one-parameter deformation:

$$(1.1) \qquad \Sigma_t \doteq \{\exp_x(tu(x)N(x)) : x \in \Sigma\}$$

for $t \in (-\epsilon, \epsilon)$ sufficiently small. Hence $\Sigma_0 = \Sigma$, and a calculation shows that:

$$\frac{d}{dt}\Big|_{t=0}\mathrm{vol}_f(\Sigma_t) = -\int_{\Sigma}(H + \frac{\partial f}{\partial N})ue^{-f}\,d\mu.$$

Here $H = \mathrm{tr}A$, and $A$ is the second fundamental form (shape operator) of the immersion, defined as $AX = -\nabla_X N$ on all tangent vectors $X$ to $\Sigma$; moreover, $\nabla$ is the Levi-Civita connection of the ambient manifold $M$. The conclusion is that $\Sigma$ is critical for the $f$-volume if and only if

$$(1.2) \qquad H + \frac{\partial f}{\partial N} = 0.$$

If one defines the $f$-mean curvature of $\Sigma$ as

$$H_f \doteq H + \frac{\partial f}{\partial N},$$

then $\Sigma$ is critical for the $f$-volume if and only if it is $f$-minimal in $M$, which means, by definition, that $H_f = 0$ identically on $\Sigma$.

If $\Sigma$ is a local minimum of the weighted area functional we say that it is $f$-stable. When $\Sigma$ is unstable, it makes sense to investigate its Morse index: this is, roughly speaking, the maximal dimension of a linear space of deformations that decrease the weighted volume up to second order. To compute it, we need to compute the second variation of the $f$-volume, which is intimately connected with the structure of the triple $M_f \doteq (M^{m+1}, g, e^{-f}d\mu)$; such structure is often termed a *weighted manifold*. In fact, if $uN$ is the normal variation used in (1.1), and $\Sigma$ is $f$-minimal, then:

$$Q_f(u,u) \doteq \frac{d^2}{dt^2}\Big|_{t=0}\mathrm{vol}_f(\Sigma_t) = \int_{\Sigma}\left(|\nabla u|^2 - (\mathrm{Ric}_f^M(N,N) + |A|^2)u^2\right)e^{-f}\,d\mu$$

where $\mathrm{Ric}_f^M = \mathrm{Ric}^M + \mathrm{Hess}f$ is the Bakry-Émery Ricci tensor of the weighted ambient manifold $M_f$.

The maximum dimension of a subspace of $C^{\infty}(\Sigma)$ on which $Q$ is negative definite is called the $f$-index of $\Sigma$, and is denoted by $\mathrm{Ind}_f(\Sigma)$. This number can also be seen as the number of negative eigenvalues of the Jacobi operator $L_f$, associated to the quadratic form $Q_f$ and acting on $L^2(\Sigma, e^{-f}d\mu)$. We will give an explicit expression of $L_f$ later.

If $\Sigma$ is compact its $f$-index is always finite. If $\Sigma$ is complete, not compact, we can define the index $\mathrm{Ind}_f(\Omega)$ of any relatively compact domain $\Omega \subseteq \Sigma$ as the maximum dimension of a subspace of $C_0^{\infty}(\Omega)$ (smooth functions with support in $\Omega$) on which $Q_f$ is negative definite; then define

$$\mathrm{Ind}_f(\Sigma) = \sup\{\mathrm{Ind}_f(\Omega) : \Omega \subset\subset \Sigma\}.$$

Obviously, $\mathrm{Ind}_f(\Sigma)$ can be infinite.

In this paper we will give lower estimates of the $f$-index of $f$-minimal hypersurfaces of the weighted manifold $(\mathbf{R}^{m+1}, g_{\mathrm{can}}, e^{-f}\, d\mu)$. When $\Sigma$ is compact and the weight $f$ is a convex function on $\mathbf{R}^{m+1}$ the lower bound will only depend on the topology of $\Sigma$ through its first Betti number $b_1(\Sigma)$.

**Remark 1.1.** The $f$-minimal equation (1.2), together with the rules of conformal change, tells us that $\Sigma^m$ is $f$-minimal in $(M^{m+1}, g)$ if and only if it is minimal (in the usual sense) in the manifold $(M^{m+1}, e^{-\frac{2f}{m}} g)$. Moreover, the $f$-index coincides with the usual index of the minimal immersion $\Sigma^m \to (M^{m+1}, e^{-\frac{2f}{m}} g)$; see [6].

## 1.2. Self-shrinkers

Perhaps the main motivation for this paper was to study an important class of $f$-minimal hypersurfaces : the *self-shrinkers* of the mean curvature flow. By definition, they are connected, orientable, isometrically immersed hypersurfaces $x : \Sigma^m \to \mathbf{R}^{m+1}$ whose mean curvature vector field $\mathbf{H}$ satisfies the equation

$$(1.3) \qquad\qquad x^{\perp} = -\mathbf{H},$$

where $(\cdot)^{\perp}$ denotes the projection on the normal bundle of $\Sigma$. Self-shrinkers play an important role in the study of singularities developed along the mean curvature flow and have been extensively studied in recent years; see e.g. [8], [5], [9], and references therein.

Taking the scalar product on both sides of (1.3) with a unit normal vector field $N$ we see that self-shrinkers are $f$-minimal hypersurfaces of $(\mathbf{R}^{m+1}, g_{\mathrm{can}}, e^{-f}\, d\mu)$ for the weight function $f(x) = \frac{|x|^2}{2}$.

Let us recall the main results regarding their index. It was proved by T. Colding and W. Minicozzi, [8], that every complete properly immersed self-shrinker is necessarily $f$-unstable (i.e it has $f$-index greater than or equal to one). Note that this result was later generalized in [14] by the first two authors, to self-shrinkers with at most exponential (intrinsic) weighted volume growth.

In the equality case, rigidity results have been proved by C. Hussey, [12], under the additional assumption of embeddedness. This last assumption was later removed in [13]. More precisely, one has that if a complete properly immersed self-shrinker in $\mathbf{R}^{m+1}$ has Morse index 1, then it has to be an hyperplane through the origin. Furthermore, if the self-shrinker is not an hyperplane through the origin, then the Morse index has to be at least $m + 2$, with equality if and only if the self-shrinker is a generalized cylinder of the form $\mathbf{R}^{m-k} \times \mathbf{S}^k_{\sqrt{k}}$ for some $1 \le k \le m$.

For other basic concepts and results about self-shrinkers and their stability properties we refer to [8] and the very recent [13].

## 1.3. Main results

It is by now a well-established guiding principle that the index of a compact minimal hypersurface of a positively curved manifold is sensitive to the topology, more

precisely, to the first Betti number of the hypersurface; hence, rich cohomology in degree one often implies high instability of the immersion. In fact, Schoen, Marques and Neves conjecture that the index of a compact minimal hypersurface of a manifold with positive Ricci curvature is bounded below by an affine function of the first Betti number.

In dimension 2, this was first shown by A. Ros in [21] for immersions in $\mathbf{R}^3$ or a quotient of it by a group of translations; in higher dimensions the third author proved this fact when the ambient manifold is the round sphere (see [23]). Recently L. Ambrozio, A. Carlotto and B. Sharp, [2] generalized the methods of Ros and Savo and verified the conjecture for a larger class of ambient spaces. For a more general result see also the very recent [18]. However, as of today, the full conjecture above is still open, although the method was employed to obtain bounds for minimal free boundary immersions ([3] and [22]) and complete minimal immersions in $\mathbf{R}^n$ ([16]).

Test-functions for the Jacobi operator are constructed as follows. One needs the family of harmonic one-forms, which are known by classical Hodge theory to represent cohomology in degree one, and a distinguished family $\mathcal{P}$ of vector fields on $\Sigma$, given by orthogonal projection onto $\Sigma$ of parallel vector fields in Euclidean space of suitable dimension. In dimension 2, Ros used the family of test-functions: $u = \omega(V)$, where $\omega$ is a harmonic one-form and $V \in \mathcal{P}$. In higher dimensions these test-functions no longer work; indeed, for immersions $\Sigma^m \to \mathbf{S}^{m+1}$ the third author introduced in [23] the test-functions $u = \omega(X_{V,W})$, where $X_{V,W} = \langle \bar{V}, N \rangle W - \langle \bar{W}, N \rangle V$; here $\bar{V}, \bar{W}$ are parallel vector fields in $\mathbf{R}^{m+2}$ and $V, W$ are their projections on $\Sigma$. It should be mentioned that harmonic vector fields were previously used by B. Palmer in [19] to prove a lower bound of the index of harmonic Gauss maps in terms of the genus.

In this paper we employ the method in [23] to prove lower bounds in the weighted case. We start with compact $f$-minimal hypersurfaces; in this case we can improve the bound by adding the number of small eigenvalues of the weighted Laplacian.

Recall that the weighted Laplacian is the operator acting on $u \in C^\infty(\Sigma)$ as follows:

$$\Delta_f u = \Delta u + \langle \nabla f, \nabla u \rangle$$

where $\Delta \doteq -\mathrm{div}(\nabla)$ is the usual Laplacian and $f$ is the weight. The weighted laplacian is self-adjoint with respect to the measure $e^{-f} d\mu$ and has a discrete spectrum $0 = \lambda_0(\Delta_f) < \lambda_1(\Delta_f) \leq \lambda_2(\Delta_f) \leq \dots$. For any positive number $a$, set

$$N_{\Delta_f}(a) = \#\{\text{positive eigenvalues of } \Delta_f \text{ which are less than } a\}.$$

Here is our first main result.

**Theorem A.** *Let $\Sigma$ be a compact $f$-minimal hypersurface of the weighted manifold $M_f \doteq (\mathbf{R}^{m+1}, g_{\mathrm{can}}, e^{-f} d\mu)$. Assume that the Bakry-Émery Ricci tensor of $M_f$ is bounded below by $\mu > 0$, that is, $\mathrm{Ric}_f \geq \mu > 0$. Then:*

$$(1.4) \qquad \mathrm{Ind}_f(\Sigma) \geq \frac{2}{m(m+1)} \Big( N_{\Delta_f}(2\mu) + b_1(\Sigma) \Big).$$

**Remark 1.2.** The theorem could also be rephrased as follows: if $\Sigma$ is a compact minimal hypersurface of the Riemannian manifold $(\mathbf{R}^{m+1}, e^{-\frac{2f}{m}} g_{\mathrm{can}})$ such that $\mathrm{Hess} f \geq \mu > 0$, then the lower bound (1.4) holds.

The theorem will be proved in Subsection 2.2 below. In particular, the $f$-index is large provided that:

- the first Betti number is large, or

- there are many small (i.e., less than $2\mu$) eigenvalues of the weighted Laplacian.

When $\Sigma$ is a self-shrinker we take $f(x) = \frac{|x|^2}{2}$. In that case $\mathrm{Hess} f = \mathrm{Ric}_f^M = g_{\mathrm{can}}$ and we get an affine lower bound.

**Theorem B.** *Let $x : \Sigma^m \to \mathbf{R}^{m+1}$ be a compact self-shrinker. Then*

$$(1.5) \qquad \mathrm{Ind}_f(\Sigma) \geq \frac{2}{m(m+1)} b_1(\Sigma) + m + 1.$$

*In particular if $\Sigma$ has dimension $m = 2$, letting $g = \mathrm{genus}(\Sigma)$, we have that*

$$\mathrm{Ind}_f(\Sigma) \geq \frac{2}{3} g + 3.$$

It should be said that self-shrinkers behave, in some respects, like closed minimal surfaces of $\mathbf{S}^3$. We remark that in that case we have (see [23]):

$$\mathrm{Ind}(\Sigma) \geq \frac{g}{2} + 4.$$

We adopt the method in [23], with a noteworthy modification. Instead of using the usual harmonic one-forms we employ the so-called $f$-*harmonic* one forms, which satisfy by definition the two conditions:

$$d\omega = \delta_f \omega = 0$$

where $\delta_f \omega = \delta \omega + \omega(\nabla f)$ is the weighted codifferential. This seems to be the natural approach in the weighted case, also because $f$-harmonic forms are those which minimize the functional

$$\omega \mapsto \int_\Sigma |\omega|^2 e^{-f} \, d\mu$$

restricted to a fixed cohomology class. By the Hodge decomposition (which continues to hold in the weighted case, see [4]) the dimension of the space of $f$-harmonic 1-forms equals the first Betti number of $\Sigma$.

It should be pointed out that P. Zhu and W. Gan also obtained an estimate in the spirit of the Theorem A above: a lower bound by the first Betti number is proved, but only assuming an additional curvature condition (see Corollary 1.2 in [24]). In that paper usual harmonic (and not $f$-harmonic) forms are being used.

Adapting to this setting the approach used in [2] we obtained similar estimates also in the case of compact $f$-minimal hypersurfaces in more general ambient weighted manifolds. The article [15] containing these results will be available soon.

In the second part of the paper we consider the case where the immersion is complete, non-compact. Denote by $\mathcal{H}_f^1(\Sigma)$ the space of $f$-harmonic one-forms $\omega$ which are square summable for the weighted measure:

$$\mathcal{H}_f^1(\Sigma) = \left\{ \omega \in \Lambda^1(\Sigma) : d\omega = \delta_f \omega = 0, \ \int_\Sigma |\omega|^2 e^{-f} \, d\mu < +\infty \right\}.$$

**Theorem C.** *Let $\Sigma$ be a complete, non-compact $f$-minimal hypersurface of a weighted manifold $(\mathbf{R}^{m+1}, g_{\mathrm{can}}, e^{-f} d\mu)$ with $\mathrm{Ric}_f \geq \mu > 0$. Then*

$$\mathrm{Ind}_f(\Sigma) \geq \frac{2}{m(m+1)} \dim \mathcal{H}_f^1(\Sigma).$$

*In particular, if $\Sigma^m$ is a complete properly immersed self-shrinker, then*

$$\mathrm{Ind}_f(\Sigma) \geq \frac{2}{m(m+1)} \Big( m + 1 + \dim \mathcal{H}_f^1(\Sigma) \Big).$$

Adapting the well-known Farkas-Kra construction (see [10]) to the weighted situation we will prove in the last section:

**Theorem D.** *Let $\Sigma$ be a two-dimensional orientable, connected, complete surface. Then, for all $f \in C^\infty(\Sigma)$:*

$$\dim \mathcal{H}_f^1(\Sigma) \geq 2g$$

*where $g$ is the genus of $\Sigma$.*

This implies the following inequalities.

**Corollary E.**

*(a) Let $\Sigma$ be a complete $f$-minimal surface in $(\mathbf{R}^3, g_{\mathrm{can}}, e^{-f} d\mu)$ with $\mathrm{Ric}_f \geq \mu > 0$. Then*

$$\mathrm{Ind}_f(\Sigma) \geq \frac{2g}{3}.$$

*In particular, stable $f$-minimal surfaces have genus zero.*

*(b) If $\Sigma^2$ is a complete self-shrinker, properly immersed in $\mathbf{R}^3$, then*

$$\mathrm{Ind}_f(\Sigma) \geq \frac{2g}{3} + 1$$

Ros proved the bound in (a) for usual complete minimal surfaces in $\mathbf{R}^3$ (see [21]); this was later improved to $\mathrm{Ind}(\Sigma) \geq \frac{2}{3}(g + r) - 1$ by O. Chodosh and D. Maximo ([7]), where $r$ is the number of ends. The inequality in b) improves the lower bound $\mathrm{Ind}_f(\Sigma) \geq \frac{g}{3}$ proved in [17] under the additional condition that $\sup_\Sigma |k_1^2 - k_2^2| \leq \delta < 1$ (here $k_1$ and $k_2$ are the principal curvatures of $\Sigma$).

**Remark 1.3.** We are grateful to the anonymous referee who suggested to add the following remark. After submission of the present article, we learned about the preprint [1] by N.S. Aiex, where it is shown that if $\Sigma^2$ is a complete self-shrinker, properly immersed in $\mathbb{R}^3$, then one has the interesting bound

$$\dim \mathcal{H}_f^1(\Sigma) \geq 2g + r - 1, \tag{1.6}$$

where $r$ is the number of asymptotically conical ends of $\Sigma$ (see Theorem 3.5 in [1]). Hence, for this class of surfaces, Theorem D can be improved. Using (1.6), Aiex improves the bound $\mathrm{Ind}_f(\Sigma) \geq \frac{g}{3}$ in [17] to

$$\mathrm{Ind}_f(\Sigma) \geq \frac{2g + r - 1}{3}$$

under the same pinching conditions (see Theorem 4.2 in [1]).

But now, inserting Aiex's bound (1.6) directly into our estimate (Theorem C) one obtains the following lower bound which improves part b) of Corollary E (this result is also stated as Corollary 4.3 in [1]).

**Corollary 1.4.** *Let $\Sigma^2$ be a properly immersed, orientable, self-shrinking surface in $\mathbb{R}^3$ with genus $g$ and $r$ asymptotically conical ends. Then*

$$\mathrm{Ind}_f(\Sigma) \geq \frac{2g + r - 1}{3} + 1.$$

In conclusion no pinching condition is in fact needed.

## 2. A comparison theorem in the compact case

In the compact case, our estimates will be a consequence of a more general comparison result between the spectrum of the stability operator and that of the $f$-Laplacian acting on 1-forms. Thus, we start by defining these operators.

### 2.1. Hodge Laplacian and Weitzenböck formula in the weighted setting

Recall that the $f$-Laplacian of a weighted manifold $(M, g, e^{-f} d\mu)$, acting on functions, is defined by:

$$\Delta_f u = \Delta u + \langle \nabla f, \nabla u \rangle.$$

In general, if one introduces the weighted divergence

$$\delta_f = \delta + i_{\nabla f},$$

then one has simply $\Delta_f u = \delta_f du$. Note that $\Delta_f$ is self-adjoint with respect to the weighted measure $e^{-f} d\mu$. More generally, we have a Hodge $f$-Laplacian acting on $p$-forms, denoted $\Delta_f^{[p]}$ and defined in the natural way:

$$\Delta_f^{[p]} = d\delta_f + \delta_f d.$$

As $M$ is compact, the Hodge Laplacian has a discrete spectrum $\{\lambda_k(\Delta_f^{[p]})\}_{k=1,2,\dots}$. The important fact is that the Hodge decomposition continues to hold in this setting; therefore the dimension of the kernel of $\Delta_f^{[p]}$ is equal to the $p$-th Betti number, which means that

$$\lambda_k(\Delta_f^{[p]}) = 0 \quad \text{for } k = 1, \dots, b_p(M).$$

The following Lemma is well-known in the field; it gives an expression of the Hodge $f$-Laplacian in terms of the connection Laplacian and the Bakry-Émery Ricci tensor. Since we haven't found a proof in literature, we provide it below.

**Lemma 2.1.** *Let* $(M, g, e^{-f}d\mu)$ *be a weighted manifold and* $\omega \in C^\infty(\Lambda^1(M))$. *Then*

$$(2.1) \qquad\qquad \Delta_f^{[1]}\omega = \nabla_f^*\nabla\omega + \mathrm{Ric}_f(\omega^\sharp),$$

*where* $\Delta_f^{[1]} = \delta_f d + d\delta_f$, $\delta_f = \delta + i_{\nabla f}$, $\nabla_f^* = \nabla^* + i_{\nabla f}$, $\mathrm{Ric}_f = \mathrm{Ric} + \mathrm{Hess} f$.

*Proof.* Recall that by the classical Weitzenböck formula, letting $\omega \in C^\infty(\Lambda^1(M))$, we have that

$$\Delta^{[1]}\omega = \nabla^*\nabla\omega + \mathrm{Ric}(\omega^\sharp).$$

Then

$$
\begin{aligned}
\Delta_f^{[1]}\omega &= (\delta_f d + d\delta_f)\,\omega = (\delta d + i_{\nabla f}d + d\delta + d\,i_{\nabla f})\,\omega \\
&= (\Delta^{[1]} + \mathcal{L}_{\nabla f})\omega = \nabla^*\nabla\omega + \mathrm{Ric}(\omega^\sharp) + \mathcal{L}_{\nabla f}\omega \\
&= \nabla_f^*\nabla\omega + \mathrm{Ric}(\omega^\sharp) - i_{\nabla f}\nabla\omega + \mathcal{L}_{\nabla f}\omega,
\end{aligned}
$$

where of course $\mathcal{L}$ is the Lie derivative. Since, for every $X \in TM$,

$$
\begin{aligned}
-(i_{\nabla f}\nabla\omega)(X) + (\mathcal{L}_{\nabla f}\omega)(X) &= -\nabla\omega(\nabla f, X) + (i_{\nabla f}d\omega)(X) + d(i_{\nabla f}\omega)(X) \\
&= -\nabla\omega(\nabla f, X) + d\omega(\nabla f, X) + X(\omega(\nabla f)) \\
&= -\nabla\omega(X, \nabla f) + \nabla_X\omega(\nabla f) + \omega(\nabla_X\nabla f) \\
&= \langle \nabla_X\nabla f, \omega^\sharp \rangle
\end{aligned}
$$

and since $\langle \nabla_X\nabla f, \omega^\sharp \rangle = \mathrm{Hess} f(X, \omega^\sharp) = \mathrm{Hess} f(\omega^\sharp, X)$ we see from the two previous facts that

$$
\begin{aligned}
\Delta_f^{[1]}\omega(X) &= \nabla_f^*\nabla\omega(X) + \mathrm{Ric}(\omega^\sharp, X) + \mathrm{Hess} f(\omega^\sharp, X) \\
&= \nabla_f^*\nabla\omega(X) + \mathrm{Ric}_f(\omega^\sharp, X)
\end{aligned}
$$

which is the assertion. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

For notational purposes we will prefer to work with vector fields instead of one-forms. If $\xi$ is a vector field on $M$ consider its dual one-form $\xi^\flat$. We define the Hodge $f$-Laplacian of $\xi$ as the unique vector field such that

$$\langle \Delta_f^{[1]}\xi, X \rangle = \Delta_f^{[1]}\xi^\flat(X)$$

for all $X \in TM$. The Weitzenböck formula becomes:

$$\Delta_f^{[1]} \xi = \nabla_f^* \nabla \xi + \mathrm{Ric}_f(\xi),$$

where $\nabla_f^* \nabla \xi = \nabla^* \nabla \xi + \nabla_{\nabla f} \xi$, and, with respect to an orthonormal basis,

$$\nabla^* \nabla \xi = -\sum_i (\nabla_{e_i} \nabla_{e_i} \xi - \nabla_{\nabla_{e_i} e_i} \xi).$$

## 2.2. A comparison theorem and the proof of Theorem A and Theorem B

Now let $\Sigma$ be a complete $f$-minimal hypersurface of the weighted manifold $M_f \doteq (\mathbf{R}^{m+1}, g_{\mathrm{can}}, e^{-f} d\mu)$. As such, it inherits a structure of weighted manifold, the weight being simply the restriction of $f$ to $\Sigma$ which, by a slight abuse of language, we keep denoting by the same letter $f$. The stability operator of $\Sigma$ is then given by

$$L_f u = \Delta_f u - (\mathrm{Ric}_f^M(N, N) + |A|^2) u,$$

where $\Delta_f$ is the weighted Laplacian of $\Sigma$ and $\mathrm{Ric}_f^M$ is the Bakry-Émery Ricci tensor of the ambient weighted manifold; of course one has

$$\mathrm{Ric}_f^M = \mathrm{Hess} f,$$

(the Hessian of $f$, computed in $\mathbf{R}^{m+1}$). Similar to the case of minimal immersions in the standard sphere, we have the following comparison theorem between the spectrum of the stability operator and that of the Hodge $f$-Laplacian acting on 1-forms.

**Theorem 2.2.** *Let $M_f = (\mathbf{R}^{m+1}, g_{\mathrm{can}}, e^{-f} d\mu)$ be a weighted manifold such that $\mathrm{Ric}_f^M \doteq \mathrm{Hess} f \geq \mu > 0$. Let $\Sigma^m \to \mathbf{R}^{m+1}$ be a compact $f$-minimal hypersurface. Then, for all $k$:*

$$\lambda_k(L_f) \leq -2\mu + \lambda_{d(k)}(\Delta_f^{[1]})$$

*where $\Delta_f^{[1]} = \delta_f d + d\delta_f$ is the Hodge $f$-Laplacian acting on 1-forms of $\Sigma$ and*

$$d(k) = \binom{m+1}{2}(k-1) + 1.$$

We will prove the theorem in the next section.

Now let

$$\beta = \#\{\text{eigenvalues of } \Delta_f^{[1]} \text{ which are less than } 2\mu\}.$$

If $k$ is the largest integer such that $d(k) \leq \beta$, one sees from the above inequality that $\lambda_k(L_f) < 0$, so that $\mathrm{Ind}_f(\Sigma) \geq k$. It is easy to estimate that $k \geq \frac{2}{m(m+1)} \beta$. Then the theorem gives:

$$(2.2) \qquad\qquad \mathrm{Ind}_f(\Sigma) \geq \frac{2}{m(m+1)} \beta.$$

*Proof of Theorem A.* Let $\gamma = N_{\Delta_f}(2\mu)$ be the number of positive eigenvalues of the $f$-Laplacian which are less than $2\mu$. Let $u_1, \ldots, u_\gamma$ be $L_f^2$-orthogonal eigenfunctions of $\Delta_f$ associated to positive eigenvalues which are less than $2\mu$. Obviously, they are all orthogonal to constants. As $\Delta_f^{[1]}$ commutes with exterior differentiation $d$, the 1-forms $du_1, \ldots, du_\gamma$ form, by Stokes formula, an orthogonal set of eigenforms of $\Delta_f^{[1]}$ associated to positive eigenvalues less than $2\mu$. As they are all orthogonal to the space of $f$-harmonic 1-forms, we see that $\beta \geq \gamma + b_1(\Sigma)$, hence, from (2.2):

$$\mathrm{Ind}_f(\Sigma) \geq \frac{2}{m(m+1)}\Big(N_{\Delta_f}(2\mu) + b_1(\Sigma)\Big)$$

which is the statement of Theorem **A**.

*Proof of Theorem B.* Reasoning as before, now let $k = \left[\frac{b_1(\Sigma) + \binom{m+1}{2} - 1}{\binom{m+1}{2}}\right]$. Then $d(k) \leq b_1(\Sigma)$ and we see from Theorem 2.2 that $\lambda_k(L_f) \leq -2$. This means that there are at least $k$ eigenvalues of $L_f$ which are less than or equal to $-2$.

Note now that by (3.6) below, for any self-shrinker, we have at least $m + 1$ eigenvalues equal to $-1$. Indeed the functions $\langle \bar{V}, N \rangle$, with $\bar{V}$ a parallel field in $\mathbf{R}^{m+1}$ are eigenfunctions of $L_f$ corresponding to the eigenvalue $-1$; moreover

$$\dim\big\{\langle \bar{V}, N \rangle : \bar{V}\,\mathrm{parallel\,field\,in}\,\mathbb{R}^{m+1}\big\} = m + 1,$$

since otherwise we could find a non-zero $\bar{V}$ such that $\langle \bar{V}, N \rangle = 0$ and this would imply that $\Sigma$ splits isometrically, which is not possible since $\Sigma$ is compact. Therefore, the index is at least $k + m + 1$ and the assertion follows. In case $m = 2$, we get the desired bound just noting that $b_1(\Sigma)$ is twice the genus of $\Sigma$.

## 3. Proof of Theorem 2.2

### 3.1. The test-functions and the main computational lemma

Then let $\Sigma$ be an hypersurface of $(\mathbf{R}^{m+1}, g_{\mathrm{can}}, e^{-f}\,d\mu)$. Here and in the rest of the paper, we will denote by $\overline{\mathcal{P}}$ the set of parallel vector fields on $\mathbf{R}^{m+1}$, and by $V$ the orthogonal projection of the parallel field $\bar{V} \in \overline{\mathcal{P}}$ on $\Sigma$, so that we have

$$\bar{V} = V + \langle \bar{V}, N \rangle N.$$

To a pair of parallel vector fields $\bar{V}, \bar{W} \in \overline{\mathcal{P}}$ we associate the vector field on $\Sigma$ defined by $X_{\bar{V}, \bar{W}} = \langle \bar{V}, N \rangle W - \langle \bar{W}, N \rangle V$. We obtain a family of test functions for the weighted Jacobi operator by pairing $X_{\bar{V}, \bar{W}}$ with a vector field $\xi \in T\Sigma$:

$$(3.1)\qquad\qquad\qquad u = \big\langle X_{\bar{V}, \bar{W}}, \xi \big\rangle.$$

Typically, $\xi$ will be a $f$-harmonic vector field, or an eigenvector field of the Hodge $f$-Laplacian. The scope of the following Lemma is to give an explicit expression of the stability operator when applied to $u$.

**Lemma 3.1.** *Let $f \in C^\infty(\mathbf{R}^{m+1})$ and let $x : \Sigma^m \to \mathbf{R}^{m+1}$ be an $f$-minimal hypersurface. Denote by $N$ the unit normal vector and by $A$ the second fundamental form of $\Sigma$. Let $\xi \in T\Sigma$ be a generic vector field on $\Sigma$ and $u$ the function defined in (3.1). Then*

$$L_f u = -u\mathrm{Hess}f(N,N) - \mathrm{Hess}f(X_{\bar{V},\bar{W}},\xi) + \left\langle X_{\bar{V},\bar{W}}, \Delta_f^{[1]}\xi \right\rangle + v,$$

*where $v = 2(\langle \nabla_{AV}\xi, W \rangle - \langle \nabla_{AW}\xi, V \rangle) - \langle W, \xi \rangle \, \mathrm{Hess}f(V,N) + \langle V, \xi \rangle \, \mathrm{Hess}f(W,N)$.*

*If $\Sigma$ is a self-shrinker (so that $\mathrm{Hess}f = g_{\mathrm{can}}$) then*

$$L_f u = -2u + \left\langle X_{\bar{V},\bar{W}}, \Delta_f^{[1]}\xi \right\rangle + v,$$

*where $v = 2(\langle \nabla_{AV}\xi, W \rangle - \langle \nabla_{AW}\xi, V \rangle)$.*

*Proof.* Recall that

$$L_f u = \Delta_f u - (\mathrm{Ric}_f^M(N,N) + |A|^2)u = \Delta_f u - (\mathrm{Hess}f(N,N) + |A|^2)u.$$

The assertion now follows from (3.9), which will be proved in the next subsection.

<div align="right">□</div>

## 3.2. The basic equations and the proof of Lemma 3.1

In the next two lemmas we collect some preliminary computations which will be used in the proof of our main results.

**Lemma 3.2.** *Let $f \in C^\infty(\mathbf{R}^{m+1})$ and let $x : \Sigma^m \to \mathbf{R}^{m+1}$ be an $f$-minimal hypersurface. Let $\bar{V} \in \overline{\mathcal{P}}$ and denote by $V$ its projection on $\Sigma$. If $X \in T\Sigma$ one has*

$$(3.2) \qquad \nabla_X V = \left\langle \bar{V}, N \right\rangle AX;$$

$$(3.3) \qquad \nabla \left\langle \bar{V}, N \right\rangle = -AV;$$

$$(3.4) \qquad \nabla_f^* \nabla V = A^2 V + \left\langle \bar{V}, N \right\rangle \mathrm{Hess}f(N)^T.$$

*Proof.* Let $\bar{\nabla}$ denote the Levi-Civita connection on $\mathbf{R}^{m+1}$. We have that

$$\nabla_X V = \left(\overline{\nabla}_X V\right)^T = -\left(\overline{\nabla}_X \left\langle \bar{V}, N \right\rangle N\right)^T = -\left\langle \bar{V}, N \right\rangle \left(\overline{\nabla}_X N\right)$$
$$= \left\langle \bar{V}, N \right\rangle AX,$$

and, for all tangent vectors $X$:

$$\left\langle \nabla \left\langle \bar{V}, N \right\rangle, X \right\rangle = X \left\langle \bar{V}, N \right\rangle = \left\langle \bar{V}, \overline{\nabla}_X N \right\rangle = -\left\langle AV, X \right\rangle.$$

As for (3.4), note that $\nabla_{e_i} V = \langle \bar{V}, N \rangle A e_i$. Then by (3.3) and Codazzi's equation,

$$\nabla^* \nabla V = -\sum_i (\nabla_{e_i}(\left\langle \bar{V}, N \right\rangle A e_i) - \left\langle \bar{V}, N \right\rangle A \nabla_{e_i} e_i)$$
$$= -A\nabla \left\langle \bar{V}, N \right\rangle - \left\langle \bar{V}, N \right\rangle \mathrm{tr}\nabla A$$
$$= A^2 V - \left\langle \bar{V}, N \right\rangle \nabla H.$$

Moreover, using the $f$-minimal equation, it is not difficult to show that

(3.5) $$\nabla H = -\mathrm{Hess} f(N)^T + A\nabla f.$$

Hence we get

$$\nabla^*\nabla V = A^2 V + \langle \bar{V}, N \rangle \,\mathrm{Hess} f(N)^T - \langle \bar{V}, N \rangle \, A\nabla f$$
$$= A^2 V + \langle \bar{V}, N \rangle \,\mathrm{Hess} f(N)^T - \nabla_{\nabla f} V.$$

$\square$

**Lemma 3.3.** *Let $f \in C^\infty(\mathbf{R}^{m+1})$ and let $x : \Sigma^m \to \mathbf{R}^{m+1}$ be an $f$-minimal hypersurface. Let $\bar{V}, \bar{W} \in \overline{\mathcal{P}}$, $V$, $W$ their projections on $\Sigma$. Then, for any $\xi \in T\Sigma$*

(3.6) $\quad \Delta_f \langle \bar{V}, N \rangle = |A|^2 \langle \bar{V}, N \rangle - \mathrm{Hess} f(V, N);$

(3.7) $\quad \Delta_f \langle V, \xi \rangle = -\mathrm{Hess} f(V, \xi) + \langle \bar{V}, N \rangle \mathrm{Hess} f(N, \xi) + 2\langle AV, A\xi \rangle$
$$- 2\langle \bar{V}, N \rangle \langle \nabla \xi, A \rangle + \left\langle \Delta_f^{[1]} \xi, V \right\rangle;$$

(3.8)
$$\Delta_f \left( \langle \bar{V}, N \rangle \langle W, \xi \rangle \right) = |A|^2 \langle \bar{V}, N \rangle \langle W, \xi \rangle + 2\langle \bar{V}, N \rangle \langle AW, A\xi \rangle + 2\langle \bar{W}, N \rangle \langle AV, A\xi \rangle$$
$$- \langle \bar{V}, N \rangle \, \mathrm{Hess} f(W, \xi) - \langle W, \xi \rangle \, \mathrm{Hess} f(V, N)$$
$$+ \langle \bar{V}, N \rangle \langle \bar{W}, N \rangle \mathrm{Hess} f(N, \xi) + 2\nabla \xi(AV, W)$$
$$- 2\langle \bar{V}, N \rangle \langle \bar{W}, N \rangle \langle \nabla \xi, A \rangle + \langle \bar{V}, N \rangle \left\langle \Delta_f^{[1]} \xi, W \right\rangle.$$

*Finally, if $X_{\bar{V}, \bar{W}} = \langle \bar{V}, N \rangle W - \langle \bar{W}, N \rangle V$ and $u = \langle X_{\bar{V}, \bar{W}}, \xi \rangle$ then:*

(3.9) $$\Delta_f u = |A|^2 u - \mathrm{Hess} f(X_{\bar{V}, \bar{W}}, \xi) + \left\langle X_{\bar{V}, \bar{W}}, \Delta_f^{[1]} \xi \right\rangle + v,$$

*where $v = 2(\langle \nabla_{AV} \xi, W \rangle - \langle \nabla_{AW} \xi, V \rangle) - \langle W, \xi \rangle \, \mathrm{Hess} f(V, N) + \langle V, \xi \rangle \, \mathrm{Hess} f(W, N)$.*

*Proof.* First note that, as a consequence of the Codazzi equation, we have that

$$\mathrm{div}(AV) = \langle \bar{V}, N \rangle |A|^2 + \langle \nabla H, V \rangle.$$

Therefore we obtain, by (3.5) and (3.3):

$$\Delta \langle \bar{V}, N \rangle = -\mathrm{div}(\nabla \langle \bar{V}, N \rangle) = \mathrm{div}(AV)$$
$$= \langle \bar{V}, N \rangle |A|^2 - \langle \mathrm{Hess} f(N), V \rangle + \langle A\nabla f, V \rangle$$
$$= \langle \bar{V}, N \rangle |A|^2 - \mathrm{Hess} f(N, V) - \langle \nabla f, \nabla \langle \bar{V}, N \rangle \rangle.$$

Equation (3.6) now follows by the definition of $\Delta_f$:

$$\Delta_f \langle \bar{V}, N \rangle = \Delta \langle \bar{V}, N \rangle + \langle \nabla f, \nabla \langle \bar{V}, N \rangle \rangle.$$

As for equation (3.7), observe that by Lemma 2.1 and (3.4)

$$\Delta_f \langle V, \xi \rangle = \langle \nabla_f^* \nabla V, \xi \rangle + \langle V, \nabla_f^* \nabla \xi \rangle - 2\langle \nabla V, \nabla \xi \rangle$$
$$= \langle AV, A\xi \rangle + \langle \bar{V}, N \rangle \, \mathrm{Hess} f(N, \xi) + \left\langle \Delta_f^{[1]} \xi, V \right\rangle - \mathrm{Ric}_f^\Sigma(V, \xi) - 2\langle \nabla V, \nabla \xi \rangle.$$

Moreover (see e.g. [14]) for an $f$-minimal hypersurface in the Euclidean space we have that

$$\mathrm{Ric}_f^\Sigma(\xi) = \mathrm{Hess}f(\xi) - A^2(\xi), \quad \xi \in T\Sigma.$$

We hence get

$$\Delta_f \langle V, \xi \rangle = -\mathrm{Hess}f(V, \xi) + \langle \bar{V}, N \rangle \mathrm{Hess}f(N, \xi) + 2\langle AV, A\xi \rangle$$
$$+ \left\langle \Delta_f^{[1]}\xi, V \right\rangle - 2\langle \nabla V, \nabla \xi \rangle.$$

Note also that, by (3.2),

$$\langle \nabla V, \nabla \xi \rangle = \langle \bar{V}, N \rangle \langle A, \nabla \xi \rangle,$$

Thus

$$\Delta_f \langle V, \xi \rangle = -\mathrm{Hess}f(V, \xi) + \langle \bar{V}, N \rangle \mathrm{Hess}f(N, \xi) + 2\langle AV, A\xi \rangle$$
$$+ \left\langle \Delta_f^{[1]}\xi, V \right\rangle - 2\langle \bar{V}, N \rangle \langle A, \nabla \xi \rangle.$$

We now note that, for any $X \in T\Sigma$:

$$\langle \nabla \langle W, \xi \rangle, X \rangle = \langle \nabla_X W, \xi \rangle + \langle W, \nabla_X \xi \rangle$$
$$= \langle \bar{W}, N \rangle \langle A\xi, X \rangle + \nabla \xi(X, W)$$

so that

(3.10)
$$\langle \nabla \langle \bar{V}, N \rangle, \nabla \langle W, \xi \rangle \rangle = -\langle AV, \nabla \langle W, \xi \rangle \rangle$$
$$= -\langle \bar{W}, N \rangle \langle AV, A\xi \rangle - \nabla \xi(AV, W)$$

As

(3.11)
$$\Delta_f \left( \langle \bar{V}, N \rangle \langle W, \xi \rangle \right) = \langle W, \xi \rangle \Delta_f \langle \bar{V}, N \rangle + \langle \bar{V}, N \rangle \Delta_f \langle W, \xi \rangle - 2\langle \nabla \langle \bar{V}, N \rangle, \nabla \langle W, \xi \rangle \rangle$$

equation (3.8) now follows by substituting in the above expression (3.6), (3.7) and (3.10).

Finally, (3.9) follows by using formula (3.8) twice. □

We also observe the following fact, which will be used later.

**Lemma 3.4.** *Let* $x : \Sigma^m \to \mathbf{R}^{m+1}$ *be a self-shrinker and* $\bar{V}$ *be a parallel vector field on* $\mathbf{R}^{m+1}$. *Then every linear function* $u = \langle \bar{V}, x \rangle$, *restricted to* $\Sigma$, *satisfies:*

$$\nabla \langle \bar{V}, x \rangle = V \quad \Delta_f \langle \bar{V}, x \rangle = \langle \bar{V}, x \rangle.$$

*Hence any such* $u$ *is an eigenfunction of the weighted Laplacian associated to the eigenvalue* $\lambda = 1$. *Moreover, if* $\Sigma$ *is complete and properly immersed then* $u \in W^{1,2}(\Sigma_f)$, *that is:*

$$\int_\Sigma (u^2 + |\nabla u|^2)e^{-f}d\mu < +\infty.$$

*Proof.* The proof of the first statement is easy. If $\Sigma$ is complete and properly immersed it is known that then it is of finite volume and, more generally, every polynomial in $|x|$ is $f$-integrable (for details we refer to [13]). The second assertion now follows because $|u| \le |x|$ and $|\nabla u|$ is bounded by a constant. □

### 3.3. End of the proof of Theorem 2.2

Select an orthonormal basis $\{\varphi_j\}$ of $L^2(\Sigma_f) = L^2(\Sigma, e^{-f}d\mu)$ given by eigenfunctions of $L_f$, where $\varphi_j$ is associated to $\lambda_j(L_f)$, and let $E^d$ be the direct sum of the first $d$ eigenspaces of $\Delta_f^{[1]}$:

$$E^d = \bigoplus_{j=1}^{d} V_{\Delta_f^{[1]}}(\lambda_j).$$

We look for vector fields $\xi \in E^d$ such that $u = \left\langle X_{\bar{V},\bar{W}}, \xi \right\rangle$ satisfies the following orthogonality relations for all choices of $\bar{V}, \bar{W} \in \bar{\mathcal{P}}$:

$$\int_{\Sigma} \left\langle X_{\bar{V},\bar{W}}, \xi \right\rangle \varphi_1 e^{-f}d\mu = \ldots = \int_{\Sigma} \left\langle X_{\bar{V},\bar{W}}, \xi \right\rangle \varphi_{k-1} e^{-f}d\mu = 0$$

As the vector space $\bar{\mathcal{P}}$ has dimension $m+1$ and since $X_{\bar{V},\bar{W}}$ is a skew symmetric bilinear function of $\bar{V}, \bar{W}$, we see that the above is a system of $\binom{m+1}{2}(k-1)$ homogeneous linear equations in the unknown $\xi \in E^d$.

If $d = d(k) = \binom{m+1}{2}(k-1)+1$, we can then find a non-trivial vector field $\xi \in E^d$ such that $u = \left\langle X_{\bar{V},\bar{W}}, \xi \right\rangle$ is $L^2(\Sigma_f)$-orthogonal to the first $k-1$ eigenfunctions of $L_f$ for all $\bar{V}, \bar{W}$. Then, by the min-max principle, we have that

$$(3.12) \qquad \lambda_k(L_f) \int_{\Sigma} u^2 e^{-f}d\mu \leq \int_{\Sigma} u L_f u \, e^{-f}d\mu.$$

Let $\bar{\mathcal{U}}$ be the family of parallel vector fields of $\mathbf{R}^{m+1}$ having unit length. As in [23], we identify $\bar{\mathcal{U}}$ with $\mathbf{S}^m$ and endow it with the measure $\hat{\mu} = \frac{m+1}{|\mathbf{S}^m|}\text{dvol}_{\mathbf{S}^m}$. Using coordinates, one verifies easily that, for all $\bar{X}, \bar{Y} \in \mathbf{R}^{m+1}$:

$$(3.13) \qquad \int_{\bar{\mathcal{U}}} \left\langle \bar{V}, \bar{X} \right\rangle \left\langle \bar{V}, \bar{Y} \right\rangle d\bar{V} = \left\langle \bar{X}, \bar{Y} \right\rangle.$$

Using the product metric on $\bar{\mathcal{U}} \times \bar{\mathcal{U}}$ we see that at each $x \in \Sigma$:

$$\int_{\bar{\mathcal{U}} \times \bar{\mathcal{U}}} u^2 d\bar{V}d\bar{W} = 2|\xi|^2;$$

$$\int_{\bar{\mathcal{U}} \times \bar{\mathcal{U}}} \text{Hess} f(X_{\bar{V},\bar{W}}, \xi) u \, d\bar{V}d\bar{W} = 2\text{Hess} f(\xi, \xi);$$

$$\int_{\bar{\mathcal{U}} \times \bar{\mathcal{U}}} \text{Hess} f(N, N) u^2 d\bar{V}d\bar{W} = 2\text{Hess} f(N, N)|\xi|^2;$$

$$\int_{\bar{\mathcal{U}} \times \bar{\mathcal{U}}} u \left\langle X_{\bar{V},\bar{W}}, \Delta_f^{[1]}\xi \right\rangle d\bar{V}d\bar{W} = 2\left\langle \xi, \Delta_f^{[1]}\xi \right\rangle;$$

$$\int_{\bar{\mathcal{U}} \times \bar{\mathcal{U}}} uv d\bar{V}d\bar{W} = 0.$$

In fact, to verify the above identities, one uses (3.13) repeatedly and proceeds as in [23]. To prove the last identity: pick an orthonormal basis $(e_1, \ldots, e_m)$ of

$T\Sigma$ and decompose the tangential part $V = \sum_{j=1}^m \langle \bar{V}, e_j \rangle e_j$; now observe that the typical term in the integral of $uv$ can be handled as follows:

$$\int_{\bar{\mathcal{U}} \times \bar{\mathcal{U}}} \langle W, \xi \rangle^2 \left\langle \bar{V}, N \right\rangle \operatorname{Hess} f(V, N) d\bar{V} d\bar{W} = |\xi|^2 \int_{\bar{\mathcal{U}}} \left\langle \bar{V}, N \right\rangle \operatorname{Hess} f(V, N) d\bar{V}$$

$$= |\xi|^2 \sum_{j=1}^m \operatorname{Hess} f(e_j, N) \int_{\bar{\mathcal{U}}} \left\langle \bar{V}, N \right\rangle \left\langle \bar{V}, e_j \right\rangle d\bar{V}$$

$$= 0.$$

The other terms can be handled similarly.

Integrating (3.12) with respect to $\left( \bar{V}, \bar{W} \right) \in \bar{\mathcal{U}} \times \bar{\mathcal{U}}$, applying the Fubini theorem and Lemma 3.1, one concludes that

$$\lambda_k(L_f) \int_\Sigma |\xi|^2 e^{-f} d\mu \leq - \int_\Sigma (\operatorname{Hess} f(\xi, \xi) + \operatorname{Hess} f(N, N)|\xi|^2) e^{-f} d\mu$$

$$+ \int_\Sigma \left\langle \xi, \Delta_f^{[1]} \xi \right\rangle e^{-f} d\mu.$$

Now note that, as $\xi$ is a linear combination of the first $d(k)$ eigenvector fields of $\Delta_f^{[1]}$, one easily verifies that

$$\int_\Sigma \left\langle \xi, \Delta_f^{[1]} \xi \right\rangle e^{-f} d\mu \leq \lambda_{d(k)}(\Delta_f^{[1]}) \int_\Sigma |\xi|^2 e^{-f} d\mu.$$

Putting together the above facts with the assumption $\operatorname{Hess} f \geq \mu$, the assertion of the Theorem follows.

## 4. Non-compact case: proofs

We now assume that the immersion $x : \Sigma^m \to \mathbf{R}^{m+1}$ is complete and non-compact, and let $f \in C^\infty(\mathbf{R}^{m+1})$ be a given weight. We let $\mathcal{H}_f^1(\Sigma)$ denote the space of $f$-square summable $f$-harmonic vector fields on $\Sigma$:

$$\mathcal{H}_f^1(\Sigma) = \{\xi \in T\Sigma : d\xi = \delta_f \xi = 0, \int_\Sigma |\xi|^2 e^{-f} d\mu < +\infty\}.$$

We want to estimate the $f$-index from below, in terms of the dimension of $\mathcal{H}_f^1(\Sigma)$. Actually, we give a slightly stronger estimate. We denote by $V_{\Delta_f}(\lambda)$ the space of $f$-square summable eigenfunctions of $\Delta_f$ associated to $\lambda$ and having finite weighted Dirichlet integral:

$$V_{\Delta_f}(\lambda) = \{u \in C^\infty(\Sigma) : \Delta_f u = \lambda u, \int_\Sigma (u^2 + |\nabla u|^2) e^{-f} d\mu < +\infty\}.$$

For a fixed $\Lambda > 0$, we let $E(\Sigma, \Lambda)$ be the vector space generated by all vector fields which are gradients of some $u \in V_{\Delta_f}(\lambda)$ with $\lambda \leq \Lambda$:

$$E(\Sigma, \Lambda) = \operatorname{span}\{\nabla u : u \in V_{\Delta_f}(\lambda), \lambda \leq \Lambda\}.$$

In the next subsection we will prove the following fact.

**Theorem 4.1.** *Assume that $\Sigma^m$ is a complete, non-compact $f$-minimal immersed hypersurface of the weighted space $M_f = (\mathbf{R}^{m+1}, g_{can}, e^{-f}\, d\mu)$, such that $\mathrm{Ric}_f \geq \mu > 0$. Assume that $\mathrm{Ind}_f(\Sigma)$ is finite. Then $\mathcal{H}^1_f(\Sigma)$ and $E(\Sigma, \Lambda)$ have finite dimensions for any $\Lambda < 2\mu$, and*

$$\mathrm{Ind}_f(\Sigma) \geq \frac{2}{m(m+1)}\Big(\dim E(\Sigma, \Lambda) + \dim \mathcal{H}^1_f(\Sigma)\Big).$$

Specializing to the case $f = \frac{1}{2}|x|^2$, we have the following consequence.

**Corollary 4.2.** *Let $\Sigma^m$ be a complete, properly immersed self-shrinker which is not a hyperplane. Then:*

$$\mathrm{Ind}_f(\Sigma) \geq \frac{2}{m(m+1)}\Big(m + 1 + \dim \mathcal{H}^1_f(\Sigma)\Big).$$

*Proof.* From Lemma 3.4 we see that each $V \in \mathcal{P}$ is the gradient of the linear function $u = \langle \bar{V}, x \rangle$, and this function belongs to $V_{\Delta_f}(1)$. Then $\mathcal{P} \subseteq E(\Sigma, 1)$. Taking $\Lambda = 1$ in the theorem (note that $1 < 2\mu$ because for a self-shrinker $2\mu = 2$), it is enough to show that

$$\dim \mathcal{P} \geq m + 1.$$

Let $\mathcal{B} = (\bar{V}_1, \ldots, \bar{V}_{m+1})$ be an orthonormal basis of $\mathbf{R}^{m+1}$, and let us denote by $V_1, \ldots, V_{m+1}$ the vector fields obtained by projection of $\mathcal{B}$ on $\Sigma$. If these vector fields were linearly dependent, we would have a parallel vector field on $\mathbf{R}^{m+1}$ which is everywhere normal to $\Sigma$: this can't happen unless $\Sigma$ is a hyperplane. $\qquad\square$

In Subsection 4.2 we will prove that if $\Sigma^2$ is a complete, connected, orientable surface of genus $g$ then $\dim \mathcal{H}^1_f(\Sigma) \geq 2g$. Hence, if $\Sigma^2$ is $f$-minimal in $\mathbf{R}^3$ and $\mathrm{Ric}_f \geq \mu > 0$ the theorem gives immediately

$$\mathrm{Ind}_f(\Sigma) \geq \frac{2g}{3}.$$

which proves the first assertion of Corollary E in the introduction. If $\Sigma^2$ is a properly immersed shrinker, then

$$\mathrm{Ind}_f(\Sigma) \geq \frac{2}{3}g + 1.$$

In fact, this is trivially true if $\Sigma^2$ is a hyperplane (in that case, in fact, the index is equal to 1); otherwise, we apply Corollary 4.2.

## 4.1. Proof of Theorem 4.1

Before giving the proof we state two lemmas.

**Lemma 4.3.** *Let $\Omega \subset \Sigma$ be a bounded domain and let $\phi \in C_0^\infty(\Omega)$. Let $u \in C^\infty(\Omega)$. If $L_f = \Delta_f + T$, is a Schrödinger operator and $T \in C^\infty(\Omega)$ is any potential, then*

$$\int_\Omega \Big(|\nabla(\phi u)|^2 + T\phi^2 u^2\Big)e^{-f}\, d\mu = \int_\Omega \phi^2 u L_f u \cdot e^{-f}\, d\mu + \int_\Omega u^2 |\nabla\phi|^2 \cdot e^{-f}\, d\mu.$$

*Proof.* The proof is obtained using integration by parts and the identity

$$\Delta_f(uv) = v\Delta_f u + u\Delta_f v - 2\langle \nabla u, \nabla v \rangle.$$

□

We use a sequence of cut-off functions defined as follows. For each positive integer $n$, let $B_n$ be the (intrinsic) geodesic ball in $\Sigma$ having radius $n$ and centered at a fixed point $x_0 \in \Sigma$. Since $\Sigma$ is complete, it is standard to obtain by Proposition 2.1 in [11] that there exists a family of smooth functions $\phi_n$ on $\Sigma$ such that $\phi_n = 1$ on $B_n$, $\phi_n$ is compactly supported on $B_{2n}$ and

$$|\nabla \phi_n| \leq \frac{c}{n}$$

for a constant $c$ depending only on $\Sigma$.

**Lemma 4.4.** *Let $V(\Sigma) = E(\Sigma, \Lambda) + \mathcal{H}_f^1(\Sigma)$. Then*

$$\dim V(\Sigma) = \dim E(\Sigma, \Lambda) + \dim \mathcal{H}_f^1(\Sigma).$$

*Moreover, for any $\eta \in V(\Sigma)$ one has:*

$$\int_\Sigma \langle \Delta_f^{[1]} \eta, \eta \rangle e^{-f} d\mu \leq \Lambda \int_\Sigma |\eta|^2 e^{-f} d\mu.$$

*Proof.* We can assume that both spaces are finite dimensional, otherwise the assertion is trivial. The first assertion follows because, if $\xi \in \mathcal{H}_f^1(\Sigma)$ and $u \in W^{1,2}(\Sigma_f)$ then $(\xi, \nabla u)_f = 0$, where $(\cdot, \cdot)_f$ is the weighted $L^2$-inner product. Hence $E(\Sigma, \Lambda)$ and $\mathcal{H}_f^1(\Sigma)$ are mutually orthogonal and the assertion follows.

For the second part, notice that we have

$$E(\Sigma, \Lambda) = V_{\Delta_f}(\lambda_1) + \cdots + V_{\Delta_f}(\lambda_k)$$

for eigenvalues $0 < \lambda_1 \leq \cdots \leq \lambda_k \leq \Lambda$. One verifies that, if $u \in V_{\Delta_f}(\lambda_i)$ and $v \in V_{\Delta_f}(\lambda_j)$ then:

$$\int_\Sigma \langle \nabla u, \nabla v \rangle e^{-f} d\mu = \lambda_i \int_\Sigma uv e^{-f} d\mu = \lambda_j \int_\Sigma uv e^{-f} d\mu.$$

This allows to construct a (finite) orthonormal basis of $E(\Sigma, \Lambda)$ by eigenfunctions of $\Delta_f$. The standard argument shows that then any $\eta \in V_\Sigma$ satisfies the assertion of the lemma. □

Let us now prove the theorem. We want to prove that $\dim V(\Sigma)$ is finite and the following inequality holds:

$$\mathrm{Ind}_f(\Sigma) \geq \binom{m+1}{2}^{-1} \dim V(\Sigma).$$

Set for short:
$$I = \mathrm{Ind}_f(\Sigma), \quad k = \dim V(\Sigma)$$
with $k$ possibly equal to $+\infty$. We have to show that

(4.1)
$$k \le \binom{m+1}{2} I.$$

Assume by contradiction that $V(\Sigma)$ contains a subspace $E^k$ of dimension $k$ satisying:
$$k > \binom{m+1}{2} I.$$

Consider the exhaustion of $\Sigma$ by relatively compact balls $\{\Omega_n\} \doteq \{B_{2n}\}$ centered at a fixed point $x_0 \in \Sigma$. As the index is finite, there exists $n_0$ such that $I = \mathrm{Ind}_f(B_{2n})$ for all $n \ge n_0$. Let $\{\phi_n\}$ be the family of cut-off functions as defined before.

We let $\bar{V}, \bar{W} \in \overline{\mathcal{P}}$ and let $V, W$ be their projections on $\Sigma$. As in the compact case, we introduce the vector field
$$X_{\bar{V},\bar{W}} = \langle \bar{V}, N \rangle W - \langle \bar{W}, N \rangle V.$$

Define $u = \langle X_{\bar{V},\bar{W}}, \xi \rangle$. For each fixed $n \ge n_0$, consider the the family of functions :
$$\{u_n\} = \{\phi_n u\}.$$
with $\xi \in E^k$ and $\bar{V}, \bar{W} \in \overline{\mathcal{P}}$. Notice that each such function is zero on the boundary of $\Omega_n \doteq B_{2n}$ and then it can be used as test-functions for the stability operator of $\Omega_n$. We now proceed exactly as in the compact case. Consider the first $I$ eigenfunctions of the stability operator on $\Omega_n$, say $\{f_1, f_2, \ldots, f_I\}$. We look for non-zero vector fields $\xi \in E^k$ such that the following orthogonality relations hold for all possible choices of $\bar{V}, \bar{W}$:

$$\int_\Sigma \phi_n \langle X_{\bar{V},\bar{W}}, \xi \rangle f_1 e^{-f} d\mu = \cdots = \int_\Sigma \phi_n \langle X_{\bar{V},\bar{W}}, \xi \rangle f_I e^{-f} d\mu = 0.$$

This is a system of $\binom{m+1}{2} I$ homogeneous linear equations in the unknown $\xi \in E^k$. Counting dimensions we see that, as $k > \binom{m+1}{2} I$ by our assumption, we can find a non-trivial vector field $\xi_n \in E^k$ which verifies all of those equations for all choices of $\bar{V}, \bar{W}$.

The notation $\xi_n$ stresses the fact that such vector field depends on $n$. We can choose $\xi_n$ so that it has unit $L^2(\Sigma_f)$-norm:
$$\int_\Sigma |\xi_n|^2 e^{-f} d\mu = 1.$$

In what follows, we make the identification:
$$\{\xi \in E^k : \int_\Sigma |\xi|^2 e^{-f} d\mu = 1\} = \mathbf{S}^{k-1},$$

in particular, we can think of $\xi_n$ as an element of $\mathbf{S}^{k-1}$.

As $I$ is the index of $\Omega_n$, we see that $\lambda_{I+1} \geq 0$ and hence

$$\int_{\Omega_n} \left( |\nabla(\phi_n u)|^2 - (|A|^2 + \mathrm{Hess} f(N, N))\phi_n^2 u^2 \right) \cdot e^{-f} d\mu \geq \lambda_{I+1} \int_{\Omega_n} \phi_n^2 u^2 \, e^{-f} d\mu \geq 0$$

for all $u = \langle X_{\bar{V}, \bar{W}}, \xi_n \rangle$.

We apply Lemma 4.3 to the above inequality with $\Omega = \Omega_n$, $\phi = \phi_n$ and $T = -(|A|^2 + \mathrm{Hess} f(N, N))$. We thus obtain

$$0 \leq \int_{\Omega_n} \phi_n^2 u L_f u \cdot e^{-f} \, d\mu + \int_{\Omega_n} u^2 |\nabla \phi_n|^2 \cdot e^{-f} \, d\mu$$

which again, is valid for all $\bar{V}, \bar{W}$. Proceeding as in the compact case, integrating with respect to $\bar{V}, \bar{W}$ we obtain:

$$0 \leq - \int_{\Omega_n} \phi_n^2 \mathrm{Hess} f(\xi_n, \xi_n) e^{-f} \, d\mu - \int_{\Omega_n} \phi_n^2 \mathrm{Hess} f(N, N) |\xi_n|^2 e^{-f} \, d\mu$$
$$+ \int_{\Omega_n} \phi_n^2 \langle \Delta_f^{[1]} \xi_n, \xi_n \rangle e^{-f} d\mu + \int_{\Omega_n} |\nabla \phi_n|^2 |\xi_n|^2 \cdot e^{-f} \, d\mu.$$

Recalling the properties of the cut-off functions $\phi_n$ (namely $\phi_n = 1$ on $B_n$), using the hypothesis $\mathrm{Hess} f \geq \mu > 0$ and the inequality in Lemma 4.4, we obtain, for all $n \geq n_0$ :

$$(2\mu - \Lambda) \int_{B_n} |\xi_n|^2 \cdot e^{-f} \, d\mu \leq \frac{c^2}{n^2} \int_{B_{2n}} |\xi_n|^2 \cdot e^{-f} \, d\mu$$

(4.2)

$$\leq \frac{c^2}{n^2}.$$

As by assumption $2\mu - \Lambda > 0$ we see

(4.3) $$\lim_{n \to \infty} \int_{B_n} |\xi_n|^2 \cdot e^{-f} \, d\mu = 0.$$

However, we will show below that this can't hold, thus getting a contradiction. The contradiction comes from the assumption $k > \binom{m+1}{2} I$. Hence

$$k \leq \binom{m+1}{2} I,$$

as asserted.

Let us then show that (4.3) can't hold. By the compactness of $\mathbf{S}^{k-1}$, the infinite set $\{\xi_n \in \mathbf{S}^{k-1} : n \geq n_0\}$ has an accumulation point, hence there exists a subsequence $\{\xi_{n_j}\}_{j=1,2,\ldots} \in \mathbf{S}^{k-1}$ which converges to $\xi \in \mathbf{S}^{k-1} \subseteq E^k$ in the $L^2(\Sigma_f)$-sense as $j \to \infty$:

$$\lim_{j \to \infty} \xi_{n_j} = \xi.$$

We will presently show that

$$(4.4) \qquad\qquad \lim_{j\to\infty} \int_{B_{n_j}} |\xi_{n_j}|^2 e^{-f}\, d\mu = 1,$$

which will contradict (4.3). Introduce the notation

$$\|\xi\|^2_\Sigma = \int_\Sigma |\xi|^2 e^{-f}\, d\mu, \quad \|\xi\|^2_{B_{n_j}} = \int_{B_{n_j}} |\xi|^2 e^{-f}\, d\mu$$

Clearly $\|\xi\|_\Sigma \geq \|\xi\|_{B_{n_j}}$ and $\lim_{j\to\infty} \|\xi_{n_j} - \xi\|_\Sigma = 0$, by assumption. Now, for all $n \geq n_0$:

$$\int_{B_{n_j}} |\xi_{n_j}|^2 e^{-f}\, d\mu = \int_{B_{n_j}} (|\xi_{n_j}|^2 - |\xi|^2) e^{-f}\, d\mu + \int_{B_{n_j}} |\xi|^2 e^{-f}\, d\mu$$

As $\lim_{n\to\infty} \int_{B_{n_j}} |\xi|^2 e^{-f}\, d\mu = 1$, for (4.4) to be true it is enough to show:

$$(4.5) \qquad\qquad \lim_{j\to\infty} \left| \int_{B_{n_j}} (|\xi_{n_j}|^2 - |\xi|^2) e^{-f}\, d\mu \right| = 0.$$

For any pair $v, w$ of unit vectors in an inner product space one has the inequality

$$\left| \|v\|^2 - \|w\|^2 \right| \leq 2\|v - w\|.$$

Therefore

$$\lim_{j\to\infty} \left| \|\xi_{n_j}\|^2_{B_{n_j}} - \|\xi\|^2_{B_{n_j}} \right| \leq 2 \lim_{j\to\infty} \|\xi_{n_j} - \xi\|_{B_{n_j}} \leq 2 \lim_{j\to\infty} \|\xi_{n_j} - \xi\|_\Sigma = 0$$

and (4.5) follows.

### 4.2. Proof of Theorem D

Recall that $\Sigma$ is a complete, connected, orientable surface having genus $g$. If $\mathcal{H}^1_f(\Sigma)$ denotes the space of $L^2_f$ $f$-harmonic one-forms on $\Sigma$, then we have to show that

$$\dim \mathcal{H}^1_f(\Sigma) \geq 2g.$$

**Step 1.** To any closed curve $\gamma$ on $\Sigma$, we associate a closed 1-form $\eta_\gamma$, whose support is close to $\gamma$.

In fact, as $\Sigma$ is orientable, we have a global unit normal vector field $\nu$ on $\gamma$. Let $U_{2\epsilon}$ be the set of points at distance less than $2\epsilon$ to $\gamma$, on the side defined by $\nu$. If $\epsilon$ is small enough, this set is an annulus, and we can construct a smooth function $f : \Sigma \setminus \gamma \to \mathbf{R}$ such that

$$f = \begin{cases} 1 & \text{on} \quad U_\epsilon \setminus \gamma \\ 0 & \text{on} \quad (\Sigma \setminus \gamma) \setminus U_{2\epsilon} \end{cases}$$

Now set:

$$\eta_\gamma = \begin{cases} df & \text{on} \quad \Sigma \setminus \gamma \\ 0 & \text{on} \quad \gamma \end{cases}$$

Then, $\eta_\gamma$ is a smooth, globally defined 1-form, supported on the compact set given by the closure of $U_{2\epsilon} \setminus U_\epsilon$. Clearly, $\eta_\gamma$ is closed.

**Step 2.** Assume that $\Sigma \setminus \gamma$ is connected; then, we can easily construct a closed curve $\tilde{\gamma}$ meeting $\gamma$ only at one point $p$, and intersecting it transversally. We fix such a curve $\tilde{\gamma}$ and call it the *dual curve* of $\gamma$. It is also easy to verify that, if $\eta_\gamma$ is the one-form associated to $\gamma$, then, for a suitable orientation of $\gamma$:

$$\int_{\tilde{\gamma}} \eta_\gamma = 1.$$

In fact, pick points $p_+, p_- \in \tilde{\gamma}$ very close to $p = \tilde{\gamma} \cap \gamma$ and laying on the two opposite sides of $\gamma$. The integral of $\eta_\gamma$ on the arc joining $p_+$ and $p_-$, not intersecting $\gamma$, is $f(p_+) - f(p_-)$, which is 1 when the two points are sufficiently close to $p$. Taking the limit shows the assertion.

**Step 3.** Since the genus of $\Sigma$ is $g$, we can find $g$ disjoint closed curves $\gamma_1, \dots, \gamma_g$ such that the set $\Sigma \setminus (\gamma_1 \cup \cdots \cup \gamma_g)$ is connected. Construct the associated 1-forms $\eta_1, \dots, \eta_g$ so that they have mutually disjoint support (this is certainly possible, as explained before). To each $\gamma_j$ we associate its dual closed curve $\tilde{\gamma}_j$, as above; looking at the process, we can do it so that this family of curves has the following properties:

- $\tilde{\gamma}_j$ intersects $\gamma_j$ only once, transversally, and
- $\tilde{\gamma}_j$ does not intersect any of the other curves $\gamma_k$, with $k \neq j$.

The first property gives $\int_{\tilde{\gamma}_j} \eta_j = 1$; the second property (after eventually restricting the support of each $\eta_j$) gives $\int_{\tilde{\gamma}_j} \eta_k = 0$ for $j \neq k$. In conclusion we have:

$$(4.6) \qquad \int_{\tilde{\gamma}_j} \eta_k = \delta_{jk} \quad \text{for all } j, k = 1, \dots, g.$$

**Step 4.** We now use the Hodge orthogonal decomposition, due to E. L. Bueler (see Theorem 5.7 in [4]).

$$L_f^2(\Lambda^1(\Sigma)) = A \oplus B_f \oplus H_f^1(\Sigma),$$

where

$$\begin{cases} A = \overline{\{dg : g \in C^\infty(\Sigma) \text{ has compact support}\}} \\ B_f = \overline{\{\delta_f \psi : \psi \in \Lambda^2(\Sigma) \text{ has compact support}\}} \\ \mathcal{H}_f^1(\Sigma) = \{\omega \in L_f^2(\Lambda^1(\Sigma)) : d\omega = \delta_f \omega = 0\} \end{cases}$$

the closure being taken in the $L_f^2$-norm. We denote by $P$ the orthogonal projection onto $\mathcal{H}_f^1(\Sigma)$.

**Step 5.** We now consider the $L_f^2$ $f$-harmonic forms:

$$\omega_j \doteq P(\eta_j); \quad j = 1, \ldots, g$$

and claim that they are linearly independent. In fact, as $\eta_j$ is closed, we have $\omega_j = \eta_j + dg_j$ with $dg_j \in A$. Hence, by (4.6),

$$\int_{\tilde{\gamma}_j} \omega_k = \delta_{jk}.$$

If $\sum_{j=1}^g a_j \omega_j = 0$, integrating both sides on the dual curves gives $a_1 = \cdots = a_g = 0$.

**Step 6.** At this point, we need $g$ more linearly independent $L_f^2$-harmonic 1-forms. In the unweighted case, one simply takes the Hodge-star dual of the forms in the previous step. Recall that, if $(e_1, e_2)$ be a positively oriented local orthonormal frame with dual frame $(\theta_1, \theta_2)$, then the classical Hodge $\star$-operator is defined by the rule:

$$\omega \wedge \star\eta = \langle \omega, \eta \rangle \theta_1 \wedge \theta_2.$$

Then one checks that $\star d\omega = \delta \star \omega$, and $d \star \omega = - \star \delta\omega$. If $f = 0$ and $\omega$ is harmonic, then $\star\omega$ is also harmonic. However, this is no longer true in the weighted case and we introduce the weighted Hodge star operator $\star_f$ as follows. For any one-form $\omega$:

$$\star_f \omega \doteq e^f \star \omega.$$

The following lemma has a straightforward proof, which we omit.

**Lemma 4.5.** *One has*

$$\delta_f \star_f \omega = \star_f d\omega, \quad d \star_{-f} \omega = - \star_{-f} \delta_f \omega.$$

**Step 7.** Define, for $j = 1, \ldots, g$,

$$\zeta_j = P(\star_f \eta_j).$$

Note that this is well-defined because $\star_f \eta_j$, being compactly supported, is in $L_f^2$. We need to show that the $f$-harmonic forms $\zeta_1, \ldots, \zeta_g$ are linearly independent. To that end, observe that, by Lemma 4.5, $\star_f \eta_j$ is $f$-coclosed, hence $\zeta_j = \star_f \eta_j + \delta_f \psi_j$ for some $\delta_f \psi_j \in B_f$. Assume $\sum_j a_j \zeta_j = 0$; then $\sum_j a_j (\star_f \eta_j) = \delta_f \psi$ for some $\delta_f \psi \in B_f$. Applying $\star_{-f}$ on both sides, using Lemma 4.5 and noting that $\star_{-f}\star_f = \star^2 = -1$ we see:

$$\sum_j a_j \eta_j = - \star_{-f} \delta_f \psi = d \star_{-f} \psi$$

that is, the form on the left is exact. Hence, integrating both sides on the dual curves $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_g$ we get $a_1 = \cdots = a_g$. The conclusion is that $\zeta_1, \ldots, \zeta_g$ are linearly independent.

**Step 8.** Now consider the subspace $E \subseteq H_f^1(\Sigma)$ given by

$$E = \mathrm{span}(\omega_1, \ldots, \omega_g, \zeta_1, \ldots, \zeta_g).$$

We prove that $\dim E = 2g$, which will imply the final assertion. It is clear that it is enough to show that

$$(\omega_j, \zeta_k)_f = 0$$

for all $j, k$. This is done as in the unweighted case.

Recall that $\omega_j = \eta_j + dg_j$, and $\zeta_k = \star_f \eta_k + \delta_f \psi_k$, for $dg_j \in A$ and $\delta_f \psi_k \in B_f$. Then:

$$(\omega_j, \zeta_k)_f = (\eta_j, \star_f \eta_k)_f + (\eta_j, \delta_f \psi_k)_f + (dg_j, \star_f \eta_k)_f + (dg_j, \delta_f \psi_k)_f.$$

Now $(dg_j, \delta_f \psi_k)_f = 0$ because the Hodge-Bueler decomposition is orthogonal. Next, a compactly supported closed form is always orthogonal to an $f$-coexact form (use Green formula) hence $(\eta_j, \delta_f \psi_k)_f = 0$. Similary, as $\star_f \eta_k$ is $f$-coclosed and compactly supported, we see that $(dg_j, \star_f \eta_k)_f = 0$. We end-up with

$$(\omega_j, \zeta_k)_f = (\eta_j, \star_f \eta_k)_f.$$

If $j = k$ this is zero because $\eta_j$ and $\star_f \eta_j$ are pointwise orthogonal. If $j \neq k$ this is zero because $\eta_j$ and $\star_f \eta_k$ have disjoint support. The assertion follows.

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