Second-Order Structured Deformations: Relaxation, Integral Representation and Applications

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SECOND-ORDER STRUCTURED DEFORMATIONS: RELAXATION, INTEGRAL REPRESENTATION AND APPLICATIONS

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ABSTRACT. Second-order structured deformations of continua provide an extension of the multiscale geometry of first-order structured deformations by taking into account the effects of submacroscopic bending and curving. We derive here an integral representation for a relaxed energy functional in the setting of second-order structured deformations. Our derivation covers inhomogeneous initial energy densities (i.e., with explicit dependence on the position); finally, we provide explicit formulas for bulk relaxed energies as well as anticipated applications.

1. INTRODUCTION

A first-order structured deformation (g, G) from a region $\Omega \subset \mathbb{R}^N$ provides not only a macroscopic deformation field $g : \Omega \to \mathbb{R}^d$ but also a field $G : \Omega \to \mathbb{R}^{d\times N}$ intended to capture the contributions at the macrolevel of smooth submacroscopic geometrical changes such as stretching, shearing, and rotation. Indeed, in a variety of settings [7, 13, 16, 33], one can prove an approximation theorem to the effect that there exist a sequence of mappings $u_n : \Omega \to \mathbb{R}^d$ that converges to g and whose gradients $\nabla u_n : \Omega \to \mathbb{R}^{d\times N}$ converge to G. In addition, one obtains a formula that identifies the difference $M := \nabla g - G = \nabla \lim_{n\to\infty} u_n - \lim_{n\to\infty} \nabla u_n$ as a limit of "disarrangements", i.e., of averages of directed jumps $[u_n] \otimes \nu_{u_n}$ in the approximating mappings (here, ν_{u_n} denotes the normal to the jump-set of u_n). These disarrangements include the formation of voids, slips, and separations occuring at submacroscopic levels. M is called the (volume) density of disarrangements, and, because $G = \lim_{n\to\infty} \nabla u_n$ does not reflect the jumps in u_n, the field G is called the deformation without disarrangements.

The additive decomposition $\nabla g = G + M$ along with the identifications above of G and M provides a richer geometrical setting in which to study mechanisms for storing mechanical energy. The main approach to assigning an energy to a continuum undergoing structured deformations (g, G) is to assume that such an assignment $E(u_n)$ is available for the approximating deformations u_n in the form of a bulk energy plus an interfacial energy, $E_B(u_n) + E_I(u_n)$, and to assign to (g, G) the relaxed energy

$$E(g, G) := \inf_{\{u_n\}} \left\{ \liminf_{n\to\infty} (E_B(u_n) + E_I(u_n)) : u_n \to g, \nabla u_n \to G \right\} \tag{1.1}$$

where the class of approximating functions and the two senses of convergence are to be specified in such a way that an appropriate version of the approximation theorem can be verified. This approach was first studied in [13], where additive decompositions

$$E(g, G) = E_{\text{bulk}}(g, G) + E_{\text{int}}(g, G)$$

of the relaxed energies as well as a variety of properties of the associated bulk and interfacial energy densities were established. In a different setting, the study [7] used similar techniques to obtain an additive decomposition of this form along with the additional decomposition $E_{\text{bulk}}(g, G) = E^1_{\text{bulk}}(M) + E^2_{\text{bulk}}(G, \nabla g)$. See the survey article [6] for details and comparisons. The article [26] addresses issues related to additional decomposition of E_{bulk} in [7], while [15] obtains detailed information about relaxed energies in the case of one-dimensional structured deformations.

Date: July 9, 2016 - revised version: March 13, 2017. Preprint SISSA: 37/MATE/2016.
2010 *Mathematics Subject Classification.* 49J45 (74G65, 74M25, 15A99) .

The various studies of relaxed energies in the case of first-order structured deformations (g, G) cited above do not account explicitly for the contributions to the energy of "gradient disarrangements", i.e., of jumps in ∇u_n, with u_n converging to g and ∇u_n converging to G. The multiscale geometry of structured deformations was broadened [29, 31], to provide additional fields capable of describing effects at the macrolevel of gradient disarrangements. A second-order structured deformation is a triple (g, G, Γ) in which (g, G) is a first-order structured deformation (with additional smoothness granted to g and G) and $\Gamma : \Omega \to \mathbb{R}^{d\times N\times N}$ is a field intended to describe the contributions at the macrolevel of smooth bending and of curving at submacroscopic levels. In [29, 31], various versions of approximation theorems are obtained that provide sequences of approximations u_n with u_n converging to g, ∇u_n converging to G, and $\nabla^2 u_n$ converging to Γ. The decomposition $\nabla g = G + M$ remains valid here and implies the higher-order decomposition

$$\nabla^2 g = \nabla M + (\nabla G - \Gamma) + \Gamma.$$

In view of the approximation theorem, we can write

$$\nabla G - \Gamma = \nabla \lim_{n\to\infty} \nabla u_n - \lim_{n\to\infty} \nabla^2 u_n.$$

As a consequence, $\nabla G - \Gamma$ can be shown to be a limit of averages of directed jumps $[\nabla u_n]\otimes\nu_{\nabla u_n}$ in analogy with the corresponding result for $\nabla g - G$, so that $\nabla G - \Gamma$ emerges as a density of gradient disarrangements.

In this article, we use this background to study the relaxation of energies in a specific mathematical setting for second-order structured deformations (g, G, Γ), the so-called "SBV^2-setting", see [17]; we also refer the reader to [12, Section 2.2] for an introduction to structured deformations in the SBV context. The results in [13] and [7] for the energetics of first-order structured deformations and those of [31] provide a guide for our analysis of energetics in the second-order case. Beyond providing an analysis in the second-order case, we broaden the scope further by following ideas in [9] in order to include in our analysis the case of "inhomogeneous energetic response", i.e., the case in which initial bulk and interfacial densities can depend explicitly on location in the body.

The overall plan of this work in the ensuing sections is as follows. In Section 2 we fix the notation and recall some auxiliary results used throughout the paper. The problem, our hypotheses and the main result, Theorem 3.2, are presented in Section 3. In Section 4 we prove some preliminary results and, in particular, show that our energy functional can be decomposed into a sum of two lower order functionals. Section 5 is devoted to the proof of Theorem 3.2, and finally, in Section 6, we give an example in which the formula in Theorem 3.2 for the bulk relaxed energy density can be calculated explicitly, thus providing an explicit formula in terms of $\nabla G - \Gamma$ for the volume density of the non-tangential part of jumps in directional derivatives of approximations. We further indicate in Section 6 applications of the energetics of second-order structured deformations in the study of elastic bodies undergoing disarrangements.

2. Preliminaries

The purpose of this section is to give a brief overview of the concepts and results that are used in the sequel. Almost all these results are stated without proofs as they can be readily found in the references given below.

2.1. **Notation.**

Throughout the text $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$, will denote an open bounded set and we will use the following notations:

- $\mathcal{O}(\Omega)$ is the family of all open subsets of Ω,
- $\mathcal{M}(\Omega)$ is the set of finite Radon measures on Ω,
- $\mathcal{M}^+(\Omega)$ is the set of finite and positive Radon measures on Ω,
- $||\mu||$ stands for the total variation of a measure $\mu \in \mathcal{M}(\Omega)$,
- S^{N-1} stands for the unit sphere in $\mathbb{R}^N$,
- e_i denotes the i^{th} element of the canonical basis of $\mathbb{R}^N$, for $i = 1, \dots, N$.

- Q denotes the unit cube centered at the origin with faces orthogonal to the coordinate axes,
- $Q(x,\delta)$ denotes a cube centered at $x \in \Omega$ with side length δ and with two of its faces orthogonal to e_N,
- $Q_\nu(x,\delta)$ is a cube centered at $x \in \Omega$ with side length δ and with two of its faces orthogonal to $\nu \in S^{N-1}$,
- $Q_\nu := Q_\nu(0,1)$,
- C represents a generic constant whose value might change from line to line,
- $\lim\limits_{n,m\to+\infty} := \lim\limits_{n\to+\infty}\lim\limits_{m\to+\infty}$ while $\lim\limits_{m,n\to+\infty} := \lim\limits_{m\to+\infty}\lim\limits_{n\to+\infty}$,

2.2. **Measure Theory.** We recall Reshetnyak's Theorem on weak convergence of vector measures (see Reshetnyak [32]; see also Ambrosio, Fusco and Pallara [4]).

Theorem 2.1. *Let μ, μ_n be $\mathbb{R}^d-$valued finite Radon measures in Ω such that $\mu_n \overset{*}{\rightharpoonup} \mu$ in Ω and such that $||\mu_n||(\Omega) \to ||\mu||(\Omega)$. Then*

$$\lim_{n\to+\infty}\int_\Omega f\left(x, \frac{\mu_n}{||\mu_n||}(x)\right) d||\mu_n||(x) = \int_\Omega f\left(x, \frac{\mu}{||\mu||}(x)\right) d||\mu||(x)$$

for every continuous and bounded function $f : \Omega \times S^{d-1} \to \mathbb{R}$.

2.3. **BV Functions.** In this section we briefly summarize some facts on functions of bounded variation that will be used throughout the paper. We refer to [4, 22, 23, 24, 34] for a detailed description of this subject.

A function $u \in L^1(\Omega;\mathbb{R}^d)$ is said to be of bounded variation, and we write $u \in BV(\Omega;\mathbb{R}^d)$, if all its first-order distributional derivatives $D_j u_i \in \mathcal{M}(\Omega)$ for $i = 1,...,d$ and $j = 1,...,N$. The matrix-valued measure whose entries are $D_j u_i$ is denoted by Du. By the Lebesgue Decomposition Theorem Du can be split into the sum of two mutually singular measures $D^a u$ and $D^s u$ (the absolutely continuous part and the singular part, respectively, of Du with respect to the Lebesgue measure $\mathcal{L}^N$). By ∇u we denote the Radon-Nikodým derivative of $D^a u$ with respect to $\mathcal{L}^N$, so that we can write

$$Du = \nabla u \mathcal{L}^N \lfloor \Omega + D^s u.$$

Let Ω_u be the set of points where the approximate limit of u exists, i.e., points $x \in \Omega$ for which there exists $z \in \mathbb{R}^N$ such that

$$\lim_{\varepsilon\to0^+} \fint_{Q(x,\varepsilon)} |u(y) - z|\, dy = 0.$$

If $x \in \Omega_u$ and $z = u(x)$ we say that u is *approximately continuous* at x (or that x is a Lebesgue point of u). The function u is approximately continuous for $\mathcal{L}^N$-a.e. $x \in \Omega_u$.

The *jump set* of the function u, denoted by S_u, is the set of points $x \in \Omega\setminus\Omega_u$ for which there exist a, $b \in \mathbb{R}^d$ and a unit vector $\nu \in S^{N-1}$, normal to S_u at x, such that $a \neq b$ and

$$\lim_{\varepsilon\to0^+} \frac{1}{\varepsilon^N}\int_{\{y\in Q_\nu(x,\varepsilon):(y-x)\cdot\nu>0\}} |u(y)-a|\, dy = 0, \qquad \lim_{\varepsilon\to0^+} \frac{1}{\varepsilon^N}\int_{\{y\in Q_\nu(x,\varepsilon):(y-x)\cdot\nu<0\}} |u(y)-b|\, dy = 0.$$

The triple (a,b,ν) is uniquely determined by the conditions above up to a permutation of (a,b) and a change of sign of ν and is denoted by $(u^+(x), u^-(x), \nu_u(x))$.

If $u \in BV(\Omega)$ it is a standard result that S_u is countably $(N-1)$-rectifiable, see [4], and the following decomposition holds

$$Du = \nabla u \mathcal{L}^N \lfloor \Omega + [u] \otimes \nu_u \mathcal{H}^{N-1} \lfloor S_u + D^c u,$$

where $[u] := u^+ - u^-$ and $D^c u$ is the Cantor part of the measure Du.

Throughout this paper we shall employ for convenience the slightly abusive notation $[f(x)]$ in place of the more accurate notation $[f](x)$ for the difference $f^+(x) - f^-(x)$.

We also recall that a measurable subset $E \subset \mathbb{R}^N$ is a *set of finite perimeter* in Ω if the characteristic function χ_E of E is a function of bounded variation. In this case, the perimeter of E in Ω is given by the total variation of χ_E in Ω, i.e., $\mathrm{Per}_\Omega(E) := |D\chi_E|(\Omega)$.

The following theorem is a variant of a well-known approximation result for sets of finite perimeter and it will be used in the proof of the upper bound inequalities in Proposition 5.6 and Theorem 5.7.

Theorem 2.2 ([8, Lemma 3.1]). *Let Ω be an open, bounded set with Lipschitz boundary and let E be a subset of Ω with $\mathrm{Per}_\Omega(E) < +\infty$. There exists a sequence $\{E_n\}$ of polyhedral sets (i.e., for each n, E_n is a bounded Lipschitz domain with $\partial E_n = H_{1,n} \cup H_{2,n} \cup \dots H_{L_n,n}$, where each $H_{j,n}$ is a closed subset of a hyperplane $\{x \in \mathbb{R}^N : x \cdot \nu_j = c_j\}$, for some $c_j \in \mathbb{R}$ and $\nu_j \in S^{N-1}$, $j = 1, \dots, L_n$, $L_n \in \mathbb{N}$) satisfying the following properties:*

(i) $\chi_{E_n} \to \chi_E$ *in* $L^1(\Omega)$, *as* $n \to +\infty$,
(ii) $\lim\limits_{n\to+\infty} \mathrm{Per}_\Omega(E_n) = \mathrm{Per}_\Omega(E)$,
(iii) $\mathcal{H}^{N-1}(\partial^* E_n \cap \partial\Omega) = 0$ *($\partial^* E$ being the reduced boundary of E, see* [4]*)*,
(iv) $\mathcal{L}^N(E_n) = \mathcal{L}^N(E)$.

If Ω is an open and bounded set with Lipschitz boundary then the outer unit normal to $\partial\Omega$ (denoted by ν) exists $\mathcal{H}^{N-1}$-a.e. and the trace for functions in $BV(\Omega;\mathbb{R}^d)$ is defined.

Lemma 2.3. *Let $u \in BV(\Omega;\mathbb{R}^d)$. There exist piecewise constant functions u_n such that $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$ and*

$$||Du||(\Omega) = \lim_{n\to+\infty} ||Du_n||(\Omega) = \lim_{n\to+\infty} \int_{S_{u_n}} |[u_n](x)| \, dH^{N-1}(x).$$

The space of *special functions of bounded variation* $SBV(\Omega;\mathbb{R}^d)$, introduced in [14] to study free discontinuity problems, is the space of functions $u \in BV(\Omega;\mathbb{R}^d)$ such that $D^c u = 0$, i.e. for which

$$Du = \nabla u \mathcal{L}^N + [u] \otimes \nu_u \mathcal{H}^{N-1} \lfloor S_u.$$

The next result is a Lusin-type theorem for gradients due to Alberti [3], and is essential for our arguments.

Theorem 2.4. *Let $f \in L^1(\Omega;\mathbb{R}^{d\times N})$. There exists $u \in SBV(\Omega;\mathbb{R}^d)$ and a Borel function $g : \Omega \to \mathbb{R}^{d\times N}$ such that*

$$Du = f\mathcal{L}^N + g\mathcal{H}^{N-1}\lfloor S_u,$$

$$\int_{S_u} |g| \, d\mathcal{H}^{N-1} \leqslant C||f||_{L^1(\Omega;\mathbb{R}^{d\times N})}.$$

Moreover,

$$||u||_{L^1(\Omega)} \leqslant C||f||_{L^1(\Omega;\mathbb{R}^{d\times N})}.$$

The following technical result is a simplified version of Lemma 4.3 in [27].

Lemma 2.5. *Let $\Omega \subset \mathbb{R}^N$ be open and bounded and let $A \in \mathbb{R}^{d\times N}$. Then there exists $u \in SBV(\Omega;\mathbb{R}^d)$ such that $u|_{\partial\Omega} = 0$ and $\nabla u = A$ a.e in Ω. In addition*

$$\|D^s u\|(\Omega) \leqslant C(N)|A|\,|\Omega|.$$

Remark 2.6. The space $SBV(\Omega;\mathbb{R}^d)$ is the right functional setting for the energetics of first-order structured deformations developed in [13], as it provides a straightforward link to the original theory developed in [16]. First-order structured deformations are defined in [16, Definition 5.1] as a triple (κ, g, G), where $\kappa \subset \Omega$ and g and G can be discontinuous on κ; in [13, Definition 2.11] structured deformations are defined as pairs $(g, G) \in SBV(\Omega;\mathbb{R}^d) \times L^1(\Omega;\mathbb{R}^{d\times N})$. The role of κ is therefore played by the jump set S_g of g.

Following [10, 11], we define

$$SBV^2(\Omega;\mathbb{R}^d) := \{v \in SBV(\Omega;\mathbb{R}^d) : \ \nabla v \in SBV(\Omega;\mathbb{R}^{d\times N})\}.$$

If $u \in SBV^2(\Omega;\mathbb{R}^d)$ we use the notation $\nabla^2 u = \nabla(\nabla u)$ to denote the absolutely continuous part of $D(\nabla u)$ with respect to the Lebesgue measure. Analogously, we let

$$BV^2(\Omega;\mathbb{R}^d) = \{v \in BV(\Omega;\mathbb{R}^d) : \ \nabla v \in BV(\Omega;\mathbb{R}^{d\times N})\}.$$

3. Statement of the problem and main result

We define a second-order structured deformation as a triplet

$$(g, G, \Gamma) \in SBV^2(\Omega;\mathbb{R}^d) \times SBV(\Omega;\mathbb{R}^{d\times N}) \times L^1(\Omega;\mathbb{R}^{d\times N\times N}).$$

The set of second-order structured deformations will be denoted in the sequel by $SD^2(\Omega;\mathbb{R}^d)$.

Given a function $u \in SBV^2(\Omega;\mathbb{R}^d)$, consider the energy defined by

$$\begin{aligned} E(u) := \int_\Omega W(x, \nabla u(x), \nabla^2 u(x))\, dx + \int_{S_u} \Psi_1(x, [u(x)], \nu_u(x))\, d\mathcal{H}^{N-1}(x) \\ + \int_{S_{\nabla u}} \Psi_2(x, [\nabla u(x)], \nu_{\nabla u}(x))\, d\mathcal{H}^{N-1}(x), \end{aligned} \tag{3.1}$$

where the densities $W : \Omega \times \mathbb{R}^{d\times N} \times \mathbb{R}^{d\times N\times N} \to [0,+\infty[$, $\Psi_1 : \Omega \times \mathbb{R}^d \times S^{N-1} \to [0,+\infty[$ and $\Psi_2 : \Omega \times \mathbb{R}^{d\times N} \times S^{N-1} \to [0,+\infty[$ satisfy the following hypotheses:

(H1) there exists $C > 0$ such that

$$\frac{1}{C}(|A| + |M|) - C \leqslant W(x, A, M) \leqslant C\big(1 + |A| + |M|\big)$$

for all $x \in \Omega, A \in \mathbb{R}^{d\times N}$ and $M \in \mathbb{R}^{d\times N\times N}$;

(H2) there exists $C > 0$ such that

$$|W(x, A_1, M_1) - W(x, A_2, M_2)| \leqslant C\big(|A_1 - A_2| + |M_1 - M_2|\big)$$

for all $x \in \Omega, A_i \in \mathbb{R}^{d\times N}$ and $M_i \in \mathbb{R}^{d\times N\times N}$, $i = 1, 2$;

(H3) for every $x_0 \in \Omega$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow |W(x, A, M) - W(x_0, A, M)| \leqslant \varepsilon C(1 + |A| + |M|),$$

for all $x \in \Omega, A \in \mathbb{R}^{d\times N}$ and $M \in \mathbb{R}^{d\times N\times N}$;

(H4) there exist $0 < \alpha < 1$ and $L > 0$ such that

$$\Big| W^\infty(x, A, M) - \frac{W(x, A, tM)}{t} \Big| \leqslant \frac{C}{t^\alpha}$$

for all $t > L$, $x \in \Omega, A \in \mathbb{R}^{d\times N}$, $M \in \mathbb{R}^{d\times N\times N}$ with $|M| = 1$, where W^∞ denotes the *recession function* of W in the variable M, i.e.,

$$W^\infty(x, A, M) = \limsup_{t\to+\infty} \frac{W(x, A, tM)}{t};$$

(H5) there exist $c_1 > 0, K_1 > 0$, such that

$$c_1|\lambda| \leqslant \Psi_1(x, \lambda, \nu) \leqslant K_1|\lambda|,$$

for all $x \in \Omega, \lambda \in \mathbb{R}^d$ and $\nu \in S^{N-1}$;
there exist $c_2 > 0, K_2 > 0$, such that

$$c_2|\Lambda| \leqslant \Psi_2(x, \Lambda, \nu) \leqslant K_2|\Lambda|,$$

for all $x \in \Omega, \Lambda \in \mathbb{R}^{d\times N}$ and $\nu \in S^{N-1}$;

(H6) for every $x_0 \in \Omega$ and for every $\varepsilon > 0$ there exist $\delta > 0$ and $C_1, C_2 > 0$ such that

$$|x - x_0| < \delta \Rightarrow |\Psi_1(x_0, \lambda, \nu) - \Psi_1(x, \lambda, \nu)| \leqslant \varepsilon C_1|\lambda|,$$
$$|x - x_0| < \delta \Rightarrow |\Psi_2(x_0, \Lambda, \nu) - \Psi_2(x, \Lambda, \nu)| \leqslant \varepsilon C_2|\Lambda|$$

for all $\lambda \in \mathbb{R}^d$, $\Lambda \in \mathbb{R}^{d\times N}$ and $\nu \in S^{N-1}$;

(H7) (homogeneity of degree one)

$$\Psi_1(x, t\lambda, \nu) = t\Psi_1(x, \lambda, \nu), \quad \Psi_2(x, t\Lambda, \nu) = t\Psi_2(x, \Lambda, \nu),$$

for all $x \in \Omega, \nu \in S^{N-1}, \lambda \in \mathbb{R}^N, \Lambda \in \mathbb{R}^{d\times N}$ and $t > 0$;

(H8) (sub-additivity)

$$\Psi_1(x, \lambda_1 + \lambda_2, \nu) \leqslant \Psi_1(x, \lambda_1, \nu) + \Psi_1(x, \lambda_2, \nu),$$
$$\Psi_2(x, \Lambda_1 + \Lambda_2, \nu) \leqslant \Psi_2(x, \Lambda_1, \nu) + \Psi_2(x, \Lambda_2, \nu),$$

for all $x \in \Omega, \nu \in S^{N-1}, \lambda_i \in \mathbb{R}^d, \Lambda_i \in \mathbb{R}^{d\times N}$, $i = 1, 2$.

Remark 3.1. (1) We extend $\Psi_i, i = 1, 2$ as homogeneous functions of degree one in the third variable to all of $\mathbb{R}^N$.

(2) The hypotheses listed above are similar to the ones in [13] and [7] where there is no explicit dependence on x, and with the hypotheses in [9] where the density functions depended explicitly on the variable x.

(3) In applications it is reasonable to expect the bulk energy to have potential wells and for this reason it is desirable to consider

$$0 \leqslant W(x, A, M) \leqslant C(1 + |A| + |M|),$$

instead of (H1). However, following the same arguments as in [13], the coercivity assumption can be removed.

(4) In the case of no explicit dependence on the position variable x, the coercivity hypothesis on the interfacial energy densities can be replaced by the extra condition that admissible sequences are bounded in BV^2-norm. This standard modification of our model covers the case of the example in Section 6.

(5) It follows immediately from the definition of the recession function and from hypotheses (H1), (H2) and (H3) that there exists $C > 0$ such that for all $x \in \Omega$, $A_i \in \mathbb{R}^{d\times N}$ and $M_i \in \mathbb{R}^{d\times N\times N}$, $i = 1, 2$

$$\frac{1}{C}|M_1| \leqslant W^\infty(x, A_1, M_1) \leqslant C|M_1|; \tag{3.2}$$

$$|W^\infty(x, A_1, M_1) - W^\infty(x, A_2, M_2)| \leqslant C|M_1 - M_2| \tag{3.3}$$

and, for every $x_0 \in \Omega$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow |W^\infty(x, A_1, M_1) - W^\infty(x_0, A_1, M_1)| \leqslant \varepsilon C|M_1|. \tag{3.4}$$

Consider now the relaxed energy

$$I(g, G, \Gamma) := \inf_{\{u_n\}\subset SBV^2(\Omega;\mathbb{R}^d)} \Big\{ \liminf_{n\to+\infty} E(u_n) : u_n \overset{L^1}{\to} g, \nabla u_n \overset{L^1}{\to} G, \nabla^2 u_n \overset{*}{\rightharpoonup} \Gamma \Big\}. \tag{3.5}$$

The main result of this work reads as follows

Theorem 3.2. *For all $(g, G, \Gamma) \in SD^2(\Omega; \mathbb{R}^d)$, under hypotheses (H1) - (H8), we have that*

$$\begin{aligned} I(g, G, \Gamma) = &\int_\Omega \{W_1(x, G(x) - \nabla g(x)) + W_2(x, G(x), \nabla G(x), \Gamma(x))\}\, dx \\ &+ \int_{S_g\cap\Omega} \gamma_1(x, [g(x)], \nu_g(x))\, d\mathcal{H}^{N-1}(x) \\ &+ \int_{S_G\cap\Omega} \gamma_2(x, G(x), [G(x)], \nu_G(x))\, d\mathcal{H}^{N-1}(x), \end{aligned} \tag{3.6}$$

where, for $x \in \Omega$, $A, \Lambda \in \mathbb{R}^{d\times N}$, $L, M \in \mathbb{R}^{d\times N\times N}$, $\lambda \in \mathbb{R}^d$ and $\nu \in S^{N-1}$,

$$W_1(x, A) = \inf_{u\in SBV^2(Q;\mathbb{R}^d)} \left\{ \int_{S_u\cap Q} \Psi_1(x, [u(y)], \nu_u(y))\, d\mathcal{H}^{N-1}(y) : \ u|_{\partial Q} = 0, \ \nabla u = A \textit{ a.e. in } Q \right\},$$

$$\begin{aligned} \gamma_1(x, \lambda, \nu) = \inf_{u\in SBV^2(Q_\nu;\mathbb{R}^d)} \Big\{ &\int_{S_u\cap Q_\nu} \Psi_1(x, [u(y)], \nu_u(y))\, d\mathcal{H}^{N-1}(y) : u|_{\partial Q_\nu} = \gamma_{(\lambda,\nu)}, \\ &\nabla u = 0 \textit{ a.e. in } Q_\nu \Big\}, \end{aligned}$$

with

$$\gamma_{(\lambda,\nu)} = \begin{cases} \lambda & \textit{if } x\cdot\nu > 0 \\ 0 & \textit{if } x\cdot\nu < 0, \end{cases}$$

and

$$W_2(x,A,L,M) = \inf_{u\in SBV(Q;\mathbb{R}^{d\times N})} \left\{ \int_Q W(x,A,\nabla u(y))\,dy + \int_{S_u\cap Q} \Psi_2(x,[u(y)],\nu_u(y))\,d\mathcal{H}^{N-1}(y) : \right. \\ \left. u|_{\partial Q}(y) = L\cdot y,\ \int_Q \nabla u(y)\,dy = M \right\},$$

$$\gamma_2(x,A,\Lambda,\nu) = \inf_{u\in SBV(Q_\nu;\mathbb{R}^{d\times N})} \left\{ \int_{Q_\nu} W^\infty(x,A,\nabla u(y))\,dy + \int_{S_u\cap Q_\nu} \Psi_2(x,[u(y)],\nu_u(y))\,d\mathcal{H}^{N-1}(y) : \right. \\ \left. u|_{\partial Q_\nu} = \gamma_{(\Lambda,\nu)},\ \int_{Q_\nu} \nabla u(y)\,dy = 0 \right\}.$$

4. PRELIMINARY RESULTS

In this section we derive some preliminary results which will be used in the proof of the main theorem.

Lemma 4.1. *Let* $(g,G,\Gamma)\in SD^2(\Omega;\mathbb{R}^d)$. *Then* $I(g,G,\Gamma)<+\infty$.

Proof. Let $(g,G,\Gamma)\in SD^2(\Omega;\mathbb{R}^d)$ be given. By applying Theorem 2.4, there exists $h\in SBV(\Omega;\mathbb{R}^{d\times N})$ such that $\nabla h=\Gamma$ a.e. in Ω and

$$\|D^s h\|(\Omega) \leqslant C||\Gamma||_{L^1(\Omega;\mathbb{R}^{d\times N\times N})}, \tag{4.1}$$

for some $C=C(N)>0$. By Lemma 2.3 there exists a sequence $\{v_n\}\subset L^1(\Omega;\mathbb{R}^d)$ of piecewise constant functions such that $v_n \overset{L^1}{\to} G-h$ and

$$\|Dv_n\|(\Omega) = \|D^s v_n\|(\Omega) \underset{n\to+\infty}{\longrightarrow} \|DG-Dh\|(\Omega). \tag{4.2}$$

Define $w_n\in SBV(\Omega;\mathbb{R}^{d\times N})$ by $w_n:=v_n+h$. We have $w_n\to G$ in $L^1(\Omega;\mathbb{R}^{d\times N})$ and $\nabla w_n=\Gamma$ a.e. in Ω. By applying again Theorem 2.4, for every $n\in\mathbb{N}$, there exists $\tilde h_n\in SBV(\Omega;\mathbb{R}^d)$ such that $\nabla\tilde h_n=w_n$ a.e. in Ω and

$$\|D^s\tilde h_n\|(\Omega) \leqslant C\|w_n\|_{L^1(\Omega;\mathbb{R}^{d\times N})}. \tag{4.3}$$

By Lemma 2.3, for every $n\in\mathbb{N}$, there exists a sequence $\{\overline{h}_{n,m}\}\subset L^1(\Omega;\mathbb{R}^d)$ of piecewise constant functions such that $\overline{h}_{n,m} \overset{L^1}{\to} g-\tilde h_n$ as $m\to+\infty$ and

$$\|D^s\overline{h}_{n,m}\|(\Omega) \underset{m\to+\infty}{\longrightarrow} \|Dg-D\tilde h_n\|(\Omega).$$

Thus, for every $n\in\mathbb{N}$, there exists $m(n)\in\mathbb{N}$ such that

$$||\overline{h}_{n,m(n)}-(g-\tilde h_n)||_{L^1(\Omega;\mathbb{R}^d)}<\frac{1}{n}, \qquad \Big|\|D^s\overline{h}_{n,m(n)}\|(\Omega)-\|Dg-D\tilde h_n\|(\Omega)\Big|<\frac{1}{n}. \tag{4.4}$$

Hence the sequence $u_n:=\tilde h_n+\overline{h}_{n,m(n)}$ is such that $u_n\to g$ in $L^1(\Omega;\mathbb{R}^d)$, $\nabla u_n=w_n\to G$ in $L^1(\Omega;\mathbb{R}^{d\times N})$ and $\nabla^2 u_n=\Gamma$, so that it is a competitor for the infimization problem (3.5).

By the growth assumptions (H1) and (H5), and (4.1), (4.2), (4.3) and (4.4), we can estimate

$$
\begin{aligned}
I(g,G,\Gamma) \leqslant & \liminf_{n\to+\infty} E(u_n) \\
\leqslant & \liminf_{n\to+\infty} \left[\int_\Omega W(x,\nabla u_n(x),\nabla^2 u_n(x))\,dx + \int_{S_{u_n}} \Psi_1(x,[u_n(x)],\nu_{u_n}(x))\,d\mathcal{H}^{N-1}(x) \right. \\
& \left. + \int_{S_{\nabla u_n}} \Psi_2(x,[\nabla u_n(x)],\nu_{\nabla u_n}(x))\,d\mathcal{H}^{N-1}(x) \right] \\
\leqslant & \liminf_{n\to+\infty} \left[C\int_\Omega (1+|\nabla u_n(x)|+|\nabla^2 u_n(x)|)\,dx + K_1 \int_{S_{u_n}} |[u_n(x)]|\,d\mathcal{H}^{N-1}(x) \right. \\
& \left. + K_2 \int_{S_{\nabla u_n}} |[\nabla u_n(x)]|\,d\mathcal{H}^{N-1}(x) \right] \\
\leqslant & \liminf_{n\to+\infty} \left[\int_\Omega C(1+|G(x)|+|\Gamma(x)|)\,dx + K_1\|D^s u_n\|(\Omega) + K_2\|D^s(\nabla u_n)\|(\Omega) \right] \qquad \textbf{(4.5)} \\
& + \limsup_{n\to+\infty} C\|w_n - G\|_{L^1(\Omega;\mathbb{R}^{d\times N})} \\
\leqslant & C\left[\mathcal{L}^N(\Omega) + \|G\|_{L^1(\Omega;\mathbb{R}^{d\times N})} + \|\Gamma\|_{L^1(\Omega;\mathbb{R}^{d\times N\times N})} + \|Dg\|(\Omega) \right. \\
& \left. + \limsup_{n\to+\infty} \|DG - Dh\|(\Omega) + \limsup_{n\to+\infty} \|w_n\|_{L^1(\Omega;\mathbb{R}^{d\times N})} \right] \\
\leqslant & C\left[\mathcal{L}^N(\Omega) + \|G\|_{L^1(\Omega;\mathbb{R}^{d\times N})} + \|\Gamma\|_{L^1(\Omega;\mathbb{R}^{d\times N\times N})} \right. \\
& \left. + \|Dg\|(\Omega) + \|DG\|(\Omega) + \limsup_{n\to+\infty} \|w_n - G\|_{L^1(\Omega;\mathbb{R}^{d\times N})} \right] \\
\leqslant & C(1+\|Dg\|(\Omega) + \|G\|_{L^1(\Omega;\mathbb{R}^{d\times N})} + \|DG\|(\Omega) + \|\Gamma\|_{L^1(\Omega;\mathbb{R}^{d\times N\times N})}).
\end{aligned}
$$

□

Remark 4.2. As the above proof shows, given $(g,G,\Gamma)\in SD^2(\Omega;\mathbb{R}^d)$ there exists a sequence $\{u_n\}\subset SBV^2(\Omega;\mathbb{R}^d)$ such that $u_n\to g$ in $L^1(\Omega;\mathbb{R}^d)$, $\nabla u_n\to G$ in $L^1(\Omega;\mathbb{R}^{d\times N})$ and $\nabla^2 u_n \overset{*}{\rightharpoonup} \Gamma$. Our proof is essentially the same as the proof of Theorem 3.2 in [31].

4.1. **Decomposition.**

Theorem 4.3. *We may decompose* $I(g,G,\Gamma)$ *as* $I(g,G,\Gamma) = I_1(g,G,\Gamma) + I_2(G,\Gamma)$, *where*

$$
\begin{aligned}
I_1(g,G,\Gamma) \quad := \quad & \inf_{\{u_n\}\subset SBV^2(\Omega;\mathbb{R}^d)} \left\{ \liminf_{n\to+\infty} \int_{S_{u_n}} \Psi_1(x,[u_n(x)],\nu_{u_n}(x))\,d\mathcal{H}^{N-1}(x) : \right. \\
& \left. u_n \xrightarrow{L^1} g,\ \nabla u_n \xrightarrow{L^1} G,\ \nabla^2 u_n \overset{*}{\rightharpoonup} \Gamma \right\}
\end{aligned}
$$

and

$$
\begin{aligned}
I_2(G,\Gamma) \quad := \quad & \inf_{\{v_n\}\subset SBV(\Omega;\mathbb{R}^{d\times N})} \left\{ \liminf_{n\to+\infty} \left[\int_\Omega W(x,v_n(x),\nabla v_n(x))\,dx \right.\right. \\
& \left.\left. + \int_{S_{v_n}} \Psi_2(x,[v_n(x)],\nu_{v_n}(x))\,d\mathcal{H}^{N-1}(x) \right] :\ v_n \xrightarrow{L^1} G,\ \nabla v_n \overset{*}{\rightharpoonup} \Gamma \right\}.
\end{aligned}
$$

Proof. It is clear that

$$I(g,G,\Gamma) \geqslant I_1(g,G,\Gamma) + I_2(G,\Gamma).$$

To show the reverse inequality let $\{u_n\} \subset SBV^2(\Omega;\mathbb{R}^d)$ be such that $u_n \xrightarrow{L^1} g$, $\nabla u_n \xrightarrow{L^1} G$, $\nabla^2 u_n \overset{*}{\rightharpoonup} \Gamma$ and

$$I_1(g,G,\Gamma) = \lim_{n\to+\infty} \int_{S_{u_n}} \Psi_1(x,[u_n(x)],\nu_{u_n}(x))\,d\mathcal{H}^{N-1}(x)$$

and let $\{v_n\} \subset SBV^2(\Omega;\mathbb{R}^{d\times N})$ be such that $v_n \xrightarrow{L^1} G$, $\nabla v_n \overset{*}{\rightharpoonup} \Gamma$ and

$$I_2(G,\Gamma) = \lim_{n\to+\infty} \left[\int_\Omega W(x,v_n(x),\nabla v_n(x))\,dx + \int_{S_{v_n}} \Psi_2(x,[v_n(x)],\nu_{v_n}(x))\,d\mathcal{H}^{N-1}(x)\right].$$

By Theorem 2.4 let $\{h_n\} \subset SBV(\Omega;\mathbb{R}^d)$ be such that $\nabla h_n = v_n - \nabla u_n$ and $\|D^s h_n\|(\Omega) \leqslant C\|v_n - \nabla u_n\|_{L^1(\Omega;\mathbb{R}^{d\times N})}$, and by Lemma 2.3 let $\tilde h_n$ be a sequence of piecewise constant functions with $\|\tilde h_n - h_n\|_{L^1} < \frac{1}{n}$ and $\big|\|D\tilde h_n\|(\Omega) - \|Dh_n\|(\Omega)\big| < \frac{1}{n}$. Define $\{w_n\} \subset SBV^2(\Omega;\mathbb{R}^d)$ by

$$w_n := u_n + h_n - \tilde h_n.$$

Then $w_n \xrightarrow{L^1} g$, $\nabla w_n = v_n \xrightarrow{L^1} G$, $\nabla^2 w_n = \nabla v_n \overset{*}{\rightharpoonup} \Gamma$ and so, by (H8) and (H5),

$$\begin{aligned}
I(g,G,\Gamma) &\leqslant \liminf_{n\to+\infty} \Bigg[\int_\Omega W(x,\nabla w_n(x),\nabla^2 w_n(x))\,dx + \int_{S_{w_n}} \Psi_1(x,[w_n(x)],\nu_{w_n}(x))\,d\mathcal{H}^{N-1}(x) \\
&\qquad + \int_{S_{\nabla w_n}} \Psi_2(x,[\nabla w_n(x)],\nu_{\nabla w_n}(x))\,d\mathcal{H}^{N-1}(x)\Bigg] \\
&\leqslant \lim_{n\to+\infty} \left[\int_\Omega W(x,\nabla v_n(x),\nabla v_n(x))\,dx + \int_{S_{v_n}} \Psi_2(x,[v_n(x)],\nu_{v_n}(x))\,d\mathcal{H}^{N-1}(x)\right] \\
&\quad + \lim_{n\to+\infty} \int_{S_{u_n}} \Psi_1(x,[u_n(x)],\nu_{u_n}(x))\,d\mathcal{H}^{N-1}(x) \\
&\quad + \limsup_{n\to+\infty} \int_{S_{h_n}\cup S_{\tilde h_n}} \Psi_1(x,[h_n-\tilde h_n](x),\nu_{h_n-\tilde h_n}(x))\,d\mathcal{H}^{N-1}(x) \\
&\leqslant I_2(G,\Gamma) + I_1(g,G,\Gamma) + \limsup_{n\to+\infty} C \int_{S_{h_n}\cup S_{\tilde h_n}} |[h_n-\tilde h_n](x)|\,d\mathcal{H}^{N-1}(x) \\
&\leqslant I_2(G,\Gamma) + I_1(g,G,\Gamma) + \limsup_{n\to+\infty} C \int_\Omega |v_n(x) - \nabla u_n(x)|\,dx \\
&= I_2(G,\Gamma) + I_1(g,G,\Gamma),
\end{aligned}$$

where we have used the properties of the functions u_n, v_n, h_n and $\tilde h_n$. □

4.2. **Localization.** In this section we localize the functionals I_1 and I_2 and show that they are Radon measures. For each $U \in \mathcal{O}(\Omega)$ we define the localized functionals

$$\begin{aligned}
I_1(g,G,\Gamma,U) \quad := \quad \inf_{\{u_n\}\subset SBV^2(U;\mathbb{R}^d)} \Bigg\{ &\liminf_{n\to+\infty} \int_{S_{u_n}\cap U} \Psi_1(x,[u_n(x)],\nu_{u_n}(x))\,d\mathcal{H}^{N-1}(x) : \\
&u_n \xrightarrow{L^1} g,\ \nabla u_n \xrightarrow{L^1} G,\ \nabla^2 u_n \overset{*}{\rightharpoonup} \Gamma \Bigg\}
\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}
I_2(G,\Gamma,U) := \inf_{\{v_n\}\subset SBV(U;\mathbb{R}^{d\times N})} \Bigg\{ &\liminf_{n\to+\infty} \Big[\int_U W(x,v_n(x),\nabla v_n(x))\,dx \\
&+ \int_{S_{v_n}\cap U} \Psi_2(x,[v_n(x)],\nu_{v_n}(x))\,d\mathcal{H}^{N-1}(x)\Big] :\ v_n \xrightarrow{L^1} G,\ \nabla v_n \overset{*}{\rightharpoonup} \Gamma \Bigg\}.
\end{aligned} \tag{4.7}$$

It is clear that localized versions of the upper bound (4.5) still hold, namely

$$I_1(g, G, \Gamma, U) \leqslant C\left[\|G\|_{L^1(U;\mathbb{R}^{d\times N})} + \|Dg\|(U)\right], \tag{4.8}$$

$$I_2(G, \Gamma, U) \leqslant C\left[1 + \|G\|_{L^1(U;\mathbb{R}^{d\times N})} + \|\Gamma\|_{L^1(U;\mathbb{R}^{d\times N\times N})} + \|DG\|(U)\right]. \tag{4.9}$$

We will now prove that $I_1(g, G, \Gamma, \cdot)\lfloor\mathcal{O}(\Omega)$ and $I_2(G, \Gamma, \cdot)\lfloor\mathcal{O}(\Omega)$ are Radon measures. For this purpose we first show that these functionals are nested subadditive.

Lemma 4.4. *Let $U, V, W \in \mathcal{O}(\Omega)$ be such that $U \subset\subset V \subset W$. Then*

$$I_1(g, G, \Gamma, W) \leqslant I_1(g, G, \Gamma, V) + I_1(g, G, \Gamma, W \setminus \overline{U}), \tag{4.10}$$

$$I_2(G, \Gamma, W) \leqslant I_2(G, \Gamma, V) + I_2(G, \Gamma, W \setminus \overline{U}). \tag{4.11}$$

Proof. We provide the details of the proof only for I_1 since for I_2 it is analogous.

Let $u_n \in SBV^2(V;\mathbb{R}^d)$ and $v_n \in SBV^2(W \setminus \overline{U};\mathbb{R}^d)$ be two sequences such that $u_n \to g$ in $L^1(V;\mathbb{R}^d)$, $\nabla u_n \to G$ in $L^1(V;\mathbb{R}^{d\times N})$, $\nabla^2 u_n \overset{*}{\rightharpoonup} \Gamma$ in $\mathcal{M}(V;\mathbb{R}^{d\times N\times N})$, $v_n \to g$ in $L^1(W \setminus \overline{U};\mathbb{R}^d)$, $\nabla v_n \to G$ in $L^1(W \setminus \overline{U};\mathbb{R}^{d\times N})$, $\nabla^2 v_n \overset{*}{\rightharpoonup} \Gamma$ in $\mathcal{M}(W \setminus \overline{U};\mathbb{R}^{d\times N\times N})$, and that, in addition,

$$I_1(g, G, \Gamma, V) = \lim_{n\to+\infty} \int_{S_{u_n}\cap V} \Psi_1(x, [u_n(x)], \nu_{u_n}(x))\, d\mathcal{H}^{N-1}(x)$$

and

$$I_1(g, G, \Gamma, W \setminus \overline{U}) = \lim_{n\to+\infty} \int_{S_{v_n}\cap(W\setminus\overline{U})} \Psi_1(x, [v_n(x)], \nu_{v_n}(x))\, d\mathcal{H}^{N-1}(x).$$

Note that

$$u_n - v_n \to 0 \ \text{ in } \ L^1(V \cap (W \setminus \overline{U});\mathbb{R}^d) \tag{4.12}$$

and

$$\nabla u_n - \nabla v_n \to 0 \ \text{ in } \ L^1(V \cap (W \setminus \overline{U});\mathbb{R}^{d\times N}),$$

$$\nabla^2 u_n - \nabla^2 v_n \overset{*}{\rightharpoonup} 0 \ \text{ in } \ \mathcal{M}(V \cap (W \setminus \overline{U});\mathbb{R}^{d\times N\times N}).$$

For $\delta > 0$ define

$$U_\delta := \{x \in V : \ \operatorname{dist}(x, U) < \delta\}.$$

For $x \in W$ let $d(x) := \operatorname{dist}(x, U)$. Since the distance function to a fixed set is Lipschitz continuous (see [34, Exercise 1.1]), we can apply the change of variables formula [22, Section 3.4.3, Theorem 2], to obtain

$$\int_{U_\delta\setminus\overline{U}} |u_n(x) - v_n(x)|\, |\det \nabla d(x)|\, dx = \int_0^\delta \left[\int_{d^{-1}(y)} |u_n(x) - v_n(x)|\, d\mathcal{H}^{N-1}(x)\right] dy$$

and, as $|\det \nabla d|$ is bounded and (4.12) holds, it follows that for almost every $\rho \in [0, \delta]$ we have

$$\lim_{n\to+\infty} \int_{d^{-1}(\rho)} |u_n(x) - v_n(x)|\, d\mathcal{H}^{N-1}(x) = \lim_{n\to+\infty} \int_{\partial U_\rho} |u_n(x) - v_n(x)|\, d\mathcal{H}^{N-1}(x) = 0. \tag{4.13}$$

Fix $\rho_0 \in [0, \delta]$ such that $\|\Gamma\chi_V\|(\partial U_{\rho_0}) = 0$, $\|\Gamma\chi_{W\setminus\overline{U}}\|(\partial U_{\rho_0}) = 0$ and such that (4.13) holds. We observe that U_{ρ_0} is a set with locally Lipschitz boundary since it is a level set of a Lipschitz function (see, e.g., [22]). Hence we can consider $u_n, v_n, \nabla u_n, \nabla v_n$ on ∂U_{ρ_0} in the sense of traces and define

$$w_n = \begin{cases} u_n & \text{in } \overline{U}_{\rho_0} \\ v_n & \text{in } W \setminus \overline{U}_{\rho_0}. \end{cases}$$

Then, by the choice of ρ_0, w_n is admissible for $I_1(g,G,\Gamma,W)$ so, by (H5), (4.12) and (4.13), we obtain

$$\begin{aligned} I_1(g,G,\Gamma,W) &\leqslant \liminf_{n\to+\infty}\int_{S_{w_n}\cap W}\Psi_1(x,[w_n(x)],\nu_{w_n}(x))\,d\mathcal{H}^{N-1}(x) \\ &\leqslant \liminf_{n\to+\infty}\Bigg[\int_{S_{u_n}\cap V}\Psi_1(x,[u_n(x)],\nu_{u_n}(x))\,d\mathcal{H}^{N-1}(x) \\ &\qquad + \int_{S_{v_n}\cap(W\setminus\overline{U})}\Psi_1(x,[v_n(x)],\nu_{v_n}(x))\,d\mathcal{H}^{N-1}(x) \\ &\qquad + \int_{S_{w_n}\cap\partial U_{\rho_0}} C|u_n(x)-v_n(x)|\,d\mathcal{H}^{N-1}(x)\Bigg] \\ &= I_1(g,G,\Gamma,V)+I_1(g,G,\Gamma,W\setminus\overline{U}), \end{aligned}$$

which concludes the proof. □

Theorem 4.5. *Assume that hypotheses (H1) and (H5) hold. Then* $I_1(g,G,\Gamma,\cdot)\lfloor\mathcal{O}(\Omega)$ *and* $I_2(G,\Gamma,\cdot)\lfloor\mathcal{O}(\Omega)$ *are Radon measures, absolutely continuous with respect to* $\mathcal{L}^N+\mathcal{H}^{N-1}\lfloor S_g$ *and to* $\mathcal{L}^N+\mathcal{H}^{N-1}\lfloor S_G$*, respectively.*

Proof. Let $u_n\in SBV^2(\Omega;\mathbb{R}^d)$ be such that $u_n\to g$ in $L^1(\Omega;\mathbb{R}^d)$, $\nabla u_n\to G$ in $L^1(\Omega;\mathbb{R}^{d\times N})$, $\nabla^2 u_n\overset{*}{\rightharpoonup}\Gamma$ in $\mathcal{M}(\Omega;\mathbb{R}^{d\times N\times N})$ and

$$I_1(g,G,\Gamma,\Omega)=\lim_{n\to+\infty}\int_{S_{u_n}\cap\Omega}\Psi_1(x,[u_n(x)],\nu_{u_n}(x))\,d\mathcal{H}^{N-1}(x).$$

For every Borel set $B\subset\overline{\Omega}$ define the sequence of measures

$$\mu_n(B) := \int_{S_{u_n}\cap B}\Psi_1(x,[u_n(x)],\nu_{u_n}(x))\,d\mathcal{H}^{N-1}(x).$$

By (H5) this sequence of non-negative Radon measures is uniformly bounded in $\mathcal{M}(\overline{\Omega})$ and thus, upon passing if necessary to a subsequence, we conclude that

$$\mu_n\overset{*}{\rightharpoonup}\mu \text{ in } \mathcal{M}(\overline{\Omega}).$$

In particular,

$$\mu(\overline{\Omega})=I_1(g,G,\Gamma,\Omega).$$

We want to show that, for all $V\in\mathcal{O}(\Omega)$,

$$\mu(V)=I_1(g,G,\Gamma,V). \tag{4.14}$$

Let $V\in\mathcal{O}(\Omega)$, let $\varepsilon>0$ and choose $W\subset\subset V$ such that $\mu(V\setminus W)<\varepsilon$. Since $W\subset\subset V\subset\overline{\Omega}$, by the nested subadditivity property it follows that

$$\begin{aligned} \mu(\overline{\Omega}) &= I_1(g,G,\Gamma,\Omega) \\ &\leqslant I_1(g,G,\Gamma,V)+I_1(g,G,\Gamma,\Omega\setminus\overline{W}) \\ &\leqslant I_1(g,G,\Gamma,V)+\mu(\overline{\Omega\setminus\overline{W}}), \end{aligned}$$

and so,

$$\begin{aligned} \mu(V) &\leqslant \mu(W)+\varepsilon \\ &= \mu(\overline{\Omega})-\mu(\overline{\Omega}\setminus W)+\varepsilon \\ &\leqslant I_1(g,G,\Gamma,\Omega)-I_1(g,G,\Gamma,\Omega\setminus\overline{W})+\varepsilon \\ &\leqslant I_1(g,G,\Gamma,V)+\varepsilon. \end{aligned}$$

Thus, letting $\varepsilon\to 0^+$, we conclude that

$$\mu(V)\leqslant I_1(g,G,\Gamma,V). \tag{4.15}$$

To prove the reverse inequality define, for $U \in \mathcal{O}(\Omega)$,

$$\lambda(U) := \int_U (|\nabla g(x)| + |G(x)|)\, dx + \|D^s g\|(U). \tag{4.16}$$

Let $K \subset\subset V$ be a compact set such that $\lambda(V \setminus K) < \varepsilon$ and choose an open set W such that $K \subset\subset W \subset\subset V$. Lemma 4.4, (4.16) and (4.8) yield

$$\begin{aligned} I_1(g, G, \Gamma, V) &\leqslant I_1(g, G, \Gamma, W) + I_1(g, G, \Gamma, V \setminus K) \\ &\leqslant \liminf_{n\to+\infty} \mu_n(W) + C\lambda(V \setminus K) \\ &\leqslant \limsup_{n\to+\infty} \mu_n(\overline{W}) + C\varepsilon \\ &\leqslant \mu(\overline{W}) + C\varepsilon \\ &\leqslant \mu(V) + C\varepsilon, \end{aligned}$$

so to conclude the result it suffices to let $\varepsilon \to 0^+$.

In the case of I_2 the proof is analogous, using hypotheses (H1) and (H5), (4.9) and the nested subadditivity property (4.11). □

We now define

$$\begin{aligned} \tilde{I}_2(G, \Gamma) \quad := \quad & \inf_{v_n \subset SBV(\Omega;\mathbb{R}^{d\times N})} \Bigg\{ \liminf_{n\to+\infty} \Bigg[\int_\Omega W(x, G(x), \nabla v_n(x))\, dx \\ & + \int_{S_{v_n}\cap\Omega} \Psi_2(x, [v_n(x)], \nu_{v_n}(x))\, d\mathcal{H}^{N-1}(x) \Bigg] : \ v_n \xrightarrow{L^1} G, \ \nabla v_n \overset{*}{\rightharpoonup} \Gamma \Bigg\}. \end{aligned}$$

Proposition 4.6. *Let $(G, \Gamma) \in BV(\Omega; \mathbb{R}^{d\times N}) \times L^1(\Omega; \mathbb{R}^{d\times N\times N})$. Then we have that*

$$I_2(G, \Gamma) = \tilde{I}_2(G, \Gamma).$$

Proof. Let $\{v_n\} \subset SBV(\Omega; \mathbb{R}^{d\times N})$ be such that $v_n \to G$ in $L^1(\Omega; \mathbb{R}^{d\times N})$, $\nabla v_n \overset{*}{\rightharpoonup} \Gamma$ and

$$I_2(G, \Gamma) = \lim_{n\to+\infty} \Bigg[\int_\Omega W(x, v_n(x), \nabla v_n(x))\, dx + \int_{S_{v_n}\cap\Omega} \Psi_2(x, [v_n(x)], \nu_{v_n}(x))\, d\mathcal{H}^{N-1}(x) \Bigg].$$

By (H2) it follows that

$$\begin{aligned} \tilde{I}_2(G, \Gamma) &\leqslant \lim_{n\to+\infty} \Bigg[\int_\Omega W(x, G(x), \nabla v_n(x))\, dx + \int_{S_{v_n}\cap\Omega} \Psi_2(x, [v_n(x)], \nu_{v_n}(x))\, d\mathcal{H}^{N-1}(x) \Bigg] \\ &\leqslant \limsup_{n\to+\infty} \Bigg[\int_\Omega W(x, G(x), \nabla v_n(x)) - W(x, v_n(x), \nabla v_n(x))\, dx \Bigg] \\ &+ \lim_{n\to+\infty} \Bigg[\int_\Omega W(x, v_n(x), \nabla v_n(x))\, dx + \int_{S_{v_n}\cap\Omega} \Psi_2(x, [v_n(x)], \nu_{v_n}(x))\, d\mathcal{H}^{N-1}(x) \Bigg] \\ &\leqslant \limsup_{n\to+\infty} C \int_\Omega |G(x) - v_n(x)|\, dx + I_2(G, \Gamma) = I_2(G, \Gamma). \end{aligned}$$

The reverse inequality is proved similarly. □

A standard diagonalization argument yields the following lower semicontinuity property of both I_1 and I_2.

Proposition 4.7. *Let $(g, G, \Gamma) \in SD^2(\Omega; \mathbb{R}^d)$ and $g_n \in SBV^2(\Omega; \mathbb{R}^d)$, $G_n \in SBV(\Omega; \mathbb{R}^{d\times N})$ be such that $g_n \to g$ in $L^1(\Omega; \mathbb{R}^d)$ and $G_n \to G$ in $L^1(\Omega; \mathbb{R}^{d\times N})$. Then*

$$I_1(g, G, \Gamma, \Omega) \leqslant \liminf_{n\to+\infty} I_1(g_n, G, \Gamma, \Omega)$$

and

$$I_2(G, \Gamma, \Omega) \leqslant \liminf_{n\to+\infty} I_2(G_n, \Gamma, \Omega).$$

4.3. **Properties of the density functions.** In order to prove the upper bound inequality for the surface energy terms of both I_1 and I_2 we need the following properties of the density functions W_1, W_2, γ_1 and γ_2.

Proposition 4.8. *i)* $W_1(x,0)=0, \forall x\in\Omega$*;*

ii) $|W_1(x,A)-W_1(x,B)|\leqslant C|A-B|, \forall x\in\Omega, \forall A,B\in\mathbb{R}^{d\times N}$.

Proof. The proof of $i)$ is immediate by noticing that the function $u=0$ is admissible for $W_1(x,0)$. To prove $ii)$ we will show that $W_1(x,B)\leqslant W_1(x,A)+C|A-B|$, $\forall x\in\Omega, \forall A,B\in\mathbb{R}^{d\times N}$; the reverse inequality follows by interchanging the roles of A and B.

Fix $\varepsilon>0$ and let $u\in SBV^2(Q;\mathbb{R}^d)$ be such that $u|_{\partial Q}=0$, $\nabla u=A$ a.e. in Q and

$$\int_{S_u\cap Q}\Psi_1(x,[u(y)],\nu_u(y))\,d\mathcal{H}^{N-1}(y)\leqslant W_1(x,A)+\varepsilon.$$

By Lemma 2.5, let $v\in SBV^2(Q;\mathbb{R}^d)$ be such that $v|_{\partial Q}=0$, $\nabla v=B-A$ a.e. in Q and $|D^s v|(Q)\leqslant C|B-A|$, and define $w=u+v$. Then w is admissible for $W_1(x,B)$ so by (H8) and (H5),

$$\begin{aligned}
W_1(x,B) &\leqslant \int_{S_w\cap Q}\Psi_1(x,[w(y)],\nu_w(y))\,d\mathcal{H}^{N-1}(y)\\
&\leqslant \int_{S_u\cap Q}\Psi_1(x,[u(y)],\nu_u(y))\,d\mathcal{H}^{N-1}(y)+\int_{S_v\cap Q}\Psi_1(x,[v(y)],\nu_v(y))\,d\mathcal{H}^{N-1}(y)\\
&\leqslant W_1(x,A)+\varepsilon+C|D^s v|(Q)\leqslant W_1(x,A)+\varepsilon+C|B-A|.
\end{aligned}$$

Hence the result follows by letting $\varepsilon\to 0^+$. □

Proposition 4.9. *i)* $\gamma_1(x,\lambda,\nu)\leqslant C|\lambda|, \forall(x,\lambda,\nu)\in\Omega\times\mathbb{R}^d\times S^{N-1}$*;*

ii) for every $x_0\in\Omega$ *and for every* $\varepsilon>0$ *there exists* $\delta>0$ *such that*

$$|x-x_0|<\delta\Rightarrow|\gamma_1(x_0,\lambda,\nu)-\gamma_1(x,\lambda,\nu)|\leqslant\varepsilon C(1+|\lambda|), \forall(x,\lambda,\nu)\in\Omega\times\mathbb{R}^d\times S^{N-1};$$

iii) $|\gamma_1(x,\lambda,\nu)-\gamma_1(x,\lambda',\nu)|\leqslant C|\lambda-\lambda'|$*,* $\forall(x,\lambda,\nu),(x,\lambda',\nu)\in\Omega\times\mathbb{R}^d\times S^{N-1}$*;*

iv) γ_1 *is upper semicontinuous in* $\Omega\times\mathbb{R}^d\times S^{N-1}$.

Proof. The proof of $i)$ follows immediately from the fact that the function $\gamma_{(\lambda,\nu)}$ is admissible for $\gamma_1(x,\lambda,\nu)$ and from hypotheses (H5).

To prove $ii)$ fix $x_0\in\Omega$ and $\varepsilon>0$. By (H6) let $\delta>0$ be such that

$$|x-x_0|<\delta\Rightarrow|\Psi_1(x_0,\lambda,\nu)-\Psi_1(x,\lambda,\nu)|\leqslant\varepsilon C|\lambda|. \tag{4.17}$$

Let $u_n\in SBV^2(Q_\nu;\mathbb{R}^d)$ be such that $u_n|_{\partial Q_\nu}=\gamma_{(\lambda,\nu)}$, $\nabla u_n=0$ a.e. in Q_ν and

$$\int_{S_{u_n}\cap Q_\nu}\Psi_1(x_0,[u_n(y)],\nu_{u_n}(y))\,d\mathcal{H}^{N-1}(y)\leqslant\gamma_1(x_0,\lambda,\nu)+\frac{1}{n}.$$

By (H5) and $i)$ we have

$$\begin{aligned}
\int_{S_{u_n}\cap Q_\nu}|[u_n(y)]|\,d\mathcal{H}^{N-1}(y) &\leqslant C\int_{S_{u_n}\cap Q_\nu}\Psi_1(x_0,[u_n(y)],\nu_{u_n}(y))\,d\mathcal{H}^{N-1}(y)\\
&\leqslant C\left(\gamma_1(x_0,\lambda,\nu)+\frac{1}{n}\right)\leqslant C(1+|\lambda|).
\end{aligned} \tag{4.18}$$

Hence, if $|x-x_0|<\delta$, it follows by (4.17) and (4.18) that

$$\begin{aligned}
&\gamma_1(x,\lambda,\nu)-\gamma_1(x_0,\lambda,\nu)\\
&\leqslant\int_{S_{u_n}\cap Q_\nu}\Psi_1(x,[u_n(y)],\nu_{u_n}(y))\,d\mathcal{H}^{N-1}(y)-\int_{S_{u_n}\cap Q_\nu}\Psi_1(x_0,[u_n(y)],\nu_{u_n}(y))\,d\mathcal{H}^{N-1}(y)+\frac{1}{n}\\
&\leqslant\int_{S_{u_n}\cap Q_\nu}\varepsilon C|[u_n(y)]|\,d\mathcal{H}^{N-1}(y)+\frac{1}{n}\\
&\leqslant\varepsilon C(1+|\lambda|)+\frac{1}{n}.
\end{aligned}$$

Letting $n \to +\infty$ we conclude that

$$\gamma_1(x,\lambda,\nu) - \gamma_1(x_0,\lambda,\nu) \leqslant \varepsilon C(1+|\lambda|).$$

Changing the roles of x and x_0 we obtain the result.

We now prove $iii)$. Let $u \in SBV^2(Q_\nu;\mathbb{R}^d)$ be such that $u|_{\partial Q_\nu} = \gamma_{(\lambda,\nu)}$, $\nabla u = 0$ a.e. in Q_ν and

$$\int_{S_u \cap Q_\nu} \Psi_1(x,[u(y)],\nu_u(y))\,d\mathcal{H}^{N-1}(y) \leqslant \gamma_1(x,\lambda,\nu) + \varepsilon.$$

Let $v = \gamma_{(\lambda',\nu)} - \gamma_{(\lambda,\nu)}$ and define $w = u + v$. Since w is admissible for $\gamma_1(x,\lambda',\nu)$ we have by (H8) and (H5),

$$\begin{aligned}
\gamma_1(x,\lambda',\nu) &\leqslant \int_{S_w \cap Q_\nu} \Psi_1(x,[w(y)],\nu_w(y))\,d\mathcal{H}^{N-1}(y)\\
&\leqslant \int_{S_u \cap Q_\nu} \Psi_1(x,[u(y)],\nu_u(y))\,d\mathcal{H}^{N-1}(y) + \int_{S_v \cap Q_\nu} \Psi_1(x,[v(y)],\nu_v(y))\,d\mathcal{H}^{N-1}(y)\\
&\leqslant \gamma_1(x,\lambda,\nu) + \varepsilon + \int_{\{y \in Q_\nu : y\cdot\nu = 0\}} \Psi_1(x,\lambda'-\lambda,\nu)\,d\mathcal{H}^{N-1}(y)\\
&\leqslant \gamma_1(x,\lambda,\nu) + \varepsilon + C|\lambda'-\lambda|,
\end{aligned}$$

so to prove the first inequality it suffices to let $\varepsilon \to 0^+$. The other inequality is obtained in a similar fashion.

To prove $iv)$, taking into account the result of $iii)$ it suffices to show that $(x,\nu) \to \gamma_1(x,\lambda,\nu)$ is upper semicontinuous, for every $\lambda \in \mathbb{R}^d$. By a change of variables argument, choosing a rotation R such that $Re_N = \nu$, it is easy to see that

$$\gamma_1(x,\lambda,\nu) = \inf_{u \in SBV^2(Q;\mathbb{R}^d)} \left\{ \int_{S_u \cap Q} \Psi_1(x,[u(y)],\nu_u(y))\,d\mathcal{H}^{N-1}(y) : u|_{\partial Q} = \gamma_{(\lambda,e_N)}, \nabla u = 0 \text{ a.e. in } Q \right\}. \tag{4.19}$$

Let $(x_n,\nu_n) \to (x,\nu)$. Given $\varepsilon > 0$, let $u_\varepsilon \in SBV^2(Q;\mathbb{R}^d)$ be such that $u_\varepsilon|_{\partial Q} = \gamma_{(\lambda,e_N)}$, $\nabla u_\varepsilon = 0$ a.e. in Q and

$$\left| \gamma_1(x,\lambda,\nu) - \int_{S_{u_\varepsilon} \cap Q} \Psi_1(x,[u_\varepsilon(y)],\nu_{u_\varepsilon}(y))\,d\mathcal{H}^{N-1}(y) \right| < \varepsilon. \tag{4.20}$$

Let K be a compact subset of Ω containing a neighborhood of x and choose $\delta > 0$ such that (H6) is satisfied uniformly in K, i.e.

$$y, y' \in K, |y-y'| < \delta \Rightarrow |\Psi_1(y,\lambda,\nu) - \Psi_1(y',\lambda,\nu)| \leqslant \varepsilon C|\lambda|, \tag{4.21}$$

for all $(\lambda,\nu) \in \mathbb{R}^d \times S^{N-1}$. Choosing rotations R_n such that $R_n e_N = \nu_n$, $R_n \to R$, by (4.21), (H5) and (4.20) we have that

$$\begin{aligned}
&\left| \int_{S_{u_\varepsilon} \cap Q} \Psi_1(x,[u_\varepsilon(y)],\nu_{u_\varepsilon}(y))\,d\mathcal{H}^{N-1}(y) - \int_{S_{u_\varepsilon} \cap Q} \Psi_1(x_n,[u_\varepsilon(y)],\nu_{u_\varepsilon}(y))\,d\mathcal{H}^{N-1}(y) \right|\\
&\leqslant \int_{S_{u_\varepsilon} \cap Q} \varepsilon C|[u_\varepsilon(y)]|\,d\mathcal{H}^{N-1}(y)\\
&\leqslant \varepsilon C \int_{S_{u_\varepsilon} \cap Q} \Psi_1(x,[u_\varepsilon(y)],\nu_{u_\varepsilon}(y))\,d\mathcal{H}^{N-1}(y)\\
&\leqslant \varepsilon C(\varepsilon + \gamma_1(x,\lambda,\nu)) = O(\varepsilon).
\end{aligned}$$

Thus, by (4.19) and (4.20),

$$\begin{aligned}
\gamma_1(x_n,\lambda,\nu_n) &\leqslant \int_{S_{u_\varepsilon} \cap Q} \Psi_1(x_n,[u_\varepsilon(y)],\nu_{u_\varepsilon}(y))\,d\mathcal{H}^{N-1}(y)\\
&\leqslant O(\varepsilon) + \int_{S_{u_\varepsilon} \cap Q} \Psi_1(x,[u_\varepsilon(y)],\nu_{u_\varepsilon}(y))\,d\mathcal{H}^{N-1}(y)\\
&\leqslant O(\varepsilon) + \gamma_1(x,\lambda,\nu).
\end{aligned}$$

Therefore, letting $\varepsilon \to 0^+$, we conclude that

$$\limsup_{n\to+\infty} \gamma_1(x_n,\lambda,\nu_n) \leqslant \gamma_1(x,\lambda,\nu).$$

$\square$

Remark 4.10. $\gamma_1(x,\lambda,\nu)$ can be extended to $\Omega\times\mathbb{R}^d\times\mathbb{R}^N$ as a positively homogeneous of degree one function in the third variable in the following way

$$\gamma_1(x,\lambda,\theta) = \begin{cases} |\theta|\gamma_1\big(x,\lambda,\frac{\theta}{|\theta|}\big), & \text{if } \theta\in\mathbb{R}^N\setminus\{0\}\\ 0, & \text{if } \theta = 0.\end{cases}$$

By Proposition 4.9 this extension is upper semicontinuous in $\Omega\times\mathbb{R}^d\times\mathbb{R}^N$ and satisfies

$$\gamma_1(x,\lambda,\theta)\leqslant C|\lambda||\theta|, \forall (x,\lambda,\theta)\in\Omega\times\mathbb{R}^d\times\mathbb{R}^N.$$

Thus there exists a non-increasing sequence of continuous functions $\gamma_1^m:\Omega\times\mathbb{R}^N\to[0,+\infty)$ such that

$$\gamma_1(x,\lambda,\theta) = \inf_m \gamma_1^m(x,\theta) = \lim_m \gamma_1^m(x,\theta)\leqslant C|\theta|, \forall (x,\theta)\in\Omega\times\mathbb{R}^N.$$

Proposition 4.11. *i)* $W_2(x,A,0,0)\leqslant W(x,A,0), \forall (x,A)\in\Omega\times\mathbb{R}^{d\times N}$*;*

ii) for every $x\in\Omega$*, every* $A_1,A_2\in\mathbb{R}^{d\times N}$*, and all* $L,M_1,M_2\in\mathbb{R}^{d\times N\times N}$ *we have that*

$$|W_2(x,A_1,L,M_1)-W_2(x,A_2,L,M_2)|\leqslant C(|A_1-A_2|+|M_1-M_2|).$$

Proof. The proof of $i)$ is immediate since the function $u=0$ is admissible for $W_2(x,A,0,0)$.

To prove $ii)$ we will show that

$$W_2(x,A_1,L,M_1)\leqslant W_2(x,A_2,L,M_2)+C(|A_1-A_2|+|M_1-M_2|),$$

$\forall x\in\Omega, \forall A_1,A_2\in\mathbb{R}^{d\times N}, \forall L,M_1,M_2\in\mathbb{R}^{d\times N\times N}$; the reverse inequality follows by interchanging the roles of A_1 and A_2 and M_1 and M_2.

Fix $\varepsilon>0$ and let $u\in SBV(Q;\mathbb{R}^{d\times N})$ be such that $u|_{\partial Q}(y)=Ly$, $\int_Q \nabla u(y)\,dy = M_2$ and

$$\int_Q W(x,A_2,\nabla u(y))\,dy + \int_{S_u\cap Q}\Psi_2(x,[u(y)],\nu_u(y))\,d\mathcal{H}^{N-1}(y)\leqslant W_2(x,A_2,L,M_2)+\varepsilon.$$

By Lemma 2.5, let $v\in SBV(Q;\mathbb{R}^{d\times N})$ be such that $v|_{\partial Q}=0$, $\nabla v = M_1-M_2$ a.e. in Q and $|D^s v|(Q)\leqslant C|M_1-M_2|$, and define $w=u+v$. Then w is admissible for $W_2(x,A_1,L,M_1)$ so by (H8), (H2) and (H5),

$$\begin{aligned} W_2(x,A_1,L,M_1) &\leqslant \int_Q W(x,A_1,\nabla w(y))\,dy + \int_{S_w\cap Q}\Psi_2(x,[w(y)],\nu_w(y))\,d\mathcal{H}^{N-1}(y)\\ &\leqslant \int_Q W(x,A_1,\nabla u(y)+M_1-M_2)\,dy\\ &\quad + \int_{S_u\cap Q}\Psi_2(x,[u(y)],\nu_u(y))\,d\mathcal{H}^{N-1}(y) + \int_{S_v\cap Q}\Psi_2(x,[v(y)],\nu_v(y))\,d\mathcal{H}^{N-1}(y)\\ &\leqslant \int_Q W(x,A_2,\nabla u(y))\,dy + C(|A_1-A_2|+|M_1-M_2|)\\ &\quad + \int_{S_u\cap Q}\Psi_2(x,[u(y)],\nu_u(y))\,d\mathcal{H}^{N-1}(y) + C|D^s v|(Q)\\ &\leqslant W_2(x,A_2,L,M_2)+\varepsilon + C(|A_1-A_2|+|M_1-M_2|), \end{aligned}$$

thus to conclude the desired inequality it suffices to let $\varepsilon\to 0^+$. $\square$

Proposition 4.12. *i)* $\gamma_2(x,A,\Lambda,\nu)\leqslant C|\Lambda|, \forall (x,A,\Lambda,\nu)\in\Omega\times\mathbb{R}^{d\times N}\times\mathbb{R}^{d\times N}\times S^{N-1}$*;*

ii) for every $x_0\in\Omega$ *and for every* $\varepsilon>0$ *there exists* $\delta>0$ *such that*

$$|x-x_0|<\delta \Rightarrow |\gamma_2(x_0,A,\Lambda,\nu)-\gamma_2(x,A,\Lambda,\nu)|\leqslant \varepsilon C(1+|\Lambda|),$$

$\forall (x,A,\Lambda,\nu)\in\Omega\times\mathbb{R}^{d\times N}\times\mathbb{R}^{d\times N}\times S^{N-1}$;

$iii)$ for every $(x, A_1, \Lambda_1, \nu), (x, A_2, \Lambda_2, \nu) \in \Omega \times \mathbb{R}^{d\times N} \times \mathbb{R}^{d\times N} \times S^{N-1}$ we have that

$$|\gamma_2(x, A_1, \Lambda_1, \nu) - \gamma_2(x, A_2, \Lambda_2, \nu)| \leqslant C(|A_1 - A_2| + |\Lambda_1 - \Lambda_2|),$$

$iv)$ γ_2 is upper semicontinuous in $\Omega \times \mathbb{R}^{d\times N} \times \mathbb{R}^{d\times N} \times S^{N-1}$.

Proof. The proof of $i)$ follows immediately from the fact that the function $\gamma_{(\Lambda,\nu)}$ is admissible for $\gamma_2(x, A, \Lambda, \nu)$, from hypotheses (H5) and since $W^\infty(x, A, 0) = 0$.

To prove $ii)$ fix $x_0 \in \Omega$ and $\varepsilon > 0$. By (3.4) and (H6) let $\delta > 0$ be such that

$$|x - x_0| < \delta \Rightarrow |W^\infty(x, A, M) - W^\infty(x_0, A, M)| \leqslant \varepsilon C|M| \tag{4.22}$$

and

$$|x - x_0| < \delta \Rightarrow |\Psi_2(x_0, \Lambda, \nu) - \Psi_2(x, \Lambda, \nu)| \leqslant \varepsilon C|\Lambda|. \tag{4.23}$$

Let $u_n \in SBV(Q_\nu; \mathbb{R}^{d\times N})$ be such that $u_n|_{\partial Q_\nu} = \gamma_{(\Lambda,\nu)}$, $\int_{Q_\nu} \nabla u_n(y)\, dy = 0$ and

$$\int_{Q_\nu} W^\infty(x_0, A, \nabla u_n(y))\, dy + \int_{S_{u_n}\cap Q_\nu} \Psi_2(x_0, [u_n(y)], \nu_{u_n}(y))\, d\mathcal{H}^{N-1}(y) \leqslant \gamma_2(x_0, A, \Lambda, \nu) + \frac{1}{n}.$$

By (3.2), (H5) and $i)$ we have

$$\begin{aligned}
&\int_{Q_\nu} |\nabla u_n(y)|\, dy + \int_{S_{u_n}\cap Q_\nu} |[u_n(y)]|\, d\mathcal{H}^{N-1}(y) \\
&\leqslant C \int_{Q_\nu} W^\infty(x_0, A, \nabla u_n(y))\, dy + C \int_{S_{u_n}\cap Q_\nu} \Psi_2(x_0, [u_n(y)], \nu_{u_n}(y))\, d\mathcal{H}^{N-1}(y) \\
&\leqslant C\left(\gamma_2(x_0, A, \Lambda, \nu) + \frac{1}{n}\right) \leqslant C(1 + |\Lambda|).
\end{aligned} \tag{4.24}$$

Hence, if $|x - x_0| < \delta$, it follows by (4.22), (4.23) and (4.24) that

$$\begin{aligned}
&\gamma_2(x, A, \Lambda, \nu) - \gamma_2(x_0, A, \Lambda, \nu) \\
&\leqslant \int_{Q_\nu} W^\infty(x, A, \nabla u_n(y))\, dy + \int_{S_{u_n}\cap Q_\nu} \Psi_2(x, [u_n(y)], \nu_{u_n}(y))\, d\mathcal{H}^{N-1}(y) \\
&\quad - \int_{Q_\nu} W^\infty(x_0, A, \nabla u_n(y))\, dy - \int_{S_{u_n}\cap Q_\nu} \Psi_2(x_0, [u_n(y)], \nu_{u_n}(y))\, d\mathcal{H}^{N-1}(y) + \frac{1}{n} \\
&\leqslant \int_{Q_\nu} \varepsilon C|\nabla u_n(y)|\, dy + \int_{S_{u_n}\cap Q_\nu} \varepsilon C|[u_n(y)]|\, d\mathcal{H}^{N-1}(y) + \frac{1}{n} \\
&\leqslant \varepsilon C(1 + |\Lambda|) + \frac{1}{n}.
\end{aligned}$$

Letting $n \to +\infty$ we conclude that

$$\gamma_2(x, A, \Lambda, \nu) - \gamma_2(x_0, A, \Lambda, \nu) \leqslant \varepsilon C(1 + |\Lambda|).$$

Changing the roles of x and x_0 we obtain the result.

We now prove $iii)$. Let $u \in SBV(Q_\nu; \mathbb{R}^{d\times N})$ be such that $u|_{\partial Q_\nu} = \gamma_{(\Lambda_1,\nu)}$, $\int_{Q_\nu} \nabla u(y)\, dy = 0$ and

$$\int_{Q_\nu} W^\infty(x, A_1, \nabla u(y))\, dy + \int_{S_u\cap Q_\nu} \Psi_2(x, [u(y)], \nu_u(y))\, d\mathcal{H}^{N-1}(y) \leqslant \gamma_2(x, A_1, \Lambda_1, \nu) + \varepsilon.$$

Let $v = \gamma_{(\Lambda_2,\nu)} - \gamma_{(\Lambda_1,\nu)}$ and define $w = u + v$. Since w is admissible for $\gamma_2(x, A_2, \Lambda_2, \nu)$ we have by (3.3), (H8) and (H5),

$$\begin{aligned}
\gamma_2(x, A_2, \Lambda_2, \nu) &\leqslant \int_{Q_\nu} W^\infty(x, A_2, \nabla w(y))\, dy + \int_{S_w \cap Q_\nu} \Psi_2(x, [w(y)], \nu_w(y))\, d\mathcal{H}^{N-1}(y) \\
&\leqslant \int_{Q_\nu} W^\infty(x, A_1, \nabla u(y))\, dy + \int_{S_u \cap Q_\nu} \Psi_2(x, [u(y)], \nu_u(y))\, d\mathcal{H}^{N-1}(y) \\
&\quad + \int_{S_v \cap Q_\nu} \Psi_2(x, [v(y)], \nu_v(y))\, d\mathcal{H}^{N-1}(y) \\
&\leqslant \gamma_2(x, A_1, \Lambda_1, \nu) + \varepsilon + \int_{\{y \in Q_\nu : y\cdot\nu = 0\}} \Psi_2(x, \Lambda_2 - \Lambda_1, \nu)\, d\mathcal{H}^{N-1}(y) \\
&\leqslant \gamma_2(x, A_1, \Lambda_1, \nu) + \varepsilon + C|\Lambda_2 - \Lambda_1|,
\end{aligned}$$

so to prove the first inequality it suffices to let $\varepsilon \to 0^+$. The other inequality is obtained in a similar fashion.

To prove $iv)$, due to the result of $iii)$ it suffices to show that $(x, \nu) \to \gamma_2(x, A, \Lambda, \nu)$ is upper semicontinuous, for every $A, \Lambda \in \mathbb{R}^{d\times N}$. By a change of variables argument, choosing a rotation R such that $Re_N = \nu$, it is easy to see that

$$\begin{aligned}
\gamma_2(x, A, \Lambda, \nu) = \inf_{u \in SBV(Q;\mathbb{R}^{d\times N})} \Bigg\{ &\int_Q W^\infty(x, A, \nabla u(y) R^T)\, dy \\
&+ \int_{S_u \cap Q} \Psi_2(x, [u(y)], \nu_u(y))\, d\mathcal{H}^{N-1}(y) : u|_{\partial Q} = \gamma_{(\Lambda, e_N)}, \\
&\int_Q \nabla u(y)\, dy = 0,\ Re_N = \nu,\ R \in SO(N) \Bigg\}.
\end{aligned} \tag{4.25}$$

Let $(x_n, \nu_n) \to (x, \nu)$. Given $\varepsilon > 0$, let $u_\varepsilon \in SBV(Q; \mathbb{R}^{d\times N})$ be such that $u_\varepsilon|_{\partial Q} = \gamma_{(\Lambda, e_N)}$, $\displaystyle\int_Q \nabla u_\varepsilon(y)\, dy = 0$ and

$$\left| \gamma_2(x, A, \Lambda, \nu) - \int_Q W^\infty(x, A, \nabla u_\varepsilon(y) R^T)\, dy - \int_{S_{u_\varepsilon} \cap Q} \Psi_2(x, [u_\varepsilon(y)], \nu_{u_\varepsilon}(y))\, d\mathcal{H}^{N-1}(y) \right| < \varepsilon. \tag{4.26}$$

Let K be a compact subset of Ω containing a neighborhood of x and choose $\delta > 0$ such that (3.4) and (H6) are satisfied uniformly in K, i.e.

$$y, y' \in K, |y - y'| < \delta \Rightarrow |W^\infty(y, A, M) - W^\infty(y', A, M)| \leqslant \varepsilon C|M|, \tag{4.27}$$

for every $(A, M) \in \mathbb{R}^{d\times N} \times \mathbb{R}^{d\times N\times N}$, and

$$y, y' \in K, |y - y'| < \delta \Rightarrow |\Psi_2(y, \Lambda, \nu) - \Psi_2(y', \Lambda, \nu)| \leqslant \varepsilon C|\Lambda|, \tag{4.28}$$

for all $(\Lambda, \nu) \in \mathbb{R}^{d\times N} \times S^{N-1}$. Choosing rotations R_n such that $R_n e_N = \nu_n$, $R_n \to R$, by (4.27), (4.28), (3.3), (3.2), (H5) and (4.26) we have that

$$\begin{aligned}
&\Bigg| \int_Q W^\infty(x, A, \nabla u_\varepsilon(y) R^T)\, dy + \int_{S_{u_\varepsilon} \cap Q} \Psi_2(x, [u_\varepsilon(y)], \nu_{u_\varepsilon}(y))\, d\mathcal{H}^{N-1}(y) \\
&\quad - \int_Q W^\infty(x_n, A, \nabla u_\varepsilon(y) R_n^T)\, dy - \int_{S_{u_\varepsilon} \cap Q} \Psi_2(x_n, [u_\varepsilon(y)], \nu_{u_\varepsilon}(y))\, d\mathcal{H}^{N-1}(y) \Bigg| \\
&\leqslant \int_Q |W^\infty(x, A, \nabla u_\varepsilon(y) R^T) - W^\infty(x_n, A, \nabla u_\varepsilon(y) R^T)|\, dy \\
&\quad + \int_Q |W^\infty(x_n, A, \nabla u_\varepsilon(y) R^T) - W^\infty(x_n, A, \nabla u_\varepsilon(y) R_n^T)|\, dy
\end{aligned}$$

$$
\begin{aligned}
&+\int_{S_{u_\varepsilon}\cap Q} \varepsilon C|[u_\varepsilon(y)]|\, d\mathcal{H}^{N-1}(y) \\
&\leqslant \int_Q \varepsilon C|\nabla u_\varepsilon(y)R^T|\, dy + \int_Q C|\nabla u_\varepsilon(y)||R_n^T - R^T|\, dy + \int_{S_{u_\varepsilon}\cap Q} \varepsilon C|[u_\varepsilon(y)]|\, d\mathcal{H}^{N-1}(y) \\
&\leqslant \varepsilon C \int_Q W^\infty(x, A, \nabla u_\varepsilon(y)R^T)\, dy + \varepsilon C \int_{S_{u_\varepsilon}\cap Q} \Psi_2(x, [u_\varepsilon(y)], \nu_{u_\varepsilon}(y))\, d\mathcal{H}^{N-1}(y) \\
&+|R_n^T - R^T| \int_Q W^\infty(x, A, \nabla u_\varepsilon(y)R^T)\, dy \\
&\leqslant \big(\varepsilon C + |R_n^T - R^T|\big)\big(\varepsilon + \gamma_2(x, A, \Lambda, \nu)\big) = O(\varepsilon) + O\big(|R_n^T - R^T|\big).
\end{aligned}
$$

Thus, by (4.25) and (4.26),

$$
\begin{aligned}
\gamma_2(x_n, A, \Lambda, \nu_n) \quad &\leqslant \quad \int_Q W^\infty(x_n, A, \nabla u_\varepsilon(y)R_n^T)\, dy + \int_{S_{u_\varepsilon}\cap Q} \Psi_2(x_n, [u_\varepsilon(y)], \nu_{u_\varepsilon}(y))\, d\mathcal{H}^{N-1}(y) \\
&\leqslant \quad \int_Q W^\infty(x, A, \nabla u_\varepsilon(y)R^T)\, dy + \int_{S_{u_\varepsilon}\cap Q} \Psi_2(x, [u_\varepsilon(y)], \nu_{u_\varepsilon}(y))\, d\mathcal{H}^{N-1}(y) \\
&\quad\quad +O(\varepsilon) + O(|R_n^T - R^T|) \\
&\leqslant \quad O(\varepsilon) + O\big(|R_n^T - R^T|\big) + \gamma_2(x, A, \Lambda, \nu).
\end{aligned}
$$

Therefore, letting $\varepsilon \to 0^+$, and passing to the limit as $n \to +\infty$, since $R_n \to R$, we conclude that

$$
\limsup_{n\to+\infty} \gamma_2(x_n, A, \Lambda, \nu_n) \leqslant \gamma_2(x, A, \Lambda, \nu). \qquad \square
$$

Remark 4.13. $\gamma_2(x, A, \Lambda, \nu)$ can be extended to $\Omega \times \mathbb{R}^{d\times N} \times \mathbb{R}^{d\times N} \times \mathbb{R}^N$ as a positively homogeneous of degree one function in the fourth variable in the following way

$$
\gamma_2(x, A, \Lambda, \theta) = \begin{cases} |\theta|\gamma_2\big(x, A, \Lambda, \frac{\theta}{|\theta|}\big), & \text{if } \theta \in \mathbb{R}^N \setminus \{0\} \\ 0, & \text{if } \theta = 0. \end{cases}
$$

By Proposition 4.12 this extension is upper semicontinuous in $\Omega \times \mathbb{R}^{d\times N} \times \mathbb{R}^{d\times N} \times \mathbb{R}^N$ and satisfies

$$
\gamma_2(x, A, \Lambda, \theta) \leqslant C|\Lambda||\theta|, \forall (x, A, \Lambda, \theta) \in \Omega \times \mathbb{R}^{d\times N} \times \mathbb{R}^{d\times N} \times \mathbb{R}^N.
$$

Thus there exists a non-increasing sequence of continuous functions $\gamma_2^m : \Omega \times \mathbb{R}^N \to [0, +\infty)$ such that

$$
\gamma_2(x, A, \Lambda, \theta) = \inf_m \gamma_2^m(x, \theta) = \lim_m \gamma_2^m(x, \theta) \leqslant C|\theta|, \forall (x, \theta) \in \Omega \times \mathbb{R}^N.
$$

5. Integral representation of $I(g, G, \Gamma)$

The proof of the integral representation of I follows along the lines of the proofs in [13] (for I_2) and in [7] (for I_1), together with arguments in [9] in order to deal with the explicit dependence on the position variable x. In what follows, we mostly restrict our attention to the integral representation of I_1 since that of I_2 can be derived in a similar manner.

5.1. **Integral representation of** $I_1(g, G, \Gamma)$**.** In this section we will prove the following result.

Theorem 5.1. *For all* $(g, G, \Gamma) \in SD^2(\Omega; \mathbb{R}^d)$*, under hypotheses (H1)–(H8), we have that*

$$
I_1(g, G, \Gamma) = \int_\Omega W_1(x, G(x) - \nabla g(x))\, dx + \int_{S_g\cap\Omega} \gamma_1(x, [g(x)], \nu_g(x))\, d\mathcal{H}^{N-1}(x).
$$

5.1.1. *The lower bound inequality.* We begin by obtaining a lower bound for $I_1(g,G,\Gamma)$.

Proposition 5.2. *For all $(g,G,\Gamma)\in SD^2(\Omega;\mathbb{R}^d)$, under hypotheses (H1) - (H8), we have that*

$$I_1(g,G,\Gamma)\geqslant \int_\Omega W_1(x,G(x)-\nabla g(x))\,dx+\int_{S_g\cap\Omega}\gamma_1(x,[g(x)],\nu_g(x))\,d\mathcal{H}^{N-1}(x).$$

Proof. Let $\{u_n\}\subset SBV^2(\Omega;\mathbb{R}^d)$ be an admissible sequence for $I_1(g,G,\Gamma)$ such that

$$\lim_{n\to+\infty}\int_{S_{u_n}\cap\Omega}\Psi_1(x,[u_n](x),\nu_{u_n}(x))\,d\mathcal{H}^{N-1}(x)<+\infty.$$

For each Borel set $B\subset\overline{\Omega}$ define the sequence of Radon measures $\{\mu_n\}$ by

$$\mu_n(B):=\int_{S_{u_n}\cap B}\Psi_1(x,[u_n](x),\nu_{u_n}(x))\,d\mathcal{H}^{N-1}(x).$$

By the choice of u_n, the sequence $\{\mu_n\}$ is bounded so there exists $\mu\in\mathcal{M}^+(\Omega)$ such that, up to a subsequence (not relabeled), $\mu_n\overset{*}{\rightharpoonup}\mu$ in the sense of measures. By the Radon-Nikodym theorem we may decompose μ as the sum of three mutually singular non-negative measures

$$\mu=\mu_a\mathcal{L}^N+\mu_j\mathcal{H}^{N-1}\lfloor S_g+\mu_s.$$

Using the blow-up method it suffices to show that, for $\mathcal{L}^N$ a.e. $x_0\in\Omega$,

$$\mu_a(x_0)=\frac{d\mu}{d\mathcal{L}^N}(x_0)\geqslant W_1(x_0,G(x_0)-\nabla g(x_0)),\tag{5.1}$$

and, for $\mathcal{H}^{N-1}$ a.e. $x_0\in S_g\cap\Omega$,

$$\mu_j(x_0)=\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S_g}(x_0)\geqslant\gamma_1(x_0,[g(x_0)],\nu_g(x_0)).\tag{5.2}$$

Assuming (5.1) and (5.2) hold, we then obtain

$$\begin{aligned}\lim_{n\to+\infty}&\int_{S_{u_n}\cap\Omega}\Psi_1(x,[u_n](x),\nu_{u_n}(x))\,d\mathcal{H}^{N-1}(x)\\&\geqslant\int_\Omega\mu_a(x)\,dx+\int_{S_g\cap\Omega}\mu_j(x)\,d\mathcal{H}^{N-1}(x)\\&\geqslant\int_\Omega W_1(x,G(x)-\nabla g(x))\,dx+\int_{S_g\cap\Omega}\gamma_1(x,[g(x)],\nu_g(x))\,d\mathcal{H}^{N-1}(x),\end{aligned}$$

and the result follows by taking the infimum over all sequences $\{u_n\}$ satisfying the above properties. ☐

The remainder of this section will be devoted to the proofs of inequalities (5.1) and (5.2).

Proposition 5.3. *For $\mathcal{L}^N$ a.e. $x_0\in\Omega$ the following inequality holds,*

$$\frac{d\mu}{d\mathcal{L}^N}(x_0)\geqslant W_1(x_0,G(x_0)-\nabla g(x_0)).$$

Proof. Let $x_0\in\Omega$ be a point of approximate differentiability of g and of approximate continuity of G. Moreover, x_0 is chosen so that $\frac{d\mu}{d\mathcal{L}^N}(x_0)$ exists and is finite. Let $\{\delta_k\}$ be a sequence of positive real numbers such that $\delta_k\to0^+$ and $\mu(\partial Q(x_0,\delta_k))=0$. Therefore,

$$\lim_{n\to+\infty}\mu_n(Q(x_0,\delta_k))=\mu(Q(x_0,\delta_k)),$$

and so

$$\begin{aligned}\frac{d\mu}{d\mathcal{L}^N}(x_0)&=\lim_{k\to+\infty}\frac{\mu(Q(x_0,\delta_k))}{\mathcal{L}^N(Q(x_0,\delta_k))}\\&=\lim_{k,n}\frac{1}{\delta_k^N}\int_{S_{u_n}\cap Q(x_0,\delta_k)}\Psi_1(x,[u_n(x)],\nu_{u_n}(x))\,d\mathcal{H}^{N-1}(x)\\&=\lim_{k,n}\frac{1}{\delta_k}\int_{Q\cap\{y:\,x_0+\delta_ky\in S_{u_n}\}}\Psi_1(x_0+\delta_ky,[u_n(x_0+\delta_ky)],\nu_{u_n}(x_0+\delta_ky))\,d\mathcal{H}^{N-1}(y).\end{aligned}\tag{5.3}$$

For $y \in Q$ define

$$v_{n,k}(y) := \frac{u_n(x_0 + \delta_k y) - g(x_0)}{\delta_k} \text{ and } v_0(y) := \nabla g(x_0)y.$$

Notice that, as x_0 is a point of approximate differentiability of g and of approximate continuity of G,

$$v_{n,k} \underset{k,n\to+\infty}{\overset{L^1}{\longrightarrow}} v_0 \text{ and } \nabla v_{k,n} \underset{k,n\to+\infty}{\overset{L^1}{\longrightarrow}} G(x_0). \tag{5.4}$$

Then, by (H7), (H6) and for k large enough, we have

$$\begin{aligned}\frac{d\mu}{d\mathcal{L}^N}(x_0) &= \lim_{k,n} \int_{Q\cap S_{v_{n,k}}} \Psi_1(x_0 + \delta_k y, [v_{n,k}(y)], \nu_{v_{n,k}}(y))\, d\mathcal{H}^{N-1}(y) \\ &\geqslant \lim_{k,n} \int_{Q\cap S_{v_{n,k}}} \Psi_1(x_0, [v_{n,k}(y)], \nu_{v_{n,k}}(y))\, d\mathcal{H}^{N-1}(y) - \varepsilon C|D^s v_{n,k}|(Q) \\ &\geqslant \lim_{k,n} \int_{Q\cap S_{v_{n,k}}} \Psi_1(x_0, [v_{n,k}(y)], \nu_{v_{n,k}}(y))\, d\mathcal{H}^{N-1}(y) + O(\varepsilon),\end{aligned} \tag{5.5}$$

where we have also used (H5) and (5.3). We must now modify $\{v_{n,k}\}$ in order to obtain a new sequence which is zero on the boundary of Q and whose gradient equals $G(x_0) - \nabla g(x_0)$. For $y \in Q$, define $w_{n,k}(y) := v_{n,k}(y) - v_0(y)$. Since $w_{n,k} \underset{k,n\to+\infty}{\overset{L^1}{\longrightarrow}} 0$, we may choose $r_{n,k} \in]0,1[$ such that $r_{n,k} \underset{k,n\to+\infty}{\longrightarrow} 1$ and

$$\lim_{k,n} \int_{\partial Q(0,r_{n,k})} |w_{n,k}(y)|\, d\mathcal{H}^{N-1}(y) = 0.$$

By Theorem 2.4, let $\rho_{n,k} \in SBV(Q;\mathbb{R}^d)$ be such that $\nabla\rho_{n,k}(y) = G(x_0) - \nabla v_{n,k}(y)$,

$$\|D^s\rho_{n,k}\|(Q(0,r_{n,k})) \leqslant C\|G(x_0) - \nabla v_{n,k}\|_{L^1},$$

and define $z_{n,k} := w_{n,k} + \rho_{n,k}$ for $y \in Q(0,r_{n,k})$. Notice that $\nabla z_{n,k}(y) = G(x_0) - \nabla g(x_0)$. Also, by (5.4), $\nabla\rho_{n,k} \underset{k,n\to+\infty}{\overset{L^1}{\longrightarrow}} 0$, so $\|D^s\rho_{n,k}\|(Q(0,r_{n,k})) \to 0$. Thus, by the continuity of the trace operator with respect to the intermediate topology it follows that

$$\lim_{k,n} \int_{\partial Q(0,r_{n,k})} |\rho_{n,k}(y)|\, d\mathcal{H}^{N-1}(y) = 0.$$

We now apply Lemma 2.5 in order to obtain a sequence $\{\eta_{n,k}\} \subset SBV(Q \setminus Q(0,r_{n,k});\mathbb{R}^d)$ such that $\nabla\eta_{n,k}(y) = G(x_0) - \nabla g(x_0)$, for $\mathcal{L}^N$ a.e. $y \in Q \setminus Q(0,r_{n,k})$, $\eta_{n,k} = 0$ on $\partial(Q \setminus Q(0,r_{n,k}))$ and $\|D^s\eta_{n,k}\|(Q \setminus Q(0,r_{n,k})) \leqslant C|Q \setminus Q(0,r_{n,k})|$. Then the sequence

$$\tilde{z}_{n,k}(y) := \begin{cases} z_{n,k}(y), & \text{if } y \in Q(0,r_{n,k}) \\ \eta_{n,k}(y), & \text{if } y \in Q \setminus Q(0,r_{n,k}) \end{cases}$$

is admissible for $W_1(x, G(x_0) - \nabla g(x_0))$ and satisfies, by (H5) and (H8),

$$\begin{aligned}&\int_{Q\cap S_{\tilde{z}_{n,k}}} \Psi_1(x_0, [\tilde{z}_{n,k}(y)], \nu_{\tilde{z}_{n,k}}(y))\, d\mathcal{H}^{N-1}(y) \\ &\leqslant \int_{Q(0,r_{n,k})\cap S_{w_{n,k}}} \Psi_1(x_0, [w_{n,k}(y)], \nu_{w_{n,k}}(y))\, d\mathcal{H}^{N-1}(y) \\ &+ \int_{Q(0,r_{n,k})\cap S_{\rho_{n,k}}} \Psi_1(x_0, [\rho_{n,k}(y)], \nu_{\rho_{n,k}}(y))\, d\mathcal{H}^{N-1}(y) \\ &+ C\int_{\partial Q(0,r_{n,k})} |z_{n,k}(y)|\, d\mathcal{H}^{N-1}(y) + C\int_{[Q\setminus Q(0,r_{n,k})]\cap S_{\eta_{n,k}}} |[\eta_{n,k}(y)]|\, d\mathcal{H}^{N-1}(y)\end{aligned}$$

$$
\begin{aligned}
&\leqslant \int_{Q\cap S_{v_{n,k}}} \Psi_1(x_0,[v_{n,k}(y)],\nu_{v_{n,k}}(y))\,d\mathcal{H}^{N-1}(y)\\
&+C\int_{Q(0,r_{n,k})\cap S_{\rho_{n,k}}} |[\rho_{n,k}(y)]|\,d\mathcal{H}^{N-1}(y)+C\int_{\partial Q(0,r_{n,k})}|w_{n,k}(y)|\,d\mathcal{H}^{N-1}(y)\\
&+C\int_{\partial Q(0,r_{n,k})}|\rho_{n,k}(y)|\,d\mathcal{H}^{N-1}(y)+C\int_{[Q\setminus Q(0,r_{n,k})]\cap S_{\eta_{n,k}}}|[\eta_{n,k}(y)]|\,d\mathcal{H}^{N-1}(y).
\end{aligned}
$$

Since the last four integrals in the above expression converge to zero as $k,n\to+\infty$ we conclude from (5.5) that

$$
\begin{aligned}
\frac{d\mu}{d\mathcal{L}^N}(x_0) \quad &\geqslant \quad \liminf_{k,n}\int_{Q\cap S_{\tilde z_{n,k}}}\Psi_1(x_0,[\tilde z_{n,k}(y)],\nu_{\tilde z_{n,k}}(y))\,d\mathcal{H}^{N-1}(y)+O(\varepsilon)\\
&\geqslant \quad W_1(x_0,G(x_0)-\nabla g(x_0))+O(\varepsilon)
\end{aligned}
$$

so to conclude the result it suffices to let $\varepsilon\to 0^+$. ☐

We proceed with the proof of (5.2).

Proposition 5.4. *For $\mathcal{H}^{N-1}$ a.e. $x_0\in S_g\cap\Omega$ we have that*

$$\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S_g}(x_0)\geqslant\gamma_1(x_0,[g(x_0)],\nu_g(x_0)).$$

Proof. Let $x_0\in S_g$ be such that $\dfrac{d\mu}{d\mathcal{H}^{N-1}\lfloor S_g}(x_0)$ exists and is finite, denote by $\nu:=\nu_g(x_0)$ and assume the point x_0 also satisfies

$$\lim_{\delta\to 0^+}\frac{\mathcal{H}^{N-1}(S_g\cap Q_\nu(x_0,\delta))}{\delta^{N-1}}=1, \tag{5.6}$$

and

$$\lim_{\delta\to 0^+}\frac{1}{\delta^{N-1}}\int_{Q_\nu(x_0,\delta)}|G(x)|\,dx=0. \tag{5.7}$$

We point out that these conditions hold for $\mathcal{H}^{N-1}$ a.e. $x_0\in S_g$. Let $\{\delta_k\}$ be a sequence of positive real numbers such that $\delta_k\to0^+$ and $\mu(\partial Q_\nu(x_0,\delta_k))=0$. Therefore,

$$\lim_{n\to+\infty}\mu_n(Q_\nu(x_0,\delta_k))=\mu(Q_\nu(x_0,\delta_k))$$

and so, by (5.6), (H6) and for k large enough, we have

$$
\begin{aligned}
&\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S_g}(x_0)=\lim_{k,n}\frac{1}{\mathcal{H}^{N-1}(S_g\cap Q_\nu(x_0,\delta_k))}\mu_n(Q_\nu(x_0,\delta_k))\\
&=\lim_{k,n}\frac{1}{\mathcal{H}^{N-1}(S_g\cap Q_\nu(x_0,\delta_k))}\int_{S_{u_n}\cap Q_\nu(x_0,\delta_k)}\Psi_1(x,[u_n(x)],\nu_{u_n}(x))\,d\mathcal{H}^{N-1}(x)\\
&=\lim_{k,n}\frac{\delta_k^{N-1}}{\mathcal{H}^{N-1}(S_g\cap Q_\nu(x_0,\delta_k))}\\
&\qquad\cdot\int_{Q_\nu\cap\{y:x_0+\delta_k y\in S_{u_n}\}}\Psi_1(x_0+\delta_k y,[u_n(x_0+\delta_k y)],\nu_{u_n}(x_0+\delta_k y))\,d\mathcal{H}^{N-1}(y)\\
&\geqslant\lim_{k,n}\int_{Q_\nu\cap\{y:x_0+\delta_k y\in S_{u_n}\}}\Psi_1(x_0,[u_n(x_0+\delta_k y)],\nu_{u_n}(x_0+\delta_k y))\,d\mathcal{H}^{N-1}(y)-O(\varepsilon)\\
&\geqslant\lim_{k,n}\int_{Q_\nu\cap S_{w_{n,k}}}\Psi_1(x_0,[w_{n,k}(y)],\nu_{w_{n,k}}(y))\,d\mathcal{H}^{N-1}(y)-O(\varepsilon),
\end{aligned} \tag{5.8}
$$

where, for $y\in Q_\nu$, we define

$$w_{n,k}(y):=u_n(x_0+\delta_k y)-g^-(x_0).$$

By definition of $g^-(x_0)$, $g^+(x_0)$, by (5.7), and since $u_n \to g$ in $L^1(\Omega;\mathbb{R}^d)$, and $\nabla u_n \to G$ in $L^1(\Omega;\mathbb{R}^{d\times N})$, one has

$$w_{n,k} \underset{k,n\to+\infty}{\overset{L^1}{\longrightarrow}} \gamma_{([g(x_0)],\nu)} \text{ and } \nabla w_{k,n} \underset{k,n\to+\infty}{\overset{L^1}{\longrightarrow}} 0. \tag{5.9}$$

We must now modify $\{w_{n,k}\}$ in order to obtain a new sequence which is equal to $\gamma_{([g(x_0)],\nu)}$ on the boundary of Q_ν and whose gradient is zero a.e. in Q_ν. For $y \in Q_\nu$, define

$$v_{n,k}(y) := w_{n,k}(y) - \gamma_{([g(x_0)],\nu)}(y).$$

Since $v_{n,k} \underset{k,n\to+\infty}{\overset{L^1}{\longrightarrow}} 0$, we may choose $r_{n,k} \in]0,1[$ such that $r_{n,k} \underset{k,n\to+\infty}{\longrightarrow} 1$ and

$$\lim_{k,n} \int_{\partial Q_\nu(0,r_{n,k})} |v_{n,k}(y)|\, d\mathcal{H}^{N-1}(y) = 0.$$

By Theorem 2.4, let $\rho_{n,k} \in SBV(Q_\nu;\mathbb{R}^d)$ be such that $\nabla\rho_{n,k}(y) = -\nabla w_{n,k}(y)$,

$$\|D^s\rho_{n,k}\|(Q_\nu(0,r_{n,k})) \leqslant C\|\nabla w_{n,k}\|_{L^1},$$

and define $z_{n,k} := w_{n,k} + \rho_{n,k}$ for $y \in Q_\nu(0,r_{n,k})$. Notice that $\nabla z_{n,k}(y) = 0$ in $Q_\nu(0,r_{n,k})$. Also, by (5.9), $\nabla\rho_{n,k} \underset{k,n\to+\infty}{\overset{L^1}{\longrightarrow}} 0$, so $\|D^s\rho_{n,k}\|(Q_\nu(0,r_{n,k})) \to 0$. Thus, by the continuity of the trace operator with respect to the intermediate topology it follows that

$$\lim_{k,n} \int_{\partial Q_\nu(0,r_{n,k})} |\rho_{n,k}(y)|\, d\mathcal{H}^{N-1}(y) = 0.$$

Then the sequence

$$\tilde{z}_{n,k}(y) := \begin{cases} z_{n,k}(y), & \text{if } y \in Q_\nu(0,r_{n,k}) \\ \gamma_{([g(x_0)],\nu)}(y), & \text{if } y \in Q_\nu \setminus Q_\nu(0,r_{n,k}) \end{cases}$$

is admissible for $\gamma_1(x_0,[g(x_0)],\nu)$ and satisfies, by (H5) and (H8),

$$\begin{aligned}
&\int_{Q_\nu\cap S_{\tilde{z}_{n,k}}} \Psi_1(x_0,[\tilde{z}_{n,k}(y)],\nu_{\tilde{z}_{n,k}}(y))\, d\mathcal{H}^{N-1}(y)\\
&\leqslant \int_{Q_\nu(0,r_{n,k})\cap S_{w_{n,k}}} \Psi_1(x_0,[w_{n,k}(y)],\nu_{w_{n,k}}(y))\, d\mathcal{H}^{N-1}(y)\\
&+\int_{Q(0,r_{n,k})\cap S_{\rho_{n,k}}} \Psi_1(x_0,[\rho_{n,k}(y)],\nu_{\rho_{n,k}}(y))\, d\mathcal{H}^{N-1}(y)\\
&+C\int_{\partial Q_\nu(0,r_{n,k})} |z_{n,k}(y)-\gamma_{([g(x_0)],\nu)}(y)|\, d\mathcal{H}^{N-1}(y)\\
&+C\int_{[Q_\nu\setminus Q_\nu(0,r_{n,k})]\cap S_{\gamma_{([g(x_0)],\nu)}}} |[g(x_0)]|\, d\mathcal{H}^{N-1}(y)\\
&\leqslant \int_{Q_\nu\cap S_{w_{n,k}}} \Psi_1(x_0,[w_{n,k}(y)],\nu_{w_{n,k}}(y))\, d\mathcal{H}^{N-1}(y)\\
&+C\int_{Q_\nu(0,r_{n,k})\cap S_{\rho_{n,k}}} |[\rho_{n,k}(y)]|\, d\mathcal{H}^{N-1}(y) + C\int_{\partial Q(0,r_{n,k})} |v_{n,k}(y)|\, d\mathcal{H}^{N-1}(y)\\
&+C\int_{\partial Q(0,r_{n,k})} |\rho_{n,k}(y)|\, d\mathcal{H}^{N-1}(y)\\
&+C\int_{[Q_\nu\setminus Q_\nu(0,r_{n,k})]\cap S_{\gamma_{([g(x_0)],\nu)}}} |[g(x_0)]|\, d\mathcal{H}^{N-1}(y).
\end{aligned}$$

Since the last four integrals in the above expression converge to zero as $k, n \to +\infty$ we conclude from (5.8) that

$$\begin{aligned}\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S_g}(x_0) &\geqslant \liminf_{k,n} \int_{Q_\nu \cap S_{\tilde{z}_{n,k}}} \Psi_1(x_0, [\tilde{z}_{n,k}(y)], \nu_{\tilde{z}_{n,k}}(y))\, d\mathcal{H}^{N-1}(y) + O(\varepsilon)\\ &\geqslant \gamma_1(x_0, [g(x_0)], \nu) + O(\varepsilon)\end{aligned}$$

so to conclude the result it suffices to let $\varepsilon \to 0^+$. □

5.1.2. *The upper bound inequality.* We now prove the upper bound inequalities for both the bulk and interfacial terms.

Proposition 5.5. *For $\mathcal{L}^N$ a.e. $x_0 \in \Omega$ we have that*

$$\frac{dI_1(g, G, \Gamma)}{d\mathcal{L}^N}(x_0) \leqslant W_1(x_0, G(x_0) - \nabla g(x_0)).$$

Proof. Let x_0 be a point of approximate continuity for G and ∇g, that is,

$$\lim_{\delta \to 0^+} \frac{1}{\delta^N} \int_{Q(x_0,\delta)} |G(x) - G(x_0)| + |\nabla g(x) - \nabla g(x_0)|\, dx = 0. \tag{5.10}$$

Given $\varepsilon > 0$ let $u \in SBV^2(\Omega; \mathbb{R}^d)$ be such that $u|_{\partial Q} = 0$, $\nabla u(x) = G(x_0) - \nabla g(x_0)$ for a.e. $x \in Q$ and

$$\int_{Q \cap S_u} \Psi_1(x_0, [u(y)], \nu_u(y))\, dy \leqslant W_1(x_0, G(x_0) - \nabla g(x_0)) + \varepsilon. \tag{5.11}$$

Extend u by periodicity to all of $\mathbb{R}^N$ and for $n \in \mathbb{N}$ and $\delta > 0$ define

$$u_{n,\delta}(x) := \frac{\delta}{n} u\left(\frac{n(x - x_0)}{\delta}\right).$$

For each $\delta > 0$, by Theorem 2.4, let $v_\delta \in SBV(Q(x_0,\delta); \mathbb{R}^{d\times N})$ be such that

$$\nabla v_\delta = \Gamma(x) - \nabla G(x), \tag{5.12}$$

for $\mathcal{L}^N$ a.e. $x \in Q(x_0, \delta)$, and

$$\|Dv_\delta\|(Q(x_0,\delta)) \leqslant C(N) \int_{Q(x_0,\delta)} |\Gamma(x) - \nabla G(x)|\, dx.$$

By Lemma 2.3 let $v_{k,\delta} : Q(x_0, \delta) \to \mathbb{R}^{d\times N}$ be a sequence of piecewise constant functions such that

$$v_{k,\delta} \underset{k\to+\infty}{\overset{L^1}{\longrightarrow}} -v_\delta, \tag{5.13}$$

and

$$\lim_{k\to+\infty} \|Dv_{k,\delta}\|(Q(x_0,\delta)) = \|Dv_\delta\|(Q(x_0,\delta)).$$

Applying once more Theorem 2.4, let $\rho_{k,\delta} \in SBV^2(Q(x_0,\delta); \mathbb{R}^d)$ be such that

$$\nabla \rho_{k,\delta}(x) = G(x) - G(x_0) + \nabla g(x_0) - \nabla g(x) + v_\delta(x) + v_{k,\delta}(x), \tag{5.14}$$

for $\mathcal{L}^N$ a.e. $x \in Q(x_0, \delta)$, and

$$\|D\rho_{k,\delta}\|(Q(x_0,\delta)) \leqslant C(N) \int_{Q(x_0,\delta)} |G(x) - G(x_0)| + |\nabla g(x) - \nabla g(x_0)| + |v_\delta(x) + v_{k,\delta}(x)|\, dx. \tag{5.15}$$

By (5.13), for each $\delta > 0$ we can choose $k = k(\delta)$ large enough so that

$$\int_{Q(x_0,\delta)} |v_\delta(x) + v_{k,\delta}(x)|\, dx \leqslant \delta^{N+1}.$$

Thus, defining $\rho_\delta := \rho_{\delta,k(\delta)}$, by (5.10) and (5.15) it follows that

$$\lim_{\delta\to 0^+} \frac{\|D\rho_\delta\|(Q(x_0,\delta))}{\delta^N} = 0. \tag{5.16}$$

Again by Lemma 2.3, let $\rho_{n,\delta}$ be a sequence of piecewise constant functions such that, for all $\delta > 0$,

$$\rho_{n,\delta} \underset{n\to+\infty}{\overset{L^1}{\longrightarrow}} -\rho_\delta \quad \text{and} \quad \|D\rho_{n,\delta}\|(Q(x_0,\delta)) \underset{n\to+\infty}{\longrightarrow} \|D\rho_\delta\|(Q(x_0,\delta)). \tag{5.17}$$

Now define, for $x \in Q(x_0,\delta)$,

$$w_{n,\delta}(x) := g(x) + u_{n,\delta}(x) + \rho_\delta(x) + \rho_{n,\delta}(x).$$

By periodicity, $w_{n,\delta} \underset{n\to+\infty}{\overset{L^1}{\longrightarrow}} g$ since,

$$\int_{Q(x_0,\delta)} |u_{n,\delta}(x)|\,dx = \frac{\delta^{N+1}}{n}\int_Q |u(y)|\,dy \underset{n\to+\infty}{\longrightarrow} 0.$$

Notice also that $\nabla^2 w_{n,\delta} = \Gamma$, and it is easy to verify that $\nabla w_{n,\delta} \underset{n\to+\infty}{\longrightarrow} G$ in $L^1(Q(x_0,\delta);\mathbb{R}^d)$. Thus the sequence $w_{n,\delta}$ is admissible for $I_1(g,G,\Gamma,Q(x_0,\delta))$ and so, by (H8), we have

$$\begin{aligned}
\frac{dI_1(g,G,\Gamma)}{d\mathcal{L}^N}(x_0) &= \lim_{\delta\to0^+}\frac{I_1(g,G,\Gamma,Q(x_0,\delta))}{\delta^N}\\
&\leqslant \lim_{\delta\to0^+}\liminf_{n\to+\infty}\left[\frac{1}{\delta^N}\int_{S_{w_{n,\delta}}\cap Q(x_0,\delta)}\Psi_1(x,[w_{n,\delta}(x)],\nu_{w_{n,\delta}}(x))\,d\mathcal{H}^{N-1}(x)\right]\\
&\leqslant \liminf_{\delta\to0^+}\liminf_{n\to+\infty}\left[\frac{1}{\delta^N}\int_{S_g\cap Q(x_0,\delta)}\Psi_1(x,[g(x)],\nu_g(x))\,d\mathcal{H}^{N-1}(x)\right.\\
&+\frac{1}{\delta^N}\int_{\{x_0+\frac{\delta}{n}S_u\}\cap Q(x_0,\delta)}\Psi_1\left(x,\frac{\delta}{n}\Big[u\Big(\frac{n(x-x_0)}{\delta}\Big)\Big],\nu_u\Big(\frac{n(x-x_0)}{\delta}\Big)\right)d\mathcal{H}^{N-1}(x)\\
&+\frac{1}{\delta^N}\int_{S_{\rho_\delta}\cap Q(x_0,\delta)}\Psi_1(x,[\rho_\delta(x)],\nu_{\rho_\delta}(x))\,d\mathcal{H}^{N-1}(x)\\
&\left.+\frac{1}{\delta^N}\int_{S_{\rho_{n,\delta}}\cap Q(x_0,\delta)}\Psi_1(x,[\rho_{n,\delta}(x)],\nu_{\rho_{n,\delta}}(x))\,d\mathcal{H}^{N-1}(x)\right].
\end{aligned}$$

Since $\dfrac{d\|D^s g\|}{d\mathcal{L}^N}(x_0) = 0$, by (H5) we conclude that

$$\begin{aligned}
\frac{1}{\delta^N}\int_{S_g\cap Q(x_0,\delta)}\Psi_1(x,[g(x)],\nu_g(x))\,d\mathcal{H}^{N-1}(x) &\leqslant \frac{1}{\delta^N}\int_{S_g\cap Q(x_0,\delta)}C|[g(x)]|\,d\mathcal{H}^{N-1}(x)\\
&\leqslant C\frac{\|D^s g\|(Q(x_0,\delta))}{\mathcal{L}^N(Q(x_0,\delta))}\underset{\delta\to0^+}{\longrightarrow} 0.
\end{aligned}$$

Moreover, once again hypothesis (H5), together with (5.16) and (5.17), also yields

$$\lim_{\delta\to0^+}\frac{1}{\delta^N}\int_{S_{\rho_\delta}\cap Q(x_0,\delta)}\Psi_1(x,[\rho_\delta(x)],\nu_{\rho_\delta}(x))\,d\mathcal{H}^{N-1}(x) = 0,$$

and

$$\lim_{\delta\to0^+}\lim_{n\to+\infty}\frac{1}{\delta^N}\int_{S_{\rho_{n,\delta}}\cap Q(x_0,\delta)}\Psi_1(x,[\rho_{n,\delta}(x)],\nu_{\rho_{n,\delta}}(x))\,d\mathcal{H}^{N-1}(x) = 0.$$

Finally, changing variables, using the periodicity of u, (H7) and (5.11), we obtain

$$\begin{aligned}
&\frac{1}{\delta^N}\int_{\{x_0+\frac{\delta}{n}S_u\}\cap Q(x_0,\delta)}\Psi_1\left(x,\frac{\delta}{n}\Big[u\Big(\frac{n(x-x_0)}{\delta}\Big)\Big],\nu_u\Big(\frac{n(x-x_0)}{\delta}\Big)\right)d\mathcal{H}^{N-1}(x)\\
&=\frac{1}{n^N}\int_{nQ\cap S_u}\Psi_1\left(x_0+\frac{\delta}{n}y,[u(y)],\nu_u(y)\right)d\mathcal{H}^{N-1}(y)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{Q\cap S_u} \Psi_1(x_0,[u(y)],\nu_u(y))\,d\mathcal{H}^{N-1}(y)\\
&\qquad + \int_{Q\cap S_u} \Psi_1\left(x_0+\frac{\delta}{n}y,[u(y)],\nu_u(y)\right) - \Psi_1(x_0,[u(y)],\nu_u(y))\,d\mathcal{H}^{N-1}(y)\\
&\leqslant W_1(x_0,G(x_0)-\nabla g(x_0))+\varepsilon\\
&\qquad + \int_{Q\cap S_u} \Psi_1\left(x_0+\frac{\delta}{n}y,[u(y)],\nu_u(y)\right) - \Psi_1(x_0,[u(y)],\nu_u(y))\,d\mathcal{H}^{N-1}(y),
\end{aligned}
$$

where, by (H6) and for δ small enough,

$$
\begin{aligned}
&\left|\int_{Q\cap S_u} \Psi_1\left(x_0+\frac{\delta}{n}y,[u(y)],\nu_u(y)\right) - \Psi_1(x_0,[u(y)],\nu_u(y))\,d\mathcal{H}^{N-1}(y)\right|\\
&\leqslant \varepsilon C\int_{Q\cap S_u} |[u(y)]|\,d\mathcal{H}^{N-1}(y) \leqslant \varepsilon C\|Du\|(Q).
\end{aligned}
$$

Thus the result follows by letting $\varepsilon\to 0^+$. □

Proposition 5.6. *For $\mathcal{H}^{N-1}$ a.e. $x_0\in S_g$ we have that*

$$
\frac{dI_1(g,G,\Gamma)}{d\mathcal{H}^{N-1}\lfloor S_g}(x_0) \leqslant \gamma_1(x_0,[g(x_0)],\nu_g(x_0)). \tag{5.18}
$$

Proof. Following an argument of Ambrosio, Mortola and Tortorelli [5], it suffices to prove (5.18) when $g=\lambda\chi_E$ where $\lambda\in\mathbb{R}^d$ and χ_E is the characteristic function of a set of finite perimeter E. We start by addressing the case where E is a polyhedral set. Let $x_0\in S_g$ be such that

$$
\lim_{\delta\to 0^+}\frac{1}{\delta^{N-1}}\int_{Q_\nu(x_0,\delta)} |G(x)|\,dx = 0, \tag{5.19}
$$

where we are denoting by $\nu:=\nu_g(x_0)$, and $[g(x_0)]=\lambda$. By definition of $\gamma_1(x_0,\lambda,\nu)$, given $\varepsilon>0$, consider $u\in SBV^2(Q_\nu;\mathbb{R}^d)$ such that $u_{|\partial Q_\nu}(x)=\gamma_{(\lambda,\nu)}(x)$, $\nabla u=0$ a.e. in Q_ν, and

$$
\int_{Q_\nu} \Psi_1(x_0,[u(y)],\nu_u(y))\,d\mathcal{H}^{N-1}(y) \leqslant \gamma_1(x_0,\lambda,\nu)+\varepsilon. \tag{5.20}
$$

For $\delta>0$ small enough, and $n\in\mathbb{N}$, define

$$
\begin{aligned}
D^n_\nu(x_0,\delta) &:= Q_\nu(x_0,\delta)\cap\left\{x : \frac{|(x-x_0)\cdot\nu|}{\delta} < \frac{1}{2n}\right\},\\
Q^+_\nu(x_0,\delta) &:= Q_\nu(x_0,\delta)\cap\left\{x : \frac{(x-x_0)\cdot\nu}{\delta} > 0\right\},\\
Q^-_\nu(x_0,\delta) &:= Q_\nu(x_0,\delta)\cap\left\{x : \frac{(x-x_0)\cdot\nu}{\delta} < 0\right\},
\end{aligned}
$$

and let

$$
u_{n,\delta}(x) := \begin{cases} \lambda & x\in Q^+_\nu(x_0,\delta)\setminus D^n_\nu(x_0,\delta),\\ u\left(\frac{n(x-x_0)}{\delta}\right) & x\in D^n_\nu(x_0,\delta),\\ 0 & x\in Q^-_\nu(x_0,\delta)\setminus D^n_\nu(x_0,\delta), \end{cases} \tag{5.21}
$$

where u has been extended by Q-periodicity to all of $\mathbb{R}^N$. Notice that, by periodicity of u,

$$
\lim_{n\to+\infty}\|u_{n,\delta}-\tilde\gamma_{(\lambda,\nu)}\|_{L^1(Q_\nu(x_0,\delta);\mathbb{R}^d)} = 0,
$$

where $\tilde\gamma_{(\lambda,\nu)}(x):=\gamma_{(\lambda,\nu)}(x-x_0)$.

By Theorem 2.4, let $v_\delta\in SBV(Q_\nu(x_0,\delta);\mathbb{R}^{d\times N})$ be such that

$$
\nabla v_\delta = \Gamma(x)-\nabla G(x), \tag{5.22}
$$

for $\mathcal{L}^N$ a.e. $x\in Q_\nu(x_0,\delta)$, and

$$
\|Dv_\delta\|(Q(x_0,\delta)) \leqslant C(N)\int_{Q_\nu(x_0,\delta)} |\Gamma(x)-\nabla G(x)|\,dx.
$$

By Lemma 2.3, let $v_{n,\delta} \in SBV(Q(x_0,\delta);\mathbb{R}^{d\times N})$ be a sequence of piecewise constant functions such that

$$v_{n,\delta} \underset{n\to+\infty}{\xrightarrow{L^1}} -v_\delta, \tag{5.23}$$

and

$$\lim_{n\to+\infty} \|Dv_{n,\delta}\|(Q_\nu(x_0,\delta)) = \|Dv_\delta\|(Q(x_0,\delta)).$$

Applying again Theorem 2.4, let $\rho_{n,\delta} \in SBV^2(Q_\nu(x_0,\delta);\mathbb{R}^d)$ be such that

$$\nabla \rho_{n,\delta}(x) = G(x) + v_\delta(x) + v_{n,\delta}(x), \tag{5.24}$$

for $\mathcal{L}^N$ a.e. $x \in Q_\nu(x_0,\delta)$, and

$$\|D\rho_{n,\delta}\|(Q_\nu(x_0,\delta)) \leqslant C(N)\int_{Q_\nu(x_0,\delta)} |G(x)| + |v_\delta(x) + v_{n,\delta}(x)|\,dx. \tag{5.25}$$

Notice that $\nabla^2\rho_{n,\delta}(x) = \Gamma(x)$. By (5.23), for each δ we can choose $n(\delta)$ such that

$$\int_{Q_\nu(x_0,\delta)} |v_\delta(x) + v_{n(\delta),\delta}(x)|\,dx \leqslant \delta^N.$$

Then, writing for simplicity ρ_δ instead of $\rho_{n(\delta),\delta}$, by (5.25) and (5.19) we have that

$$\lim_{\delta\to 0^+} \frac{\|D\rho_\delta\|(Q_\nu(x_0,\delta))}{\delta^{N-1}} = 0. \tag{5.26}$$

By Lemma 2.3, let $\tilde\rho_{n,\delta} \in SBV(Q_\nu(x_0,\delta);\mathbb{R}^d)$ be a sequence of piecewise constant functions such that, for all $\delta > 0$,

$$\tilde\rho_{n,\delta} \underset{n\to+\infty}{\xrightarrow{L^1}} -\rho_\delta \quad\text{and}\quad \lim_{n\to+\infty}\|D\tilde\rho_{n,\delta}\|(Q_\nu(x_0,\delta)) = \|D\rho_\delta\|(Q_\nu(x_0,\delta)). \tag{5.27}$$

Now, for $x \in Q_\nu(x_0,\delta)$, define the sequence

$$w_{n,\delta}(x) := u_{n,\delta}(x) + \rho_\delta(x) + \tilde\rho_{n,\delta}(x).$$

We point out that

$$\lim_{n\to+\infty}\|w_{n,\delta} - \tilde\gamma_{(\lambda,\nu)}\|_{L^1(Q_\nu(x_0,\delta);\mathbb{R}^d)} = \lim_{n\to+\infty}\|w_{n,\delta} - g\|_{L^1(Q_\nu(x_0,\delta);\mathbb{R}^d)} = 0,$$

that

$$\lim_{n\to+\infty}\|\nabla w_{n,\delta} - G\|_{L^1(Q_\nu(x_0,\delta);\mathbb{R}^{d\times N})} = 0,$$

and that $\nabla^2 w_{n,\delta} = \Gamma$, hence the sequence $w_{n,\delta}$ is admissible for $I_1(g,G,\Gamma,Q_\nu(x_0,\delta))$. Therefore we have, by (H8) and (H5),

$$\begin{aligned}
&\frac{dI_1(g,G,\Gamma)}{d\mathcal{H}^{N-1}\lfloor S_g}(x_0) = \lim_{\delta\to0^+}\frac{I_1(g,G,\Gamma,Q_\nu(x_0,\delta))}{\delta^{N-1}}\\
&\leqslant \lim_{\delta\to0^+}\liminf_{n\to+\infty}\frac{1}{\delta^{N-1}}\int_{S_{w_{n,\delta}}\cap Q_\nu(x_0,\delta)}\Psi_1(x,[w_{n,\delta}(x)],\nu_{w_{n,\delta}}(x))\,d\mathcal{H}^{N-1}(x)\\
&\leqslant \limsup_{\delta\to0^+}\limsup_{n\to+\infty}\frac{1}{\delta^{N-1}}\int_{S_{u_{n,\delta}}\cap Q_\nu(x_0,\delta)}\Psi_1(x,[u_{n,\delta}(x)],\nu_{(u_{n,\delta})}(x))\,d\mathcal{H}^{N-1}(x)\\
&+\limsup_{\delta\to0^+}\limsup_{n\to+\infty}\frac{1}{\delta^{N-1}}\int_{S_{\rho_\delta+\tilde\rho_{n,\delta}}\cap Q_\nu(x_0,\delta)}\Psi_1(x,[\rho_\delta(x)+\tilde\rho_{n,\delta}(x)],\nu_{\rho_\delta+\tilde\rho_{n,\delta}}(x))\,d\mathcal{H}^{N-1}(x)\\
&\leqslant \limsup_{\delta\to0^+}\limsup_{n\to+\infty}\frac{1}{\delta^{N-1}}\int_{\{x:\frac{n(x-x_0)}{\delta}\in S_u\}\cap D_\nu^n(x_0,\delta)}\Psi_1\left(x,\left[u\Big(\frac{n(x-x_0)}{\delta}\Big)\right],\nu_u\Big(\frac{n(x-x_0)}{\delta}\Big)\right)d\mathcal{H}^{N-1}(x)\\
&+\limsup_{\delta\to0^+}\limsup_{n\to+\infty}\frac{1}{\delta^{N-1}}\int_{S_{\rho_\delta+\tilde\rho_{n,\delta}}\cap Q(x_0,\delta)} C|[\rho_\delta(x)+\tilde\rho_{n,\delta}(x)]|\,d\mathcal{H}^{N-1}(x).
\end{aligned}$$

By (5.26) and (5.27) the integral in the last line vanishes in the limit, while by changing variables setting $y := \frac{n(x-x_0)}{\delta}$, we obtain by (H6), for δ small enough,

$$\begin{aligned}
&\limsup_{\delta\to 0^+}\limsup_{n\to+\infty}\frac{1}{\delta^{N-1}}\int_{\{x:\frac{n(x-x_0)}{\delta}\in S_u\}\cap D^n_\nu(x_0,\delta)}\Psi_1\bigg(x,\Big[u\Big(\frac{n(x-x_0)}{\delta}\Big)\Big],\nu_u\Big(\frac{n(x-x_0)}{\delta}\Big)\bigg)\,d\mathcal{H}^{N-1}(x)\\
&\leqslant \limsup_{\delta\to 0^+}\limsup_{n\to+\infty}\frac{1}{n^{N-1}}\int_{S_u\cap\{y\in nQ_\nu:|y\cdot\nu|\leqslant\frac12\}}\Psi_1\left(x_0+\frac{\delta}{n}y,[u(y)],\nu_u(y)\right)\,d\mathcal{H}^{N-1}(y)\\
&\leqslant \limsup_{\delta\to 0^+}\limsup_{n\to+\infty}\frac{1}{n^{N-1}}\left[\int_{S_u\cap\{y\in nQ_\nu:|y\cdot\nu|\leqslant\frac12\}}\Psi_1(x_0,[u(y)],\nu_u(y))\,d\mathcal{H}^{N-1}(y)\right.\\
&\qquad\qquad\left.+\int_{S_u\cap\{y\in nQ_\nu:|y\cdot\nu|\leqslant\frac12\}}\varepsilon C|[u(y)]|\,d\mathcal{H}^{N-1}(y)\right]\\
&=\int_{S_u\cap Q_\nu}\Psi_1(x_0,[u(y)],\nu_u(y))\,d\mathcal{H}^{N-1}(y)+\int_{S_u\cap Q_\nu}\varepsilon C|[u(y)]|\,d\mathcal{H}^{N-1}(y)\\
&\leqslant \gamma_1(x_0,\lambda,\nu)+O(\varepsilon),
\end{aligned}$$

where we have used the periodicity of u and (5.20). The conclusion follows by the arbitrariness of ε.

We now assume that $g=\lambda\chi_E$ where E is an arbitrary set of finite perimeter. Let $x_0\in S_g$ be such that

$$\lim_{\delta\to 0^+}\frac{1}{\delta^{N-1}}\int_{Q_\nu(x_0,\delta)}|G(x)|\,dx=0, \tag{5.28}$$

where we are denoting by $\nu:=\nu_g(x_0)$. By Theorem 2.2, let E_n be a sequence of polyhedral sets such that $\lim\limits_{n\to+\infty}\mathit{Per}_\Omega(E_n)=\mathit{Per}_\Omega(E)$, $\mathcal{L}^N(E_n)=\mathcal{L}^N(E)$ and $\chi_{E_n}\to\chi_E$ in $L^1(\Omega)$, as $n\to+\infty$. Let $g_n=\lambda\chi_{E_n}$, then $\lim\limits_{n\to+\infty}g_n=g$ in $L^1(\Omega;\mathbb{R}^d)$. Hence, given $U\in\mathcal{O}(\Omega)$ by Propositions 4.7 and 4.8, we have

$$\begin{aligned}
I_1(g,G,\Gamma,U) &\leqslant \liminf_{n\to+\infty}I_1(g_n,G,\Gamma,U)\\
&\leqslant \liminf_{n\to+\infty}\left[\int_U W_1(x,G(x))\,dx+\int_{U\cap S_{g_n}}\gamma_1(x,[g_n(x)],\nu_{g_n}(x))\,d\mathcal{H}^{N-1}(x)\right]\\
&\leqslant C\int_U|G(x)|\,dx+\limsup_{n\to+\infty}\int_{U\cap S_{g_n}}\gamma_1(x,[g_n(x)],\nu_{g_n}(x))\,d\mathcal{H}^{N-1}(x).
\end{aligned}\tag{5.29}$$

Recall that by Remark 4.10 there exists a non-increasing sequence of continuous functions $\gamma_1^m:\Omega\times\mathbb{R}^N\to[0,+\infty)$ such that

$$\gamma_1(x,\lambda,\theta)=\inf_m\gamma_1^m(x,\theta)=\lim_m\gamma_1^m(x,\theta)\leqslant C|\theta|,\forall(x,\theta)\in\Omega\times\mathbb{R}^N.$$

Thus, by Theorem 2.1, it follows from (5.29) that

$$\begin{aligned}
I_1(g,G,\Gamma,U) &\leqslant C\int_U|G(x)|\,dx+\limsup_{n\to+\infty}\int_{U\cap S_{g_n}}\gamma_1^m(x,\nu_{g_n}(x))\,d\mathcal{H}^{N-1}(x)\\
&\leqslant C\int_U|G(x)|\,dx+\int_{U\cap S_g}\gamma_1^m(x,\nu_g(x))\,d\mathcal{H}^{N-1}(x).
\end{aligned}$$

Letting $m\to+\infty$ and using the monotone convergence theorem we conclude that

$$I_1(g,G,\Gamma,U)\leqslant C\int_U|G(x)|\,dx+\int_{U\cap S_g}\gamma_1(x,\nu_g(x))\,d\mathcal{H}^{N-1}(x).$$

Using (5.28) and Proposition 4.9 we finally obtain

$$\begin{aligned}\frac{dI_1(g,G,\Gamma)}{d\mathcal{H}^{N-1}\lfloor S_g}(x_0) &= \lim_{\delta\to 0^+}\frac{1}{\delta^{N-1}}I_1(g,G,\Gamma,Q_\nu(x_0,\delta))\\ &\leqslant \lim_{\delta\to 0^+}\frac{1}{\delta^{N-1}}\int_{Q_\nu(x_0,\delta)\cap S_g}\gamma_1(x,\nu_g(x))\,d\mathcal{H}^{N-1}(x)\\ &\leqslant \gamma_1(x_0,[g(x_0)],\nu_g(x_0))+O(\varepsilon),\end{aligned}$$

and the result follows by letting $\varepsilon\to 0^+$. □

5.2. **Integral representation of I_2.**

Theorem 5.7. *Under hypotheses (H1)-(H8) we have*

$$I_2(G,\Gamma)=\int_\Omega W_2(x,G(x),\nabla G(x),\Gamma(x))\,dx+\int_{S_G\cap\Omega}\gamma_2(x,G(x),[G(x)],\nu_G(x))\,d\mathcal{H}^{N-1}(x).$$

Proof. The proof of the above integral representation for I_2 is similar to that of I_1 so we will only outline the proof.

In order to obtain a lower bound for the bulk term we start by fixing a point x_0, which is chosen to be a point of approximate differentiability of G and of approximate continuity of Γ. Starting from a sequence v_n for which

$$\lim_{n\to+\infty}\left[\int_\Omega W(x,G(x),\nabla v_n(x))\,dx+\int_{S_{v_n}\cap\Omega}\Psi_2(x,[v_n(x)],\nu_{v_n}(x))\,d\mathcal{H}^{N-1}(x)\right]<+\infty \tag{5.30}$$

we construct a new sequence $u_{n,k}$ so that

$$\begin{aligned}&\frac{dI_2(G,\Gamma)}{d\mathcal{L}^N}(x_0)\\ &\geqslant \lim_{k,n}\left[\int_Q W(x_0,G(x_0),\nabla u_{n,k}(x))\,dx+\int_{S_{u_{n,k}}\cap Q}\Psi_2(x_0,[u_{n,k}(x)],\nu_{u_{n,k}}(x))\,d\mathcal{H}^{N-1}(x)\right]+O(\varepsilon),\end{aligned}$$

where we use hypotheses (H2) and (H6) to fix x_0 and $G(x_0)$. We further modify $u_{n,k}$ in order to obtain a sequence $z_{n,k}$ which is admissible for $W_2(x_0,G(x_0),\nabla G(x_0),\Gamma(x_0))$. This is achieved by setting $z_{n,k}(x)$ equal to $\nabla G(x_0)\cdot x$ near the boundary of Q and equal to $u_{n,k}(x)+C_{n,k}\cdot x$ in a smaller cube of the form $Q(0,r_{n,k})$, where $C_{n,k}$ is chosen so that

$$\int_Q \nabla z_{n,k}(x)\,dx=\Gamma(x_0).$$

Hypotheses (H2) and (H5) and a careful selection of the side-length of the smaller cube $r_{n,k}$ guarantee that the energy does not increase when $u_{n,k}$ is replaced by $z_{n,k}$ so the result follows by letting $\varepsilon\to 0^+$.

Regarding the lower bound for the interfacial term we fix a point x_0, which is chosen to be a point of approximate continuity of G, and such that

$$\lim_{\delta\to 0^+}\frac{\mathcal{H}^{N-1}(S_G\cap Q_\nu(x_0,\delta))}{\delta^{N-1}}=1$$

and

$$\lim_{\delta\to 0^+}\frac{1}{\delta^{N-1}}\int_{Q_\nu(x_0,\delta)}|\Gamma(x)|\,dx=0,$$

where $\nu:=\nu_G(x_0)$. Starting from the sequence v_n in (5.30), the properties of x_0, together with hypotheses (H2) and (H6), yield a new sequence $w_{n,k}$ satisfying

$$\begin{aligned}&\frac{dI_2(G,\Gamma)}{d\mathcal{H}^{N-1}\lfloor S_G}(x_0)\\ &\geqslant \lim_{k,n}\left[\int_{Q_\nu} W^\infty(x_0,G(x_0),\nabla w_{n,k}(x))\,dx+\int_{S_{w_{n,k}}\cap Q_\nu}\Psi_2(x_0,[w_{n,k}(x)],\nu_{w_{n,k}}(x))\,d\mathcal{H}^{N-1}(x)\right]+O(\varepsilon),\end{aligned}$$

in this step hypothesis (H4) comes into play. As above, $w_{n,k}$ is further modified in order to obtain a sequence $z_{n,k}$ which is admissible for $\gamma_2(x_0, G(x_0), [G(x_0)], \nu_G(x_0))$. This is achieved by setting $z_{n,k}(x)$ equal to $\gamma_{([G(x_0)],\nu)}$ near the boundary of Q_ν and equal to $w_{n,k}(x) + C_{n,k} \cdot x$ in a smaller cube of the form $Q(0, r_{n,k})$, where $C_{n,k}$ is chosen so that

$$\int_{Q_\nu} \nabla z_{n,k}(x)\, dx = 0.$$

Due to hypotheses (H2) and (H5), the replacement of $w_{n,k}$ by $z_{n,k}$ does not translate into an increase in energy, so the result follows by letting $\varepsilon \to 0^+$.

For the upper bound for the bulk term we fix a point x_0 of approximate continuity of both G and Γ and, for $\varepsilon > 0$, we let $v \in SBV(Q; \mathbb{R}^{d\times N})$ be such that $v(x) = \nabla G(x_0) \cdot x$ on ∂Q, $\int_Q \nabla v(x)\, dx = \Gamma(x_0)$ and

$$\begin{aligned}\int_Q W(x_0, G(x_0), \nabla v(x))\, dx + \int_{S_v \cap Q} \Psi_2(x_0, [v(x)], \nu_v(x))\, d\mathcal{H}^{N-1}(x)\\ \leqslant W_2(x_0, G(x_0), \nabla G(x_0), \Gamma(x_0)) + \varepsilon.\end{aligned} \tag{5.31}$$

Extending v by periodicity to all of $\mathbb{R}^N$ and using Theorem 2.4 and Lemma 2.3 we construct a sequence $w_{n,\delta}$ so that

$$\begin{aligned}&\frac{dI_2(G,\Gamma)}{d\mathcal{L}^N}(x_0)\\ &\leqslant \limsup_{\delta\to 0^+} \limsup_{n\to+\infty} \frac{1}{\delta^N}\left[\int_{Q(x_0,\delta)} W(x, G(x), \nabla w_{n,\delta}(x))\, dx\right.\\ &\qquad\qquad \left. + \int_{Q(x_0,\delta)\cap S_{w_{n,\delta}}} \Psi_2(x, [w_{n,\delta}(x)], \nu_{w_{n,\delta}}(x))\, d\mathcal{H}^{N-1}(x)\right]\\ &\leqslant \int_Q W(x_0, G(x_0), \nabla v(x))\, dx + \int_{S_v\cap Q} \Psi_2(x_0, [v(x)], \nu_v(x))\, d\mathcal{H}^{N-1}(x) + O(\varepsilon),\end{aligned}$$

where we use hypotheses (H2) and (H6) to fix x_0 and $G(x_0)$, and periodicity arguments. Hence, by (5.31) and given the arbitrariness of ε, we conclude the desired inequality.

As in the case of I_1, the upper bound for the interfacial term of I_2 is proved in two steps, first for $G = \Lambda\chi_E$, where E is a polyhedral set, and then generalized to an arbitrary set of finite perimeter E, by using Theorem 2.2, Propositions 4.7, 4.11, 4.13, as well as Remark 4.13 and Theorem 2.1.

To prove the first step, fix a point x_0 such that

$$\lim_{\delta\to 0^+} \frac{1}{\delta^{N-1}} \int_{Q_\nu(x_0,\delta)} |\Gamma(x)| + |G(x)| + |\nabla G(x)|\, dx = 0,$$

where $\nu := \nu_G(x_0)$. Given $\varepsilon > 0$, we let $v \in SBV(Q_\nu; \mathbb{R}^{d\times N})$ be such that $v(x) = \gamma_{(\Lambda,\nu)}$ on ∂Q_ν, $\int_{Q_\nu} \nabla v(x)\, dx = 0$ and

$$\begin{aligned}\int_{Q_\nu} W^\infty(x_0, G(x_0), \nabla v(x))\, dx + \int_{S_v\cap Q_\nu} \Psi_2(x_0, [v(x)], \nu_v(x))\, d\mathcal{H}^{N-1}(x)\\ \leqslant \gamma_2(x_0, G(x_0), \Lambda, \nu) + \varepsilon.\end{aligned} \tag{5.32}$$

Extending v by periodicity to all of $\mathbb{R}^N$, and using the usual combination of Theorem 2.4 and Lemma 2.3, we construct a sequence $w_{n,\delta}$ so that

$$\begin{aligned}&\frac{dI_2(G,\Gamma)}{d\mathcal{H}^{N-1}\lfloor S_G}(x_0)\\ &\leqslant \limsup_{\delta\to 0^+} \limsup_{n\to+\infty} \frac{1}{\delta^{N-1}}\left[\int_{Q_\nu(x_0,\delta)} W(x, G(x), \nabla w_{n,\delta}(x))\, dx\right.\end{aligned}$$

$$+\int_{Q_\nu(x_0,\delta)\cap S_{w_{n,\delta}}} \Psi_2(x,[w_{n,\delta}(x)],\nu_{w_{n,\delta}}(x))\,d\mathcal{H}^{N-1}(x)\Bigg]$$

$$\leqslant \int_{Q_\nu} W^\infty(x_0,G(x_0),\nabla v(x))\,dx + \int_{S_v\cap Q_\nu} \Psi_2(x_0,[v(x)],\nu_v(x))\,d\mathcal{H}^{N-1}(x) + O(\varepsilon),$$

where we use hypotheses (H2) and (H6) to fix x_0 and $G(x_0)$, (H4) to pass from W to W^∞, and periodicity arguments. Thus, letting $\varepsilon \to 0^+$, the desired inequality follows by (5.32). □

6. Example and Applications

6.1. **An example.** We provide an example in which the initial energy depends only on jumps in gradients through a specific initial interfacial energy Ψ_2 and in which an explicit formula for the bulk relaxed energy density emerges. Consider the initial energy E in (3.1) with $W = 0, \Psi_1 = 0$ and, for $a \in \mathbb{R}^N$ a fixed unit vector,

$$\Psi_2(x,J,\nu) = |\nu \cdot Ja|\,, \tag{6.1}$$

for all $x \in \Omega$, $J \in \mathbb{R}^{N\times N}$, and $\nu \in S^{N-1}$.

From Theorem 3.2, and in view of Remark 3.1(4), we have that $W_1 = 0$, and we have the following cell formula for the bulk part $W_1 + W_2 = W_2$ of the relaxed energy in this setting: for almost every $x \in \Omega$, $A \in \mathbb{R}^{N\times N}$, $L, M \in \mathbb{R}^{N\times N\times N}$

$$\begin{aligned} W_2(x,A,L,M) = \inf_{u\in SBV(Q;\mathbb{R}^{N\times N})} \Bigg\{ &\int_{Q\cap S_u} |\nu_u(y)\cdot[u](y)a|\;d\mathcal{H}^{N-1}(y) : \\ & u|_{\partial Q}(y) = Ly, \int_Q \nabla u(y)\,dy = M\Bigg\}. \end{aligned} \tag{6.2}$$

Consequently, W_2 does not depend upon x and A, and we omit these variables. It is helpful in what follows to use the fact that each element $M \in \mathbb{R}^{N\times N\times N}$ can be identified with a bilinear mapping from $\mathbb{R}^N$ into $\mathbb{R}^N$

$$\mathbb{R}^N \times \mathbb{R}^N \ni (y,z) \longmapsto M(y,z) \in \mathbb{R}^N \tag{6.3}$$

where we have used the same symbol for the matrix and its associated bilinear mapping. Specifically, we may put

$$M(y,z)_i = \sum_{j,k=1}^N M_{ijk}y_jz_k \quad \text{for all } y,z \in \mathbb{R}^N.$$

We denote the set of bilinear mappings on $\mathbb{R}^N$ with values in $\mathbb{R}^N$ by $Lin^2(\mathbb{R}^N)$, and we note that for each $M \in Lin^2(\mathbb{R}^N)$ the mapping $M(\cdot,a)$ is a linear mapping on $\mathbb{R}^N$ with values in $\mathbb{R}^N$, i.e., $M(\cdot,a) \in Lin(\mathbb{R}^N)$.

Our main result here is the following explicit formula for W_2 in (6.2): for all $L, M \in Lin^2(\mathbb{R}^N)$

$$W_2(L,M) = |tr(L(\cdot,a) - M(\cdot,a))| \tag{6.4}$$

where tr denotes the trace operation on $Lin(\mathbb{R}^N)$. In terms of the associated elements of $\mathbb{R}^{N\times N\times N}$ the formula (6.4) reads

$$W_2(L,M) = \left|\sum_{i,j=1}^N (L_{iij} - M_{iij})a_j\right|. \tag{6.5}$$

With reference to Theorem 3.2, when $W = 0, \Psi_1 = 0$, and Ψ_2 is given by (6.1), we conclude that, for all $(g,G,\Gamma) \in SD^2(\Omega;\mathbb{R}^d)$, the bulk part of the relaxed energy $I(g,G,\Gamma)$ is given by the integral

$$\int_\Omega W_2(\nabla G(x),\Gamma(x))d\mathcal{L}^N(x) = \int_\Omega |tr((\nabla G(x) - \Gamma(x))(\cdot,a))|\,d\mathcal{L}^N(x). \tag{6.6}$$

This formula shows explicitly how the volume density of gradient disarrangements $\nabla G - \Gamma$ determines the bulk relaxed energy associated with the purely interfacial initial energy density

$$E(u) = \int_{S_{\nabla u}\cap\Omega} |\nu_{\nabla u} \cdot [\nabla u]a| \, d\mathcal{H}^{N-1}. \tag{6.7}$$

It is worth noting that the initial energy density $E(u)$ measures the non-tangential part of the jumps in the directional derivative $(\nabla u)a$, so that the integrand in (6.6) provides for the second-order structured deformation (g, G, Γ) an optimal volume density that accounts for non-tangential jumps in the directional derivative $(\nabla u)a$ of approximating deformations u.

To verify (6.4), we use Theorem 2 of [30] and follow the strategy in the proof of Lemma 2 in that article. As in their proof, a simple argument based on the triangle inequality and the Divergence Theorem for functions of bounded variation shows that $|tr(L(\cdot,a) - M(\cdot,a))|$ is a lower bound for $W_2(L,M)$. To show the opposite inequality, we first consider the case in which the linear mapping $L(\cdot,a) - M(\cdot,a)$ is in the set $\mathcal{S} \subset Lin(\mathbb{R}^N)$ of linear mappings with N distinct eigenvalues each having non-zero real part and each with trace non-zero. According to Theorem 1 in [30], $\mathcal{S}$ is dense in $Lin(\mathbb{R}^N)$. Let $R \subset Q$ be in the set $\mathcal{A}$ of all sets of finite perimeter having non-zero volume and compactly contained in Q. We define $u_R : Q \longrightarrow \mathbb{R}^{N\times N}$ by

$$u_R(x) = \begin{cases} Lx & \text{if } x \in Q\backslash R \\ |R|^{-1}(M - (1-|R|)L)x & \text{if } x \in R \end{cases} \tag{6.8}$$

and note that $u_R \in SBV(Q, \mathbb{R}^{N\times N})$, its jump set S_{u_R} is included in $\partial^* R$ (the essential boundary of R, see [4]), and

$$[u_R](x) = |R|^{-1}(L-M)x \quad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial R.$$

These properties of u_R and the arbitrariness of R imply that for all $R \in \mathcal{A}$

$$W_2(L,M) \leqslant |R|^{-1} \int_{\partial R} |\nu_{u_R}(x) \cdot ((L-M)x)a| \, d\mathcal{H}^{N-1}(x)$$

so that $W_2(L,M)$ does not exceed the infimum of the right-hand side with respect to $R \in \mathcal{A}$. Because $L(\cdot,a) - M(\cdot,a)$ is in the set $\mathcal{S}$ we may apply Theorem 2 of [30] to conclude

$$W_2(L,M) \leqslant |tr(L(\cdot,a) - M(\cdot,a))|$$

which implies the equality (6.4) when $L(\cdot,a) - M(\cdot,a) \in \mathcal{S}$.

In order to verify (6.4) for arbitrary $L, M \in Lin^2(\mathbb{R}^N)$, we first note that for each $z \in \mathbb{R}^N$ we may write $z = (z\cdot a)a + z_\perp$ where $z_\perp \cdot a = 0$. Now put $\Delta = L - M$ and notice that, by the linearity of $\Delta(y,\cdot)$, there holds

$$\begin{aligned} \Delta(y,z) &= \Delta(y,(z\cdot a)a + z_\perp) \\ &= (z\cdot a)\Delta(y,a) + \Delta(y,z_\perp). \end{aligned}$$

Since $\Delta(\cdot,a) \in Lin(\mathbb{R}^N)$ and $\mathcal{S}$ is dense in $Lin(\mathbb{R}^N)$, we may choose a sequence $n \longmapsto A_n \in \mathcal{S}$ such that $\lim_{n\to\infty} A_n = \Delta(\cdot,a)$. We set

$$\Delta_n(y,z) = (z\cdot a)A_n y + \Delta(y,z_\perp)$$

and observe that $\Delta_n \in Lin^2(\mathbb{R}^N)$ and for all $y,z \in \mathbb{R}^N$

$$\begin{aligned} \lim_{n\to\infty} \Delta_n(y,z) &= (z\cdot a)\Delta(y,a) + \Delta(y,z_\perp) = \Delta(y,z) \\ &= (L-M)(y,z). \end{aligned}$$

Putting $M_n = L - \Delta_n$, we conclude that $\lim_{n\to\infty} M_n = L - \lim_{n\to\infty}\Delta_n = M$ as well as

$$\begin{aligned} (L-M_n)(y,a) &= \Delta_n(y,a) \\ &= (a\cdot a)A_n y + \Delta(y,a_\perp) = A_n y, \end{aligned}$$

so that $(L-M_n)(\cdot,a) \in \mathcal{S}$. (In the last step we have used the fact that $a_\perp = 0$.) Therefore, $W_2(L,M_n) = |trA_n|$, and letting $n \to \infty$ and using the continuity of $W_2(L,\cdot)$ established in Proposition 4.11 and of the trace operator we conclude that

$$\begin{aligned} W_2(L,M) &= \lim_{n\to\infty} W_2(L,M_n) = \lim_{n\to\infty} |trA_n| = \left|tr \lim_{n\to\infty} A_n\right| \\ &= |tr(L-M)(\cdot,a)| \end{aligned}$$

and thereby complete the verification of (6.4).

6.2. **Applications.** For the case $\Omega \subset \mathbb{R}^3$ the relaxed energies for first-order structured deformations $(g,G) \in SBV^2(\Omega,\mathbb{R}^3) \times SBV(\Omega,\mathbb{R}^{3\times 3})$ studied in [7] provide a means of capturing the effect of both submacroscopically smooth changes and of submacroscopically non-smooth geometrical changes (disarrangements) on the bulk energy stored in a three-dimensional body. In particular, the bulk relaxed energy density $(A,B) \longmapsto W_1(A-B)$ of [7] provides the portion

$$I_{dis}(g,G) = \int_\Omega W_1(\nabla g(x) - G(x))d\mathcal{L}^3(x)$$

of the bulk part of the relaxed energy that arises from disarrangements. This interpretation of $I_{dis}(g,G)$ is justified by considering a sequence $\{u_n\}$ in $SBV^2(\Omega,\mathbb{R}^3)$ with $u_n \to g$ and $\nabla u_n \to G$ both in L^1 and by writing

$$\begin{aligned} \nabla g\, \mathcal{L}^3 + [g]\otimes \nu_g\, \mathcal{H}^2 &= Dg = D\lim_{n\to\infty} u_n = \lim_{n\to\infty} Du_n \\ &= \lim_{n\to\infty} (\nabla u_n\, \mathcal{L}^3 + [u_n]\otimes \nu_{u_n}\, \mathcal{H}^2) \\ &= G\mathcal{L}^3 + \lim_{n\to\infty}([u_n]\otimes \nu_{u_n}\, \mathcal{H}^2), \end{aligned} \tag{6.9}$$

showing that $M\mathcal{L}^3 := (\nabla g - G)\,\mathcal{L}^3$ is the absolutely continuous part of the limit of the singular measures $[u_n]\otimes \nu_{u_n}\, \mathcal{H}^2$ that capture the submacroscopic disarrangements associated with (g,G). Moreover, the energy density $(A,L) \longmapsto W_2(A,L)$ of [7] provides the remaining portion

$$I_\setminus(g,G) = \int_\Omega W_2(G(x),\nabla G(x))d\mathcal{L}^3(x)$$

of the bulk part of the relaxed energy, namely, the portion that arises without disarrangements.

The availability in [7] (or, alternatively, directly from the results of [13]) of such refined bulk energies provides connections to the research [18] that attempts to broaden classical, finite elasticity into the setting of first-order structured deformations through the field theory "elasticity with disarrangements". (That theory requires the specification at the outset of a bulk energy in the form $\int_\Omega \Psi(G(x),\nabla g(x))d\mathcal{L}^3(x)$, so that, for applications of energy relaxation to elasticity with disarrangements, the dependence of the bulk density W_2 on the third-order tensor field ∇G in the formula for $I_\setminus(g,G)$ can be dropped). Elasticity with disarrangements [18] has been applied to the study of granular materials [19, 20, 21], with G representing the smooth deformation of grains and with g representing the macroscopic deformation of the aggregate of grains, and this broadened version of finite elasticity has provided a setting in which no-tension materials with non-linear response in compression arise in a natural way.

While the scope of elasticity with disarrangements is broad enough to capture some energetic effects of disarrangements, its setting in the context of first-order structured deformations precludes its capturing directly the effects of "gradient disarrangements", i.e., of jumps in the gradients of deformations that approximate geometrical changes at the smaller length scale. The theory of second-order structured deformations $(g,G,\Gamma) \in SBV^2(\Omega,\mathbb{R}^3) \times SBV(\Omega,\mathbb{R}^{3\times 3}) \times L^1(\Omega,\mathbb{R}^{3\times 3\times 3})$ guarantees the existence of a sequence $n \longmapsto u_n \in SBV^2(\Omega,\mathbb{R}^3)$ such that $u_n \to g$ and $\nabla u_n \to G$ in L^1 while $\nabla^2 u_n$ tends to Γ weakly in the sense of measures.

Following the idea of the calculation (6.9) we have

$$\begin{aligned}
\nabla^2 g\,\mathcal{L}^3 + [\nabla g]\otimes\nu_{\nabla g}\,\mathcal{H}^2 &= D\nabla g \\
&= D(\nabla g - G) + DG = DM + D\lim_{n\to\infty}\nabla u_n \\
&= DM + \lim_{n\to\infty} D\nabla u_n \\
&= \nabla M\mathcal{L}^3 + [M]\otimes\nu_{[M]}\,\mathcal{H}^2 \\
&\quad + \lim_{n\to\infty}(\nabla^2 u_n\,\mathcal{L}^3 + [\nabla u_n]\otimes\nu_{\nabla u_n}\,\mathcal{H}^2) \\
&= (\nabla M + \Gamma)\mathcal{L}^3 + [M]\otimes\nu_{[M]}\,\mathcal{H}^2 \\
&\quad + \lim_{n\to\infty}([\nabla u_n]\otimes\nu_{\nabla u_n}\,\mathcal{H}^2),
\end{aligned} \tag{6.10}$$

which shows that $\nabla^2 g - \nabla M - \Gamma = \nabla^2 g - \nabla(\nabla g - G) - \Gamma = \nabla G - \Gamma$ is the absolutely continuous part of the distributional limit as $n\to\infty$ of the singular measures $[\nabla u_n]\otimes\nu_{\nabla u_n}\,\mathcal{H}^2$.

We conclude that each second-order structured deformation (g, G, Γ) provides the field $\nabla G - \Gamma \in L^1(\Omega, \mathbb{R}^{3\times3\times3})$ that serves as a volume density of gradient disarrangements. Moreover, since the initial pair (g, G) in the triple (g, G, Γ) is a first-order structured structured deformation, the field $\nabla g - G \in SBV(\Omega, \mathbb{R}^{3\times3})$ remains available as a volume density of disarrangements. Consequently, the results in the present paper on relaxation in the context of second-order structured deformations capture the influence both of disarrangements and of gradient-disarrangements on relaxed energies and provide the starting point for broadening elasticity with disarrangements to the richer multiscale geometry of second-order structured deformations. Initial steps toward such a broadening have been taken [28] in the context of second-order structured deformations. A physical context of significance – phase-transitions in metals [1, 2, 25] – provides a setting in which deformations can be approximately piecewise homogeneous at small length scales. In this setting it is appropriate to assume that there are approximating piecewise smooth deformations u_n with the property $\Gamma = \lim_{n\to\infty}\nabla^2 u_n = 0$. Second-order structured deformations of the form $(g, G, 0)$ are called *submacroscopically affine,* and, for them, the gradient-disarrangement density $\nabla G - \Gamma$ reduces to ∇G, i.e., the "strain-gradient" quantity ∇G measures the volume density of jumps in gradients of approximating piecewise affine deformations. The results of the present paper provide in particular an energetics of bodies undergoing submacroscopically affine structured deformations and, looking ahead, will provide the constitutive input for the field theory "elasticity with gradient disarrangements" applied to bodies undergoing deformations that are approximately piecewise homogeneous at small length scales.

Acknowledgements. The authors warmly thank the Center for Nonlinear Analysis (NSF grants no. DMS-0405343 and DMS-0635983) at Carnegie Mellon University, Pittsburgh, USA, CAMGSD at Instituto Superior Técnico, Lisbon, Portugal, and SISSA, Trieste, Italy, where this research was carried out. MM is a member of the Progetto di Ricerca GNAMPA-INdAM 2015 "Fenomeni critici nella meccanica dei materiali: un approccio variazionale". The research of ACB, JM, and MM was partially supported by the Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) through the CMU-Portugal Program under grant FCT-UTA_CMU/MAT/0005/2009 "Thin Structures, Homogenization, and Multiphase Problems". The research of ACB was partially supported by the Fundação para a Ciência e a Tecnologia through grant UID/MAT/04561/2013. The research of MM was partially supported by the European Research Council through the ERC Advanced grant "QuaDynEvoPro", grant agreement no. 290888 and the ERC Starting grant "High-Dimensional Sparse Optimal Control", grant agreement no. 306274. The research of JM was partially supported by the Fundação para a Ciência e a Tecnologia through grant UID/MAT/04459/2013.

Conflict of Interest: The authors declare that they have no conflict of interest.

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