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ON THE SUM OF DIGITS OF THE FACTORIAL

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Abstract. Let $b \geq 2$ be an integer and denote by $s_b(m)$ the sum of the digits of the positive integer m when is written in base b. We prove that $s_b(n!) > C_b \log n \log\log\log n$ for each integer $n > e$, where C_b is a positive constant depending only on b. This improves of a factor $\log\log\log n$ a previous lower bound for $s_b(n!)$ given by Luca. We prove also the same inequality but with $n!$ replaced by the least common multiple of $1, 2, \dots, n$.

1. Introduction

Let $b \geq 2$ be an integer and denote by $s_b(m)$ the sum of the digits of the positive integer m when is written in base b. Lower bounds for $s_b(m)$ when m runs through the member of some special sequence of natural numbers (e.g., linear recurrence sequences [Ste80] [Luc00] and sequences with combinatorial meaning [LS10] [LS11] [KL12] [Luc12]) have been studied before.

In particular, Luca [Luc02] showed that the inequality

$$s_b(n!) > c_b \log n, \tag{1}$$

holds for all the positive integers n, where c_b is a positive constant, depending only on b. He also remarked that (1) remains true if one replaces $n!$ by

$$\Lambda_n := \mathrm{lcm}(1, 2, \dots, n),$$

the least common multiple of $1, 2, \dots, n$. We recall that Λ_n has an important role in elementary proofs of the Chebyshev bounds $\pi(x) \asymp x/\log x$, for the prime counting function $\pi(x)$ [Nai82].

In this paper, we give a slight improvement of (1) by proving the following

Theorem 1.1. *For each integer $n > e$, it results*

$$s_b(n!),\ s_b(\Lambda_n) > C_b \log n \log\log\log n,$$

where C_b is a positive constant, depending only on b.

2. Preliminaries

In this section, we discuss a few preliminary results needed in our proof of Theorem 1.1. Let φ be the Euler's totient function. We prove an asymptotic formula for the maximum of the preimage of $[1, x]$ through φ, as $x \to +\infty$. Although the cardinality of the set $\varphi^{-1}([1, x])$ is well studied [Bat72] [BS90] [BT98], in the literature we have found no results about $\max(\varphi^{-1}([1, x]))$ as our next lemma.

Lemma 2.1. *For each $x \geq 1$, let $m = m(x)$ be the greatest positive integer such that $\varphi(m) \leq x$. Then $m \sim e^\gamma x \log\log x$, as $x \to +\infty$, where γ is the Euler–Mascheroni constant.*

Proof. Since $\varphi(n) \leq n$ for each positive integer n, it results $m \geq \lfloor x \rfloor$. In particular, $m \to +\infty$ as $x \to +\infty$. Therefore, since the minimal order of $\varphi(n)$ is $e^{-\gamma} n/\log\log n$ (see [Ten95, Chapter I.5, Theorem 4]), we obtain

$$(e^{-\gamma} + o(1)) \frac{m}{\log\log m} \leq \varphi(m) \leq x,$$

as $x \to +\infty$. Now $\varphi(n) \geq \sqrt{n}$ for each integer $n \geq 7$, thus $m \leq x^2$ for $x \geq 7$. Hence,

$$m \leq (e^\gamma + o(1))\, x \log\log m \leq (e^\gamma + o(1))\, x \log\log(x^2) = (e^\gamma + o(1))\, x \log\log x,$$

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as $x \to +\infty$.

On the other hand, let $p_1 < p_2 < \cdots$ be the sequence of all the (natural) prime numbers and let $a_1 < a_2 < \cdots$ be the sequence of all the 3-smooth numbers, i.e., the natural numbers of the form $2^a 3^b$, for some integers $a, b \geq 0$. Moreover, let $s = s(x)$ be the greatest positive integer such that

$$(p_1 - 1) \cdots (p_s - 1) \leq \sqrt{x},$$

and let $t = t(x)$ be the greatest positive integer such that

$$a_t (p_1 - 1) \cdots (p_s - 1) \leq x.$$

Note that $s, t \to +\infty$ as $x \to +\infty$. Now we have (see [Ten95, Chapter I.1, Theorem 4])

$$\sqrt{x} < (p_1 - 1) \cdots (p_{s+1} - 1) < p_1 \cdots p_{s+1} \leq 4^{p_{s+1}},$$

hence

$$p_s > \tfrac{1}{2} p_{s+1} > \tfrac{1}{4 \log 4} \log x, \tag{2}$$

from Bertrand's postulate. Put $m' := a_t p_1 \cdots p_s$, so that for $s \geq 2$ we get

$$\varphi(m') = a_t (p_1 - 1) \cdots (p_s - 1) \leq x,$$

and thus $m \geq m'$. By a result of Pólya [Pól18], $a_t / a_{t+1} \to 1$ as $t \to +\infty$. Therefore, from our hypothesis on s and t, Mertens' formula [Ten95, Chapter I.1, Theorem 11] and (2) it follows that

$$\begin{aligned} m \geq m' &= \frac{a_t}{a_{t+1}} \cdot a_{t+1} \prod_{i=1}^{s} (p_i - 1) \cdot \prod_{i=1}^{s} \left(1 - \frac{1}{p_i}\right)^{-1} > (1 + o(1)) \cdot x \cdot \frac{\log p_s}{e^{-\gamma} + o(1)} \\ &> (e^{\gamma} + o(1))\, x \log\log x, \end{aligned}$$

as $x \to +\infty$. □

Actually, we do not make use of Lemma 2.1. We need more control on the factorization of a "large" positive integer m such that $\varphi(m) \leq x$, even at the cost of having only a lower bound for m and not an asymptotic formula.

Lemma 2.2. *For each $x \geq 1$ there exists a positive integer $m = m(x)$ such that: $\varphi(m) \leq x$; $m = 2^t Q$, where t is a nonnegative integer and Q is an odd squarefree number; and*

$$m \geq \left(\tfrac{1}{2} e^{\gamma} + o(1)\right) x \log\log x,$$

as $x \to +\infty$.

Proof. The proof proceeds as the second part of the proof of Lemma 2.1, but with $a_k := 2^{k-1}$ for each positive integer k. So instead of $a_t / a_{t+1} \to 1$, as $t \to +\infty$, we have $a_t / a_{t+1} = 1/2$ for each t. We leave the remaining details to the reader. □

To study Λ_n is useful to consider the positive integers as a poset ordered by the divisibility relation $\mid$. Thus, obviously, Λ_n is a monotone nondecreasing function, i.e., $\Lambda_m \mid \Lambda_n$ for each positive integers $m \leq n$. The next lemma says that Λ_n is also super-multiplicative.

Lemma 2.3. *We have $\Lambda_m \Lambda_n \mid \Lambda_{mn}$, for any positive integers m and n.*

Proof. It is an easy exercise to prove that

$$\Lambda_n = \prod_{p \leq n} p^{\lfloor \log_p n \rfloor},$$

for each positive integer n, where p runs over all the prime numbers not exceeding n. Therefore, the claim follows since

$$\lfloor \log_p m \rfloor + \lfloor \log_p n \rfloor \leq \lfloor \log_p m + \log_p n \rfloor = \lfloor \log_p mn \rfloor,$$

for each prime number p. □

We recall some basic facts about cyclotomic polynomials. For each positive integer n, the n-th cyclotomic polynomial $\Phi_n(x)$ is defined by

$$\Phi_n(x) := \prod_{\substack{1 \le k \le n \\ \gcd(k,n)=1}} \left(x - e^{2\pi i k/n}\right).$$

It results that $\Phi_n(x)$ is a polynomial with integer coefficients and that it is irreducible over the rationals, with degree $\varphi(n)$. We have the polynomial identity

$$x^n - 1 = \prod_{d \mid n} \Phi_d(x),$$

where d runs over all the positive divisors of n. Moreover, it holds $\Phi_n(a) \le (a+1)^{\varphi(n)}$, for all $a \ge 0$. The next lemma regards when $\Phi_m(a)$ and $\Phi_n(a)$ are not coprime.

Lemma 2.4. *Suppose that $\gcd(\Phi_m(a), \Phi_n(a)) > 1$ for some integers $m, n, a \ge 1$. Then m/n is a prime power, i.e., $m/n = p^k$ for a prime number p and an integer k.*

Proof. See [Ge08, Theorem 7]. □

Finally, we state an useful lower bound for the sum of digits of the multiples of $b^m - 1$.

Lemma 2.5. *For each positive integers m and q it results $s_b((b^m-1)q) \ge m$.*

Proof. See [BD12, Lemma 1]. □

3. Proof of Theorem 1.1

Without loss of generality, we can assume n sufficiently large. Put $x := \frac{1}{8}\log_{b+1} n \ge 1$. Thanks to Lemma 2.2, we know that there exists a positive integer m such that $\varphi(m) \le x$ and

$$m > \tfrac{1}{3} e^\gamma x \log\log x > C_b \log n \log\log\log n, \tag{3}$$

where $C_b > 0$ is a constant depending only on b. Precisely, we can assume that $m = 2^t Q$, where t is a nonnegative integer and Q is an odd squarefree number. Fix a nonnegative integer $j \le t$. For each positive divisor d of Q, we have $\varphi(2^{t-j}d) \mid \varphi(m/2^j)$ and so, a fortiori, $\varphi(2^{t-j}d) \le \varphi(m/2^j)$. Therefore,

$$\Phi_{2^{t-j}d}(b) \le (b+1)^{\varphi(2^{t-j}d)} \le (b+1)^{\varphi(m/2^j)} \le (b+1)^{\varphi(m)/2^{j-1}} \le n^{1/2^{j+2}}. \tag{4}$$

Let μ be the Möbius function. Now from (4) and Lemma 2.4 we have that the $\Phi_{2^{t-j}d}(b)$'s, where d runs over the positive divisors of Q such that $\mu(d) = 1$, are pairwise coprime and not exceeding $n^{1/2^{j+2}}$, thus

$$\prod_{\substack{d \mid Q \\ \mu(d)=1}} \Phi_{2^{t-j}d}(b) = \operatorname{lcm}\{\Phi_{2^{t-j}d}(b) : d \mid Q,\ \mu(d) = 1\} \mid \Lambda_{\lfloor n^{1/2^{j+2}} \rfloor}. \tag{5}$$

Similarly, the same result holds for the divisors d such that $\mu(d) = -1$. Clearly, we have

$$b^m - 1 = \prod_{d \mid m} \Phi_d(b) = \prod_{\substack{0 \le j \le t \\ r \in \{-1,+1\}}} \prod_{\substack{d \mid Q \\ \mu(d) = r}} \Phi_{2^{t-j}d}(b).$$

Moreover,

$$\left(\prod_{0 \le j \le t} \lfloor n^{1/2^{j+2}} \rfloor\right)^2 \le \prod_{0 \le j \le t} n^{1/2^{j+1}} \le n.$$

As a consequence, from (5) and Lemma 2.3, we obtain

$$b^m - 1 \mid \left(\prod_{0 \le j \le t} \Lambda_{\lfloor n^{1/2^{j+2}} \rfloor}\right)^2 \mid \Lambda_n.$$

Thus $b^m - 1 \mid \Lambda_n$ and also $b^m - 1 \mid n!$, since obviously $\Lambda_n \mid n!$. In conclusion, from Lemma 2.5 and (3), we get

$$s_b(\Lambda_n),\ s_b(n!) \geq m > C_b \log n \log\log\log n,$$

which is our claim, this completes the proof.

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