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## A factor of integer polynomials with minimal integrals

par CARLO SANNA

RÉSUMÉ. Pour chaque entier positif  $N$ , soit  $S_N$  l'ensemble des polynômes  $P(x) \in \mathbb{Z}[x]$  de degré inférieur à  $N$  et d'intégrale positive non-nulle minimale sur  $[0, 1]$ . Ces polynômes sont liés à la répartition des nombres premiers puisque, si l'on note  $\psi$  la fonction de Tchebychev, on a  $\int_0^1 P(x) dx = \exp(-\psi(N))$ . Nous démontrons que, pour tout nombre entier positif  $N$ , il existe  $P(x) \in S_N$  tel que le polynôme  $(x(1-x))^{\lfloor N/3 \rfloor}$  divise  $P(x)$  dans  $\mathbb{Z}[x]$ . Nous montrons en fait que l'exposant  $\lfloor N/3 \rfloor$  ne peut pas être amélioré. Ce résultat est analogue à celui obtenu par Aparicio concernant les polynômes de  $\mathbb{Z}[x]$  de norme  $L^\infty$  non-nulle minimale sur  $[0, 1]$ . En outre, il est en quelque sorte l'amélioration d'un résultat de Bazzanella, qui considérait  $x^{\lfloor N/2 \rfloor}$  et  $(1-x)^{\lfloor N/2 \rfloor}$  au lieu de  $(x(1-x))^{\lfloor N/3 \rfloor}$ .

ABSTRACT. For each positive integer  $N$ , let  $S_N$  be the set of all polynomials  $P(x) \in \mathbb{Z}[x]$  with degree less than  $N$  and minimal positive integral over  $[0, 1]$ . These polynomials are related to the distribution of prime numbers since  $\int_0^1 P(x) dx = \exp(-\psi(N))$ , where  $\psi$  is the second Chebyshev function. We prove that for any positive integer  $N$  there exists  $P(x) \in S_N$  such that  $(x(1-x))^{\lfloor N/3 \rfloor}$  divides  $P(x)$  in  $\mathbb{Z}[x]$ . In fact, we show that the exponent  $\lfloor N/3 \rfloor$  cannot be improved. This result is analog to a previous of Aparicio concerning polynomials in  $\mathbb{Z}[x]$  with minimal positive  $L^\infty$  norm on  $[0, 1]$ . Also, it is in some way a strengthening of a result of Bazzanella, who considered  $x^{\lfloor N/2 \rfloor}$  and  $(1-x)^{\lfloor N/2 \rfloor}$  instead of  $(x(1-x))^{\lfloor N/3 \rfloor}$ .

### 1. Introduction

It is well-known that the celebrated Prime Number Theorem is equivalent to the assertion:

$$\psi(x) \sim x, \text{ as } x \rightarrow +\infty.$$

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Here  $\psi(x)$  is the second Chebyshev function, defined for  $x \geq 0$  as

$$\psi(x) := \sum_{p^m \leq x} \log p,$$

where the sum is extended over all the prime numbers  $p$  and all the positive integers  $m$  such that  $p^m \leq x$ .

In 1936, Gelfond and Shnirelman proposed an elementary and clever method to obtain lower bounds for  $\psi(x)$  (see Gelfond's comments in [5, pp. 285–288]). In 1982, the same method was rediscovered and developed by Nair [9, 10].

The main idea of the Gelfond–Shnirelman–Nair method is the following: Given a positive integer  $N$ , let  $P_N(x)$  be a polynomial with integer coefficients and degree less than  $N$ , say

$$P_N(x) = \sum_{n=0}^{N-1} a_n x^n,$$

with  $a_0, \dots, a_{N-1} \in \mathbb{Z}$ . Now consider the integral of  $P_N(x)$  over  $[0, 1]$ , that is

$$I(P_N) := \int_0^1 P_N(x) dx = \sum_{n=0}^{N-1} \frac{a_n}{n+1}.$$

Clearly,  $I(P_N)$  is a rational number whose denominator divides

$$d_N := \text{lcm}\{1, 2, \dots, N\},$$

hence  $d_N |I(P_N)|$  is an integer. In particular, if we suppose  $I(P_N) \neq 0$ , then  $d_N |I(P_N)| \geq 1$ . Now  $d_N = \exp(\psi(N))$ , so we get

$$(1.1) \quad \psi(N) \geq \log \left( \frac{1}{|I(P_N)|} \right).$$

Finally, from the trivial upper bound

$$|I(P_N)| = \left| \int_0^1 P_N(x) dx \right| \leq \int_0^1 |P_N(x)| dx \leq \max_{x \in [0,1]} |P_N(x)| =: \|P_N\|,$$

we obtain

$$(1.2) \quad \psi(N) \geq \log \left( \frac{1}{\|P_N\|} \right).$$

At this point, if we choose  $P_N$  to have a sufficiently small norm  $\|P_N\|$ , then a lower bound for  $\psi(x)$  follows from (1.2). For example, the choice

$$P_N(x) = (x(1-x))^{2\lfloor (N-1)/2 \rfloor}$$

gives the lower bound

$$\psi(N) \geq \log 2 \cdot (N-2) > 0.694 \cdot (N-2).$$

This motivates the study of the quantities

$$\ell_N := \min\{\|P\| : P(x) \in \mathbb{Z}[x], \deg(P) < N, \|P\| > 0\},$$

$$C_N := \frac{1}{N} \log\left(\frac{1}{\ell_N}\right),$$

and the set of polynomials

$$T_N := \{P(x) \in \mathbb{Z}[x] : \deg(P) < N, \|P\| = \ell_N\};$$

the so-called Integer Chebyshev Problem [4].

In particular, Aparicio [1] proved the following theorem about the structure of polynomials in  $T_N$ .

**Theorem 1.1.** *Given any sufficiently large positive integer  $N$ , for all  $P \in T_N$  it holds*

$$(x(1-x))^{\lfloor \lambda_1 N \rfloor} (2x-1)^{\lfloor \lambda_2 N \rfloor} (5x^2-5x+1)^{\lfloor \lambda_3 N \rfloor} \mid P(x)$$

in  $\mathbb{Z}[x]$ , where

$$\lambda_1 \in [0.1456, 0.1495], \quad \lambda_2 \in [0.0166, 0.0187], \quad \lambda_3 \in [0.0037, 0.0053]$$

are some constants.

It is known that  $C_N$  converges to a limit  $C$ , as  $N \rightarrow +\infty$  (see [8, Chapter 10]). Furthermore, Pritsker [11, Theorem 3.1] showed that

$$C \in ]0.85991, 0.86441[,$$

and this is the best estimate of  $C$  known to date.

As a consequence of Pritsker’s result, the Gelfond–Shnirelman–Nair method cannot lead to a lower bound better than

$$\psi(x) \geq 0.86441 \cdot x,$$

which is quite far from what is expected by the Prime Number Theorem.

To deal with this problem, Bazzanella [2, 3] suggested to study the polynomials  $P_N$  such that  $|I(P_N)|$  is nonzero and minimal, or, without loss of generality, such that  $I(P_N)$  is positive and minimal.

We recall the following elementary lemma about the existence of solutions of some linear diophantine equations.

**Lemma 1.2.** *Fix some integers  $c_1, \dots, c_k$ . Then the diophantine equation*

$$\sum_{i=1}^k c_i x_i = 1$$

has a solution  $x_1, \dots, x_k \in \mathbb{Z}$  if and only if  $\gcd\{c_1, \dots, c_k\} = 1$ . Moreover, if a solution exists, then there exist infinitely many solutions.

On the one hand, because of the above considerations, we know that if  $I(P_N) > 0$  then  $I(P_N) \geq 1/d_N$ . On the other hand,  $I(P_N) = 1/d_N$  if and only if

$$\sum_{n=0}^{N-1} \frac{d_N}{n+1} \cdot a_n = 1,$$

and it is easy to see that each of the coefficients  $d_N/(n+1)$  is an integer and

$$\gcd\left\{\frac{d_N}{n+1} : n = 0, \dots, N-1\right\} = 1.$$

Hence, by Lemma 1.2, there exist infinitely many polynomials  $P_N$  such that  $I(P_N) = 1/d_N$ , so that (1.1) holds with the equality.

This leads to define the following set of polynomials

$$S_N := \{P(x) \in \mathbb{Z}[x] : \deg(P) < N, I(P) = 1/d_N\}.$$

Bazzanella proved some results about the roots of the polynomials in  $S_N$ . In particular, regarding the multiplicity of the roots  $x = 0$  and  $x = 1$ , he gave the following theorem [2, Theorem 1], which is vaguely similar to Theorem 1.1.

**Theorem 1.3.** *For each positive integer  $N$ , there exists  $P(x) \in S_N$  such that*

$$x^{\lfloor N/2 \rfloor} \mid P(x)$$

*in  $\mathbb{Z}[x]$ . Moreover, the exponent  $\lfloor N/2 \rfloor$  cannot be improved, i.e., there exist infinitely many positive integers  $N$  such that*

$$x^{\lfloor N/2 \rfloor + 1} \nmid P(x)$$

*for all  $P(x) \in S_N$ . The same results hold if the polynomial  $x^{\lfloor N/2 \rfloor}$  is replaced by  $(1-x)^{\lfloor N/2 \rfloor}$ .*

Actually, what Bazzanella proved is that the maximum nonnegative integer  $K(N)$  such that there exists a polynomial  $P(x) \in S_N$  divisible by  $x^{K(N)}$ , respectively by  $(1-x)^{K(N)}$ , is given by

$$K(N) = \min\{p^m - 1 : p \text{ prime}, m \geq 1, p^m > N/2\},$$

so that Theorem 1.3 follows quickly.

Despite the similarity between Theorems 1.1 and 1.3, note that the statement of Theorem 1.1 holds “for all  $P(x) \in T_N$ ”, while Theorem 1.3 only says that “there exists  $P(x) \in S_N$ ”. However, this distinction is unavoidable, indeed: On the one hand,  $T_N$  is a finite set, even conjectured to be a singleton for any sufficiently large  $N$  [4, §5 Q2]. On the other hand,  $S_N$  is an infinite set and if  $P(x) \in S_N$  then  $(d_N + 1)P(x) - 1 \in S_N$ , hence the elements of  $S_N$  have no common nontrivial factor in  $\mathbb{Z}[x]$ .

The purpose of this paper is to move another step further in the direction of a stronger analog of Theorem 1.1 for the set of polynomials  $S_N$ . For we prove the following theorem.

**Theorem 1.4.** *For each positive integer  $N$ , there exist infinitely many  $P(x) \in S_N$  such that*

$$(x(1-x))^{\lfloor N/3 \rfloor} \mid P(x)$$

*in  $\mathbb{Z}[x]$ . Moreover, the exponent  $\lfloor N/3 \rfloor$  cannot be improved, i.e., there exist infinitely many positive integers  $N$  such that*

$$(x(1-x))^{\lfloor N/3 \rfloor + 1} \nmid P(x),$$

*for all  $P(x) \in S_N$ .*

We leave the following informal question to the interested readers:

**Question.** *Let  $\{Q_N(x)\}_{N \geq 1}$  be a sequence of “explicit” integer polynomials such that for each positive integer  $N$  it holds  $Q_N(x) \mid P(x)$  in  $\mathbb{Z}[x]$ , for some  $P(x) \in S_N$ . In light of Theorems 1.3 and 1.4, three examples of such sequences are given by  $\{x^{\lfloor N/2 \rfloor}\}_{N \geq 1}$ ,  $\{(1-x)^{\lfloor N/2 \rfloor}\}_{N \geq 1}$ , and  $\{(x(1-x))^{\lfloor N/3 \rfloor}\}_{N \geq 1}$ .*

*How big can be*

$$\delta := \liminf_{N \rightarrow +\infty} \frac{\deg(Q_N)}{N} ?$$

*Can  $\delta$  be arbitrary close to 1, or even equal to 1?*

Note that the sequences of Theorem 1.3 give  $\delta = 1/2$ , while the sequence of Theorem 1.4 gives  $\delta = 2/3$ .

## 2. Preliminaries

In this section, we collect a number of preliminary results needed to prove Theorem 1.4. The first is a classic theorem of Kummer [7] concerning the  $p$ -adic valuation of binomial coefficients.

**Theorem 2.1.** *For all integers  $u, v \geq 0$  and any prime number  $p$ , the  $p$ -adic valuation of the binomial coefficient  $\binom{u+v}{v}$  is equal to the number of carries that occur when  $u$  and  $v$  are added in the base  $p$ .*

Now we can prove the following lemma.

**Lemma 2.2.** *For any positive integer  $N$ , and for all integers  $u, v \geq 0$  with  $u + v < N$ , we have that*

$$(2.1) \quad \frac{d_N}{(u+v+1) \binom{u+v}{u}}$$

*is an integer.*

*Proof.* We have to prove that for any prime number  $p \leq N$  the  $p$ -adic valuation of the denominator of (2.1) does not exceed  $\nu_p(d_N) = \lfloor \log_p N \rfloor$ . Write  $u + v + 1$  in base  $p$ , that is

$$u + v + 1 = \sum_{i=i_0}^s d_i p^i,$$

where  $i_0 := \nu_p(u + v + 1)$  and  $d_{i_0}, \dots, d_s \in \{0, \dots, p - 1\}$ , with  $d_{i_0}, d_s > 0$ . Hence, the expansion of  $u + v$  in base  $p$  is

$$(2.2) \quad u + v = \sum_{i=i_0+1}^s d_i p^i + (d_{i_0} - 1)p^{i_0} + \sum_{i=0}^{i_0-1} (p - 1)p^i.$$

In particular, by (2.2), we have that  $u + v$  written in base  $p$  has exactly  $s + 1$  digits, of which the  $i_0$  least significant are all equal to  $p - 1$ . Therefore, in the sum of  $u$  and  $v$  in base  $p$  there occur at most  $s - i_0$  carries. Since, thanks to Theorem 2.1, we know that  $i_1 := \nu_p\left(\binom{u+v}{v}\right)$  is equal to the number of carries occurring in the sum of  $u$  and  $v$  in base  $p$ , it follows that  $i_1 \leq s - i_0$ .

In conclusion,

$$\nu_p\left((u + v + 1)\binom{u + v}{v}\right) = i_0 + i_1 \leq s \leq \lfloor \log_p N \rfloor,$$

where the last inequality holds since  $u + v + 1 \leq N$ . □

We recall the value of a well-known integral (see, e.g., [6, §11.1.7.1, Eq. 2]).

**Lemma 2.3.** *For all integers  $u, v \geq 0$ , it holds*

$$\int_0^1 x^u(1 - x)^v dx = \frac{1}{(u + v + 1)\binom{u+v}{v}}.$$

We conclude this section with a lemma that will be fundamental in the proof of Theorem 1.4.

**Lemma 2.4.** *Let  $N$  and  $m$  be integers such that  $N \geq 1$  and  $0 \leq m \leq (N - 1)/2$ . The following statements are equivalent:*

- (1) *There exist infinitely many  $P(x) \in S_N$  such that  $(x(1 - x))^m \mid P(x)$  in  $\mathbb{Z}[x]$ .*
- (2) *For each prime number  $p \leq N$ , there exists an integer  $h_p$  such that  $h_p \in [m, N - m - 1]$  and*

$$\nu_p\left((h_p + m + 1)\binom{h_p + m}{m}\right) = \lfloor \log_p N \rfloor.$$



*Proof.* Let  $P(x) \in \mathbb{Z}[x]$  be such that  $\deg(P) < N$  and

$$(x(1-x))^m \mid P(x)$$

in  $\mathbb{Z}[x]$ . Hence,

$$P(x) = (x(1-x))^m \sum_{h=m}^{N-m-1} b_h x^{h-m},$$

for some  $b_m, \dots, b_{N-m-1} \in \mathbb{Z}$ . Then, by Lemma 2.3, it follows that

$$I(P) = \sum_{h=m}^{N-m-1} b_h \int_0^1 x^h (1-x)^m dx = \sum_{h=m}^{N-m-1} \frac{b_h}{(h+m+1) \binom{h+m}{m}}.$$

Now we have  $P(x) \in S_N$  if and only if  $I(P) = 1/d_N$ , i.e., if and only if

$$\sum_{h=m}^{N-m-1} \frac{d_N}{(h+m+1) \binom{h+m}{m}} \cdot b_h = 1.$$

Therefore, thanks to Lemma 2.2 and Lemma 1.2, we get infinitely many  $P(x) \in S_N$  if and only if

$$\gcd \left\{ \frac{d_N}{(h+m+1) \binom{h+m}{m}} : h = m, \dots, N-m-1 \right\} = 1.$$

At this point, recalling that  $\nu_p(d_N) = \lfloor \log_p N \rfloor$  for each prime number  $p$ , the equivalence of (1) and (2) follows easily.  $\square$

### 3. Proof of Theorem 1.4

We are ready to prove Theorem 1.4. Put  $m := \lfloor N/3 \rfloor$ ,  $s := \lfloor \log_p N \rfloor$ , and pick a prime number  $p \leq N$ . In light of Lemma 2.4, in order to prove the first part of Theorem 1.4 we have to show the existence of an integer  $h_p \in [m, N-m-1]$  such that

$$(3.1) \quad \nu_p \left( (h_p + m + 1) \binom{h_p + m}{m} \right) = s.$$

Let us write  $N = \ell p^s + r$ , for some  $\ell \in \{1, \dots, p-1\}$  and  $r \in \{0, \dots, p^s - 1\}$ . We split the proof in three cases:

*Case  $\ell \geq 2$ .* It is enough to take  $h_p := \ell p^s - m - 1$ . In fact, on the one hand, it is straightforward that (3.1) holds. On the other hand, since  $\ell \geq 2$ , we have

$$h_p = \ell p^s - m - 1 \geq \frac{2}{3}(\ell + 1)p^s - m - 1 > \frac{2}{3}N - m - 1 \geq m - 1,$$

while clearly  $h_p \leq N - m - 1$ , hence  $h_p \in [m, N - m - 1]$ , as desired.

Case  $m < p^{s-1}$ . It holds

$$\frac{p^s}{3} \leq \frac{N}{3} < m + 1 \leq p^{s-1},$$

hence  $p = 2$ . Now it is enough to take  $h_2 := 2^s - m - 1$ . In fact, on the one hand, it is again straightforward that (3.1) holds. On the other hand, since  $m < 2^{s-1}$ , we have

$$h_2 = 2^s - m - 1 > 2^s - 2^{s-1} - 1 = 2^{s-1} - 1 \geq m,$$

while obviously  $h_2 \leq N - m - 1$ , hence  $h_2 \in [m, N - m - 1]$ , as desired.

Case  $\ell = 1$  and  $m \geq p^{s-1}$ . This case requires more effort. We have

$$p^{s-1} \leq m \leq \frac{N}{3} = \frac{p^s + r}{3} < \frac{2p^s}{3} < p^s,$$

hence the expansion of  $m$  in base  $p$  is

$$m = \sum_{i=0}^{s-1} d_i p^i,$$

for some  $d_0, \dots, d_{s-1} \in \{0, \dots, p-1\}$ , with  $d_{s-1} > 0$ .

Let  $i_1$  be the least nonnegative integer not exceeding  $s$  such that

$$(3.2) \quad d_i \geq \frac{p-1}{2}, \quad \forall i \in \mathbb{Z}, i_1 \leq i < s.$$

Moreover, let  $i_2$  be the greatest integer such that  $i_1 \leq i_2 \leq s$  and

$$d_i = \frac{p-1}{2}, \quad \forall i \in \mathbb{Z}, i_1 \leq i < i_2.$$

Note that, by the definitions of  $i_1$  and  $i_2$ , we have

$$(3.3) \quad d_i > \frac{p-1}{2}, \quad \forall i \in \mathbb{Z}, i_2 \leq i < s.$$

Clearly, it holds

$$(3.4) \quad m = \sum_{i_2 \leq i < s} d_i p^i + \sum_{i_1 \leq i < i_2} \frac{p-1}{2} p^i + \sum_{0 \leq i < i_1} d_i p^i.$$

Define now

$$(3.5) \quad h_p := \sum_{i_2 \leq i < s} d_i p^i + \sum_{i_1 \leq i < i_2} \frac{p-1}{2} p^i + \sum_{0 \leq i < i_1} (p - d_i - 1) p^i.$$

Note that (3.5) is actually the expansion of  $h_p$  in base  $p$ , that is, all the coefficients of the powers  $p^i$  belong to the set of digits  $\{0, \dots, p-1\}$ . At this point, looking at (3.4) and (3.5), and taking into account (3.3), it follows

easily that in the sum of  $h_p$  and  $m$  in base  $p$  there occur exactly  $s - i_2$  carries. Therefore, by Theorem 2.1 we have

$$(3.6) \quad \nu_p \left( \binom{h_p + m}{m} \right) = s - i_2.$$

Furthermore, from (3.4) and (3.5) we get

$$(3.7) \quad \begin{aligned} h_p + m + 1 &= 2 \sum_{i_2 \leq i < s} d_i p^i + \sum_{0 \leq i < i_2} (p - 1) p^i + 1 \\ &= 2 \sum_{i_2 \leq i < s} d_i p^i + p^{i_2}, \end{aligned}$$

hence

$$(3.8) \quad \nu_p(h_p + m + 1) = i_2.$$

Therefore, putting together (3.6) and (3.8) we obtain (3.1).

It remains only to prove that  $h_p \in [m, N - m - 1]$ . If  $i_2 = s$ , then from (3.7) it follows that

$$h_p + m + 1 = 0 + p^s \leq N,$$

hence  $h_p \leq N - m - 1$ . If  $i_2 < s$ , then from (3.2) it follows  $d_{i_2} \geq (p - 1)/2$ , hence  $d_{i_2} \geq 1$  and from (3.7) and (3.4) we obtain

$$h_p + m + 1 \leq 2 \sum_{i_2 \leq i < s} d_i p^i + d_{i_2} p^{i_2} \leq 2m + m = 3m \leq N,$$

so that again  $h_p \leq N - m - 1$ . If  $i_1 = 0$ , then by (3.4) and (3.5) we have immediately that  $h_p = m$ . If  $i_1 > 0$ , then by the definition of  $i_1$ , we have  $d_{i_1-1} < (p - 1)/2$ , i.e.,  $d_{i_1-1} < p - d_{i_1-1} - 1$ , thus looking at the expansions (3.4) and (3.5) we get that  $h_p > m$ . Hence, in conclusion we have  $h_p \in [m, N - m - 1]$ , as desired.

Regarding the second part of Theorem 1.4, take  $N := 3q$ , where  $q > 3$  is a prime number. Put  $m := \lfloor N/3 \rfloor + 1 = q + 1$ , and let  $h \in [m, N - m - 1]$  be an integer. On the one hand, it is straightforward that  $q \nmid h + m + 1$ . On the other, it is also easy to see that in the sum of  $h$  and  $m$  in base  $q$  there is no carry, hence, by Theorem 2.1, we have that  $q \nmid \binom{h+m}{m}$ . Therefore,

$$\nu_q \left( (h + m + 1) \binom{h + m}{m} \right) = 0 < 1 = \lfloor \log_q N \rfloor,$$

so that, thanks to Lemma 2.4, we have  $(x(1 - x))^m \nmid P(x)$  in  $\mathbb{Z}[x]$ , for all  $P(x) \in S_N$ . This completes the proof.  $\square$

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