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THE MOMENTS OF THE LOGARITHM OF A G.C.D. RELATED TO LUCAS SEQUENCES

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Abstract. Let $(u_n)_{n \geq 0}$ be a nondegenerate Lucas sequence satisfying $u_n = a_1 u_{n-1} + a_2 u_{n-2}$ for all integers $n \geq 2$, where a_1 and a_2 are some fixed relatively prime integers; and let g_u be the arithmetic function defined by $g_u(n) := \gcd(n, u_n)$, for all positive integers n. Distributional properties of g_u have been studied by several authors, also in the more general context where $(u_n)_{n \geq 0}$ is a linear recurrence. We prove that for each positive integer λ it holds

$$\sum_{n \leq x} (\log g_u(n))^\lambda \sim M_{u,\lambda} x$$

as $x \to +\infty$, where $M_{u,\lambda} > 0$ is a constant depending only on a_1, a_2, and λ. More precisely, we provide an error term for the previous asymptotic formula and we show that $M_{u,\lambda}$ can be written as an infinite series.

1. Introduction

Let $(u_n)_{n \geq 0}$ be an integral linear recurrence, that is, $(u_n)_{n \geq 0}$ is a sequence of integers and there exist $a_1, \dots, a_k \in \mathbb{Z}$, with $a_k \neq 0$, such that

$$u_n = a_1 u_{n-1} + a_2 u_{n-2} + \cdots + a_k u_{n-k},$$

for all integers $n \geq k$. We recall that $(u_n)_{n \geq 0}$ is said to be *nondegenerate* if none of the ratios α_i / α_j $(i \neq j)$ is a root of unity, where $\alpha_1, \dots, \alpha_v \in \mathbb{C}$ are all the pairwise distinct roots of the *characteristic polynomial*

$$\psi_u(X) = X^k - a_1 X^{k-1} - a_2 X^{k-2} - \cdots - a_k.$$

Moreover, $(u_n)_{n \geq 0}$ is said to be a *Lucas sequence* if $u_0 = 0$, $u_1 = 1$, and $k = 2$. In particular, the Lucas sequence with $a_1 = a_2 = 1$ is known as the *Fibonacci sequence*. We refer the reader to [9, Chapter 1] for the basic terminology and theory of linear recurrences.

Let g_u be the arithmetic function defined by $g_u(n) := \gcd(n, u_n)$, for all positive integers n. Many researchers have studied the properties of g_u. For instance, the set of fixed points of g_u, that is, the set of positive integers n such that $n \mid u_n$, has been studied by Alba González, Luca, Pomerance, and Shparlinski [1], under the mild hypotheses that $(u_n)_{n \geq 0}$ is nondegenerate and that its characteristic polynomial has only simple roots; and by André-Jeannin [2], Luca and Tron [16], Sanna [21], and Somer [26], when $(u_n)_{n \geq 0}$ is a Lucas sequence or the Fibonacci sequence. This topic can be regarded as a generalization of the study of *Fermat pseudoprimes*. Indeed, when the linear recurrence is given by $u_n = a^{n-1} - 1$, for some fixed integer $a \geq 2$, then the composite integers $n \geq 2$ such that $g_u(n) = n$ are exactly the Fermat pseudoprimes to base a [8, Definition 9.9]. Also, it can be considered as the easiest nontrivial instance of the problem of studying when $v_n \mid u_n$ for "many" positive integers n, where $(u_n)_{n \geq 0}$ and $(v_n)_{n \geq 0}$ are fixed integral linear recurrences. This problem is due to Pisot and the major results have been given by van der Poorten [28], Corvaja and Zannier [6, 7]. (See also [20] for a proof of the last remark in [7].) Furthermore, upper bounds for the generalization of g_u defined by $g_{u,v}(n) := \gcd(u_n, v_n)$, for all positive integers n, have been proved by Bugeaud, Corvaja, and Zannier [4], and by Fuchs [10], for large classes of linear recurrences $(u_n)_{n \geq 0}$ and $(v_n)_{n \geq 0}$.

On the other hand, Sanna and Tron [22, 24] have investigated the fiber $g_u^{-1}(y)$, when $(u_n)_{n \geq 0}$ is nondegenerate and $y = 1$, and when $(u_n)_{n \geq 0}$ is the Fibonacci sequence and y is an arbitrary

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positive integer; while the image $g_u(\mathbb{N})$ have been studied by Leonetti and Sanna [14], in the case in which $(u_n)_{n\geq 0}$ is the Fibonacci sequence.

Moreover, fixed points and fibers of g_u have been studied also when $(u_n)_{n\geq 0}$ is an elliptic divisibility sequence [11, 12, 25], the orbit of 0 under a polynomial map [5], and the sequence of central binomial coefficients [17, 23].

In light of these results, which regard the two extremal values 1 and n of $g_u(n)$, a natural question is asking about the average value of g_u and, more generally, its moments.

Question 1.1. *Given a positive integer λ, can we find an asymptotic formula for*
$$\sum_{n\leq x}(g_u(n))^\lambda$$
as $x\to+\infty$?

An even more ambitious problem is estimating the distribution function of g_u.

Question 1.2. *Can we find upper and lower bounds, or even better an asymptotic formula, for the quantity $\#\{n\leq x : g_u(n)>y\}$, holding for a large range of values of x,y ?*

Probably, both Questions 1.1 and 1.2 are easier in the case in which $(u_n)_{n\geq 0}$ is a Lucas sequence. Unfortunately, even in this particular case, we have not been able to answer the questions, which are left as open problems for the interested readers. However, we have succeeded in obtaining a precise asymptotic formula for the moments of the logarithm of g_u. In turn, this result gives as a corollary a partial answer to Question 1.2.

Hereafter, we assume that $(u_n)_{n\geq 0}$ is a nondegenerate Lucas sequence with a_1 and a_2 relatively prime integers. Our main result is the following:

Theorem 1.1. *Fix a positive integer λ and some $\varepsilon>0$. Then, for all sufficiently large x, how large depending on a_1, a_2, λ, and ε, we have*
$$\sum_{n\leq x}(\log g_u(n))^\lambda = M_{u,\lambda}x + E_{u,\lambda}(x),$$
where $M_{u,\lambda}>0$ is a constant depending on a_1, a_2, and λ, while the bound
$$E_{u,\lambda}(x)\ll_{u,\lambda} x^{(1+3\lambda)/(2+3\lambda)+\varepsilon}$$
holds.

Indeed, $M_{u,\lambda}$ can be expressed by an infinite series, but before doing so we need to introduce some notations. For each positive integer m relatively prime with a_2, let $z_u(m)$ be the *rank of appearance* of m in the Lucas sequence $(u_n)_{n\geq 0}$, that is, $z_u(m)$ is the smallest positive integer n such that m divides u_n. It is well known that $z_u(m)$ exists (see, e.g., [18]). Also, put $\ell_u(m):=\operatorname{lcm}(m,z_u(m))$. Furthermore, for each positive integer λ and for each integer $m>1$ with prime factorization $m=q_1^{h_1}\cdots q_s^{h_s}$, where $q_1<\cdots<q_s$ are prime numbers and $h_1,\dots,h_s$ are positive integers, define
$$\rho_\lambda(m):=\lambda!\sum_{\lambda_1+\cdots+\lambda_s=\lambda}\prod_{i=1}^{s}\frac{(h_i^{\lambda_i}-(h_i-1)^{\lambda_i})(\log q_i)^{\lambda_i}}{\lambda_i!},$$
where the sum is extended over all the s-tuples ($s\geq 1$) of positive integers $(\lambda_1,\dots,\lambda_s)$ such that $\lambda_1+\cdots+\lambda_s=\lambda$. In particular, note that if $s>\lambda$ then $\rho_\lambda(m)=0$, since the sum is empty. For the sake of convenience, put also $\rho_\lambda(1):=0$.

Theorem 1.2. *For all positive integers λ, we have*
$$M_{u,\lambda}=\sum_{(m,a_2)=1}\frac{\rho_\lambda(m)}{\ell_u(m)},$$
where m runs over all positive integers relatively prime to a_2.

We conclude this section with the following corollary of Theorem 1.1.

Corollary 1.3. *For each positive integer λ, we have*

$$\#\{n \le x : g_u(n) > y\} \ll_{u,\lambda} \frac{x}{(\log y)^\lambda},$$

for all $x, y > 1$.

Proof. Clearly, we can assume x sufficiently large, depending on λ. Then, thanks to Theorem 1.1, we have

$$\#\{n \le x : g_u(n) > y\} < \sum_{n \le x} \left(\frac{\log g_u(n)}{\log y}\right)^\lambda \ll_{u,\lambda} \frac{x}{(\log y)^\lambda},$$

for all $y > 1$, as claimed. This is an application of Markov's inequality for higher moments. □

Notation. We employ the Landau–Bachmann "Big Oh" and "little oh" notations O and o, as well as the associated Vinogradov symbols $\ll$ and $\gg$, with their usual meanings. Any dependence of the implied constants is explicitly stated or indicated with subscripts. In particular, notations like O_u and $\ll_u$ are shortcuts for O_{a_1,a_2} and $\ll_{a_1,a_2}$, respectively. For any set of positive integers $\mathcal{S}$, we put $\mathcal{S}(x) := \mathcal{S} \cap [1, x]$ for all $x > 0$. Throughout, the letters p and q are reserved for prime numbers. We write $(n_1, \ldots, n_s)$ and $[n_1, \ldots, n_s]$ to denote the greatest common divisor and least common multiple of the integers $n_1, \ldots, n_s$, respectively. The first notation should not be mistaken for the s-tuple notation $(n_1, \ldots, n_s)$, which we also use. We write $\tau(n)$, $\omega(n)$, and $P(n)$, for the number of divisors, the number of prime factors, and the greatest prime factor, of a positive integer n, respectively.

2. Preliminaries

In this section, we collect some preliminary results needed in later proofs. From now on, let $(u_n)_{n\ge0}$ be a nondegenerate Lucas sequence with $(a_1, a_2) = 1$. Also, let $\Delta_u := a_1^2 + 4a_2$ be the discriminant of the characteristic polynomial ψ_u. Note that $\Delta_u \neq 0$ since $(u_n)_{n\ge0}$ is nondegenerate and therefore, in particular, $\alpha_1 \neq \alpha_2$.

We begin with a lemma concerning several elementary properties of the functions z_u, ℓ_u, and g_u, which will be implicitly used later without further mention.

Lemma 2.1. *For all positive integers m, n, j and for all prime numbers $p \nmid a_2$, we have:*

(i) $m \mid u_n$ *if and only if* $(m, a_2) = 1$ *and* $z_u(m) \mid n$.

(ii) $[z_u(m), z_u(n)] = z_u([m, n])$, *whenever* $(mn, a_2) = 1$.

(iii) $z_u(p) \mid p - (-1)^{p-1}\eta_u(p)$, *where*

$$\eta_u(p) := \begin{cases} +1 & \text{if } p \nmid \Delta_u \text{ and } \Delta \equiv x^2 \pmod p \text{ for some } x \in \mathbb{Z}, \\ -1 & \text{if } p \nmid \Delta_u \text{ and } \Delta \not\equiv x^2 \pmod p \text{ for all } x \in \mathbb{Z}, \\ 0 & \text{if } p \mid \Delta_u. \end{cases}$$

(iv) $z_u(p^j) = p^{e_u(p)} z_u(p)$, *where* $e_u(p)$ *is some nonnegative integer less than* j.

(v) $m \mid g_u(n)$ *if and only if* $(m, a_2) = 1$ *and* $\ell_u(m) \mid n$.

(vi) $[\ell_u(m), \ell_u(n)] = \ell_u([m, n])$, *whenever* $(mn, a_2) = 1$.

(vii) $\ell_u(p^j) = p^j z_u(p)$ *if* $p \nmid \Delta_u$, *and* $\ell_u(p^j) = p^j$ *if* $p \mid \Delta_u$.

Proof. (i)–(iv) are well-known properties of the rank of appearance of a Lucas sequence (see, e.g., [18], [19, Chapter 1], or [21, §2]). On the other hand, (v)–(vii) can be easily deduced from the definitions of ℓ_u, g_u, and from (i)–(iv). □

For all $\gamma > 0$, define the following set of prime numbers

$$\mathcal{Q}_\gamma := \{p : p \nmid a_2,\ z_u(p) \le p^\gamma\}.$$

The next lemma belongs to the folklore.

Lemma 2.2. *For all $x, \gamma > 0$, we have* $\#\mathcal{Q}_\gamma(x) \ll_u x^{2\gamma}$.

Proof. It is well known that the generalized Binet's formula

$$u_n = \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}$$

holds for all positive integers n. As a consequence, since α_1/α_2 is not a root of unity, we have $u_n \neq 0$ for all positive integers n. Furthermore, it follows easily that $|u_n| \leq C^n$ for all positive integers n, where $C := |\alpha_1| + |\alpha_2|$. Therefore, from

$$2^{\#\mathcal{Q}_\gamma(x)} \leq \prod_{p \,\in\, \mathcal{Q}_\gamma(x)} p \,\Bigg|\, \prod_{n \,\leq\, x^\gamma} |u_n| \leq C^{\sum_{n \,\leq\, x^\gamma} n} \leq C^{x^{2\gamma}},$$

we obtain that

$$\#\mathcal{Q}_\gamma(x) \leq \frac{\log C}{\log 2} \cdot x^{2\gamma} \ll_u x^{2\gamma},$$

as claimed. □

For each positive integer λ and for all $x, y \geq 0$, define

$$\Phi_\lambda(x, y) := \#\{n \leq x : \omega(n) \leq \lambda,\ P(n) \leq y\}.$$

We need the following easy estimate.

Lemma 2.3. *For each integer $\lambda \geq 1$ and all $x \geq 2$, $y \geq 0$, we have $\Phi_\lambda(x, y) \ll_\lambda (y \log x)^\lambda$.*

Proof. Each of the positive integers n counted by $\Phi_\lambda(x, y)$ can be written as $n = p_1^{a_1} \cdots p_\lambda^{a_\lambda}$, where $p_1, \dots, p_\lambda$ are prime numbers not exceeding y, and $a_1, \dots, a_\lambda$ are nonnegative integers. Clearly, there are at most y choices for each p_i, and at most $1 + \log x/\log 2$ choices for each a_i. Therefore,

$$\Phi_\lambda(x, y) \leq \left(y\left(1 + \frac{\log x}{\log 2}\right)\right)^\lambda \ll_\lambda (y \log x)^\lambda,$$

as claimed. □

The next lemma is an upper bound for the arithmetic function ρ_λ.

Lemma 2.4. *For all positive integers λ and m, we have $\rho_\lambda(m) \leq (\lambda \log m)^\lambda$.*

Proof. For $m = 1$ the claim is trivial, since $\rho_\lambda(m) = 0$ by definition. Hence, suppose $m > 1$ and let $m = q_1^{h_1} \cdots q_s^{h_s}$ be the prime factorization of m, with prime numbers $q_1 < \cdots < q_s$ and positive integers $h_1, \dots, h_s$. Assume also that $s \leq \lambda$, since otherwise $\rho_\lambda(m) = 0$, as we previously observed. By the inequality of (weighted) arithmetic and geometric means, if $\lambda_1, \dots, \lambda_s$ are positive integers such that $\lambda_1 + \cdots + \lambda_s = \lambda$, then

$$\prod_{i=1}^s (h_i \log q_i)^{\lambda_i} \leq \left(\frac{1}{\lambda}\sum_{i=1}^s \lambda_i h_i \log q_i\right)^\lambda \leq \left(\sum_{i=1}^s h_i \log q_i\right)^\lambda = (\log m)^\lambda.$$

Therefore,

$$\begin{aligned}\rho_\lambda(m) &\leq \sum_{\lambda_1 + \cdots + \lambda_s = \lambda} \frac{\lambda!}{\lambda_1! \cdots \lambda_s!} \prod_{i=1}^s (h_i \log q_i)^{\lambda_i} \leq \sum_{\lambda_1 + \cdots + \lambda_s = \lambda} \frac{\lambda!}{\lambda_1! \cdots \lambda_s!} (\log m)^\lambda \\ &\leq (s \log m)^\lambda \leq (\lambda \log m)^\lambda,\end{aligned}$$

as desired. □

Now we give two upper bounds for series over the reciprocals of the $\ell_u(m)$'s. The methods employed are somehow similar to those used to prove the result of [13]. (See also [3] for a wide generalization of that result.)

Lemma 2.5. *We have*

$$\sum_{\substack{(m,a_2)=1 \\ P(m)\geq y}} \frac{1}{\ell_u(m)} \ll_u \frac{1}{y^{1/3-\varepsilon}},$$

for all $\varepsilon \in \,]0,1/4]$ and $y \gg_{u,\varepsilon} 1$.

Proof. Assume y sufficiently large, depending on a_1, a_2, and ε. Let $m > 1$ be an integer relatively prime with a_2, and put $p := P(m)$. Clearly, $\operatorname{lcm}(m, z_u(p))$ is divisible by $\ell_u(p)$. Hence, we can write $\operatorname{lcm}(m, z_u(p)) = \ell_u(p)m'$, where m' is a positive integer such that $P(m') \leq p+1$. Also, if p and $\operatorname{lcm}(m, z_u(p))$ are known, then m can be chosen in at most $\tau(z_u(p))$ ways. Therefore,

$$\sum_{\substack{(m,a_2)=1 \\ P(m)\geq y}} \frac{1}{\ell_u(m)} \leq \sum_{\substack{(m,a_2)=1 \\ P(m)\geq y}} \frac{1}{\operatorname{lcm}(m, z_u(P(m)))} \leq \sum_{p\geq y} \frac{\tau(z_u(p))}{p z_u(p)} \sum_{P(m')\leq p+1} \frac{1}{m'}.$$

On the one hand, by Mertens' formula [27, Chapter I.1, Theorem 11], we have

$$\sum_{P(m')\leq p+1} \frac{1}{m'} = \prod_{q\leq p+1} \left(1 - \frac{1}{q}\right)^{-1} \ll \log p,$$

for all prime numbers p. On the other hand, it is well known [27, Chapter I.5, Corollary 1.1] that $\tau(n) = o(n^\varepsilon)$ as $n \to +\infty$. Hence, $\tau(z_u(p)) \log p \leq p^\varepsilon$ for all sufficiently large prime numbers p, depending on ε. Thus we have found that

$$\sum_{\substack{(m,a_2)=1 \\ P(m)\geq y}} \frac{1}{\ell_u(m)} \ll \sum_{p\geq y} \frac{1}{p^{1-\varepsilon} z_u(p)}. \tag{1}$$

Put $\gamma := 1/3$. On the one hand, by partial summation and by Lemma 2.2, we have

$$\begin{aligned} \sum_{\substack{p\geq y \\ p\in\mathcal{Q}_\gamma}} \frac{1}{p^{1-\varepsilon} z_u(p)} &\leq \sum_{\substack{p\geq y \\ p\in\mathcal{Q}_\gamma}} \frac{1}{p^{1-\varepsilon}} = \frac{\#\mathcal{Q}_\gamma(t)}{t^{1-\varepsilon}}\bigg|_{t=y}^{+\infty} + (1-\varepsilon)\int_y^{+\infty} \frac{\#\mathcal{Q}_\gamma(t)}{t^{2-\varepsilon}}\,\mathrm{d}t \\ &\ll_u \int_y^{+\infty} \frac{\mathrm{d}t}{t^{2-2\gamma-\varepsilon}} \ll \frac{1}{y^{1-2\gamma-\varepsilon}}, \end{aligned} \tag{2}$$

since $1 - 2\gamma - \varepsilon \geq 1/12$. On the other hand, by the definition of $\mathcal{Q}_\gamma$, we have

$$\sum_{\substack{p\geq y \\ p\notin\mathcal{Q}_\gamma}} \frac{1}{p^{1-\varepsilon} z_u(p)} < \sum_{p\geq y} \frac{1}{p^{1+\gamma-\varepsilon}} \ll \frac{1}{y^{\gamma-\varepsilon}}. \tag{3}$$

Hence, putting together (1), (2), and (3), we get the claim. □

Lemma 2.6. *We have*

$$\sum_{\substack{(m,a_2)=1 \\ m>w}} \frac{\rho_\lambda(m)}{\ell_u(m)} \ll_{u,\lambda} \frac{1}{w^{1/(1+3\lambda)-\varepsilon}},$$

for all integers $\lambda \geq 1$, $\varepsilon \in \,]0,1/5]$, and $w \gg_{u,\lambda,\varepsilon} 1$.

Proof. Put $y := w^{3/(1+3\lambda)}$. By Lemma 2.3 and by partial summation, we have

$$\begin{aligned} \sum_{\substack{(m,a_2)=1 \\ \omega(m)\leq\lambda \\ P(m)\leq y \\ m>w}} \frac{1}{\ell_u(m)} &\leq \sum_{\substack{\omega(m)\leq\lambda \\ P(m)\leq y \\ m>w}} \frac{1}{m} = \frac{\Phi_\lambda(t,y)}{t}\bigg|_{t=w}^{+\infty} + \int_w^{+\infty} \frac{\Phi_\lambda(t,y)}{t^2}\,\mathrm{d}t \\ &\ll_\lambda y^\lambda \int_w^{+\infty} \frac{(\log t)^\lambda}{t^2}\,\mathrm{d}t \ll \frac{y^\lambda}{w^{1-\varepsilon}} = \frac{1}{w^{1/(1+3\lambda)-\varepsilon}}, \end{aligned}$$

for all $w \gg_{\lambda,\varepsilon} 1$. This together with Lemma 2.5 implies that

$$S(w) := \sum_{\substack{(m,a_2)=1 \\ \omega(m) \leq \lambda \\ m > w}} \frac{1}{\ell_u(m)} \ll_{u,\lambda} \frac{1}{w^{1/(1+3\lambda)-\varepsilon}}.$$

At this point, by the fact that $\rho_\lambda(m) = 0$ whenever $\omega(m) > \lambda$, by Lemma 2.4, and by partial summation, we obtain

$$\begin{aligned}
\sum_{\substack{(m,a_2)=1 \\ m > w}} \frac{\rho_\lambda(m)}{\ell_u(m)} &\ll_\lambda \sum_{\substack{(m,a_2)=1 \\ \omega(m) \leq \lambda \\ m > w}} \frac{(\log m)^\lambda}{\ell_u(m)} = -\, S(t)(\log t)^\lambda \Big|_{t=w}^{+\infty} + \int_w^{+\infty} S(t) \frac{\lambda(\log t)^{\lambda-1}}{t} \,\mathrm{d}t \\
&\ll_{u,\lambda} \frac{(\log w)^\lambda}{w^{1/(1+3\lambda)-\varepsilon}} + \int_w^{+\infty} \frac{(\log t)^{\lambda-1}}{t^{1+1/(1+3\lambda)-\varepsilon}} \,\mathrm{d}t \ll \frac{1}{w^{1/(1+3\lambda)-\varepsilon/2}},
\end{aligned}$$

as desired.

$\square$

3. Proof of Theorems 1.1 and 1.2

Throughout this section, the letter p, with or without subscript, denotes a prime number not dividing a_2, while the letter j, with or without subscript, denotes a positive integer.

First, we have that

$$\log g_u(n) = \sum_{p^j \,||\, g_u(n)} j \log p = \sum_{p^j \,|\, g_u(n)} \log p = \sum_{\ell_u(p^j) \,|\, n} \log p,$$

for all positive integers n.

Consequently, for any positive integer λ and for all $x > 0$, we have

$$\begin{aligned}
\sum_{n \leq x} (\log g_u(n))^\lambda &= \sum_{n \leq x} \Bigg(\sum_{\ell_u(p^j) \,|\, n} \log p \Bigg)^\lambda \\
&= \sum_{n \leq x} \sum_{\ell_u(p_1^{j_1}) \,|\, n, \,\ldots,\, \ell_u(p_\lambda^{j_\lambda}) \,|\, n} \log p_1 \cdots \log p_\lambda \\
&= \sum_{n \leq x} \sum_{\ell_u([p_1^{j_1}, \ldots, p_\lambda^{j_\lambda}]) \,|\, n} \log p_1 \cdots \log p_\lambda \\
&= \sum_{p_1^{j_1}, \ldots, p_\lambda^{j_\lambda}} \log p_1 \cdots \log p_\lambda \sum_{\substack{n \leq x \\ \ell_u([p_1^{j_1}, \ldots, p_\lambda^{j_\lambda}]) \,|\, n}} 1 \\
&= \sum_{p_1^{j_1}, \ldots, p_\lambda^{j_\lambda}} \log p_1 \cdots \log p_\lambda \left\lfloor \frac{x}{\ell_u([p_1^{j_1}, \ldots, p_\lambda^{j_\lambda}])} \right\rfloor \\
&= \sum_{(m,a_2)=1} \left\lfloor \frac{x}{\ell_u(m)} \right\rfloor \sum_{m = [p_1^{j_1}, \ldots, p_\lambda^{j_\lambda}]} \log p_1 \cdots \log p_\lambda.
\end{aligned} \tag{4}$$

Now we need some combinatorial reasoning. Given an integer $m > 1$ relatively prime to a_2 and with prime factorization $m = q_1^{h_1} \cdots q_s^{h_s}$, where $q_1 < \cdots < q_s$ are prime numbers and $h_1, \ldots, h_s$ are positive integers, we have to consider the λ-tuples $(p_1^{j_1}, \ldots, p_\lambda^{j_\lambda})$ satisfying $m = [p_1^{j_1}, \ldots, p_\lambda^{j_\lambda}]$. Clearly, we must have $\{p_1, \ldots, p_\lambda\} = \{q_1, \ldots, q_s\}$. Fix some positive integers $\lambda_1, \ldots, \lambda_s$ such that $\lambda_1 + \cdots + \lambda_s = \lambda$. Then, the number of λ-tuples $(p_1, \ldots, p_\lambda)$

such that each q_i appears exactly λ_i times among the entries of $(p_1, \dots, p_\lambda)$ is given by the multinomial coefficient

$$\frac{\lambda!}{\lambda_1! \cdots \lambda_s!}.$$

Furthermore, in the λ-tuples $(p_1^{j_1}, \dots, p_\lambda^{j_\lambda})$ the number of possible exponents for the prime powers whose bases are equal to q_i is exactly $h_i^{\lambda_i} - (h_i - 1)^{\lambda_i}$, since all those exponents are not exceeding h_i and at least one of them is equal to h_i. As a consequence,

$$\sum_{m = [p_1^{j_1}, \dots, p_\lambda^{j_\lambda}]} \log p_1 \cdots \log p_\lambda = \sum_{\lambda_1 + \cdots + \lambda_s = \lambda} \frac{\lambda!}{\lambda_1! \cdots \lambda_s!} \prod_{i=1}^{s} (h_i^{\lambda_i} - (h_i - 1)^{\lambda_i})(\log q_i)^{\lambda_i} = \rho_\lambda(m).$$

Hence, recalling (4), we obtain

$$\sum_{n \le x} (\log g_u(n))^\lambda = \sum_{(m, a_2) = 1} \rho_\lambda(m) \left\lfloor \frac{x}{\ell_u(m)} \right\rfloor = M_{u,\lambda} x + E_{u,\lambda}(x),$$

where

$$M_{u,\lambda} := \sum_{(m,a_2)=1} \frac{\rho_\lambda(m)}{\ell_u(m)} \tag{5}$$

and

$$E_{u,\lambda}(x) := - \sum_{(m,a_2)=1} \rho_\lambda(m) \left\{ \frac{x}{\ell_u(m)} \right\}.$$

Note that the series in (5) converges thanks to Lemma 2.6. Thus, it remains to prove the claimed bound for $E_{u,\lambda}(x)$. Fix some $\varepsilon \in \,]0, 1/5]$ and put $w := x^{(1+3\lambda)/(2+3\lambda)}$. By Lemma 2.4 and Lemma 2.6, we have

$$\begin{aligned} |E_{u,\lambda}(x)| = \sum_{(m,a_2)=1} \rho_\lambda(m) \left\{ \frac{x}{\ell_u(m)} \right\} &\ll_\lambda (\log w)^\lambda w + x \sum_{\substack{(m,a_2)=1 \\ m > w}} \frac{\rho_\lambda(m)}{\ell_u(m)} \\ &\ll_{u,\lambda} (\log w)^\lambda w + \frac{x}{w^{1/(1+3\lambda) - \varepsilon}} \ll x^{(1+3\lambda)/(2+3\lambda)+\varepsilon}, \end{aligned}$$

for all sufficiently large x, depending on a_1, a_2, λ, and ε. The proof is complete.

Remark 3.1. A function somehow similar to the last sum of (4) have been studied in [15, Lemma 2].

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