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An Asymptotic Homogenization Approach to the Microstructural Evolution of Heterogeneous Media

Ariel Ramírez-Torres^a, Salvatore Di Stefano^a, Alfio Grillo^a,
Reinaldo Rodríguez-Ramos^b, José Merodio^c, Raimondo Penta^{d,*}

^a*Dipartimento di Scienze Matematiche “G. L. Lagrange”,
Politecnico di Torino, Torino, 10129, Italy*

^b*Departamento de Matemáticas, Facultad de Matemática y Computación,
Universidad de La Habana, La Habana, CP 10400, Cuba*

^c*Departamento de Mecánica de los Medios Continuos y T. Estructuras,
E.T.S. de Caminos, Canales y Puertos,
Universidad Politécnica de Madrid, Madrid, CP 28040, Spain*

^d*School of Mathematics and Statistics, Mathematics and Statistics Building,
University of Glasgow, University Place, Glasgow G12 8QQ, UK*

Abstract

In the present work, we apply the asymptotic homogenization technique to the equations describing the dynamics of a heterogeneous material with evolving micro-structure, thereby obtaining a set of upscaled, effective equations. We consider the case in which the heterogeneous body comprises two hyper-elastic materials and we assume that the evolution of their micro-structure occurs through the development of plastic-like distortions, the latter ones being accounted for by means of the Bilby-Kröner-Lee (BKL) decomposition. The asymptotic homogenization approach is applied simultaneously to the linear momentum balance law of the body and to the evolution law for the plastic-like distortions. Such evolution law models a stress-driven production of inelastic distortions, and stems from phenomenological observations done on cellular aggregates. The whole study is also framed within the limit of small elastic distortions, and provides a robust framework that can be readily generalized to growth and remodeling of nonlinear composites. Finally, we complete our theoretical model by performing numerical simulations.

Keywords: Asymptotic homogenization, heterogeneous media, remodeling, BKL decomposition, two-scale plasticity, nonlinear composites

*Corresponding author

Email address: raimondo.penta@glasgow.ac.uk (Raimondo Penta)

1. Introduction

The study of material growth, remodeling and aging is of great importance in Biomechanics, specially when the tissue, in which these processes occur, features a very complex structure, with different scales of observation and various constituents.

In the literature, the study of heterogeneous materials follows several approaches. In this work we focus on the multi-scale asymptotic homogenization technique [4, 5, 8, 14, 77], which exploits the information available at the smallest scale characterizing the considered medium or phenomenon to obtain an effective description of the medium or phenomenon itself valid at its largest scale. This is achieved by expanding in asymptotic series the equations constituting the mathematical model formulated at the lowest scale. As a result, the coefficients of the effective governing equations encode the information on the other hierarchical levels, as they are to be computed solving microstructural problems at the smaller scales. The multi-scale asymptotic homogenization approach has been successfully applied to investigate various physical systems due to its potentiality in decreasing the complexity of the problem at hand. Biomechanical applications of asymptotic homogenization may be found mainly in nanomedicine [81], biomaterials modeling, such as the bone [58, 65], tissue engineering [24], poroelasticity [63], and active elastomers [64]. Most of the literature concerning applications of the asymptotic homogenization technique focuses on linearized governing equations, as in this case it is possible to obtain, under a number of simplifying assumptions, a full decoupling between scales, which leads to a dramatic reduction in the computational complexity, as also noted for example in [64]. In fact, homogenization in nonlinear mechanics is usually tackled via average field approaches based on representative volume elements or Eshelby-based techniques (see e.g. [41] for a comparison between the latter and asymptotic homogenization), as done for example in [11]. These homogenization approaches are typically well-suited when seeking for suitable bounds for the coefficients of the model, such as the elastic moduli, while asymptotic homogenization can provide a precise characterization of the coefficients under appropriate regularity assumptions (namely, *local periodicity*).

However, to the best of our knowledge and understanding, there exists only a few examples, e.g. [15, 68, 74, 75], dealing with the asymptotic homogenization in the case of media undergoing large deformations. In [68], the static microstructural effects of periodic hyperelastic composites at finite

54 strain are investigated. In [74], the interactions between large deforming solid
55 and fluid media at the microscopic level are described by using the two-scale
56 homogenization technique and the updated Lagrangian formulation. In [15],
57 the effective equations describing the flow, elastic deformation and transport
58 in an active poroelastic medium were obtained. Therein, the authors consid-
59 ered the spatial homogenization of a coupled transport and fluid-structure
60 interaction model, incorporating details of the microscopic system and ad-
61 mitting finite growth and deformation at the pore scale. Some works can be
62 also found dealing with homogenization in the case of elastic perfectly plastic
63 constituents [79, 83].

64 Here we embrace the asymptotic homogenization approach and consider
65 a heterogeneous body composed of two hyperelastic solid constituents sub-
66 jected to the evolution of their internal structure. We refer to this phe-
67 nomenon as to material remodeling and we interpret it with the production
68 of plastic-like distortions. The wording “material remodeling” is used as a
69 synonym of “evolution of the internal structure” of a tissue, and is intended in
70 the sense of [16], who states that “*biological systems can adapt their structure*
71 *[...] to accommodate a changed mechanical load environment*”. In this case,
72 always in the terminology of [16] and [80], one speaks of *epigenetic* adap-
73 tation (or material remodeling). In the framework of the manuscript, such
74 adaptation is assumed to occur through plastic-like distortions that represent
75 processes like the redistribution of the adhesion bonds among the tissue cells.

76 It is worth to recall in which sense the concept of “plastic distortions”,
77 conceived in the context of the Theory of Plasticity (cf. e.g. [50, 55]),
78 and originally referred to non-living materials such as metals or soils, can
79 be imported to describe the structural evolution of biological tissues. To
80 this end, it is important to emphasize that the wording “plastic distortions”
81 is understood as the result of a complex of transformations that conducts
82 to the reorganization of the internal structure of a material, and that —
83 as anticipated in the Introduction— such reorganization is referred to as
84 “remodeling” in the biomechanical context.

85 The ways in which the structural transformations may take place in a
86 given material depend on the structural properties of the material itself. For
87 this reason, the plasticity in metals is markedly different from that occurring
88 in amorphous materials. In the case of metals, indeed, for which the internal
89 structure is granular and characterized by the arrangement of the atomic lat-
90 tice within each grain, plastic distortions are the *macroscopic* manifestation
91 of the formation and evolution of lattice defects. As reported in [55], such

defects can be due, for example, to edge dislocations, wedge disclinations, missing atoms at some lattice sites, or to the presence of atoms in the lattice interstices. To describe how the defects evolve, thereby giving rise to the plastic distortions, one should compare the real lattice at the current instant of time with an ideal lattice, and decompose the overall deformation (i.e., shape change *and* structural transformation) into an elastic and an inelastic contribution [55]. The elastic contribution describes the part of deformation that is recoverable by completely relaxing mechanical stress, whereas the inelastic contribution represents the structural variation, which, in general, is of irreversible nature.

Clearly, metals have structural features markedly different from those of living matter. Still, some of the fundamental mechanisms that trigger the reorganization of their internal structure can be adapted to describe the remodeling of biological tissues.

For instance, in the case of bones, plastic-like phenomena are due to the formation of microcracks that, in turn, favors the gliding of the material along the direction of the opening of the cracks [17, 86]. Lastly, as anticipated above, in the case of biological tissues such as cellular aggregates, the phenomenon analogous to the generation of dislocations is the rearrangement of the adhesion bonds among the cells or the reorganization of the extracellular matrix due to the reorientation of the collagen fibers or their deposition and resorption, as is the case for blood vessels [48]. Also in all these situations, the comparison of the real configuration of the tissue with an “ideal” one, taken as reference, permits the separation of the overall deformation into an elastic part and a structure-related, “plastic-like” part.

Here, taking inspiration from the theory of finite Elastoplasticity [55, 78, 34], we describe the plastic-like distortions by invoking the Bilby-Kröner-Lee (BKL) decomposition of the deformation gradient tensor, and rephrasing it in a scale-dependent fashion. We remark that, at each of the medium’s characteristic scales, a tensor of plastic distortions is introduced, which accounts for the fact that the structural variations of the medium cannot be expressed, in general, in terms of compatible deformations. Our study is conducted within a purely mechanical framework and under the assumption of negligible inertial forces. These hypotheses imply that the model equations reduce to a set comprising a scale-dependent, quasi-static law of balance of linear momentum and an evolution law for the tensor of plastic-like distortions. The latter one is assumed to obey a phenomenological flow rule driven by stress.

The manuscript is organized as follows. In Section 2, we introduce the

130 fundamental notions related to the separation of scales, kinematics, and the
 131 Bilby-Kröner-Lee decomposition for the heterogeneous material. Therein,
 132 the kinematics of the considered medium is discussed, which has to account
 133 for the different length-scales characterizing the heterogeneities and results
 134 into the definition of a scale-dependent deformation gradient tensor. In Sec-
 135 tion 3, the problem to be solved is formulated, and in Section 4, the two-
 136 scales asymptotic homogenization technique is applied to obtain the local
 137 and the homogenized sub-problems. In Section 5, we prescribe a constitutive
 138 equation for the response of the material and, independently, an evolution
 139 equation for the tensor of plastic-like distortions. In that respect, the local
 140 and homogenized problems derived in Section 4 are formulated by consid-
 141 ering the De Saint-Venant strain energy density and we demonstrate the
 142 relationship between our new model and the classical ones. In Section 6 we
 143 outline a computational scheme to solve the resulting up-scaled model and,
 144 in Section 7, we address the numerical results of our simulations. Finally,
 145 some concluding remarks on the ongoing work, along with suggestions for
 146 future research, are summarized in Section 8. We highlight the novelty of
 147 our approach, and we explain how it may contribute to the understanding of
 148 the mechanics of heterogeneous media with evolving micro-structure.

149 2. Theoretical background

150 2.1. Separation of scales

151 The homogenization of a highly heterogeneous medium is only possible
 152 when the characteristic length of the the local structure (ℓ_0) and the char-
 153 acteristic length of the material, or of the phenomenon, of interest (L_0) are
 154 well separated. This condition of separation of scales can be expressed as

$$\varepsilon_0 := \frac{\ell_0}{L_0} \ll 1. \quad (1)$$

155 There may exist more than two coexisting scales and, if they are well sepa-
 156 rated from each other, a homogenization approach is possible. In this case,
 157 we then move from the smallest scale to the largest one by homogenization
 158 [1, 8, 51, 82, 69].

159 Condition (1) is taken as a base assumption for all homogenization pro-
 160 cesses. The two characteristic length scales ℓ_0 and L_0 introduce two di-
 161 mensionless spatial variables in the reference configuration, $\tilde{Y} = X/\ell_0$ and
 162 $\tilde{X} = X/L_0$, where X is said to be the *physical spatial variable*, whereas \tilde{Y}

163 and \tilde{X} represent the microscopic and the macroscopic non-dimensional spa-
 164 tial variables, respectively. By using (1), \tilde{Y} and \tilde{X} can be related through
 165 the expression

$$\tilde{Y} = \varepsilon_0^{-1} \tilde{X}. \quad (2)$$

166 Given a field Φ defined over the region of interest of the heterogeneous
 167 medium, the separation of scales allows to rephrase the space dependence of
 168 Φ as $\Phi(X) = \check{\Phi}(\tilde{X}(X), \tilde{Y}(X))$, and the spatial derivative of Φ takes thus the
 169 form

$$\text{Grad}_X \Phi = L_0^{-1} (\text{Grad}_{\tilde{X}} \check{\Phi} + \varepsilon_0^{-1} \text{Grad}_{\tilde{Y}} \check{\Phi}). \quad (3)$$

170 By following this approach, all equations should be written in non-dimensional
 171 form. In the literature, the switch to the auxiliary variables \tilde{X} and \tilde{Y} is often
 172 omitted. However, as shown for example in [4], both paths are equivalent,
 173 provided that the dimensional formulation of the problem consistently ac-
 174 counts for any asymptotic behavior of the involved fields and parameters
 175 (see e.g. [62] and the discussion therein concerning problems where such a
 176 behavior is actually deduced via a non-dimensional analysis). By exploiting
 177 this result, in what follows, our analysis is carried out directly in a system of
 178 physical variables X and Y . Moreover, by adopting the approach usually fol-
 179 lowed in asymptotic multiscale analysis, we assume that each field and each
 180 material property characterizing the considered medium are functions of both
 181 X and Y , with $Y = \varepsilon_0^{-1} X$. Roughly speaking, the dependence on X captures
 182 the behavior of a given physical quantity over the largest length-scale, while
 183 the dependence on Y captures the behavior over the smallest one. We express
 184 this property by introducing the notation $\Phi^\varepsilon(X) = \Phi(X, \varepsilon_0^{-1} X) = \Phi(X, Y)$
 185 [66]. Moreover, for a fixed X , we assume that $\Phi(X, Y)$ is periodic with
 186 respect to Y .

187 In the classical theory of two-scale asymptotic homogenization [5, 8, 14],
 188 the small scaling dimensionless parameter ε_0 is constant. However, in the
 189 case of a composite material subjected to deformation and change of internal
 190 structure (as is the case, for instance, when plastic-like distortions occur),
 191 the characteristic macroscopic and microscopic lengths, which refer to the
 192 body and to its heterogeneities, respectively, depend on X and t , and should
 193 thus be denoted by $\ell(X, t)$ and $L(X, t)$. Therefore, the corresponding scaling
 194 parameter, obtained as the ratio $\varepsilon(X, t) = \ell(X, t)/L(X, t)$, is also a func-
 195 tion of X and t , which need not be equal to ε_0 in general. This variability

196 notwithstanding, if $\varepsilon(X, t)$ is bounded from above for all X and for all t ,
 197 and if the upper bound is much smaller than unity, we can indicate such
 198 upper bound with ε , and use this constant as a scaling parameter for our
 199 asymptotic analysis.

200 2.2. Kinematics

201 Let us denote by \mathcal{B}^ε a continuum body with periodic microstructure, and
 202 by \mathcal{S} the three-dimensional Euclidean space. Furthermore, we denote by
 203 $\mathcal{B}_0^\varepsilon$ the reference, unloaded configuration of \mathcal{B}^ε , in which the body's periodic
 204 micro-structure is reproduced. Now, let us assume that $\chi^\varepsilon : \mathcal{B}_0^\varepsilon \times \mathcal{T} \rightarrow \mathcal{S}$
 205 describes the motion of the heterogeneous body, where $\mathcal{T} = [t_0, t_f[$ is an
 206 interval of time. Then, the region occupied by the body at time $t \in \mathcal{T}$
 207 is $\mathcal{B}_t^\varepsilon := \chi^\varepsilon(\mathcal{B}_0^\varepsilon, t) \subset \mathcal{S}$ and is said to be its current configuration. Each
 208 point $x \in \mathcal{B}_t^\varepsilon$ is such that $x = \chi^\varepsilon(X, t)$, with $X \in \mathcal{B}_0^\varepsilon$ being the point's
 209 reference placement. The deformation from $\mathcal{B}_0^\varepsilon$ to $\mathcal{B}_t^\varepsilon$ is characterized by the
 210 deformation gradient, $\mathbf{F}^\varepsilon(X, t)$, which is defined as $\mathbf{F}^\varepsilon(X, t) = T\chi^\varepsilon(X, t)$
 211 [53], with $T\chi^\varepsilon$ being the tangent map of the motion χ^ε , defined from the
 212 tangent space $T_X\mathcal{B}_0^\varepsilon$ into $T_x\mathcal{S}$. In the sequel, however, since our focus is on
 213 Homogenization Theory, we find it convenient to use the less formal definition

$$\mathbf{F}^\varepsilon = \mathbf{I} + \text{Grad}\mathbf{u}^\varepsilon, \quad (4)$$

214 where \mathbf{I} is the second-order identity tensor and $\text{Grad}\mathbf{u}^\varepsilon$ denotes the gradient
 215 operator of the displacement \mathbf{u}^ε . The condition $J^\varepsilon = \det\mathbf{F}^\varepsilon > 0$ must be
 216 satisfied in order for χ^ε to be admissible. The symmetric, positive definite,
 217 second-order tensor $\mathbf{C}^\varepsilon = (\mathbf{F}^\varepsilon)^T \mathbf{F}^\varepsilon$ is the right Cauchy-Green deformation
 218 tensor induced by \mathbf{F}^ε . For our purposes, we partition $\mathcal{B}_0^\varepsilon$ into two sub-
 219 domains \mathcal{B}_0^1 and \mathcal{B}_0^2 , such that $\bar{\mathcal{B}}_0^1 \cup \bar{\mathcal{B}}_0^2 = \bar{\mathcal{B}}_0^\varepsilon$ and $\bar{\mathcal{B}}_0^1 \cap \mathcal{B}_0^2 = \mathcal{B}_0^1 \cap \bar{\mathcal{B}}_0^2 = \emptyset$,
 220 where the bar over a set denotes its closure. We let Γ_0^ε stand for the interface
 221 between \mathcal{B}_0^1 and \mathcal{B}_0^2 . Particularly, \mathcal{B}_0^1 denotes the matrix of \mathcal{B}^ε (also referred
 222 to as *host phase*) and \mathcal{B}_0^2 a collection of N disjoint inclusions. The periodic
 223 cell in the reference configuration is denoted by \mathcal{Y}_0 . The portion of matrix
 224 contained in \mathcal{Y}_0 is indicated by \mathcal{Y}_0^1 , while \mathcal{Y}_0^2 is the inclusion in \mathcal{Y}_0 . In each
 225 cell, \mathcal{Y}_0^1 and \mathcal{Y}_0^2 are such that $\bar{\mathcal{Y}}_0^1 \cup \bar{\mathcal{Y}}_0^2 = \bar{\mathcal{Y}}_0$ and $\bar{\mathcal{Y}}_0^1 \cap \mathcal{Y}_0^2 = \mathcal{Y}_0^1 \cap \bar{\mathcal{Y}}_0^2 = \emptyset$. The
 226 symbol Γ_0 indicates the interface between \mathcal{Y}_0^1 and \mathcal{Y}_0^2 . In the present work, we
 227 assume that the periodicity of the body's micro-structure is preserved even
 228 though the body evolves by both changing its shape and varying its internal
 229 structure. In general, however, this is not the case. Clearly, our hypothesis is

unrealistic in several circumstances, but it might be helpful to describe those situations in which the breaking of the material symmetries occurs at a scale different from those of interest, as is the case, for instance, when the plastic distortions occur in a tissue with evolving material properties [49], that are not directly related to the change of the tissue's micro-geometry. On the other hand, for nonperiodic media, the macro model is still valid when one assumes local boundedness. In that case, the coefficients are simply to be retrieved experimentally, as the “cell” problem is no longer to be computed on the cell but on the whole micro domain, which would be more complex than the original problem.

Moreover, we define $\chi_1^\varepsilon := \chi^\varepsilon|_{\mathcal{B}_0^1} : \mathcal{B}_0^1 \times \mathcal{T} \rightarrow \mathcal{S}$ such that $\mathcal{B}_t^1 := \chi_1^\varepsilon(\mathcal{B}_0^1, t)$ denotes the host phase at the current configuration and $\chi_2^\varepsilon := \chi^\varepsilon|_{\mathcal{B}_0^2} : \mathcal{B}_0^2 \times \mathcal{T} \rightarrow \mathcal{S}$, with $\mathcal{B}_t^2 := \chi_2^\varepsilon(\mathcal{B}_0^2, t)$ denoting the inclusions. Specifically, we enforce the condition $\bar{\mathcal{B}}_t^1 \cup \bar{\mathcal{B}}_t^2 = \bar{\mathcal{B}}_t^\varepsilon$, with $\bar{\mathcal{B}}_t^1 \cap \bar{\mathcal{B}}_t^2 = \mathcal{B}_t^1 \cap \bar{\mathcal{B}}_t^2 = \emptyset$, and denote by Γ_t^ε the interface between \mathcal{B}_t^1 and \mathcal{B}_t^2 . In addition, we let \mathcal{Y}_t indicate the periodic cell in the current configuration, with $\bar{\mathcal{Y}}_t^1 \cup \bar{\mathcal{Y}}_t^2 = \bar{\mathcal{Y}}_t$, $\bar{\mathcal{Y}}_t^1 \cap \bar{\mathcal{Y}}_t^2 = \mathcal{Y}_t^1 \cap \bar{\mathcal{Y}}_t^2 = \emptyset$, and with Γ_t being the interface between \mathcal{Y}_t^1 and \mathcal{Y}_t^2 (see Fig. 1). We emphasize that \mathcal{Y}_t^1 is the portion of matrix and \mathcal{Y}_t^2 is the inclusion in \mathcal{Y}_t . We note that inside a single cell it can be present also a collection of inclusions and, in such a case, we should consider multiple interface conditions [60].

2.3. Multiplicative decomposition

When the body \mathcal{B}^ε is subjected to a system of external loads, the change of its shape could be accompanied by a rearrangement of its intrinsic structure. This process is generally inelastic and may not be described just in terms of deformation. Moreover, when mechanical agencies are removed, the body is generally unable to recover the unloaded configuration $\mathcal{B}_0^\varepsilon$, and may occupy a configuration characterized by the presence of residual stresses and strains. To bring the body into a fully relaxed state, an ideal tearing process has to be introduced [55]. More specifically, for each material point $X \in \mathcal{B}^\varepsilon$, we individuate a small neighborhood of X , referred to as *body element*, we ideally cut it out from the body, and we let it relax until it reaches a stress-free state. Such state is the *ground state* of the relaxed body element and is called *natural state*. This concept, originally used in the theory of elastoplasticity (see [50, 55]), has been used in the biomechanical context by various authors like, for instance, [23, 76, 30, 26, 27, 42, 44, 18, 55, 34, 19]. Before going further with the use of the BKL decomposition, we mention that, in the literature, there exist other approaches to the issue of residual stresses in

267 biological tissues, which call neither for the multiplicative decomposition of
 268 the deformation gradient tensor, nor for the introduction of an “intermediate,
 269 relaxed configuration”. One recent publication adhering to this philosophy
 270 is for example [13], in which the authors warn that the intermediate config-
 271 uration may “*not exist in physical reality and must be postulated a priori*”.
 272 Although we are aware of the fact that a framework based on the BKL-
 273 decomposition may lead in some cases to assume unrealistic results —as any
 274 other framework would do—, we prefer here to adhere to the BKL approach
 275 for consistency with previous works of ours.

276 By performing the ideal process described above for all the body points, a
 277 collection of relaxed body pieces is obtained, in which each piece finds itself
 278 in its natural state. We denote such collection by $\mathcal{B}_\nu^\varepsilon$. In the language of
 279 continuum mechanics, these physical considerations lead to the BKL decom-
 280 position [55, 34]. Although summarizing these theoretical results is useful for
 281 sake of completeness, the consequences of the BKL decomposition are well-
 282 known, as it is one the pillars of Elastoplasticity. For this reason, we do not
 283 fuss over its theoretical justification, and we highlight, rather, the fact that
 284 one of the purposes of this work is to investigate the use of a scale-dependent
 285 BKL decomposition. In detail, by referring to Figure 1, we invoke a multi-
 286 plicative decomposition of the deformation gradient \mathbf{F}^ε that is parameterized
 287 by the scaling ratio ε , i.e.,

$$\mathbf{F}^\varepsilon = \mathbf{F}_e^\varepsilon \mathbf{F}_p^\varepsilon, \quad (5)$$

288 where the tensors \mathbf{F}_e^ε and \mathbf{F}_p^ε describe, respectively, the elastic and the in-
 289 elastic distortions contributing to \mathbf{F}^ε . Along with (5), we also define the
 290 determinants $J_e^\varepsilon = \det(\mathbf{F}_e^\varepsilon)$ and $J_p^\varepsilon = \det(\mathbf{F}_p^\varepsilon)$, which are both strictly pos-
 291 itive. Consistently with the notation introduced above, it holds true that
 292 $\mathbf{F}_e^\varepsilon(X) = \mathbf{F}_e(X, Y)$, $\mathbf{F}_p^\varepsilon(X) = \mathbf{F}_p(X, Y)$, and $\mathbf{F}^\varepsilon(X) = \mathbf{F}(X, Y)$ as well as
 293 $J_e^\varepsilon(X) = J_e(X, Y)$ and $J_p^\varepsilon(X) = J_p(X, Y)$.

294 In this work, we focus on remodeling, i.e., plastic-like distortions that
 295 occur to modify the internal structure of \mathcal{B}^ε . Although this phenomenon is
 296 not visible, it could lead to the alteration of the mechanical properties of \mathcal{B}^ε .

297 3. Formulation of the problem

298 We consider a composite material comprising two solid constituents, whose
 299 point-wise constitutive response is hyperelastic. Therefore, to model its me-
 300 chanical behavior, we introduce the scale-dependent strain energy function,

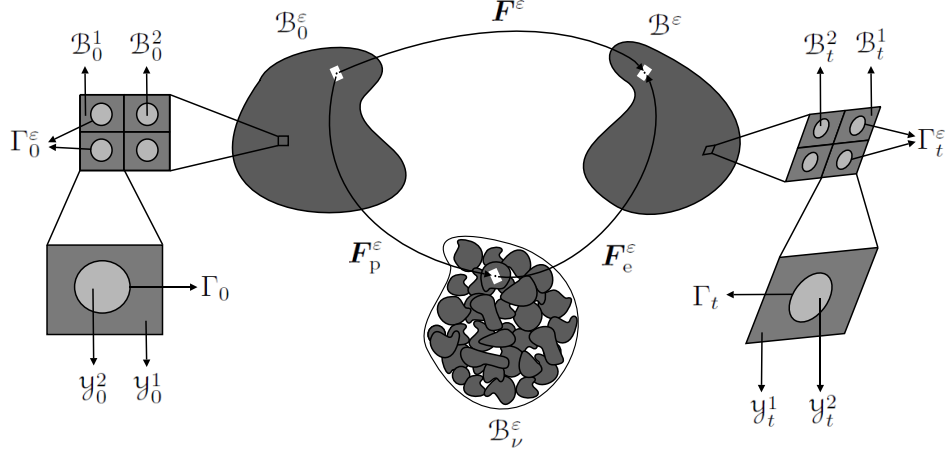


Figure 1: Schematic of a composite material with periodic internal micro-structure and subjected to inelastic remodeling distortions. From left to right: Magnification of an excerpt of material and description of its nested, periodic micro-structure. Change of shape of the body from the reference to the current configuration, and definition of the conglomerate of relaxed body pieces, each in its natural state. Magnification of an excerpt of material, taken from the body's current configuration, and description of its deformed, and remodeled, micro-structure.

301 defined per unit volume of the natural state,

$$\check{\psi}_\nu(X, t) = \psi_\nu^\epsilon(\mathbf{F}_e^\epsilon(X, t), i^\epsilon(X, t)) = \psi_\nu(\mathbf{F}_e(X, Y, t), i(X, Y, t)), \quad (6)$$

302 where i is defined by the expression $i(X, Y, t) = (X, Y)$, i.e., i extracts the
 303 spatial pair (X, Y) from the triplet (X, Y, t) . From (6) we can derive the first
 304 Piola-Kirchoff stress tensor,

$$\mathbf{T}^\epsilon = J_p^\epsilon \frac{\partial \psi_\nu^\epsilon}{\partial \mathbf{F}_e^\epsilon} (\mathbf{F}_p^\epsilon)^{-T}, \quad (7)$$

305 where $J_p^\epsilon = \det \mathbf{F}_p^\epsilon$. In particular, if we neglect body forces and inertial terms,
 306 the balance of linear momentum reads,

$$\begin{cases} \text{Div } \mathbf{T}^\epsilon = \mathbf{0}, & \text{in } \mathcal{B}_0^\epsilon \setminus \Gamma_0^\epsilon \times \mathcal{T}, \\ \mathbf{T}^\epsilon \cdot \mathbf{N} = \bar{\mathbf{T}}, & \text{on } \partial_T \mathcal{B}_0^\epsilon \times \mathcal{T}, \\ \mathbf{u}^\epsilon = \bar{\mathbf{u}}, & \text{on } \partial_u \mathcal{B}_0^\epsilon \times \mathcal{T}, \end{cases} \quad (8)$$

307 where $\bar{\mathbf{T}}$ and $\bar{\mathbf{u}}$ are, respectively, the prescribed traction and displacement
 308 on the boundary $\partial \mathcal{B}_0^\epsilon = \partial_T \mathcal{B}_0^\epsilon \cup \partial_u \mathcal{B}_0^\epsilon$ with $\overline{\partial_T \mathcal{B}_0^\epsilon} \cap \partial_u \mathcal{B}_0^\epsilon = \partial_T \mathcal{B}_0^\epsilon \cap \overline{\partial_u \mathcal{B}_0^\epsilon} = \emptyset$

309 and \mathbf{N} is the outward unit vector normal to the surface $\partial\mathcal{B}_0^\varepsilon$. Continuity
 310 conditions for displacement and traction are imposed,

$$\llbracket \mathbf{u}^\varepsilon \rrbracket = \mathbf{0} \quad \text{and} \quad \llbracket \mathbf{T}^\varepsilon \cdot \mathbf{N}_\gamma \rrbracket = \mathbf{0}, \quad \text{on } \Gamma_0 \times \mathcal{T}, \quad (9)$$

311 where $\llbracket \bullet \rrbracket$ denotes the jump across the interface between the two constituents
 312 and \mathbf{N}_γ defines the unit outward normal to Γ_0 . Moreover, problem (8)
 313 must be supplemented with an appropriate evolution law for \mathbf{F}_p^ε . It is worth
 314 mentioning that the homogenization process can be performed regardless of
 315 the particular choice of *external* boundary conditions (Dirichlet-Neumann
 316 in this case). This means that the formulation presented in this work is
 317 potentially applicable also to other external boundary conditions, such as
 318 e.g. those of Robin-type. This is due to the fact that, as pointed out in [69],
 319 also in the present study the homogenization is applied in regions sufficiently
 320 far away from the outer boundary of the considered medium. For problems
 321 in which it is necessary to homogenize also close to the outer heterogeneous
 322 boundaries, we refer to [8, 57, 46].

323 **Remark 1.** *In the present work, we impose conditions (9) for displacements*
 324 *and tractions just to exemplify the homogenization technique applied to het-*
 325 *erogeneous media with evolving microstructure. In other words, we assume*
 326 *that the contact interface between the constituents is ideal. This means that*
 327 *the displacements are congruent, and thus continuous, and that linear mo-*
 328 *mentum is conserved across the interface, which in our context implies the*
 329 *continuity of the tractions. However, the hypothesis of the ideal interface can*
 330 *be relaxed in some biological situations. For instance, in cancerous tissues,*
 331 *there exist cross-links between normal and malignant cells, whose density and*
 332 *strength determine a spring constant that relates the normal stresses on each*
 333 *cell surface, thereby making it non-ideal [47, 37]. Another example of non-*
 334 *ideal interface is the periodontal ligament, which represents the thin layer*
 335 *between the cementum of the tooth to the adjacent alveolar bone [28]. In the*
 336 *context of composite materials, when non-ideal interfaces are accounted for,*
 337 *the interface conditions are suitably reformulated [38, 39, 7, 6]. In particular,*
 338 *the asymptotic homogenization technique has been applied for linear elastic*
 339 *periodic fiber reinforced composites with imperfect contact between matrix and*
 340 *fibers (see e.g. [36]).*

341 4. Asymptotic homogenization of the balance of linear momentum

342 A formal two-scale asymptotic expansion is performed for the displace-
343 ment \mathbf{u}^ε , which thus reads

$$\mathbf{u}^\varepsilon(X, t) = \mathbf{u}^{(0)}(X, t) + \sum_{k=1}^{+\infty} \mathbf{u}^{(k)}(X, Y, t) \varepsilon^k, \quad (10)$$

344 where, for all $k \geq 1$, $\mathbf{u}^{(k)}$ is periodic with respect to Y . Following [68] we
345 consider the leading order term of the expansion (10) to be independent
346 of the fast variable Y . From formula (4), the expansion (10), and taking
347 into account the property of scale separation, it follows that the deformation
348 gradient tensor can be written as

$$\mathbf{F}^\varepsilon(X, t) = \sum_{k=0}^{+\infty} \mathbf{F}^{(k)}(X, Y, t) \varepsilon^k, \quad (11)$$

349 with the notation

$$\mathbf{F}^{(0)} := \mathbf{I} + \text{Grad}_X \mathbf{u}^{(0)} + \text{Grad}_Y \mathbf{u}^{(1)}, \quad (12a)$$

$$\mathbf{F}^{(k)} := \text{Grad}_X \mathbf{u}^{(k)} + \text{Grad}_Y \mathbf{u}^{(k+1)}, \quad \forall k \geq 1, \quad (12b)$$

350 where Grad_X and Grad_Y are the gradient operators with respect to X and Y ,
351 respectively. Now, the following two-scale asymptotic expansion is proposed
352 for the first Piola-Kirchhoff stress tensor \mathbf{T}^ε ,

$$\mathbf{T}^\varepsilon(X, t) = \sum_{k=0}^{+\infty} \mathbf{T}^{(k)}(X, Y, t) \varepsilon^k, \quad (13)$$

353 where the fields $\mathbf{T}^{(k)}$ are periodic with respect to Y . By substituting the
354 power series representation (13) into (8), using the scale separation condition,
355 and multiplying the result by ε , the following multi-scale system is obtained

$$\text{Div} \mathbf{T}^\varepsilon = \sum_{k=0}^{+\infty} \mathfrak{D}^{(k)} \varepsilon^k = \mathbf{0}, \quad (14)$$

356 with

$$\mathfrak{D}^{(0)} := \text{Div}_Y \mathbf{T}^{(0)}, \quad (15a)$$

$$\mathfrak{D}^{(k)} := \text{Div}_X \mathbf{T}^{(k-1)} + \text{Div}_Y \mathbf{T}^{(k)}, \quad \forall k \geq 1. \quad (15b)$$

357 We require that the equilibrium equation (14) is satisfied at every ε , which
 358 amounts to impose the conditions

$$\text{Div}_Y \mathbf{T}^{(0)} = \mathbf{0}, \quad (16a)$$

$$\text{Div}_X \mathbf{T}^{(k-1)} + \text{Div}_Y \mathbf{T}^{(k)} = \mathbf{0}, \quad \forall k \geq 1. \quad (16b)$$

359 At this point we introduce the average operator over the microscopic cell, i.e.

$$\langle \bullet \rangle = \frac{1}{|\mathcal{Y}_t|} \int_{\mathcal{Y}_t} \bullet \, dY, \quad (17)$$

360 where $|\mathcal{Y}_t|$ represents the volume of the periodic cell \mathcal{Y}_t at time t . Indeed,
 361 because of the deformations and distortions to which the microscopic, refer-
 362 ence periodic cell is subjected, \mathcal{Y}_t is different at every time instant. Averaging
 363 (16b) over the microscopic cell yields, for $k = 1$,

$$\langle \text{Div}_X \mathbf{T}^{(0)} \rangle + \frac{1}{|\mathcal{Y}_t|} \int_{\partial \mathcal{Y}_t} \mathbf{T}^{(1)} \cdot \mathbf{N} \, dY = \mathbf{0}, \quad (18)$$

364 where, on the left-hand side, we have applied the divergence theorem. Since
 365 the contributions on the periodic cell boundary $\partial \mathcal{Y}_t$ cancel due to the Y -
 366 periodicity, the integral over \mathcal{Y}_t is equal to zero, and (18) becomes

$$\langle \text{Div}_X \mathbf{T}^{(0)} \rangle = \mathbf{0}. \quad (19)$$

367 Here, we restrict our analysis to the particular case in which the periodic
 368 cell can be uniquely chosen independently of X , which implies that the in-
 369 tegration over \mathcal{Y}_t and the computation of the divergence commute. This
 370 assumption is also referred to as *macroscopic uniformity*, see also [9, 40, 59]
 371 for examples dealing with non-macroscopically uniform media in the context
 372 of poroelasticity and diffusion. Therefore, Equation (19) can be recast as

$$\text{Div}_X \langle \mathbf{T}^{(0)} \rangle = \mathbf{0}. \quad (20)$$

373 Equations (16a) and (20) represent, respectively, the local and the homoge-
 374 nized equation associated with the original one, stated in (8). Both equations
 375 still need to be supplemented with the corresponding interface, boundary, and

376 initial conditions. Note that, although both problems feature no time deriva-
 377 tive, initial conditions are required because $\mathbf{T}^{(0)}$ depends on the variable $\mathbf{F}_p^{(0)}$,
 378 which satisfies an evolution equation in time.

379 We remark that the leading term $\mathbf{T}^{(0)} = \mathbf{T}^{(0)}(X, Y, t)$ of the multi-scale
 380 expansion (13) is the unknown, both in (16a) and in (20). To identify $\mathbf{T}^{(0)}$,
 381 we propose here to expand \mathbf{F}_p^ε and ψ_ν^ε as

$$\mathbf{F}_p^\varepsilon(X, t) = \sum_{k=0}^{+\infty} \mathbf{F}_p^{(k)}(X, Y, t) \varepsilon^k, \quad (21a)$$

$$\psi_\nu^\varepsilon(X, t) = \sum_{k=0}^{+\infty} \psi_\nu^{(k)}(\mathbf{F}_e(X, Y, t), X, Y) \varepsilon^k, \quad (21b)$$

382 where $\mathbf{F}_p^{(k)}$ and $\psi_\nu^{(k)}$ are periodic in Y . By using (5), (11) and (21a), we can
 383 deduce a series expansion for \mathbf{F}_e^ε in powers of ε , where the leading order term
 384 $\mathbf{F}_e^{(0)}$ is given by

$$\mathbf{F}_e^{(0)} = \mathbf{F}^{(0)}(\mathbf{F}_p^{(0)})^{-1}. \quad (22)$$

385 Following [15] and [68], $\mathbf{T}^{(0)}$ is therefore supplied constitutively as

$$\mathbf{T}^{(0)} = J_p^{(0)} \frac{\partial \psi_\nu^{(0)}}{\partial \mathbf{F}_e^{(0)}} (\mathbf{F}_p^{(0)})^{-T}, \quad (23)$$

386 with $\psi_\nu^{(0)} = \psi_\nu^{(0)}(\mathbf{F}_e^{(0)}(X, Y, t), X, Y)$ and $J_p^{(0)} = \det \mathbf{F}_p^{(0)}$. To obtain the
 387 *cell problem*, equation (14) must be supplemented with the corresponding
 388 interface conditions. This is done by substituting the asymptotic expansions
 389 of \mathbf{u}^ε and of \mathbf{T}^ε into the interface conditions $\llbracket \mathbf{u}^\varepsilon \rrbracket = \mathbf{0}$ and $\llbracket \mathbf{T}^\varepsilon \cdot \mathbf{N}_y \rrbracket = \mathbf{0}$.
 390 Both conditions are satisfied at any order of ε . At the order ε^0 , we simply
 391 obtain $\llbracket \mathbf{T}^{(0)} \cdot \mathbf{N}_y \rrbracket = \mathbf{0}$ for the stresses, and that the condition $\llbracket \mathbf{u}^{(0)} \rrbracket = \mathbf{0}$ is
 392 trivially satisfied, because $\mathbf{u}^{(0)}$ depends solely on X and t . Thus, the interface
 393 condition on the displacements is written only for $\mathbf{u}^{(1)}$ and reads, $\llbracket \mathbf{u}^{(1)} \rrbracket = \mathbf{0}$.
 394 By summarizing these results, the cell problem at zero order of the epsilon
 395 parameter can be stated as

$$\begin{cases} \operatorname{Div}_Y \mathbf{T}^{(0)} = \mathbf{0}, & \text{in } \mathcal{Y}_0 \setminus \Gamma_0 \times \mathcal{T}, \\ \llbracket \mathbf{u}^{(1)} \rrbracket = \mathbf{0}, & \text{on } \Gamma_0 \times \mathcal{T}, \\ \llbracket \mathbf{T}^{(0)} \cdot \mathbf{N}_y \rrbracket = \mathbf{0}, & \text{on } \Gamma_0 \times \mathcal{T}. \end{cases} \quad (24)$$

396 Together with the cell problem, we also need to formulate the macro-scopic
 397 homogenized problem. To this end, we take equation (20) and complete it
 398 with a set of boundary conditions. This is done by substituting the asymp-
 399 totic expansions of \mathbf{T}^ε and \mathbf{u}^ε into the boundary conditions $\mathbf{T}^\varepsilon \cdot \mathbf{N} = \bar{\mathbf{T}}$
 400 and $\mathbf{u}^\varepsilon = \bar{\mathbf{u}}$, respectively. Thus, equating the coefficients at order ε^0 , and
 401 averaging the results over the unit cell, we find the *homogenized problem*,

$$\begin{cases} \operatorname{Div}_X \langle \mathbf{T}^{(0)} \rangle = \mathbf{0}, & \text{in } \mathcal{B}_h \times \mathcal{T}, \\ \langle \mathbf{T}^{(0)} \rangle \cdot \mathbf{N} = \bar{\mathbf{T}}, & \text{on } \partial_T \mathcal{B}_h \times \mathcal{T}, \\ \mathbf{u}^{(0)} = \bar{\mathbf{u}}, & \text{on } \partial_u \mathcal{B}_h \times \mathcal{T}, \end{cases} \quad (25)$$

402 where \mathcal{B}_h denotes the homogeneous macro-scale domain in which the homog-
 403 enized equations are defined.

404 The problem (25) has to be solved along with a homogenized evolution
 405 equation for $\mathbf{F}_p^{(0)}$ and the initial condition associated with it. In addition, we
 406 remark that, according to (25), the boundary tractions acting on $\partial_T \mathcal{B}_h$ are
 407 balanced *only* by the normal component of the average of the leading order
 408 stress, $\mathbf{T}^{(0)}$, and *only* the leading order displacement, $\mathbf{u}^{(0)}$, has to be equal
 409 to the displacement $\bar{\mathbf{u}}$, imposed on $\partial_u \mathcal{B}_h$.

410 **Remark 2.** *In the medical scientific literature, there exist studies that iden-*
 411 *tify the existence of anatomical boundary layers interposed between the brain*
 412 *surface and tumors (see e.g. [72]). Here we do not address boundary layer*
 413 *phenomena, which are usually neglected in the asymptotic homogenization*
 414 *literature. The homogenization process described in this work is fine for re-*
 415 *gions far enough away from the boundary so that its effect is not felt because,*
 416 *close to the boundaries, the material will not behave as an effective material*
 417 *with homogenized coefficients. To properly account for boundary effects, the*
 418 *so-called boundary-layer technique could be used [8, 57].*

419 5. Constitutive framework and evolution law

420 In this section, we prescribe a constitutive equation for the response of the
 421 material and, independently, an evolution equation for the tensor of plastic-
 422 like distortions.

423 5.1. Constitutive law

424 In the following, we formulate the local and homogenized problems for a
 425 specific constitutive law. In general, this process can be rather cumbersome

426 for complicated strain energy densities, and it becomes even more involved
 427 when plastic-like distortions are accounted for. To reduce complexity, we
 428 choose a very simple constitutive law for ψ_ν^ε , such as the De Saint-Venant
 429 strain energy density,

$$\psi_\nu^\varepsilon = \frac{1}{2} \mathbf{E}_e^\varepsilon : \mathcal{C}^\varepsilon : \mathbf{E}_e^\varepsilon, \quad (26)$$

430 where $\mathbf{E}_e^\varepsilon = \frac{1}{2} ((\mathbf{F}_e^\varepsilon)^T \mathbf{F}_e^\varepsilon - \mathbf{I})$ is the elastic Green-Lagrange strain tensor and
 431 $\mathcal{C}^\varepsilon(X) = \mathcal{C}(X, Y)$ is the positive definite fourth-order elasticity tensor, which
 432 satisfies both major and minor symmetries, i.e. $\mathcal{C}_{ijkl} = \mathcal{C}_{jikl} = \mathcal{C}_{ijlk} = \mathcal{C}_{klij}$.
 433 Particularly, we consider that the constituents of the heterogeneous material
 434 are isotropic, and thus

$$\mathcal{C}^\varepsilon = 3\kappa^\varepsilon \mathcal{K} + 2\mu^\varepsilon \mathcal{M}, \quad (27)$$

435 where $\kappa^\varepsilon(X) = \kappa(X, Y)$ is the bulk modulus, $\mu^\varepsilon(X) = \mu(X, Y)$ is the shear
 436 modulus, and the fourth-order tensors $\mathcal{K} = \frac{1}{3}(\mathbf{I} \otimes \mathbf{I})$ and $\mathcal{M} = \mathcal{I} - \mathcal{K}$
 437 extract the spherical and the deviatoric part, respectively, of a symmetric
 438 second-order tensor \mathbf{A} , i.e., $\mathcal{K} : \mathbf{A} = \frac{1}{3} \text{tr}(\mathbf{A}) \mathbf{I}$ and $\mathcal{M} : \mathbf{A} = \mathbf{A} - \frac{1}{3} \text{tr}(\mathbf{A}) \mathbf{I} :=$
 439 $\text{dev}(\mathbf{A})$ [84, 85]. We remark that the fourth-order identity tensor \mathcal{I} is the
 440 identity operator over the linear subspace of symmetric second-order tensors.
 441 Indeed, for every \mathbf{A} such that $\mathbf{A} = \mathbf{A}^T$, it holds that $\mathcal{I} : \mathbf{A} = \mathbf{A}$. In
 442 terms of \mathbf{I} , an explicit expression of \mathcal{I} is given by $\mathcal{I} = \frac{1}{2} [\mathbf{I} \underline{\otimes} \mathbf{I} + \mathbf{I} \overline{\otimes} \mathbf{I}]$ (in
 443 components: $\mathcal{I}_{ijkl} = \frac{1}{2} [I_{ik} I_{jl} + I_{il} I_{jk}]$ [17]).

444 We can identify the leading order term in the expansion of the constitutive
 445 law (26), which reads

$$\psi_\nu^{(0)} = \frac{1}{2} \mathbf{E}_e^{(0)} : \mathcal{C} : \mathbf{E}_e^{(0)}, \quad (28)$$

446 with $\mathbf{E}_e^{(0)} = \frac{1}{2} ((\mathbf{F}_e^{(0)})^T \mathbf{F}_e^{(0)} - \mathbf{I})$. We recall that, although the expression of
 447 $\psi_\nu^{(0)}$ in (28) depends only on $\mathbf{E}_e^{(0)}$, the material coefficient \mathcal{C} is still a two-
 448 scale function and should be thus interpreted as $\mathcal{C}(X, Y)$. As a consequence,
 449 $\psi_\nu^{(0)}$ is not homogenized yet.

450 By taking into account the major and minor symmetries of \mathcal{C} , we obtain

$$\mathbf{S}_\nu^{(0)} = \frac{\partial \psi_\nu^{(0)}}{\partial \mathbf{E}_e^{(0)}} = \mathcal{C} : \mathbf{E}_e^{(0)} = \lambda \text{tr}(\mathbf{E}_e^{(0)}) \mathbf{I} + 2\mu \mathbf{E}_e^{(0)}, \quad (29)$$

451 where $\mathbf{S}_\nu^{(0)}$ is the leading order term of the second Piola-Kirchhoff stress
 452 tensor written with respect to the natural state, $\lambda = \kappa - \frac{2}{3}\mu$ is Lamé's
 453 constant, and $\mathbf{E}_e^{(0)}$ is given by

$$\mathbf{E}_e^{(0)} = (\mathbf{F}_p^{(0)})^{-T} \left(\mathbf{E}^{(0)} - \mathbf{E}_p^{(0)} \right) (\mathbf{F}_p^{(0)})^{-1}, \quad (30)$$

454 with $\mathbf{E}^{(0)} = \frac{1}{2} \left((\mathbf{F}^{(0)})^T \mathbf{F}^{(0)} - \mathbf{I} \right)$ and $\mathbf{E}_p^{(0)} = \frac{1}{2} \left((\mathbf{F}_p^{(0)})^T \mathbf{F}_p^{(0)} - \mathbf{I} \right)$.

455 By pulling $\mathbf{S}_\nu^{(0)}$ back to the reference configuration, and recalling that the
 456 plastic-like distortions are assumed to be isochoric in our framework, (i.e.
 457 $J_p^\varepsilon = 1$), we obtain the second Piola-Kirchhoff stress tensor

$$\mathbf{S}^{(0)} = \mathcal{C}_R : (\mathbf{E}^{(0)} - \mathbf{E}_p^{(0)}), \quad (31)$$

458 where

$$\begin{aligned} \mathcal{C}_R &= (\mathbf{F}_p^{(0)})^{-1} \underline{\otimes} (\mathbf{F}_p^{(0)})^{-1} : \mathcal{C} : (\mathbf{F}_p^{(0)})^{-T} \underline{\otimes} (\mathbf{F}_p^{(0)})^{-T} \\ &= 3\lambda \mathcal{K}_p^{(0)} + 2\mu \mathcal{J}_p^{(0)}, \end{aligned} \quad (32)$$

459 is the elasticity tensor pulled-back to the reference configuration through
 460 $\mathbf{F}_p^{(0)}$, and, upon setting $\mathbf{B}_p^{(0)} = (\mathbf{F}_p^{(0)})^{-1} (\mathbf{F}_p^{(0)})^{-T}$, we employed the notation

$$\mathcal{K}_p^{(0)} = \frac{1}{3} \mathbf{B}_p^{(0)} \otimes \mathbf{B}_p^{(0)}, \quad (33a)$$

$$\mathcal{J}_p^{(0)} = \frac{1}{2} \left[\mathbf{B}_p^{(0)} \underline{\otimes} \mathbf{B}_p^{(0)} + \mathbf{B}_p^{(0)} \overline{\otimes} \mathbf{B}_p^{(0)} \right]. \quad (33b)$$

461 We remark that $\mathcal{K}_p^{(0)}$ extracts the “volumetric part” of a generic second-
 462 order tensor, taken with respect to the inverse plastic metric tensor $\mathbf{B}_p^{(0)}$ i.e.
 463 for all $\mathbf{A} = \mathbf{A}^T$, it holds that $\mathcal{K}_p^{(0)} : \mathbf{A} = \frac{1}{3} \text{tr}(\mathbf{B}_p^{(0)} \mathbf{A}) \mathbf{B}_p^{(0)}$. Furthermore,
 464 $\mathcal{J}_p^{(0)}$ transforms \mathbf{A} into $\mathcal{J}_p^{(0)} : \mathbf{A} = \mathbf{B}_p^{(0)} \mathbf{A} \mathbf{B}_p^{(0)}$ and $\mathcal{M}_p^{(0)} = \mathcal{J}_p^{(0)} - \mathcal{K}_p^{(0)}$
 465 extracts the “deviatoric part” of \mathbf{A} with respect to the metric tensor $\mathbf{B}_p^{(0)}$,
 466 i.e. $\mathcal{M}_p^{(0)} : \mathbf{A} = \mathbf{B}_p^{(0)} \mathbf{A} \mathbf{B}_p^{(0)} - \frac{1}{3} \text{tr}(\mathbf{B}_p^{(0)} \mathbf{A}) \mathbf{B}_p^{(0)}$. We note that similar results
 467 have been obtained in the case of non-linear elasticity in [25].

468 Next, we notice that $\mathbf{F}^{(0)}$ can be written as

$$\mathbf{F}^{(0)} = \mathbf{I} + \mathbf{H}, \quad (34)$$

469 with $\mathbf{H} = \text{Grad}_X \mathbf{u}^{(0)} + \text{Grad}_Y \mathbf{u}^{(1)}$. Thus, by substituting (34) in $\mathbf{E}_e^{(0)}$,
 470 the result into (31), and retaining only the terms linear in \mathbf{H} , $\mathbf{S}^{(0)}$ can be
 471 linearized as

$$\mathbf{S}_{\text{lin}}^{(0)} = \mathcal{C}_R : (\text{sym} \mathbf{H} - \mathbf{E}_p^{(0)}). \quad (35)$$

472 We recall now that, at the leading order, the first Piola-Kirchhoff stress tensor
 473 reads $\mathbf{T}^{(0)} = \mathbf{F}^{(0)} \mathbf{S}^{(0)}$. Hence, its linearized form is given by

$$\mathbf{T}_{\text{lin}}^{(0)} = \mathcal{C}_{\text{R}} : \text{sym} \mathbf{H} - (\mathbf{I} + \mathbf{H})(\mathcal{C}_{\text{R}} : \mathbf{E}_{\text{p}}^{(0)}). \quad (36)$$

474 Looking at the definition of \mathcal{C}_{R} in (32), it can be noticed that our model re-
 475 solves at the macro-scale the structural evolution of the considered medium
 476 through the dependence of \mathcal{C}_{R} on $\mathbf{F}_{\text{p}}^{(0)}$, which indeed describes the produc-
 477 tion of material inhomogeneities [21, 22, 23]. Additionally, our model is also
 478 capable of simultaneously resolving the material heterogeneities at both the
 479 micro- and macro-scale through the dependence of \mathcal{C}_{R} on X and Y . The lat-
 480 ter dependence in fact, keeps track of the variability of the elastic coefficient
 481 at both scales.

482 Because of Equations (33a) and (33b), \mathcal{C}_{R} possesses the same symmetry
 483 properties of \mathcal{C} , i.e.

$$(\mathcal{C}_{\text{R}})_{IJKL} = (\mathcal{C}_{\text{R}})_{JIKL} = (\mathcal{C}_{\text{R}})_{IJLK} = (\mathcal{C}_{\text{R}})_{KLIJ}, \quad (37)$$

484 and therefore, $\mathbf{T}_{\text{lin}}^{(0)}$ can be written as

$$\mathbf{T}_{\text{lin}}^{(0)} = \mathcal{C}_{\text{R}} : \mathbf{H} - (\mathbf{I} + \mathbf{H})(\mathcal{C}_{\text{R}} : \mathbf{E}_{\text{p}}^{(0)}). \quad (38)$$

485 *Local problem.* Substituting (38) in the equation of the local problem (24),
 486 the linear momentum balance law is rephrased as

$$\text{Div}_Y [\mathcal{C}_{\text{R}} : \mathbf{H} - (\mathbf{I} + \mathbf{H})(\mathcal{C}_{\text{R}} : \mathbf{E}_{\text{p}}^{(0)})] = \mathbf{0}, \quad (39)$$

487 or, equivalently,

$$\begin{aligned} \text{Div}_Y [\mathcal{C}_{\text{R}} : \text{Grad}_Y \mathbf{u}^{(1)} - \text{Grad}_Y \mathbf{u}^{(1)} (\mathcal{C}_{\text{R}} : \mathbf{E}_{\text{p}}^{(0)})] = \\ - \text{Div}_Y [\mathcal{C}_{\text{R}} : \text{Grad}_X \mathbf{u}^{(0)} - (\mathbf{I} + \text{Grad}_X \mathbf{u}^{(0)}) (\mathcal{C}_{\text{R}} : \mathbf{E}_{\text{p}}^{(0)})]. \end{aligned} \quad (40)$$

488 In the absence of plastic distortions, i.e., when $\mathbf{F}_{\text{p}}^{\varepsilon} = \mathbf{I}$, Equation (40) coin-
 489 cides with the equation of the classical cell problem encountered in the ho-
 490 mogeneization of linear elasticity, which is known to admit a unique solution,
 491 up to a Y -constant function, if the average over the cell of the right-hand-side
 492 vanishes identically (in the jargon of Homogenization Theory, this condition
 493 is referred to as *solvability condition* or *compatibility condition*) [5]. In our

case, since the pulled-back elasticity tensor \mathcal{C}_R is periodic in Y , while $\mathbf{u}^{(0)}$ is independent of Y , the solvability condition is satisfied, i.e.,

$$\langle \text{Div}_Y [\mathcal{C}_R : \text{Grad}_X \mathbf{u}^{(0)} - (\mathbf{I} + \text{Grad}_X \mathbf{u}^{(0)}) (\mathcal{C}_R : \mathbf{E}_p^{(0)})] \rangle = \mathbf{0}. \quad (41)$$

Exploiting the linearity of equation (40) in $\mathbf{u}^{(1)}$, we make the *ansatz*

$$\mathbf{u}^{(1)}(X, Y, t) = \boldsymbol{\xi}(X, Y, t) : \text{Grad}_X \mathbf{u}^{(0)}(X, t) + \boldsymbol{\omega}(X, Y, t), \quad (42)$$

where $\boldsymbol{\xi}$ and $\boldsymbol{\omega}$ are a third-order tensor field and a vector field, both periodic in Y .

We now require that $\boldsymbol{\xi}$ and $\boldsymbol{\omega}$ satisfy two independent cell problems. The cell problem for $\boldsymbol{\xi}$ reads

$$\begin{cases} \text{Div}_Y [\mathcal{C}_R : T\text{Grad}_Y \boldsymbol{\xi} - T\text{Grad}_Y \boldsymbol{\xi} (\mathcal{C}_R : \mathbf{E}_p^{(0)})] \\ \quad = \text{Div}_Y [-\mathcal{C}_R + \mathbf{I} \otimes (\mathcal{C}_R : \mathbf{E}_p^{(0)})], & \text{in } \mathcal{Y}_0 \setminus \Gamma_0 \times \mathcal{T}, \\ \llbracket \boldsymbol{\xi} \rrbracket = \mathbf{0}, & \text{on } \Gamma_0 \times \mathcal{T}, \\ \llbracket [\mathcal{C}_R : T\text{Grad}_Y \boldsymbol{\xi} - T\text{Grad}_Y \boldsymbol{\xi} (\mathcal{C}_R : \mathbf{E}_p^{(0)}) \\ \quad + \mathcal{C}_R - \mathbf{I} \otimes (\mathcal{C}_R : \mathbf{E}_p^{(0)})] \cdot \mathbf{N}_Y \rrbracket = \mathbf{0}, & \text{on } \Gamma_0 \times \mathcal{T}. \end{cases} \quad (43)$$

Before going further, some words of explanation on the notation are necessary. First, we notice that $\text{Grad}_Y \boldsymbol{\xi}$ is a fourth-order tensor function, which admits the representation $\text{Grad}_Y \boldsymbol{\xi} = (\partial \xi_{ABC}) / (\partial Y_D) \mathbf{e}_A \otimes \mathbf{e}_B \otimes \mathbf{e}_C \otimes \mathbf{e}_D$. Then, $T\text{Grad}_Y \boldsymbol{\xi}$ is a fourth-order tensor function obtained by ordering the indices of $\text{Grad}_Y \boldsymbol{\xi}$ in the following fashion

$$\begin{aligned} T\text{Grad}_Y \boldsymbol{\xi} &= (T\text{Grad}_Y \boldsymbol{\xi})_{ABCD} \mathbf{e}_A \otimes \mathbf{e}_B \otimes \mathbf{e}_C \otimes \mathbf{e}_D \\ &= (\text{Grad}_Y \boldsymbol{\xi})_{ACDB} \mathbf{e}_A \otimes \mathbf{e}_B \otimes \mathbf{e}_C \otimes \mathbf{e}_D \\ &= \frac{\partial \xi_{ACD}}{\partial Y_B} \mathbf{e}_A \otimes \mathbf{e}_B \otimes \mathbf{e}_C \otimes \mathbf{e}_D. \end{aligned} \quad (44)$$

The cell problem for $\boldsymbol{\omega}$ is given by

$$\begin{cases} \text{Div}_Y [\mathcal{C}_R : \text{Grad}_Y \boldsymbol{\omega} - \text{Grad}_Y \boldsymbol{\omega} (\mathcal{C}_R : \mathbf{E}_p^{(0)})] \\ \quad = \text{Div}_Y [\mathcal{C}_R : \mathbf{E}_p^{(0)}], & \text{in } \mathcal{Y}_0 \setminus \Gamma_0 \times \mathcal{T}, \\ \llbracket \boldsymbol{\omega} \rrbracket = \mathbf{0}, & \text{on } \Gamma_0 \times \mathcal{T}, \\ \llbracket (\mathcal{C}_R : \text{Grad}_Y \boldsymbol{\omega} - \text{Grad}_Y \boldsymbol{\omega} (\mathcal{C}_R : \mathbf{E}_p^{(0)}) \\ \quad - \mathcal{C}_R : \mathbf{E}_p^{(0)}) \cdot \mathbf{N}_Y \rrbracket = \mathbf{0}, & \text{on } \Gamma_0 \times \mathcal{T}. \end{cases} \quad (45)$$

By virtue of the linearization process, we obtain two auxiliary cell problems where the macroscopic term $\text{Grad}_X \mathbf{u}^{(0)}$ is not explicitly present. Indeed, this is in general possible only when accounting for the linearized deformations' regime, see also [15]. Then, the dependence of the macro-scale variable is given through the tensor $\mathbf{F}_p^{(0)}$, which describes the plastic-like distortions. Moreover, if $\mathbf{F}_p^{(0)}$ only depends on time, as is the case in [2], the cell problems are also decoupled in the spatial micro- and macro-variables provided that the elasticity tensor solely depends on the microscale variable. The cell problems are in any case time-dependent, as they encode the evolution of the material response and its link with the plastic-like distortions.

Homogenized problem. From (36) and (42), the homogenized problem rewrites

$$\begin{cases} \text{Div}_X [\hat{\mathcal{C}}_R : \text{Grad}_X \mathbf{u}^{(0)}] = -\text{Div}_X [\hat{\mathbf{D}}_R], & \text{in } \mathcal{B}_h \times \mathcal{T}, \\ (\hat{\mathcal{C}}_R : \text{Grad}_X \mathbf{u}^{(0)}) \cdot \mathbf{N} + \hat{\mathbf{D}}_R \cdot \mathbf{N} = \bar{\mathbf{T}}, & \text{on } \partial_T \mathcal{B}_h \times \mathcal{T}, \\ \mathbf{u}^{(0)} = \bar{\mathbf{u}}, & \text{on } \partial_u \mathcal{B}_h \times \mathcal{T}, \end{cases} \quad (46)$$

where

$$\hat{\mathcal{C}}_R = \langle \mathcal{C}_R + \mathcal{C}_R : T\text{Grad}_Y \boldsymbol{\xi} - T\text{Grad}_Y \boldsymbol{\xi} (\mathcal{C}_R : \mathbf{E}_p^{(0)}) - \mathbf{I} \otimes (\mathcal{C}_R : \mathbf{E}_p^{(0)}) \rangle, \quad (47a)$$

$$\hat{\mathbf{D}}_R = \langle \mathcal{C}_R : \text{Grad}_Y \boldsymbol{\omega} - \text{Grad}_Y \boldsymbol{\omega} (\mathcal{C}_R : \mathbf{E}_p^{(0)}) - \mathcal{C}_R : \mathbf{E}_p^{(0)} \rangle. \quad (47b)$$

Remark 3. In the absence of distortions, that is for $\mathbf{F}_p^\varepsilon = \mathbf{I}$, the cell problems (43) and (45) reduce to one single cell problem,

$$\begin{cases} \text{Div}_Y [\mathcal{C} + \mathcal{C} : T\text{Grad}_Y \boldsymbol{\xi}] = \mathbf{0}, & \text{in } \mathcal{Y}_0 \setminus \Gamma_0 \times \mathcal{T}, \\ \llbracket \boldsymbol{\xi} \rrbracket = \mathbf{0}, & \text{on } \Gamma_0 \times \mathcal{T}, \\ \llbracket (\mathcal{C} + \mathcal{C} : T\text{Grad}_Y \boldsymbol{\xi}) \cdot \mathbf{N}_Y \rrbracket = \mathbf{0}, & \text{on } \Gamma_0 \times \mathcal{T}. \end{cases} \quad (48)$$

This is due to the fact that the symmetric tensor $\mathbf{E}_p^{(0)}$ appearing in (40) is equal to zero. On the other hand, the homogenized problem is rewritten as follows,

$$\begin{cases} \text{Div}_X [\hat{\mathcal{C}} : \text{Grad}_X \mathbf{u}^{(0)}] = \mathbf{0}, & \text{in } \mathcal{B}_h \times \mathcal{T}, \\ (\hat{\mathcal{C}} : \text{Grad}_X \mathbf{u}^{(0)}) \cdot \mathbf{N} = \bar{\mathbf{T}}, & \text{on } \partial_T \mathcal{B}_h \times \mathcal{T}, \\ \mathbf{u}^{(0)} = \bar{\mathbf{u}}, & \text{on } \partial_u \mathcal{B}_h \times \mathcal{T}, \end{cases} \quad (49)$$

524 where $\hat{\mathcal{C}} = \langle \mathcal{C} + \mathcal{C} : T\text{Grad}_Y \boldsymbol{\xi} \rangle$ is the effective elasticity tensor. Formula-
525 tions (48) and (49) are the counterparts of (24) and (25), respectively, when
526 plastic-like distortions are neglected and a linearized approach for the defor-
527 mations is considered. Particularly, (48) and (49) identify identically with
528 classical results in the asymptotic homogenization literature [5, 77].

529 5.2. Evolution law

530 Several procedures can be adopted to establish a proper evolution law
531 for the inelastic distortions. One choice is to follow a phenomenological
532 approach, which should be based on experimental evidences and comply with
533 suitable constitutive requirements [29]. On the other hand, one could invoke
534 some general principles, such as the invariance of the evolution law with
535 respect to a class of transformations and thermodynamic constraints [21, 22,
536 23]. Within the latter approach, and adapting the theoretical framework
537 explored in [21, 22, 23, 29], an evolution equation for the inelastic distortions
538 has been studied in [19]. Therein, the plastic-like distortions describe a
539 remodeling process with the following assumptions: (i) \mathbf{F}_p is restricted by the
540 constraint $J_p = 1$, (ii) the solid phase exhibits hyperelastic behavior, and (iii)
541 the considered system remodels when the stress induced by external loading
542 exceeds a characteristic threshold. An evolution law for \mathbf{F}_p satisfying these
543 conditions, and compatible with the Dissipation inequality [12, 32, 33, 34],
544 is given by

$$\text{sym} \left(\mathbf{C} \mathbf{F}_p^{-1} \dot{\mathbf{F}}_p \right) = \gamma \left[\|\text{dev} \boldsymbol{\sigma}\| - \sqrt{\frac{2}{3}} \sigma_y \right]_+ \frac{\text{dev}(\boldsymbol{\Sigma}) \mathbf{C}}{\|\text{dev} \boldsymbol{\sigma}\|}, \quad (50)$$

545 where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\text{dev}(\boldsymbol{\Sigma}) = \boldsymbol{\Sigma} - \frac{1}{3} \text{tr}(\boldsymbol{\Sigma}) \mathbf{I}$, is the deviatoric
546 part of the Mandell stress tensor $\boldsymbol{\Sigma} = \mathbf{C} \mathbf{S}$ being the Mandel stress tensor,
547 and $\mathbf{S} = \mathbf{F}^{-1} \mathbf{T}$ the second Piola-Kirchhoff stress tensor. Moreover, γ is a
548 strictly positive model parameter, $\sigma_y > 0$ is the yield, or threshold, stress,
549 and the operator $[A]_+$ is such that, for any real number A , $[A]_+ = A$, if $A > 0$,
550 and $[A]_+ = 0$ otherwise. As anticipated in the Introduction, in the present
551 context the physical meaning of the plastic-like distortions, represented by
552 \mathbf{F}_p , is that of structural reorganization, i.e. remodeling, as is the case in
553 biological tissues when the adhesion bonds among cells or the structure of
554 the ECM reorganize themselves.

555 Although Equation (50) has been successfully used to describe some bi-
556 ological situations in which the onset of remodeling is subordinated to the

557 excess of the yield stress σ_y , the homogenization of the evolution law (50) is
 558 too complicated. For this reason, in this work, we replace (50) with a much
 559 easier law of the type

$$\text{sym} \left(\mathbf{C}(\mathbf{F}_p)^{-1} \dot{\mathbf{F}}_p \right) = \gamma \text{dev}(\boldsymbol{\Sigma}) \mathbf{C}, \quad (51)$$

560 according to which no stress-activation criterion is supplied. Clearly, this
 561 choice may turn out to be unrealistic in many circumstances, but it can
 562 still be useful to understand the essence of some stress-driven remodeling
 563 processes.

564 We need to clarify that, although in some sentences of this work we
 565 mentioned growth, our model focuses on *pure* remodeling. This is reflected
 566 by the condition $\det \mathbf{F}_p = 1$, and, more importantly, by the fact that the
 567 evolution laws (50) and (51) are triggered and controlled exclusively by me-
 568 chanical factors. On the one hand, the requirement $\det \mathbf{F}_p = 1$ means that
 569 the plastic-like distortions are isochoric and, thus, unable to describe volu-
 570 metric growth. On the other hand, the evolution laws for \mathbf{F}_p , i.e., Eqs. (50)
 571 or (51), imply that remodeling is viewed as a consequence of the mechanical
 572 environment only: When mechanical stress exceeds a given threshold (see
 573 also [29, 34]), the internal structure of the tissue starts to vary. In other
 574 words, in the present framework, no biochemical phenomena are accounted
 575 for as possible activators of remodeling. This is a remarkable difference with
 576 growth, which, in contrast, occurs only when the concentration of nutrients
 577 is above a certain threshold value [2, 10, 3, 26, 52]. Our results do not apply
 578 to growth as they stand, nonetheless, the theory can be adapted to model
 579 growth by doing some necessary modifications. This is the reason why in
 580 the abstract we stated that our study offers “*a robust framework that can be*
 581 *readily generalized to growth and remodeling of nonlinear composites*”.

582 To homogenize (51), the first step is to rewrite it as

$$\text{sym} \left(\mathbf{C}^\varepsilon(\mathbf{F}_p^\varepsilon)^{-1} \dot{\mathbf{F}}_p^\varepsilon \right) = \gamma^\varepsilon \text{dev}(\boldsymbol{\Sigma}^\varepsilon) \mathbf{C}^\varepsilon, \quad (52)$$

583 by admitting that $\gamma^\varepsilon(X) = \gamma(X, Y)$ is a rapidly oscillating strictly positive
 584 function. Moreover, by performing the power expansion for $\boldsymbol{\Sigma}^\varepsilon$,

$$\boldsymbol{\Sigma}^\varepsilon(X, t) = \sum_{k=0}^{+\infty} \boldsymbol{\Sigma}^{(k)}(X, Y, t) \varepsilon^k, \quad (53)$$

585 and using (31), the leading order term of Σ^ε is

$$\Sigma^{(0)} = \mathbf{C}^{(0)}[\mathcal{C}_R : (\mathbf{E}^{(0)} - \mathbf{E}_p^{(0)})]. \quad (54)$$

586 In the limit of small elastic deformations, in (54) we must neglect non-linear
587 terms in \mathbf{H} . Therefore, $\Sigma^{(0)}$ is approximated with

$$\Sigma_{\text{lin}}^{(0)} = \mathcal{C}_R : \text{sym} \mathbf{H} - (\mathbf{I} + 2\text{sym} \mathbf{H})(\mathcal{C}_R : \mathbf{E}_p^{(0)}).$$

588 By virtue of (12a), $\text{sym} \mathbf{H}$ splits additively as the sum of

$$\text{sym} \mathbf{H} = \mathbf{E}_X^{(0)} + \mathbf{E}_Y^{(1)}, \quad (55)$$

589 where, for $k = 0, 1$, and $j_k = X, Y$,

$$\mathbf{E}_{j_k}^{(k)} = \frac{1}{2}[\text{Grad}_{j_k} \mathbf{u}^{(k)} + (\text{Grad}_{j_k} \mathbf{u}^{(k)})^T]. \quad (56)$$

590 By using (55) and (42), we can now rewrite $\Sigma_{\text{lin}}^{(0)}$ as

$$\Sigma_{\text{lin}}^{(0)} = \mathcal{A}_R : \text{Grad}_X \mathbf{u}^{(0)} + \mathcal{B}_R : \text{Grad}_Y \boldsymbol{\omega} - \mathcal{C}_R : \mathbf{E}_p^{(0)}, \quad (57)$$

591 with

$$\begin{aligned} \mathcal{A}_R &= \mathcal{C}_R + \mathcal{C}_R : T\text{Grad}_Y \boldsymbol{\xi} - \mathbf{I} \underline{\otimes} (\mathcal{C}_R : \mathbf{E}_p^{(0)}) \\ &\quad + [\mathbf{I} \underline{\otimes} (\mathcal{C}_R : \mathbf{E}_p^{(0)})] : [T\text{Grad}_Y \boldsymbol{\xi} + {}^t(T\text{Grad}_Y \boldsymbol{\xi})], \end{aligned} \quad (58a)$$

$$\mathcal{B}_R = \mathcal{C}_R + \mathbf{I} \underline{\otimes} (\mathcal{C}_R : \mathbf{E}_p^{(0)}). \quad (58b)$$

592 In Equation (58a), the symbol ${}^t(\bullet)$ transposes the fourth-order tensor to
593 which it is applied by exchanging the order of its first pair of indices only,
594 i.e., given an arbitrary fourth-order tensor $\mathcal{T} = \mathcal{T}_{ABCD} \mathbf{e}_A \otimes \mathbf{e}_B \otimes \mathbf{e}_C \otimes \mathbf{e}_D$,
595 ${}^t \mathcal{T}$ reads

$${}^t \mathcal{T} = \mathcal{T}_{BACD} \mathbf{e}_A \otimes \mathbf{e}_B \otimes \mathbf{e}_C \otimes \mathbf{e}_D. \quad (59)$$

596 Note that in the calculations performed to obtain \mathcal{A}_R and \mathcal{B}_R in (57), we
597 employed the following properties: given two second-order tensors \mathbf{A} and \mathbf{U} ,
598 with \mathbf{A} being symmetric, it holds that

$$\mathbf{U} \mathbf{A} = (\mathbf{I} \underline{\otimes} \mathbf{A}) : \mathbf{U}, \quad (60a)$$

$$\mathbf{U}^T \mathbf{A} = (\mathbf{I} \overline{\otimes} \mathbf{A}) : \mathbf{U}. \quad (60b)$$

599 Finally, by substituting the expansions of Σ^ε and \mathbf{F}_p^ε in (52), equating
600 the leading order terms, excluding non-linear terms of \mathbf{H} and averaging, the
601 homogenized evolution law for the plastic-like distortions is

$$\text{sym}[\langle \mathbf{C}_{\text{lin}}^{(0)}(\mathbf{F}_p^{(0)})^{-1} \dot{\overline{\mathbf{F}_p^{(0)}}} \rangle] = -\langle \gamma \text{dev}(\Sigma_{\text{lin}}^{(0)}) \rangle - \langle \gamma(\mathcal{C}_R : \mathbf{E}_p^{(0)})(\mathbf{C}_{\text{lin}}^{(0)} - \mathbf{I}) \rangle, \quad (61)$$

602 where $\Sigma_{\text{lin}}^{(0)}$ is given in (57) and

$$\begin{aligned} \mathbf{C}_{\text{lin}}^{(0)} &= \mathbf{I} + 2\text{sym}\mathbf{H} \\ &= \mathbf{I} + 2(\mathcal{J} + \mathcal{J} : T\text{Grad}_Y \boldsymbol{\xi}) : \text{Grad}_X \mathbf{u}^{(0)} + 2\mathcal{J} : \text{Grad}_Y \boldsymbol{\omega}. \end{aligned} \quad (62)$$

603 We note that, to compute $\mathbf{C}_{\text{lin}}^{(0)}$, we must first determine $\boldsymbol{\xi}$ and $\boldsymbol{\omega}$, which is
604 done by solving the local problems (43) and (45). Furthermore, Equation
605 (61) needs to be supplemented with an initial condition for $\mathbf{F}_p^{(0)}$.

606 **Remark 4.** *In the linearized theory of elasticity, even when the individual*
607 *constituents of a given composite material are isotropic, the effective elas-*
608 *tic coefficients may turn out to be anisotropic, depending on the geometric*
609 *properties of the micro-structure. In fact, when the Homogenization Theory*
610 *is applied, the anisotropy arises quite naturally due to the solution of the*
611 *local cell problems [5, 8]. In fact, the homogenized material is anisotropic*
612 *also in the case of rather simple cells, see for instance [61], where an ex-*
613 *PLICIT deviation-from- isotropy function is introduced in the context of cubic*
614 *symmetric elasticity tensors arising from asymptotic homogenization. This*
615 *has noticeable repercussions also on the evolution law that should be chosen*
616 *for a correct description of remodeling. To see this, we first notice that, for*
617 *an isotropic medium, the evolution law of the plastic-like distortions can be*
618 *formulated in terms of tensor \mathbf{B}_p , since the constitutive framework is such*
619 *that \mathbf{F}_p does not feature explicitly in any constitutive function (see e.g. [78]).*
620 *In such cases, a possible evolution law for \mathbf{B}_p may be given in the form*

$$\dot{\mathbf{B}}_p = \gamma \mathbf{B}_p \text{dev}(\Sigma). \quad (63)$$

621 Equation (63) is, in fact, in harmony with the symmetry properties of the
622 material Mandel stress tensor, Σ , i.e., $\mathbf{B}_p \Sigma = (\mathbf{B}_p \Sigma)^T$ [54]. However, if
623 one writes an equation of the same type as (63) at the scale of a cell problem
624 (which seems to be a justified choice, because the material is isotropic at
625 that scale), and then homogenizes, one ends up with a material for which

the Mandel stress tensor Σ no longer obeys the symmetry condition $\mathbf{B}_p \Sigma = (\mathbf{B}_p \Sigma)^T$. This is because the material is not isotropic at the macroscale and, thus, the description of remodeling based on \mathbf{B}_p becomes inadequate. Therefore, if one wants to homogenize, one should start with evolution laws at the microscale, which have to be suitable to account for anisotropy, even though the single constituents are isotropic at that scale. These considerations lead us to Equation (52), as suggested in [22, 23], and subsequently employed in [19].

Remark 5. Equations (50) and (51) can be obtained by adhering to the philosophy presented in [12, 18], and subsequently adopted, for example, in [3] for growth, in [44] for growth and remodeling, and in [31, 32] for remodeling only. Accordingly, \mathbf{F}_p is regarded as the kinematic descriptor of the structural degrees of freedom of the medium, and $\dot{\mathbf{F}}_p$ as the generalized velocity with which the structural changes occur. Within this setting, it can be proven that for growth and remodeling problems, the dissipation inequality reads

$$\mathcal{D} = \mathbf{Y}_\nu : \mathbf{L}_p + \mathcal{D}_{\text{nc}} \geq 0, \quad (64)$$

where $\mathcal{D}_{\text{mech}} := \mathbf{Y}_\nu : \mathbf{L}_p$ is the mechanical contribution to dissipation, with \mathbf{Y}_ν being the dissipative part of a generalized internal force, dual to \mathbf{L}_p . In our work, however, \mathbf{Y}_ν can be identified with the tensor $\mathbf{Y}_\nu \equiv J_p^{-1} \mathbf{F}_p^{-T} \Sigma \mathbf{F}_p^T$, so that $\mathcal{D}_{\text{mech}}$ coincides with the mechanical dissipation encountered in the standard formulation of Elastoplasticity, i.e., $\mathcal{D}_{\text{mech}} = J_p^{-1} \mathbf{F}_p^{-T} \Sigma \mathbf{F}_p^T : \mathbf{L}_p = J_p^{-1} \Sigma : \mathbf{F}_p^{-1} \dot{\mathbf{F}}_p$.

In the terminology of [45, 30], \mathcal{D}_{nc} is referred to as “non-compliant” contribution to the overall dissipation. Physically, it summarizes a class of phenomena that are not —or cannot be— resolved in terms of mechanical power at the scale at which the dissipation inequality is written. For instance, in the case of growth, \mathcal{D}_{nc} may represent biochemical effects contributing to the overall dissipation.

The inequality (64) can be studied in several ways, depending on the problem at hand. First, we consider a growth problem. To this end, we assume that \mathcal{D}_{nc} can be written as $\mathcal{D}_{\text{nc}} = r\mathcal{A}$, where r is the rate at which mass is added or depleted from the system (its units are given by the reciprocal of time), and \mathcal{A} is the energy density (per unit volume) associated with the introduction or uptake of mass. In this setting, it is possible to conceive a particular state of the system in which the mechanical stress is null, i.e., $\Sigma = \mathbf{0}$, while r and \mathcal{A} are generally nonzero. When this occurs, the system

grows without mechanical dissipation, i.e., $\mathcal{D}_{\text{mech}} = 0$, whereas the overall dissipation of the system reduces to the non-compliant one:

$$\mathcal{D} \equiv \mathcal{D}_{\text{nc}} = r\mathcal{A} \geq 0. \quad (65)$$

The second case addresses the situation of pure remodeling, for which we set $\mathcal{D}_{\text{nc}} = 0$, so that the dissipation inequality (64) becomes

$$\mathcal{D} = \mathcal{D}_{\text{mech}} = \mathbf{Y}_\nu : \mathbf{L}_p = J_p^{-1} \boldsymbol{\Sigma} : \mathbf{F}_p^{-1} \dot{\mathbf{F}}_p \geq 0. \quad (66)$$

It is possible to show that the evolution laws (50) and (51) are in harmony with (66).

6. A computational scheme for small deformations

The macro-scale model given by the problems (46) and (61), together with the auxiliary cell problems (43) and (45), requires dedicated numerical schemes which are subject of our current investigations. The main computational challenge is due to the fact that the local problems depend on the macro-scale in a time-dependent way. Therefore, at each time, there is a different cell problem at each macroscopic point $X \in \mathcal{B}_h$. Moreover, one has to transfer the information (represented by the geometry, material coefficients, and unknowns of the problem) from the cell problems to the homogenized problem in the domain \mathcal{B}_h , and vice versa.

Here, as a first step towards the numerical study of this kind of problems, we propose an algorithm adapted from [31] that could be useful in our case. In [31] it is introduced a computational algorithm, named Generalised Plasticity Algorithm (GPA), to study the mechanical response of a biological tissue that undergoes large deformations and remodeling of its internal structure. Following [31], the discrete and linearized version of the problem constituted by Equations (43), (45), (46) and (61) is formulated in three steps.

First step. The weak form of the cell problems (43) and (45), and of the homogenized problem (46) can be *formally* rewritten as

$$\mathcal{L}_1^w(\boldsymbol{\xi}, \mathbf{F}_p^{(0)}, \tilde{\boldsymbol{\xi}}) = 0, \quad (67a)$$

$$\mathcal{L}_2^w(\boldsymbol{\omega}, \mathbf{F}_p^{(0)}, \tilde{\boldsymbol{\omega}}) = 0, \quad (67b)$$

$$\mathcal{H}_1^w(\mathbf{u}^{(0)}, \mathbf{F}_p^{(0)}, \tilde{\mathbf{u}}^{(0)}) = 0, \quad (67c)$$

686 where $\tilde{\boldsymbol{\xi}}$, $\tilde{\boldsymbol{\omega}}$ and $\tilde{\boldsymbol{u}}^{(0)}$ are test functions defined in certain Sobolev spaces, and
 687 \mathcal{L}_1^w , \mathcal{L}_2^w and \mathcal{H}_1^w are suitable integral operators. Together with (67a)-(67c),
 688 we rewrite in operatorial form also the homogenized problem (61) as

$$\mathcal{H}_2(\boldsymbol{\xi}, \boldsymbol{\omega}, \boldsymbol{u}^{(0)}, \boldsymbol{F}_p^{(0)}) = \mathbf{0}. \quad (68)$$

689 Note that (68) is not a weak form because the corresponding equation does
 690 not involved spatial derivatives of $\boldsymbol{F}_p^{(0)}$.

691 *Second step.* We perform a backward Euler method [78] for discretizing the
 692 evolution law for $\boldsymbol{F}_p^{(0)}$ given by (68), thereby ending up with the following
 693 system of time-discrete equations,

$$\mathcal{L}_{1[n]}^w(\boldsymbol{\xi}_{[n]}, \boldsymbol{F}_{p[n]}^{(0)}, \tilde{\boldsymbol{\xi}}) = 0, \quad (69a)$$

$$\mathcal{L}_{2[n]}^w(\boldsymbol{\omega}_{[n]}, \boldsymbol{F}_{p[n]}^{(0)}, \tilde{\boldsymbol{\omega}}) = 0, \quad (69b)$$

$$\mathcal{H}_{1[n]}^w(\boldsymbol{u}_{[n]}^{(0)}, \boldsymbol{F}_{p[n]}^{(0)}, \tilde{\boldsymbol{u}}^{(0)}) = 0, \quad (69c)$$

$$\mathcal{H}_{2[n]}(\boldsymbol{\xi}_{[n]}, \boldsymbol{\omega}_{[n]}, \boldsymbol{u}_{[n]}^{(0)}, \boldsymbol{F}_{p[n]}^{(0)}) = \mathbf{0}, \quad (69d)$$

694 where $n = 1, \dots, N$ enumerates the nodes of a suitable time grid. We notice
 695 that an explicit time discrete method could be also used. However, when
 696 dealing with problems in Elastoplasticity, this election could lead to a less
 697 accurate solution.

698 *Third step.* The operators $\mathcal{L}_{1[n]}^w$, $\mathcal{L}_{2[n]}^w$, $\mathcal{H}_{1[n]}^w$ and $\mathcal{H}_{2[n]}$, are linear in $\boldsymbol{\xi}_{[n]}$, $\boldsymbol{\omega}_{[n]}$
 699 and $\boldsymbol{u}_{[n]}^{(0)}$, respectively, but they are nonlinear in $\boldsymbol{F}_{p[n]}^{(0)}$. Thus, to search the
 700 solution to (69a)-(69d), we linearize at each time step according to Newton's
 701 method (with a linesearch). Therefore, at the k th iteration, $k \in \mathbb{N}$, $k \geq 1$,
 702 $\boldsymbol{F}_{p[n,k]}^{(0)}$ is written as

$$\boldsymbol{F}_{p[n,k]}^{(0)} = \boldsymbol{F}_{p[n,k-1]}^{(0)} + \boldsymbol{\Psi}_{[n,k]}, \quad (70)$$

703 where $\boldsymbol{F}_{p[n,k-1]}^{(0)}$ is known and $\boldsymbol{\Psi}_{[n,k]}$ represents the unknown increment. We
 704 introduce the notation

$$\mathcal{L}_{1[n,k-1]}^w(\boldsymbol{\xi}_{[n]}, \tilde{\boldsymbol{\xi}}) = \mathcal{L}_{1[n]}^w(\boldsymbol{\xi}_{[n]}, \boldsymbol{F}_{p[n,k-1]}^{(0)}, \tilde{\boldsymbol{\xi}}), \quad (71a)$$

$$\mathcal{L}_{2[n,k-1]}^w(\boldsymbol{\omega}_{[n]}, \tilde{\boldsymbol{\omega}}) = \mathcal{L}_{2[n]}^w(\boldsymbol{\omega}_{[n]}, \boldsymbol{F}_{p[n,k-1]}^{(0)}, \tilde{\boldsymbol{\omega}}), \quad (71b)$$

$$\mathcal{H}_{1[n,k-1]}^w(\boldsymbol{u}_{[n]}^{(0)}, \tilde{\boldsymbol{u}}^{(0)}) = \mathcal{H}_{1[n]}^w(\boldsymbol{u}_{[n]}^{(0)}, \boldsymbol{F}_{p[n,k-1]}^{(0)}, \tilde{\boldsymbol{u}}^{(0)}). \quad (71c)$$

705 Now, for each time step, and at the k th iteration, we solve

$$\mathcal{L}_{1[n,k-1]}^w(\boldsymbol{\xi}_{[n]}, \tilde{\boldsymbol{\xi}}) = 0, \quad (72a)$$

$$\mathcal{L}_{2[n,k-1]}^w(\boldsymbol{\omega}_{[n]}, \tilde{\boldsymbol{\omega}}) = 0, \quad (72b)$$

$$\mathcal{H}_{1[n,k-1]}^w(\mathbf{u}_{[n]}^{(0)}, \tilde{\mathbf{u}}^{(0)}) = 0, \quad (72c)$$

706 and obtain the “temporary” solutions $\boldsymbol{\xi}_{[n,k-1]}$, $\boldsymbol{\omega}_{[n,k-1]}$, and $\mathbf{u}_{[n,k-1]}^{(0)}$, respec-
707 tively. Then, upon setting

$$\mathcal{H}_{2[n,k-1]} = \mathcal{H}_{2[n]}(\boldsymbol{\xi}_{[n,k-1]}, \boldsymbol{\omega}_{[n,k-1]}, \mathbf{u}_{[n,k-1]}^{(0)}, \mathbf{F}_{p[n,k-1]}^{(0)}), \quad (73a)$$

$$\mathcal{H}_{[n,k-1]} = \mathcal{H}_{[n]}(\boldsymbol{\xi}_{[n,k-1]}, \boldsymbol{\omega}_{[n,k-1]}, \mathbf{u}_{[n,k-1]}^{(0)}, \mathbf{F}_{p[n,k-1]}^{(0)}), \quad (73b)$$

708 we linearize (69d), i.e.,

$$\mathcal{H}_{2[n,k-1]} + \mathcal{H}_{[n,k-1]} : \boldsymbol{\Psi}_{[n,k]} = \mathbf{0}, \quad (74)$$

709 where $\mathcal{H}_{[n,k-1]}$ is a fourth-order tensor given by the Gâteaux derivative
710 of $\mathcal{H}_{2[n]}$, computed with respect to its fourth argument, and evaluated in
711 $\mathbf{F}_{p[n,k-1]}^{(0)}$.

712 If the residuum $\mathbf{F}_{p[n,k]}^{(0)}$ for k greater than, or equal to, a certain k_* is less
713 than a tolerance $\delta > 0$, then we set $\mathbf{F}_{p[n]}^{(0)} \equiv \mathbf{F}_{p[n,k_*]}^{(0)} = \mathbf{F}_{p[n,k_*-1]}^{(0)} + \boldsymbol{\Psi}_{[n,k_*]}$ and
714 we regard it as the solution of Newton’s method. Thus, we compute $\boldsymbol{\xi}_{[n]}$, $\boldsymbol{\omega}_{[n]}$
715 and $\mathbf{u}_{[n]}^{(0)}$.

716 These three steps are summarized in the algorithm 1.

717 7. Numerical results

718 In this section, the potentiality of our model, which is given by Equations
719 (43), (45), (46) and (61), is shown by performing numerical simulations. In
720 particular, we make the following considerations.

721 **(i) Geometry.** We consider the composite body \mathcal{B}^ε to have a layered three-
722 dimensional structure, and we assume that the layers are orthogonal to the
723 direction $\boldsymbol{\mathcal{E}}_3$, where $\{\boldsymbol{\mathcal{E}}_A\}_{A=1}^3$ is an orthonormal basis of a system of Cartesian
724 coordinates $\{X_A\}_{A=1}^3$. In this particular case, the material properties of
725 the heterogeneous body only change along the $\boldsymbol{\mathcal{E}}_3$ direction and, thus, they
726 depend solely on the coordinate X_3 . Consequently, the benchmark test at

Algorithm 1

```

1: procedure
2:   for  $n = 1, \dots, N$  do
3:     State  $k = 1$ 
4:     while  $e > \delta$  do (Known  $\mathbf{F}_{p[n,k-1]}^{(0)}$ )
5:       Solve  $\mathcal{L}_{1[n,k-1]}^w$  and  $\mathcal{L}_{2[n,k-1]}^w$  (To find  $\xi_{[n,k-1]}$  and  $\omega_{[n,k-1]}$ )
6:       Solve  $\mathcal{H}_{1[n,k-1]}^w$  (To find  $\mathbf{u}_{[n,k-1]}^{(0)}$ )
7:       Solve  $\mathcal{H}_{1[n,k-1]}^w$  (To find  $\Psi_{[n,k]}$ )
8:        $\mathbf{F}_{p[n,k-1]}^{(0)} \leftarrow \mathbf{F}_{p[n,k-1]}^{(0)} + \Psi_{[n,k]}$ 
9:       Compute  $e$ 
10:       $k = k + 1$ 
11:    end while
12:     $\mathbf{F}_{p[n]}^{(0)} = \mathbf{F}_{p[n,k-1]}^{(0)} + \Psi_{[n,k]}$ 
13:    Solve  $\mathcal{L}_{1[n]}^w$  and  $\mathcal{L}_{2[n]}^w$  (To find  $\xi_{[n]}$  and  $\omega_{[n]}$ )
14:    Solve  $\mathcal{H}_{1[n]}^w$  (To find  $\mathbf{u}_{[n]}^{(0)}$ )
15:    Update micro and macro geometries
16:  end for
17: end procedure

```

727 hand can be recast into a one dimensional problem, that is, the reference
 728 configuration of the periodic cell and the body are considered to be the
 729 unidimensional domains $\mathcal{Y}_0 = [0, \ell]$ and $\mathcal{B}_h = [0, L]$, respectively. We denote
 730 with ℓ and L , respectively, the dimension of the periodic cell and the body
 731 along the direction \mathcal{E}_3 . Moreover, we suppose that the interface Γ_0 is the
 732 middle point $\ell/2$, so that, each material under consideration has the same
 733 volume in the microscopic cell \mathcal{Y}_0 .

734 **(ii) Material properties.** We prescribe the elasticity tensor \mathcal{C}^ε to be in-
 735 dependent on the macroscale variable X_3 , i.e. $\mathcal{C}^\varepsilon(X_3) = \mathcal{C}(X_3, Y_3) \equiv \mathcal{C}(Y_3)$,
 736 where $\{Y_A\}_{A=1}^3$ is a system of microscale Cartesian coordinates. In addition,
 737 as stated above, we consider that the constituents of the heterogeneous ma-
 738 terial are isotropic, which implies that the non zero components of the 6×6
 739 symmetric matrix representation of \mathcal{C} are given by

$$[\mathcal{C}]_{11} = [\mathcal{C}]_{22} = [\mathcal{C}]_{33} = \lambda + 2\mu, \quad (75a)$$

$$[\mathcal{C}]_{12} = [\mathcal{C}]_{13} = [\mathcal{C}]_{23} = \lambda, \quad (75b)$$

$$[\mathcal{C}]_{44} = [\mathcal{C}]_{55} = [\mathcal{C}]_{66} = \frac{1}{2}([\mathcal{C}]_{11} - [\mathcal{C}]_{12}) = \mu, \quad (75c)$$

740 where λ and μ are Lamé's parameters. We suppose that \mathcal{C} is piece-wise
 741 constant, which means that λ and μ are defined as

$$\lambda(Y_3) = \begin{cases} \lambda_1, & \text{in } \mathcal{Y}_0^1 \\ \lambda_2, & \text{in } \mathcal{Y}_0^2 \end{cases} \quad \text{and} \quad \mu(Y_3) = \begin{cases} \mu_1, & \text{in } \mathcal{Y}_0^1 \\ \mu_2, & \text{in } \mathcal{Y}_0^2 \end{cases}. \quad (76)$$

742 Furthermore, we consider that γ has the same value in both constituents,
 743 which means that it is already averaged.

744 **(iii) Plastic-like distortions.** We assume that the matrix representa-
 745 tion of the tensor $\mathbf{F}_p^{(0)}$ is diagonal with non-zero components $[\mathbf{F}_p^{(0)}]_{11} = \frac{1}{\sqrt{p}}$,
 746 $[\mathbf{F}_p^{(0)}]_{22} = \frac{1}{\sqrt{p}}$ and $[\mathbf{F}_p^{(0)}]_{33} = p$, where p is defined as the remodeling pa-
 747 rameter. Furthermore, we restrict our investigation to the simpler case of
 748 $\mathbf{F}_p^{(0)}$ depending solely on X_3 . This means that, the plastic-like distortions of
 749 order ε^0 are, in a sense, already averaged, and thus variable from one cell
 750 to the other, not inside them. In other words, we are interested in the pro-
 751 duction of distortions in the tissue starting from the cell scale, rather than
 752 from the cell's microstructure. This, of course, does not mean that the cell's
 753 microstructure does not change.

754 Together with assumption (ii), we find that the 6×6 matrix represen-
 755 tation of the elasticity tensor, pulled-back to the reference configuration,
 756 is symmetric, and its non-zero components are given by

$$[\mathcal{C}_R]_{11} = [\mathcal{C}_R]_{22} = (\lambda + 2\mu)p^2, \quad [\mathcal{C}_R]_{33} = (\lambda + 2\mu)p^{-4}, \quad (77a)$$

$$[\mathcal{C}_R]_{12} = \lambda p^2, \quad [\mathcal{C}_R]_{44} = [\mathcal{C}_R]_{55} = \mu p^{-1}, \quad (77b)$$

$$[\mathcal{C}_R]_{13} = [\mathcal{C}_R]_{23} = \lambda p^{-1}, \quad [\mathcal{C}_R]_{66} = \mu p^2. \quad (77c)$$

757 We remark that \mathcal{C}_R depends on X_3 and time through p , whereas it inherits
 758 the dependence of \mathcal{C} on the micro-scale variable, Y_3 .

759 **(iv) Initial and boundary conditions.** In the present context, we im-
 760 pose Dirichlet conditions for $\mathbf{u}^{(0)}$ on the whole boundary $\partial\mathcal{B}_h$, i.e. we do not
 761 consider a Neumann condition and therefore, $\partial_u\mathcal{B}_h \equiv \partial\mathcal{B}_h$. We note that,
 762 although the homogenization process was developed for mixed boundary con-
 763 ditions, the whole procedure stands, since the type of boundary conditions
 764 does not play a role in the derivation of the homogenized model. In par-
 765 ticular, we set $[\mathbf{u}^{(0)}]_3 = 0$ at $X_3 = 0$, and $[\mathbf{u}^{(0)}]_3 = \frac{u_L t}{t_f}$ at $X_3 = L$, where
 766 u_L is a target value for the displacement in the direction \mathcal{E}_3 . Moreover,

we enforce an initial spatial distribution for the remodeling parameter p as $p_{\text{in}}(X_3) = \alpha + \beta \cos(\frac{\pi}{L}X_3)$, where α and β are constants, such that $p_{\text{in}}(X_3)$ is always strictly positive.

7.1. Discussion of the numerical results

Given the above considerations, we solve the following homogenized equations for $\mathbf{u}^{(0)}$ and p ,

$$-\frac{\partial}{\partial X_3}([\hat{\mathcal{C}}_{\text{R}}]_{i3n3} \frac{\partial[\mathbf{u}^{(0)}]_n}{\partial X_3}) = \frac{\partial[\hat{\mathbf{D}}_R]_{i3}}{\partial X_3}, \quad \text{for } i = 1, 2, 3, \quad (78a)$$

$$\langle[\mathbf{C}_{\text{lin}}^{(0)}]_{33}\rangle \frac{\partial p}{\partial t} = \frac{\gamma}{3} \langle \text{dev}(\boldsymbol{\Sigma}_{\text{lin}}^{(0)}) \rangle p - \gamma \langle [\mathcal{C}_{\text{R}}]_{33nn}[\mathbf{E}_{\text{p}}]_{nn}([\mathbf{C}_{\text{lin}}^{(0)}]_{33} - 1) \rangle p, \quad (78b)$$

The coefficients $[\hat{\mathcal{C}}_{\text{R}}]_{ijkl}$, $[\hat{\mathbf{D}}_R]_{ij}$ and $[\mathbf{C}_{\text{lin}}^{(0)}]_{ij}$ are given by Equations (47a), (47b) and (62), respectively, and are to be found by solving the auxiliary cell problems for $\boldsymbol{\xi}$ and $\boldsymbol{\omega}$, given by

$$-\frac{\partial}{\partial Y_3}([\mathcal{Q}]_{i3i3} \frac{\partial[\boldsymbol{\xi}]_{ik3}}{\partial Y_3}) = \frac{\partial[\mathcal{Q}]_{i3i3}}{\partial Y_3} \delta_{ik}, \quad \text{for } i, k = 1, 2, 3, \quad (79a)$$

$$-\frac{\partial}{\partial Y_3}([\mathcal{Q}]_{i3i3} \frac{\partial[\boldsymbol{\omega}]_i}{\partial Y_3}) = -\frac{\partial[\mathbf{Q}]_{33}}{\partial Y_3} \delta_{i3}, \quad \text{for } i = 1, 2, 3, \quad (79b)$$

with

$$[\mathcal{Q}]_{i3i3} = [\mathcal{C}_{\text{R}}]_{i3i3} - [\mathbf{Q}]_{33}, \quad [\mathbf{Q}]_{33} = [\mathcal{C}_{\text{R}}]_{33nn}[\mathbf{E}_{\text{p}}]_{nn}. \quad (80a)$$

In this work, we are not interested to address a real world situation. Our aim is, instead, to show how the present theoretical framework can be numerically simulated. For this reason, the parameters used in our computations are arbitrarily chosen (see Table 1).

In Fig. 2, it is plotted the time evolution of the remodeling parameter p at two different points of the macroscopic domain, that is at $X_3 = 7$ cm and $X_3 = 21$ cm. We observe that the evolution of p is quite different at these two points. Indeed, at $X_3 = 21$ cm, p increases and it is always greater than one. On the contrary, at $X_3 = 7$ cm, it is monotonically decreasing and tends to be lower than one. In Fig. 3, we show the spatial profile of the effective coefficients $[\hat{\mathcal{C}}]_{33}$, $[\hat{\mathcal{C}}_{\text{R}}]_{33}$ and $[\hat{\mathbf{D}}_R]_{33}$. The effective coefficient $[\hat{\mathcal{C}}]_{33}$ (see Remark 3) can be computed by using the analytical formula (see e.g. [56, 69]),

Parameter	Unit	Value	Parameter	Unit	Value
L	[cm]	28.000	λ_1	[Pa]	1.00
u_L	[cm]	1.0000	λ_2	[Pa]	2.00
γ	[1/s]	1.0000	μ_1	[Pa]	0.10
α	[—]	1.0035	μ_2	[Pa]	0.06
β	[—]	−0.0035	t_0	[s]	0.00
N	[—]	4.0000	t_f	[s]	10.0

Table 1: Parameters used in the numerical simulations.

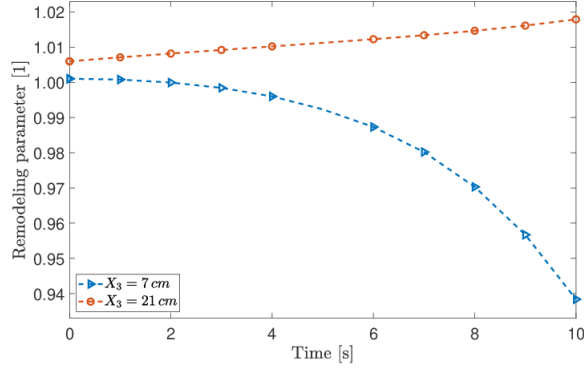


Figure 2: Evolution of the remodeling parameter p at two different points ($X_3 = 7$ cm and $X_3 = 21$ cm) of the macroscopic domain.

$$\begin{aligned}
[\hat{\mathcal{C}}]_{ijkl} = & \langle [\mathcal{C}]_{ijkl} - [\mathcal{C}]_{ijp3}([\mathcal{C}]_{p3s3})^{-1}[\mathcal{C}]_{s3kl} \rangle \\
& + \langle [\mathcal{C}]_{ijp3}([\mathcal{C}]_{p3s3})^{-1} \rangle \langle ([\mathcal{C}]_{s3t3})^{-1} \rangle^{-1} \langle ([\mathcal{C}]_{t3m3})^{-1}[\mathcal{C}]_{m3kl} \rangle. \quad (81)
\end{aligned}$$

790 We observe that even if a loading ramp condition has been imposed on $\mathbf{u}^{(0)}$
791 at the border $X_3 = L$, the effective coefficient $[\hat{\mathcal{C}}]_{33}$ does not vary on time.
792 This is because, in contrast to the case in which the plastic-like distortions
793 are accounted for, the cell and homogenized problems (cf. (48) and (49)) are
794 decoupled. On the other hand, the pulled-back effective coefficients $[\hat{\mathcal{C}}_{\text{R}}]_{33}$
795 and $[\hat{\mathbf{D}}_{\text{R}}]_{33}$, given by Equations (47a) and (47b), respectively, do change in
796 time since their equations are coupled with an evolution one and, as it can
797 be observed, they are strongly influenced by the initial distribution of p . In
798 fact, at the spatial point $X_3 = 21$ cm, that is, when $p > 1$, $[\hat{\mathcal{C}}_{\text{R}}]_{33}$ decreases

799 and $[\hat{\mathbf{D}}_{\mathbf{R}}]_{33}$ increases with time. The contrary occurs at $X_3 = 7$ cm, i.e. when
800 $p < 1$.

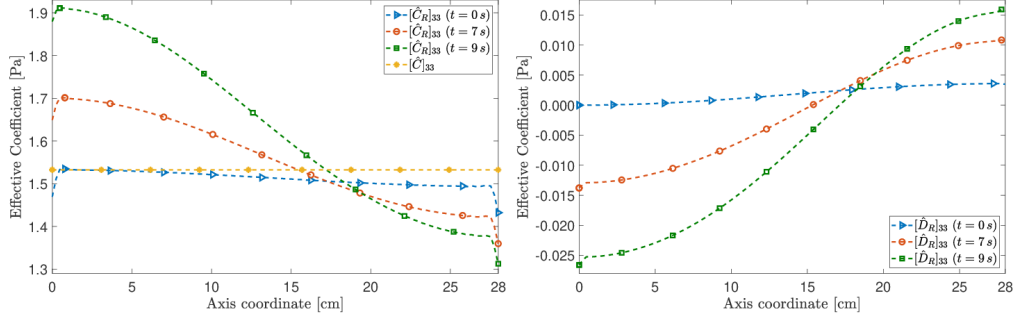


Figure 3: Spatial distribution of the effective coefficients $[\hat{\mathcal{C}}]_{33}$, $[\hat{\mathcal{D}}_{\mathbf{R}}]_{33}$ and $[\hat{\mathbf{D}}_{\mathbf{R}}]_{33}$ at different time instants.

801 Additionally, in Fig. 4 it is illustrated the third component of the macroscopic leading order term of the displacement \mathbf{u}^ε at three different time
802 instants. Particularly, we plot the numerical solution of the homogenized
803 problems (46) and (49), represented with $[\mathbf{u}_{\mathbf{R}}^{(0)}]_3$ and $[\mathbf{u}^{(0)}]_3$, respectively. We
804 note that, as expected from our election of the boundary condition, the dis-
805 placement component increases monotonically in time. However, we notice
806 that the introduction of the plastic-like distortions has a direct impact on the
807 displacement distribution in the interior macroscopic points. Specifically, in
808 these points the displacement has a higher magnitude.
809

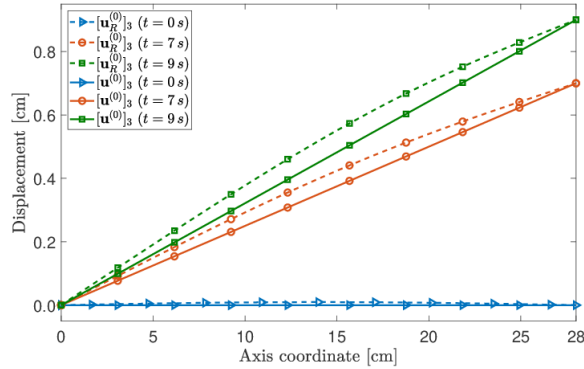


Figure 4: Spatial distribution of the macroscopic leading order term of the displacement with remodeling ($[\mathbf{u}_{\mathbf{R}}^{(0)}]_3$) and without remodeling ($[\mathbf{u}^{(0)}]_3$).

810 The situation described in our numerical simulations, although simplified,

811 could be a good starting point in the study of the remodeling of biological
812 tissues. For example, the geometrical properties of bone's osteons permit to
813 model them as layered composites (see e.g. [69]).

814 **8. Concluding remarks**

815 In the present work, we studied the dynamics of a heterogeneous material,
816 constituted by two hyperelastic media with evolving micro-structure, by the
817 application of the asymptotic homogenization technique. The evolution of
818 the micro-structure of the composite media was characterized through the
819 development of plastic-like distortions, which were described by means of the
820 BKL decomposition.

821 The asymptotic homogenization method was applied to a set of problems
822 comprising a scale-dependent, quasi-static law of balance of linear momentum
823 and an evolution law for the tensor of plastic-like distortions. After obtaining
824 the local and homogenized problems, we rewrote them by considering the De
825 Saint-Venant strain energy density within the limit of small deformations.
826 Although the selection of the strain energy density was due to its simplicity,
827 it is helpful for the description of remodeling processes undergoing small
828 deformations. For instance, this could be the case for describing bone aging.
829 Then, the theoretical setting developed in the present work is applicable
830 (Elastoplasticity is actually quite appropriate to model the bone [73]). In
831 such a case, appropriate constitutive laws describing the progression of the
832 material properties should be found based on experimental literature (e.g.
833 [35]). Nevertheless, for studying a larger range of problems, we need to select
834 nonlinear constitutive laws and write the corresponding cell and homogenized
835 problems.

836 As a consequence of the introduction of the tensor of plastic distortions,
837 two independent cell problems were inferred, which reduce to the classical cell
838 problems encountered in the homogenization of linear problems in elastostat-
839 ics. Moreover, we proposed an evolution equation for the inelastic distortions
840 describing a remodeling process. Such evolution law models a stress-driven
841 production of inelastic distortions, as the one that is often encountered in
842 studies of inelastic processes constructed on the decomposition given by (5)
843 [78]. The evolution law is suitable for the case of finite strain Elastoplastic-
844 ity, and for the case of remodeling of biological tissues. Finally, we outlined
845 a computational procedure in order to solve the up-scaled problems and we
846 performed numerical simulations for a particular case of a layered composite

body. Besides, we assumed that the leading order term of the asymptotic expansion of the tensor of plastic distortions, $\mathbf{F}_p^{(0)}$, depends only on the macro-scale variable X . This consideration, however, might be relaxed by allowing $\mathbf{F}_p^{(0)}$ to take into account the heterogeneities of the composite material through the microscopic spatial variable Y . The numerical results showed the influence of the plastic-like distortions on both the effective coefficients and the macroscopic leading order term of the displacement.

As future work, we intend to deal with the resolution of a particular problem, like for instance the modeling of bones [49], tumor growth [67, 2, 43, 52, 70, 71], or tissue aging [20]. A further step could be the study, with the aid of the Homogenization Theory, of the coupling between the results presented in this work and the fluid flow in a hydrated tissue, or in the case of wavy laminar structures.

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Declaration of interest

The Authors declare that they have no conflict of interest.

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