POLITECNICO DI TORINO
Repository ISTITUZIONALE

If the primes are finite, then all of them divide the number one

Original
If the primes are finite, then all of them divide the number one / Cerruti, Umberto; Murru, Nadir. - In: THE AMERICAN MATHEMATICAL MONTHLY. - ISSN 0002-9890. - 124:(2017), pp. 969-969.

Availability:
This version is available at: 11583/2719242 since: 2018-12-03T12:21:37Z

Publisher:
Mathematical America Association

Published
DOI:

Terms of use:

This article is made available under terms and conditions as specified in the corresponding bibliographic description in the repository

Publisher copyright

(Article begins on next page)

23 April 2024

If the primes are finite, then all of them divide the number one

We propose a novel proof of the infinitude of the primes based on elementary considerations of Legendre's function ϕ, defined in [**1**, p. 153] as

$$\phi(x,y) = |\{1 \le n \le x : \text{integer } n \text{ has no prime factors} \le y\}|,$$

where x and y are positive integers. The reader can see that

$$\pi(x) = \pi(\sqrt{x}) + \phi(x, \sqrt{x}) - 1,$$

where $\pi(\cdot)$ is the prime-counting function. Let $p_1, ..., p_s$ be the prime numbers less than or equal to y. Using the inclusion-exclusion principle, it can be proved that

$$\phi(x,y) = x - \sum_{1 \le i \le s} \left[\frac{x}{p_i}\right] + \sum_{1 \le i,j \le s} \left[\frac{x}{p_i p_j}\right] + ... + (-1)^s \left[\frac{x}{p_1 \cdots p_s}\right],$$

where $[\cdot]$ is the floor function. Pinasco [**2**] also used this principle for proving the infinitude of the primes, but his proof is remarkably different from ours.

Suppose that $\{p_1, ..., p_s\}$ is the set of all prime numbers. Consider $N = p_1 \cdots p_s$. Then $\phi(N^2, N) = 1$. On the other hand, we have

$$\phi(N^2, N) = N^2 - \sum_{1 \le i \le s} \left[\frac{N^2}{p_i}\right] + \sum_{1 \le i,j \le s} \left[\frac{N^2}{p_i p_j}\right] + ... + (-1)^s N.$$

Hence, $\phi(N^2, N) = mN$ for some integer m, i.e., every prime number divides 1. This means that all primes, and, consequently, all non–zero natural numbers, are invertible in $\mathbb{N}$, i.e., we find that $\mathbb{N}$ is a field. This completes the proof.

REFERENCES

1. R. Crandall, C. Pomerance, *Prime numbers. A computational perspective*, Springer–Verlag, New York, 2nd Edition, 2005.
2. J. P. Pinasco, New proofs of Euclid's and Euler's theorems, *Amer. Math. Monthly* **116** (2009) 172–174.

—Submitted by Umberto Cerruti and Nadir Murru

http://dx.doi.org/10.XXXX/amer.math.monthly.122.XX.XXX
MSC: Primary 11A41, Secondary 11A99