ON THE p-ADIC DENSENESS OF THE QUOTIENT SET OF A POLYNOMIAL IMAGE

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Abstract. The *quotient set*, or *ratio set*, of a set of integers A is defined as

$$R(A) := \{a/b : a, b \in A,\ b \neq 0\}.$$

We consider the case in which A is the image of $\mathbb{Z}^+$ under a polynomial $f \in \mathbb{Z}[X]$, and we give some conditions under which $R(A)$ is dense in $\mathbb{Q}_p$. Then, we apply these results to determine when $R(S_m^n)$ is dense in $\mathbb{Q}_p$, where S_m^n is the set of numbers of the form $x_1^n + \cdots + x_m^n$, with $x_1, \dots, x_m \geq 0$ integers. This allows us to answer a question posed in [Garcia *et al.*, p-adic quotient sets, Acta Arith. **179**, 163–184]. We end leaving an open question.

1. Introduction

The *quotient set*, also known as *ratio set*, of a set of integers A is defined as

$$R(A) := \left\{ \frac{a}{b} : a, b \in A,\ b \neq 0 \right\}.$$

The question of when $R(A)$ is dense in $\mathbb{R}^+$ is a classical topic and has been studied by many researchers (see, e.g., [1, 2, 3, 7, 8, 9, 11, 15]).

Recently, some authors approached the study of the denseness of $R(A)$ in the field of p-adic numbers $\mathbb{Q}_p$. Garcia and Luca [6] proved that the quotient set of the Fibonacci numbers is dense in $\mathbb{Q}_p$, and Sanna [12] extended this result to the k-generalized Fibonacci numbers. In [5], the denseness of $R(A)$ in $\mathbb{Q}_p$ is studied when A is the set of values of a Lucas sequence, the set of positive integers which are sum of k squares, respectively k cubes, or the union of two geometric progressions. Moreover, Miska and Sanna [10] proved that, given any partition $A_1, \dots, A_k$ of $\mathbb{Z}^+$, for all prime numbers p but at most $\lfloor \log_2 k \rfloor$ exceptions at least one of $R(A_1), \dots, R(A_k)$ is dense in $\mathbb{Q}_p$.

In this paper, we focus on the study of the denseness of $R(A)$ in $\mathbb{Q}_p$ when A is the image of $\mathbb{Z}^+$ under a polynomial $f \in \mathbb{Z}[X]$. For the sake of notation, we put $R_f := R(f(\mathbb{Z}^+))$ for any function $f : \mathbb{Z} \to \mathbb{Q}_p$. The following easy lemma provides a necessary condition under which R_f is dense in $\mathbb{Q}_p$.

Lemma 1.1. *Let $f : \mathbb{Z}_p \to \mathbb{Q}_p$ be a continuous function. If R_f is dense in $\mathbb{Q}_p$, then f has a zero in $\mathbb{Z}_p$.*

Proof. Since R_f is dense in $\mathbb{Q}_p$, there exists a sequence of integers $(x_n)_{n \geq 0}$ such that $f(x_n) \to 0$ (in the p-adic topology) as $n \to \infty$. By the compactness of $\mathbb{Z}_p$, there exists a subsequence $(x_{n_k})_{k \geq 0}$ converging to some $x_\infty \in \mathbb{Z}_p$. Since f is continuous, we get $f(x_\infty) = 0$, as desired. □

Our first result is a sufficient condition under which R_f is dense in $\mathbb{Q}_p$. We postpone its proof to Section 2.

Theorem 1.2. *Let $f : \mathbb{Z}_p \to \mathbb{Q}_p$ be an analytic function and let $z_1, z_2 \in \mathbb{Z}_p$ be two (not necessarily distinct) zeros of f of multiplicities μ_1, μ_2, respectively. If μ_1, μ_2 are coprime, then R_f is dense in $\mathbb{Q}_p$.*

As an immediate consequence we have the following corollary.

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Corollary 1.3. *If $f : \mathbb{Z}_p \to \mathbb{Q}_p$ is an analytic function with a simple zero in $\mathbb{Z}_p$, then R_f is dense in $\mathbb{Q}_p$.*

The above results make possible to completely characterize the linear and quadratic polynomials f for which R_f is dense in $\mathbb{Q}_p$.

Proposition 1.4. *Let $f \in \mathbb{Z}[X]$ be a polynomial of degree 1 or 2. Then, R_f is dense in $\mathbb{Q}_p$ if and only if f has a simple zero in $\mathbb{Z}_p$.*

Proof. When f has degree 1, the thesis follows immediately from Lemma 1.1 and Corollary 1.3. Assume f has degree 2. If f has a simple zero in $\mathbb{Z}_p$, then R_f is dense in $\mathbb{Q}_p$ by Corollary 1.3. On the other hand, if f has no simple zeros in $\mathbb{Z}_p$, then we have two cases. In the first case, f has no zeros in $\mathbb{Z}_p$. Then, by Lemma 1.1, R_f is not dense in $\mathbb{Q}_p$. In the second case, f has a zero in $\mathbb{Z}_p$ with multiplicity 2, i.e., $f(x) = a(x - z)^2$, for some $a, z \in \mathbb{Z}_p$ with $a \neq 0$. Consequently, R_f is not dense in $\mathbb{Q}_p$, since the p-adic valuation of each element of R_f is divisible by 2. □

For polynomials of higher degrees, we can not exploit Lemma 1.1 and Corollary 1.3 to determine if R_f is dense in $\mathbb{Q}_p$. For instance, consider the case of a polynomial of degree 3 with a double root in $\mathbb{Z}_p$ and the other root not in $\mathbb{Z}_p$. However, if we consider polynomials having all their roots in $\mathbb{Z}_p$, then we have the following result.

Proposition 1.5. *Let $f \in \mathbb{Z}[X]$ be a nonconstant polynomial splitting in $\mathbb{Z}_p$ and of degree less than 31. Then, R_f is not dense in $\mathbb{Q}_p$ if and only if there exists an integer $n > 1$ which divides the multiplicity of each root of f.*

Proof. Let $\mu_1, \dots, \mu_s$ be the multiplicities of the roots of f. If there exists an integer $n > 1$ dividing all $\mu_1, \dots, \mu_s$, then $f = ag^n$, for some $a \in \mathbb{Z} \setminus \{0\}$ and $g \in \mathbb{Z}[X]$. Consequently, R_f is not dense in $\mathbb{Q}_p$, since the p-adic valuation of each element of R_f is divisible by n. Now suppose that there exists no integer $n > 1$ dividing all $\mu_1, \dots, \mu_s$. We shall prove that $\gcd(\mu_i, \mu_j) = 1$ for some i, j. In this way, by Theorem 1.2, it follows that R_f is dense in $\mathbb{Q}_p$. For the sake of contradiction, assume $\gcd(\mu_i, \mu_j) > 1$ for all i, j. In particular, we have $s \geq 3$, and that each μ_i has at least two distinct prime factors. Also, at least one of $\mu_1, \dots, \mu_s$ is odd. Without loss of generality, we can assume μ_1 odd. Thus $\mu_1 \in \{15, 21\}$, and at least one of $\mu_2, \dots, \mu_s$ is not divisible by 3. Without loss of generality, we can assume μ_2 not divisible by 3. Thus $\mu_2 \in \{10, 14\}$. Since μ_3 has at least two distinct prime factors, $\mu_3 \geq 6$ and consequently $\deg f = \mu_1 + \dots + \mu_s > 30$, absurd. □

Remark 1.6. Proposition 1.5 is optimal in the sense that there exists a polynomial $f \in \mathbb{Z}[X]$ of degree 31, splitting in $\mathbb{Z}_p$, with the greatest common divisor of the multiplicities of its roots equal to 1, but such that R_f is not dense in $\mathbb{Q}_p$. Indeed, consider

$$f(X) = (X + 1)^6 (X + 2)^{10} (X + 3)^{15}.$$

Then, for $p > 2$ (respectively $p = 2$) the p-adic valuation of each element of $f(\mathbb{Z}^+)$ is of the form $6n$, $10n$, or $15n$ (respectively $10n$, $6n+15$, or $15n+6$), for some integer $n \geq 0$. Therefore, no element of R_f has p-adic valuation equal to 1 (respectively 2), and R_f is not dense in $\mathbb{Q}_p$.

Remark 1.7. Using the same reasonings as in the proof of Proposition 1.5, one can prove a slightly more general statement: Given $f = gh$, where $g, h \in \mathbb{Z}[X]$ are such that g splits in $\mathbb{Z}_p$, $1 \leq \deg g \leq 30$, and the p-adic valuation of h is constant, we have that R_f is not dense in $\mathbb{Q}_p$ if and only if there does not exist an integer $n > 1$ dividing all the multiplicities of the roots of g.

For integers $m, n \geq 2$, define the set

$$S_m^n := \{x_1^n + \dots + x_m^n : x_1, \dots, x_m \in \mathbb{Z}_{\geq 0}\}.$$

The authors of [5] considered $n = 2, 3$ and proved the following results [5, Theorems 4.1 and 4.2]. (Actually, there is a small error, here corrected, in [5, Theorem 4.2], see Remark 1.15 below.)

Theorem 1.8. *For all prime numbers p, we have:*

(a) *$R(S_2^2)$ is dense in $\mathbb{Q}_p$ if and only if $p \equiv 1 \pmod 4$.*

(b) *$R(S_m^2)$ is dense in $\mathbb{Q}_p$ for all integers $m \geq 3$.*

(c) *$R(S_m^3)$ is dense in $\mathbb{Q}_p$ for all integers $m \geq 2$.*

For all integers $n, b \geq 2$, let $\gamma(n, b)$ denote the smallest positive integer g such that for every $a \in \mathbb{Z}$ the equation

$$X_1^n + \cdots + X_g^n \equiv a \pmod b \tag{1}$$

has a solution. Furthermore, let $\theta(n, b)$ be the smallest positive integer g such that for $a = 0$ the equation (1) has a solution with at least one of $X_1, \ldots, X_g$ coprime with b. The quantities $\gamma(n, b)$, $\theta(n, b)$ have been studied in regard to analogs of Waring's problem modulo p (see, e.g., [13, 14]).

We give an effective criterion to establish if $R(S_m^n)$ is dense in $\mathbb{Q}_p$. We postpone its proof to Section 3.

Theorem 1.9. *Let $m, n \geq 2$ be integers, let p be a prime number, and put $k := \nu_p(n)$.*

(a) *If $m \geq \theta(n, p^{2k+1})$, then $R(S_m^n)$ is dense in $\mathbb{Q}_p$.*

(b) *If $m < \theta(n, p^{2k+1})$ and $(n, p) \notin \{(2, 2), (4, 2), (8, 2), (16, 2)\}$, then $R(S_m^n)$ is not dense in $\mathbb{Q}_p$.*

(c) *$R(S_m^2)$ is dense in $\mathbb{Q}_2$ if and only if $m \geq 3$.*

(d) *$R(S_m^4)$ is dense in $\mathbb{Q}_2$ if and only if $m \geq 8$.*

(e) *$R(S_m^8)$ is dense in $\mathbb{Q}_2$ if and only if $m \geq 16$.*

(f) *$R(S_m^{16})$ is dense in $\mathbb{Q}_2$ if and only if $m \geq 64$.*

Example 1.10. Let us consider the denseness of $R(S_m^6)$ in $\mathbb{Q}_{11}$. In order to apply Theorem 1.9, we have to compute $\theta(6, 11)$. The nonzero sixth powers modulo 11 are 1, 3, 4, 5, and 9. Hence, the minimum positive integer g such that the equation $X_1^6 + \cdots + X_g^6 \equiv 0 \pmod{11}$ has a solution, with at least one of $X_1, \ldots, X_g$ not divisible by 11, is $\theta(6, 11) = 3$. Consequently, by points (a) and (b) of Theorem 1.9, we have that $R(S_m^6)$ is dense in $\mathbb{Q}_{11}$ if and only if $m \geq 3$.

Example 1.11. Let us consider the denseness of $R(S_m^{10})$ in $\mathbb{Q}_2$. In order to apply Theorem 1.9, we have to compute $\theta(10, 8)$. We have $x^{10} \equiv 1 \pmod 8$ for each odd integer x. Hence, it follows easily that $\theta(10, 8) = 8$. Consequently, by points (a) and (b) of Theorem 1.9, we have that $R(S_m^{10})$ is dense in $\mathbb{Q}_2$ if and only if $m \geq 8$.

For $m = 2$, we have the following corollary.

Corollary 1.12. *Let $n \geq 2$ be an integer, let p be a prime number, and put $k = \nu_p(n)$. Then $R(S_2^n)$ is dense in $\mathbb{Q}_p$ if and only if -1 is an nth power modulo p^{2k+1}. In particular, $R(S_2^n)$ is dense in $\mathbb{Q}_p$ whenever n is odd.*

Proof. First, assume $p = 2$ and $n \in \{2, 4, 8, 16\}$. Then, it can be easily checked that -1 is not an nth power modulo p^{2k+1}. By Theorem 1.8, $R(S_2^2)$ is not dense in $\mathbb{Q}_p$ and, since $S_2^n \subseteq S_2^2$, we get that $R(S_2^n)$ is not dense in $\mathbb{Q}_p$. Now assume $(n, p) \notin \{(2, 2), (4, 2), (8, 2), (16, 2)\}$. By Theorem 1.9, we have that $R(S_2^n)$ is dense in $\mathbb{Q}_p$ if and only if there exist integers $0 \leq x_1, x_2 < p^{2k+1}$, not both divisible by p, such that $x_1^n + x_2^n$ is divisible by p^{2k+1}. It easy to see that this last condition is equivalent to the -1 being an nth power modulo p^{2k+1}. $\square$

In [5, Problem 4.3] it is asked about the denseness in $\mathbb{Q}_p$ of $R(S_m^4)$ and $R(S_m^5)$. From Corollary 1.12, we have that $R(S_m^5)$ is dense in $\mathbb{Q}_p$ for all integers $m \geq 2$ and prime numbers p. Regarding $R(S_m^4)$, the situation is more complicated. Theorem 1.9(d) already covers the case $p = 2$. For $p > 2$ we have the following result.

Theorem 1.13. *For all prime numbers $p > 2$, we have:*

(a) *$R(S_2^4)$ is dense in $\mathbb{Q}_p$ if and only if $p \equiv 1 \pmod 8$.*

(b) *$R(S_3^4)$ is dense in $\mathbb{Q}_p$ if and only if $p \neq 5, 29$.*

(c) *$R(S_4^4)$ is dense in $\mathbb{Q}_p$ if and only if $p \neq 5$.*

(d) *$R(S_m^4)$ is dense in $\mathbb{Q}_p$ for all integers $m \geq 5$.*

Proof. By Corollary 1.12, $R(S_2^4)$ is dense in $\mathbb{Q}_p$ if and only if -1 is a fourth power modulo p. In turn, this is well known to be equivalent to $p \equiv 1 \pmod 8$. Hence, (a) is proved. Substituting $a = -1$ into (1), the bound $\theta(n, b) \leq \gamma(n, b) + 1$ follows. From [13, Theorem 3′], we have that $\gamma(4, p) = 2$ for all prime numbers $p > 41$. Hence, $\theta(4, p) \leq 3$ for all prime numbers $p > 41$. Then, a computation shows that $\theta(4, p) \leq 3$ for all prime numbers $p \neq 5, 29$. Precisely, $\theta(4, 5) = 5$ and $\theta(4, 29) = 4$. Now the claims (b), (c), and (d) follow from Theorem 1.9. □

We leave the following general question to the readers.

Question 1.14. *Given a prime number p and a polynomial $f \in \mathbb{Z}[X]$, is there an effective criterion to establish if R_f is dense in $\mathbb{Q}_p$? What about multivariate polynomials?*

Remark 1.15. In [5, Theorem 4.2] it is stated that $R(S_2^3)$ is not dense in $\mathbb{Q}_3$. This is not correct, since $R(S_2^3)$ is dense in $\mathbb{Q}_3$ in light of Corollary 1.12. The mistake in the proof of [5, Theorems 4.2] is when, at point (b2), it is asserted that: "If $x/y \in R(S_2^3)$ is sufficiently close to 3 in $\mathbb{Q}_3$, then $\nu_3(x) = \nu_3(y)+1$. Without loss of generality, we may suppose that $\nu_3(x) = 1$ and $\nu_3(y) = 0$." This is not true, because if y is the sum of two cubes, then there is no guarantee that $y/3^{\nu_3(y)}$ is still the sum of two cubes. For instance, if $y = 1^3 + 5^3$ then $y/3^{\nu_3(y)} = 14$ is not the sum of two cubes.

Notation. For each prime number p, let ν_p denote the usual p-adic valuation, with the convention $\nu_p(0) := +\infty$. For integers a and $m > 0$, we write $(a \bmod m)$ for the unique integer $r \in \,]-b/2, b/2]$ such that $a - r$ is divisible by m.

2. Proof of Theorem 1.2

We have to prove that for all $r \in \mathbb{Q}_p$ and $u > 0$ there exist $x_1, x_2 \in \mathbb{Z}^+$ such that $f(x_2) \neq 0$ and

$$\nu_p\left(\frac{f(x_1)}{f(x_2)} - r\right) > u.$$

Clearly, since $\mathbb{Q}_p^*$ is dense in $\mathbb{Q}_p$, it is enough to consider $r \neq 0$. Furthermore, since $\mathbb{Z}^+$ is dense in $\mathbb{Z}_p$ and f is continuous, we can assume, less restrictively, $x_1, x_2 \in \mathbb{Z}_p$. By hypothesis, for $i = 1, 2$, we have $f(X) = (X - z_i)^{\mu_i} g_i(X)$, where $g_i : \mathbb{Z}_p \to \mathbb{Q}_p$ is an analytic function such that $g_i(z_i) \neq 0$. Put $x_i := y_i p^{k_i} + z_i$, for $i = 1, 2$, where $y_1, y_2 \in \mathbb{Z}_p \setminus \{0\}$ and $k_1, k_2 \in \mathbb{Z}^+$ will be chosen later. Without loss of generality, we can assume $\nu_p(g_1(z_1)) \leq \nu_p(g_2(z_2))$. Thus, setting $G := g_2(z_2)/g_1(z_1)$, we have $G \in \mathbb{Z}_p \setminus \{0\}$. Since g_1, g_2 are continuous, for sufficiently large k_1, k_2 we have

$$\nu_p\left(G \cdot \frac{g_1(x_1)}{g_2(x_2)} - 1\right) > u - \nu_p(r), \tag{2}$$

In particular, it is implicit that $g(x_2) \neq 0$ and consequently $f(x_2) \neq 0$. We fix k_1, k_2 such that

$$k_1\mu_1 - k_2\mu_2 = \nu_p(r),$$

and (2) holds. This is possible thanks to the condition $\gcd(\mu_1, \mu_2) = 1$. Indeed, by Bézout's lemma, the quantity $k_1\mu_1 - k_2\mu_2$ can be equal to any integer with k_1 and k_2 arbitrarily large (if $k_1\mu_1 - k_2\mu_2 = a$, then $(k_1 + K\mu_2)\mu_1 - (k_2 + K\mu_1)\mu_2 = a$, for any integer K).

Again by Bézout's lemma, there exist integers $h_1, h_2 \geq 0$ such that $h_1\mu_1 - h_2\mu_2 = 1$. We set $y_i = s^{h_i}$, for $i = 1, 2$, where $s := p^{-\nu_p(r)} rG$. Note that $y_1, y_2 \in \mathbb{Z}_p \setminus \{0\}$, as required.

Hence, we have

$$\begin{aligned}\frac{f(x_1)}{f(x_2)} &= \frac{(x_1 - z_1)^{\mu_1}}{(x_2 - z_2)^{\mu_2}} \cdot \frac{g_1(x_1)}{g_2(x_2)} = p^{k_1\mu_1 - k_2\mu_2} \cdot \frac{y_1^{\mu_1}}{y_2^{\mu_2}} \cdot \frac{g_1(x_1)}{g_2(x_2)} \\ &= p^{\nu_p(r)} \cdot s^{h_1\mu_1 - h_2\mu_2} \cdot \frac{g_1(x_1)}{g_2(x_2)} = p^{\nu_p(r)} \cdot s \cdot \frac{g_1(x_1)}{g_2(x_2)} = rG \cdot \frac{g_1(x_1)}{g_2(x_2)},\end{aligned}$$

so that, recalling (2), we get

$$\nu_p\left(\frac{f(x_1)}{f(x_2)} - r\right) = \nu_p\left(r\left(G \cdot \frac{g_1(x_1)}{g_2(x_2)} - 1\right)\right) > u,$$

as desired.

3. Proof of Theorem 1.9

(a) Suppose that there exist integers $0 \leq x_1, \ldots, x_m < p^{2k+1}$, not all divisible by p, such that $x_1^n + \cdots + x_m^n$ is divisible by p^{2k+1}. Up to reordering $x_1, \ldots, x_m$, we can assume that $p \nmid x_1$. Put $f(X) = X^n + x_2^n + \cdots + x_m^n$, so that $f'(X) = nX^{n-1}$. In particular, all the roots of f are simple. Since $p \nmid x_1$, we have

$$\nu_p(f(x_1)) \geq 2k + 1 > 2k = 2\nu_p(f'(x_1)),$$

so that, by Hensel's lemma [4, Ch. 4, Lemma 3.1], f has a simple root in $\mathbb{Z}_p$. Hence, by Corollary 1.3, R_f is dense in $\mathbb{Q}_p$. Clearly, $R_f \subseteq R(S_m^n)$, so that $R(S_m^n)$ is dense in $\mathbb{Q}_p$.

(b) Suppose that there are no integers $x_1, \ldots, x_m$ as before, and that

$$(n, p) \notin \{(2, 2), (4, 2), (8, 2), (16, 2)\}. \tag{3}$$

We shall prove that $4k+1 < n$. For the sake of contradiction, suppose $4k+1 \geq n$. Since $n \geq 2$, we have $k \geq 1$. Also, we have $4k + 1 \geq p^k$, which implies $p \leq 5$. Now, taking into account (3), it can be readily checked that

$$(n, p) \in \{(3, 3), (9, 3), (5, 5)\}.$$

But $3^3 \mid (1^3 + 8^3)$, $3^5 \mid (1^9 + 26^9)$, and $5^3 \mid (1^5 + 24^5)$, contradicting the nonexistence of $x_1, \ldots, x_m$.

Let $y_1, \ldots, y_m \geq 0$ be integers, not all equal to zero. Put $\mu := \min\{\nu_p(y_i) : i = 1, \ldots, m\}$, $I := \{i : \nu_p(y_i) = \mu\}$, and $J := \{1, \ldots, m\} \setminus I$. Also, put $z_i := y_i/p^\mu$ for $i \in I$, so that z_i is an integer not divisible by p. The nonexistence of $x_1, \ldots, x_m$ implies that

$$\nu_p\left(\sum_{i \in I} z_i^n\right) \leq 2k. \tag{4}$$

Therefore, since $2k < n$, we have

$$\nu_p\left(\sum_{i \in I} y_i^n\right) = \mu n + \nu_p\left(\sum_{i \in I} z_i^n\right) \leq \mu n + 2k < (\mu + 1)n \leq \nu_p\left(\sum_{j \in J} y_j^n\right),$$

and consequently

$$\nu_p(y_1^n + \cdots + y_m^n) = \nu_p\left(\sum_{i \in I} y_i^n\right) = \mu n + \nu_p\left(\sum_{i \in I} z_i^n\right),$$

which in turn, by (4), implies that

$$(\nu_p(y_1^n + \cdots + y_m^n) \bmod n) \in \{0, \ldots, 2k\}.$$

Thus, for each $a \in R(S_m^n) \setminus \{0\}$ we have

$$(\nu_p(a) \bmod n) \in \{-2k, \ldots, 2k\},$$

that is, the p-adic valuations of the nonzero elements of $R(S_m^n)$ belong to at most $4k + 1$ residue classes modulo n. Since $4k + 1 < n$, at least one residue class modulo n is missing and, a fortiori, $R(S_m^n)$ is not dense in $\mathbb{Q}_p$.

(c) The claim follows immediately from Theorem 1.8.

From now on, assume $n = 2^k$, with $k \in \{2, 3, 4\}$. Let T_m^n be the topological closure of S_m^n in $\mathbb{Q}_2$. Clearly, we have

$$T_m^n = \{x_1^n + \cdots + x_m^n : x_1, \ldots, x_m \in \mathbb{Z}_2\}.$$

It is a standard exercise showing that the nonzero nth powers of $\mathbb{Z}_2^*$ are exactly the elements of the form $1 + 4ny$, with $y \in \mathbb{Z}_2$. As a consequence,

$$T_1^n = \{2^{nv}(1 + 4ny) : v \in \mathbb{Z}_{\geq 0},\, y \in \mathbb{Z}_2\} \cup \{0\}.$$

Let $v_1, v_2 \geq 0$, $j \geq 1$ be integers and $y_1, y_2 \in \mathbb{Z}_2$. If $v_1 = v_2$, then

$$2^{nv_1}(j + 4ny_1) + 2^{nv_2}(1 + 4ny_2) = 2^{nv_1}(j + 1 + 4nz),$$

where $z := y_1 + y_2 \in \mathbb{Z}_2$. If $v_1 < v_2$, then

$$2^{nv_1}(j + 4ny_1) + 2^{nv_2}(1 + 4ny_2) = 2^{nv_1}(j + 4nz),$$

where $z := y_1 + 2^{n(v_2 - v_1) - k - 2}(1 + 4ny_2) \in \mathbb{Z}_2$, since $n = 2^k \geq k + 2$. If $v_1 > v_2$, then

$$2^{nv_1}(j + 4ny_1) + 2^{nv_2}(1 + 4ny_2) = 2^{nv_2}(1 + 4nz),$$

where $z := 2^{n(v_1 - v_2) - k - 2}(j + 4ny_1) + y_2 \in \mathbb{Z}_2$, again since $n \geq k + 2$.

Therefore, it follows easily by induction on m that

$$T_m^n = \{2^{nv}(j + 4ny) : v \in \mathbb{Z}_{\geq 0},\, j \in \{1, \ldots, m\},\, y \in \mathbb{Z}_2\} \cup \{0\}. \tag{5}$$

(d) On the one hand, using (5), it can be checked quickly that $15 \notin R(T_7^4)$. Hence, $R(S_7^4)$ is not dense in $\mathbb{Q}_2$. On the other hand, we have

$$2^{4v+r}(1 + 2y) = \frac{2^{4v}(8 + 16y)}{2^{4\cdot 0}(2^{3-r} + 16 \cdot 0)} \in R(T_8^4),$$

for all $v \in \mathbb{Z}_{\geq 0}$, $r \in \{0, 1, 2, 3\}$, and $y \in \mathbb{Z}_2$. Hence, $\mathbb{Z}_p \subseteq R(T_8^4)$ and, since $R(T_8^4)$ is closed by inversion, we get that $R(T_8^4) = \mathbb{Q}_p$. Thus $R(S_8^4)$ is dense in $\mathbb{Q}_p$.

(e) On the one hand, by (5), the 2-adic valuation of each nonzero element of T_{15}^8 is congruent to 0, 1, 2, or 3 modulo 8. Hence, $R(T_{15}^8)$ contains no element with 2-adic valuation equal to 4, and consequently $R(S_{15}^8)$ is not dense in $\mathbb{Q}_2$. On the other hand, we have

$$2^{8v+r}(1 + 2y) = \frac{2^{8v}(16 + 32y)}{2^{8\cdot 0}(2^{4-r} + 32 \cdot 0)} \in R(T_{16}^8)$$

and

$$2^{8v+r+4}(1 + 2y) = \frac{2^{8(v+1)}(2^r + 32 \cdot 0)}{2^{8\cdot 0}(16 + 32\frac{-y}{1+2y})} \in R(T_{16}^8)$$

for all $v \in \mathbb{Z}_{\geq 0}$, $r \in \{0, 1, 2, 3, 4\}$, and $y \in \mathbb{Z}_2$. Hence, $\mathbb{Z}_p \subseteq R(T_{16}^8)$ and, since $R(T_{16}^8)$ is closed by inversion, we get that $R(T_{16}^8) = \mathbb{Q}_p$. Thus $R(S_{16}^8)$ is dense in $\mathbb{Q}_p$.

(f) On the one hand, by (5), the 2-adic valuation of each nonzero element of T_{63}^{16} is congruent to 0, 1, 2, 3, 4, or 5 modulo 16. Hence, $R(T_{63}^{16})$ contains no element with 2-adic valuation equal to 6, and consequently $R(S_{63}^{16})$ is not dense in $\mathbb{Q}_2$. On the other hand, 2^9 divides $5^{16} + 1^{16} + \cdots + 1^{16}$ (63 times 1^{16}). Hence, by point (a), we get that $R(S_{64}^{16})$ is dense in $\mathbb{Q}_2$.

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References

[1] B. Brown, M. Dairyko, S. R. Garcia, B. Lutz, and M. Someck, *Four quotient set gems*, Amer. Math. Monthly **121** (2014), no. 7, 590–599.

[2] J. Bukor, T. Šalát, and J. T. Tóth, *Remarks on R-density of sets of numbers*, Tatra Mt. Math. Publ. **11** (1997), 159–165, Number theory (Liptovský Ján, 1995).

[3] J. Bukor and J. T. Tóth, *On accumulation points of ratio sets of positive integers*, Amer. Math. Monthly **103** (1996), no. 6, 502–504.

[4] J. W. S. Cassels, *Local fields*, London Mathematical Society Student Texts **3**, Cambridge University Press, 1986.

[5] S. R. Garcia, Y. X. Hong, F. Luca, E. Pinsker, C. Sanna, E. Schechter, and A. Starr, *p-adic quotient sets*, Acta Arith. **179** (2017), no. 2, 163–184.

[6] S. R. Garcia and F. Luca, *Quotients of Fibonacci numbers*, Amer. Math. Monthly **123** (2016), no. 10, 1039–1044.

[7] S. R. Garcia, V. Selhorst-Jones, D. E. Poore, and N. Simon, *Quotient sets and Diophantine equations*, Amer. Math. Monthly **118** (2011), no. 8, 704–711.

[8] S. Hedman and D. Rose, *Light subsets of $\mathbb{N}$ with dense quotient sets*, Amer. Math. Monthly **116** (2009), no. 7, 635–641.

[9] D. Hobby and D. M. Silberger, *Quotients of primes*, Amer. Math. Monthly **100** (1993), no. 1, 50–52.

[10] P. Miska and C. Sanna, *p-adic denseness of members of partitions of $\mathbb{N}$ and their ratio sets*, (preprint) https://doi.org/10.13140/RG.2.2.28136.57608.

[11] T. Šalát, *On ratio sets of sets of natural numbers*, Acta Arith. **15** (1968/1969), 273–278.

[12] C. Sanna, *The quotient set of k-generalised Fibonacci numbers is dense in $\mathbb{Q}_p$*, Bull. Aust. Math. Soc. **96** (2017), no. 1, 24–29.

[13] C. Small, *Solution of Waring's problem* mod n, Amer. Math. Monthly **84** (1977), no. 5, 356–359.

[14] C. Small, *Waring's problem* mod n, Amer. Math. Monthly **84** (1977), no. 1, 12–25.

[15] O. Strauch and J. T. Tóth, *Asymptotic density of $A \subset \mathbf{N}$ and density of the ratio set $R(A)$*, Acta Arith. **87** (1998), no. 1, 67–78.

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