Singular Edge and Corner Basis Functions for Scattering from Conducting Plates

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***Abstract* — The Method of Moments (MoM) is an efficient way of obtaining solutions of integral equations for 2D and 3D electromagnetic structures by subdividing them into simple shapes such as triangles and rectangles and using suitable polynomial basis functions to describe fields or currents. In the presence of sharp edges and corners, the currents may be unbounded and the accuracy of the solution may be poor due to the inappropriate model provided by a polynomial basis. Attempts to improve the accuracy by increasing the number of cells or the polynomial order of the basis functions may fail as a result. In this paper new basis functions are proposed with unbounded behavior, to more efficiently model edge and corner singularities for quadrilateral cells.**

***Keywords* — Method of Moments, scattering, numerical methods.**

I. Introduction

Modern electromagnetic design relies on numerical field solvers to predict field behavior in a wide variety of problems. Solvers based on integral equations usually treat the surface current density as the primary unknown to be determined. However, in the vicinity of conducting or penetrable edges and corners of structures of interest, the surface current density is often singular and sometimes infinite [1]. Although this behavior is localized at the edge or corner, it reduces the quality of the solution and degrades the efficiency of the numerical solvers. Attempts to adaptively refine the cell sizes (h-refinement) in the neighborhood of the singular region [2] can improve the numerical results but are seldom used with integral equation formulations because of the cost of error estimation and repeatedly re-solving the dense MoM system matrices. An alternate approach known as p-refinement involves selectively increasing the polynomial order of the representation in certain parts of the domain [3], [4]. The p-refinement approach requires hierarchical bases, where the functions used for order $p-1$ are a subset of those used for order p. However, p-refinement usually fails in the vicinity of singularities if the expansion is based entirely on polynomials. In this paper a different approach is proposed based on the use of singular basis functions to more appropriately represent the unbounded currents and fields. The authors' long-term goals include the development of hierarchical representations that can accurately model singular behavior while providing exponential convergence through the use of adaptive p-refinement algorithms. In the present paper, basis functions are proposed for quadrilateral cell shapes with that goal in mind.

Singular basis functions have been used since the 1970s [5], [6], and most existing singular basis functions can be classified as either *substitutive* or *additive* functions. Substitutive expansions involve replacing some of the polynomial bases with new functions containing appropriate singularities. The additive representation is obtained by retaining the full polynomial basis set and augmenting it with additional independent degrees of freedom to model the singular field behavior [7], [8], [9]. In the scalar case, it has been demonstrated that high-order basis functions of the additive kind provide improved accuracy and additional flexibility compared to substitutive bases, since one can model appropriate field behavior even if the expected singularity is not excited by the source. The new functions under consideration are vector bases; hierarchical polynomial families of vector bases have been previously developed by the authors and in this work we extend the idea to singular vector bases. We observe that there are ancillary issues associated with using this type of basis that must be addressed, including integration techniques used to compute the MoM matrix entries for singular functions, and the possible deterioration of the matrix condition number due to the presence of singular basis functions in the representation.

Although the basis functions are defined for general wedge angles, we restrict our examples to conducting plates and disks, modeled by (curved) quadrilateral cells. The circular disk exhibits an edge singularity around its circumference, while the plate exhibits both edge and corner singularities. Edge singularities are well-characterized by the exact solution of the wedge problem [10], which for the case of plates and disks involves only wedge angles of zero. Corner singularities may be modeled by the behavior at the tip of a plane angular sector (a flattened elliptic cone) [11]. Singularity exponents for rectangular corners have been computed from the eigenvalues of Lame equations and are reported in [12]. Incidentally, the disk offers an exact solution that can be used for comparison purposes [13].

A. Singular Basis for Edges

The representation order of the proposed additive quadrilateral basis used to provide the edge singularity is defined by three integers $[p, s, m]$, where p is the degree of the hierarchical vector polynomial basis subset, s is the number of fractional exponents included in the singular part of the representation (the so-called Meixner subset), and m is the order of the Meixner subset [5]. The fractional exponents are those associated with a wedge of some interior angle (zero for plates and disks); in general these can be expressed as an ordered series of *non-integer* singularity coefficients

$$\nu = \{\nu_1, \nu_2, \ldots, \nu_j, \ldots, \nu_s\} \tag{1}$$

The coefficients in (1) are the smallest non-integer exponents that appear in the infinite set that forms the complete expansion of the electromagnetic field near the singular edge. These are used to build singular scalar factors of the form

$$f_a(\nu) = \nu\, \xi_i^{\nu-1} - 1 + Pa_{s-1}(\boldsymbol{\xi}) \tag{2}$$

$$f_b(\nu) = \xi_i^{\nu-1} - 1 + Pb_{s-1}(\boldsymbol{\xi}) \tag{3}$$

which in turn are used to construct the basis functions belonging to the Meixner subset. In the preceding equations, ξ_i is the parent cell variable that vanishes on the singular edge (the edge lying along the coordinate-line $\xi_i = 0$). Polynomials $Pa_{s-1}(\boldsymbol{\xi})$ and $Pb_{s-1}(\boldsymbol{\xi})$ have maximum degree $(s-1)$ and are introduced to orthogonalize f_a and f_b.

The third integer, m, specifies the maximum order of the orthogonal polynomials of the second independent parent variable (in the direction orthogonal to ξ_i) used to construct the Meixner functions. The vector functions of the Meixner set are then obtained by multiplying singular scalar factors with the zeroth-order regular vector basis functions [5]. The polynomial and the Meixner vector subsets are actually built using orthogonal polynomials to maintain the linear independence of the basis functions and improve the condition number of the MoM matrices.

B. *Singular Basis for Corners*

We assume that all the quadrilateral vector basis functions of the rectangular patch are used to provide the proper polynomial behavior and the proper edge singularities; now we also add the dominant corner singularities in the form of two singular vector basis functions defined on a triangle having two edges and the singular tip in common with the quadrilateral cell. The variable ξ_i is the parent cell variable on the triangle that vanishes on the diagonal of the quadrilateral and has unit value at the tip. The current is divided into an even component (vanishing at the tip) and an odd component (unbounded at the tip); these two components introduce two different non-integer exponents into the representation of the current density at the corner [1], [11], [12].

1) *Even Function for Corner Singularities*

The even component Λ_{ce} that represents the singular radial current is proportional to:

$$\Lambda_{ce} = f_e(\boldsymbol{\xi})\,(1-\xi_i)^{\nu_e}\Lambda_i \tag{4}$$

where

$$\Lambda_i = \frac{1}{\mathcal{J}}\,(\xi_{i+1}\ell_{i-1} - \xi_{i-1}\ell_{i+1}) \tag{5}$$

$f_e(\boldsymbol{\xi})$ is a radial function and $(1-\xi_i)$ is the radial distance from the tip. The term ν_e depends on the corner angle and can be determined by the procedure in [11]; Fig. 1 shows the behavior of the even component of the currents on the tip.

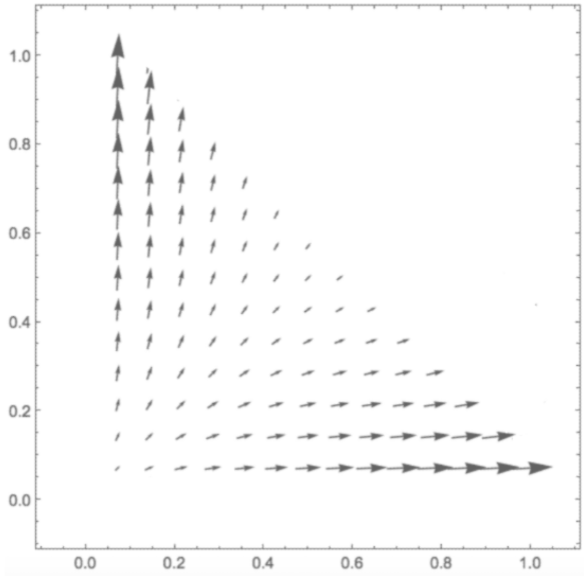

Fig. 1. Even singular function for the corner

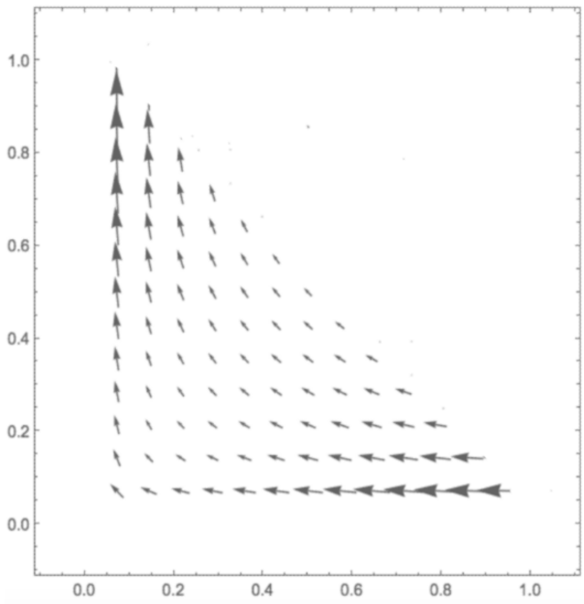

Fig. 2. Odd singular function for the corner

2) *Tangential Function for Corner Singularities*

The odd component, which "turns around the tip" is modeled by:

$$\Lambda_{co} = \xi_i\, f_o(\boldsymbol{\xi})\,(1-\xi_i)^{\nu_o - 1} \tag{6}$$

where $f_o(\boldsymbol{\xi})$ is the tangential function. The term ν_o depends on the corner angle and can be derived from [1]; Fig. 2 shows the behavior of the odd component of the currents on the tip.

Note that the integrals of these corner functions needed to compute the MoM system must account for the unbounded nature of the current density in (6) and the charge density associated with (4). The functions $f_e(\boldsymbol{\xi})$, $f_o(\boldsymbol{\xi})$ and $(1-\xi_i)^{\nu_o-1}$ are all singular.

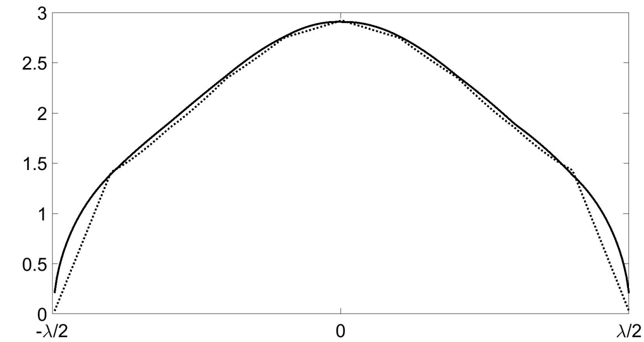

Fig. 3. J_x component along $x = 0$. In solid line results obtained with the $[1, 1, 0]$-order base, and in dotted line results obtained with rooftop only base.

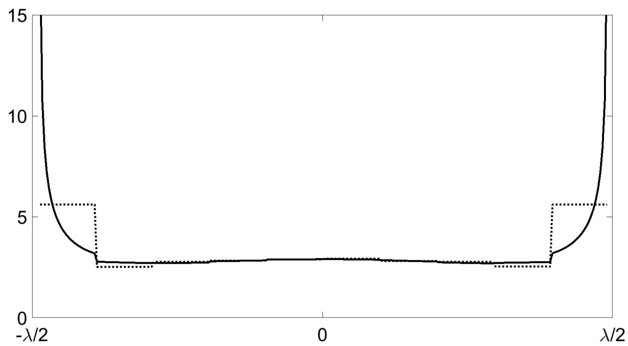

Fig. 4. J_x component along $y = 0$. In solid line results obtained with the $[1, 1, 0]$-order base, and in dotted line results obtained with rooftop only base.

II. Results

A. Square Plate

An infinitely thin metal $1\lambda \times 1\lambda$ square plate centered in the $(x,\, y)$-plane was modeled with a 10×10 mesh of square cells and illuminated by a plane wave propagating along z-axis with unit magnetic field on y-axis. Fig. 3 shows the J_x component computed at the $x = 0$ line, while Fig. 4 shows the J_x component computed at the $y = 0$ line. The solid line represents the results obtained with polynomial order 1 plus one singular function ($\nu = 0.5$, the dominant), that is the $[1, 1, 0]$ order base, while dotted line represents the results for a pure polynomial basis of order 0 (i.e. rooftop functions). The reader can see that the J_x component along $x = 0$ which is unbounded cannot be represented correctly with rooftop basis. Fig. 5 shows the unbounded currents at $y = \pm 0.5\lambda$. The price paid is the increase of the MoM matrix condition number from 40 to 6600.

B. Circular Disk

A circular, perfectly conducting disk is one of only a few three-dimensional geometries amenable to exact electromagnetic (EM) analysis [12]. As such, the disk offers the potential to serve as a benchmark for validating EM modeling software, and specifically for studying the performance of special numerical techniques for accurately modeling edge singularities. A disk of radius a can be discretized by (curved) rectangular cells as shown in Fig. 6.

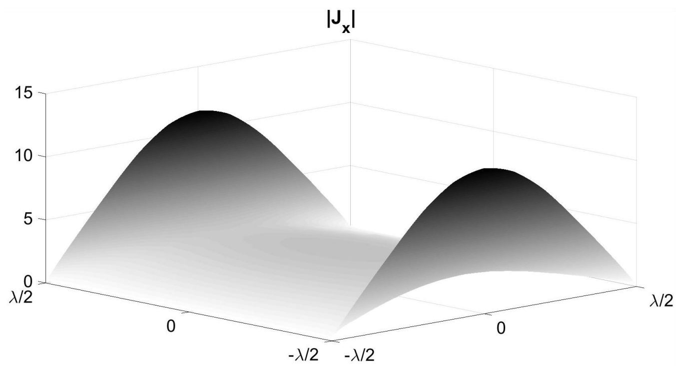

Fig. 5. Magnitude of the J_x-component on a $1\lambda \times 1\lambda$ square plate as induced by a normally incident plane-wave with $H_y = 1$ A/m.

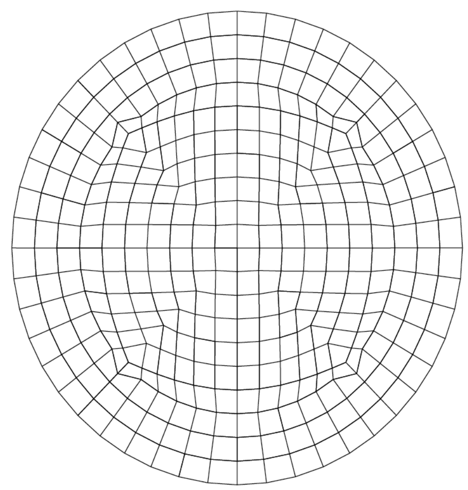

Fig. 6. A typical quadrilateral-cell mesh used to model the disk.

Fig. 7 shows the normalized magnitude of the co-polarized current component induced on disk with $ka = 4\pi$ by a normally incident plane wave with unity magnitude H-field, for two cuts through the disk center along the x and y axes. The results, compared with the exact solution, are obtained with a zeroth order pure polynomial basis subset (roof top basis functions) augmented with singular basis functions that model only the first (dominant) singular coefficient $\nu = 1/2$. This figure clearly shows that very good results are obtainable with low p-order basis by adding just the functions that model only the first dominant singularity. The good convergence to accurate results is shown also in Fig. 8 that reports the backscatter RCS for a disk with $ka = 4\pi$, for a range of mesh sizes while using a zeroth-order basis functions subset.

III. Conclusions

Hierarchical vector basis functions are proposed for modeling edge singularities in quadrilateral cells. The functions are very efficient, and results for perfectly conducting square plates and circular disks are used to illustrate the improved accuracy and efficiency of the bases. The increase in matrix condition number resulting from the use of singular functions is also reported.

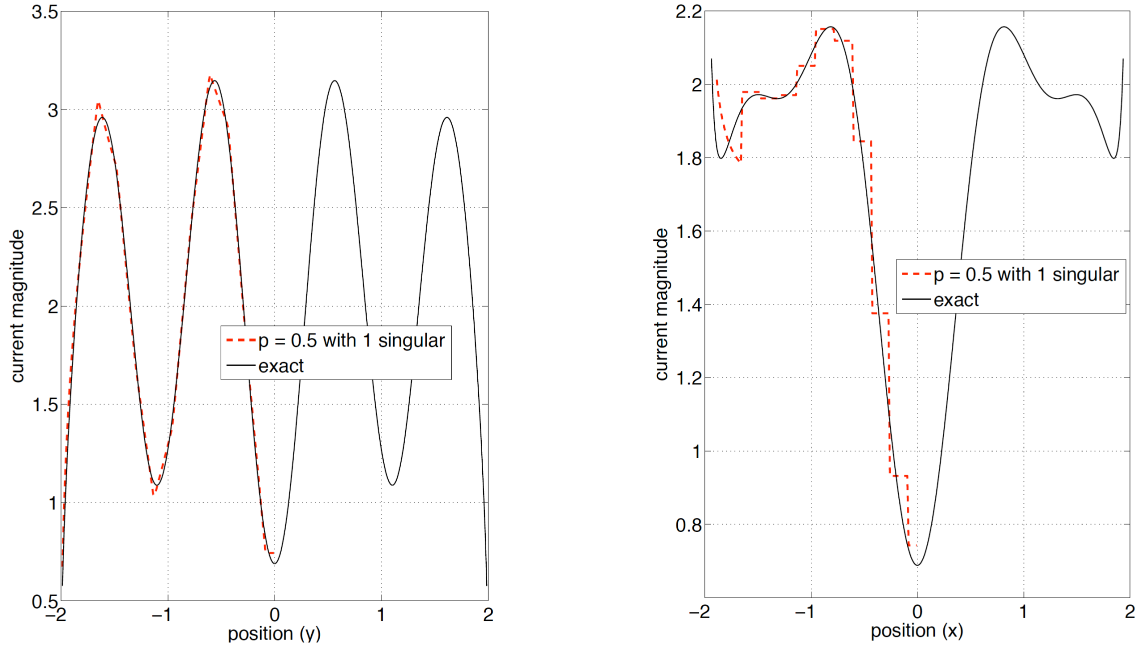

Fig. 7. The figures show the co-polarized current induced on disk with $ka = 4\pi$ by a normally incident plane wave with unity magnitude H-field. The numerical result incorporating singular basis functions in the edge cells for a 341-cell mesh is shown over half the range, for a cut through the disk center along the y-axis (at top) and x-axis (at bottom).

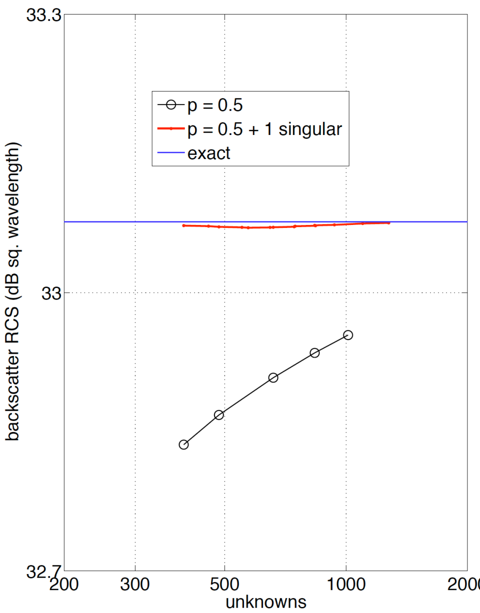

Fig. 8. Backscatter RCS for a disk with $ka = 4$, for a range of mesh sizes. Numerical results based on polynomial basis functions ($p = 0.5$) and a mixture of polynomial and singular basis functions ($p = 0.5 + 1$ singular) is compared to the exact solution.

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