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# $f$ -domain Analysis of Linear Circuits with Time-Varying Parameters via Integral Equations

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**Abstract**—This paper focuses on a preliminary study on the spectral analysis of linear time-varying circuits for the prediction of their steady-state and transient behavior. The proposed approach is based on the integral representation of the circuit equations in either the Fourier or Laplace domains. It provides the readers with an elegant and robust tool for the frequency-domain simulation of this class of circuits, enabling the direct inclusion of frequency-dependent elements. The feasibility and strength of the method are demonstrated on two illustrative examples consisting of a periodically linear time-varying parallel resonators and a simple time-varying circuit.

**Index Terms**—Spectral analysis, transient analysis, linear time-varying circuits, integral equations.

## I. INTRODUCTION

In the past years, the importance of time-varying networks is grown. A well known example is provided by the power engineering domain and the switched mode power converters which are massively used in almost any electrical and electronic equipment [1]. This class of circuits generates a possibly complex dynamical behavior that must be properly predicted during the early design phase [2]. The above need stimulated a fruitful research activity focused on the development of numerical techniques for the steady-state and transient solution of circuits governed by coupled differential equations with time-varying coefficients. To this aim a number of advanced methods has been developed to provide a robust and accurate simulation framework for the specific case of periodically linear time varying (PLTV) circuits [3]–[11], with emphasis on the steady-state response only.

For the general case of linear time-varying (LTV) circuits, a standard time-domain analysis is probably the most effective solution. However, the inclusion in the simulation framework of native frequency-dependent blocks such as lossy transmission-lines and measured characteristics of multiport elements (e.g., via admittance or scattering parameters) is troublesome and unavoidably requires a non-negligible overhead in recasting these blocks into time-domain circuit equivalents.

To overcome the above limitation, this paper focuses on a different interpretation of the solution of a LTV network in the spectral domain, via integral equations [12]. The proposed solution provides a unified picture which holds for both aperiodic and periodic time-varying circuits. What is more important, operating in frequency domain allows accounting for frequency-dependent multiport elements without any additional effort. The mathematical framework is applied to the

analysis of two illustrative examples, namely a PLTV and a LTV circuit.

## II. ADOPTED FRAMEWORK FOR LTV NETWORKS

This Section briefly introduces two well known tools for the analysis of LTV networks along with their integral formulation in the spectral domain.

### A. Modified Nodal Analysis

The generic form of the modified nodal analysis (MNA) [13] for a LTV circuit with  $n$  nodes and  $l$  current controlled elements reads:

$$(\mathbf{m}_0(t) + \dot{\mathbf{m}}_1(t))\mathbf{w}(t) + \mathbf{m}_1(t)\dot{\mathbf{w}}(t) = \mathbf{j}(t) \quad (1)$$

where  $\mathbf{m}_0(t), \mathbf{m}_1(t) \in \mathbb{R}^{(n-1+l) \times (n-1+l)}$  are matrices defining the time-varying parameters of the circuit,  $\mathbf{w}(t) = [\mathbf{v}(t), \mathbf{i}(t)]^T \in \mathbb{R}^{(n-1+l)}$  is a column vector collecting the  $(n-1)$  nodal voltages and the currents flowing through the  $l$  current controlled elements. Vector  $\mathbf{j}(t) \in \mathbb{R}^{(n-1+l)}$  collects the independent current and voltage sources.

The above relationship can be converted in the spectral domain by applying the Laplace transform  $\mathcal{L}(\cdot)$  to both sides of the equation, leading to the following coupled Fredholm integral equation:

$$\int_{\sigma-j\infty}^{\sigma+j\infty} [\mathbf{M}_0(s-s') + s\mathbf{M}_1(s-s')] \mathbf{W}(s') ds' = \mathbf{J}(s), \quad (2)$$

where  $s = \sigma + j\omega$  is the Laplace variable defined for  $\sigma > 0$ .

### B. State Space Representation

The state space representation of a LTV network containing  $d$  dynamical elements in time-domain reads:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \quad (3)$$

where  $\mathbf{A}(t), \mathbf{B}(t) \in \mathbb{R}^{d \times d}$  are time-varying matrices, and  $\mathbf{x}(t), \mathbf{u}(t) \in \mathbb{R}^d$  are column vectors defining the states and the excitation of the circuit, respectively.

The  $s$ -domain formulation of (3) is given by the following Fredholm integral equation:

$$s\mathbf{X}(s) = \int_{\sigma-j\infty}^{\sigma+j\infty} \mathbf{A}(s-s')\mathbf{X}(s')ds' + \int_{\sigma-j\infty}^{\sigma+j\infty} \mathbf{B}(s-s')\mathbf{U}(s')ds'. \quad (4)$$

### III. EXAMPLE 1: PLTV CIRCUIT

This Section summarizes the results of the application of the proposed spectral technique to the analysis of the simple illustrative PLTV parallel resonator shown in Fig. 1. An MNA based formulation is used to compute both the steady-state and the transient behavior of the circuit.

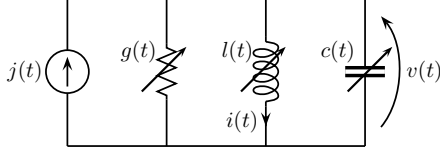


Fig. 1. First illustrative example consisting of a PLTV resonator with parameters  $g(t) = 1 + 0.5 \sin(\omega_c t)$  S,  $c(t) = 5 + 2.5 \sin(\omega_c t)$  mF and  $l(t) = 5 + 2.5 \sin(\omega_c t)$  mH, and  $\omega_c/2\pi = 300$  Hz.

The MNA matrices and vectors in (1) take the following values:

$$\mathbf{w} = [v(t), i(t)]^T, \quad \mathbf{j}(t) = [j(t), 0]^T \quad (5)$$

and

$$\mathbf{m}_0(t) = \begin{bmatrix} g(t) & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{m}_1 = \begin{bmatrix} c(t) & 0 \\ 0 & -l(t) \end{bmatrix}. \quad (6)$$

Without loss of generality, the integral equation arising from the first row of (5) and (6) reads:

$$\int_{\sigma-j\infty}^{\sigma+j\infty} \underbrace{[G(s-s') + sC(s-s')]}_{K(s-s')} V(s')ds' + I(s) = J(s). \quad (7)$$

A similar derivation holds for the second row (not included for the sake of compactness).

#### A. Steady-State Analysis

Let's start by considering the  $f$ -domain representation of the periodic time-varying parameters of the circuit in terms of their Fourier expansions, which reads, e.g., for  $c(t)$ :

$$c(t) = \sum_{n=-\infty}^{+\infty} C_n \exp(jn\omega_c t), \quad \forall t \in (-\infty, +\infty) \quad (8)$$

where  $\omega_c = 2\pi f_c$  is the angular frequency related to the period  $T_c = 1/f_c$  and  $C_n$  are the coefficients of the Fourier series. The above equation leads to the following Fourier transform:

$$C(\omega) = \sum_{k=-\infty}^{+\infty} C_k \delta(\omega - k\omega_c). \quad (9)$$

According to [9], the steady-state responses of a generic circuit variable such as the voltage  $v(t)$  to a quasi-periodic excitation  $j(t) = \sum_p J_p \exp(j\omega_p t)$  containing a finite set of harmonics  $\mathcal{S} \in \{\omega_{-P}, \dots, \omega_P\}$  reads:

$$V(\omega) = \sum_{p=-P}^{+P} \sum_{n=-\infty}^{+\infty} V_{(p,n)} \delta(\omega - n\omega_c - \omega_p) \quad (10)$$

where  $V_{(p,n)}$  represents the  $n$ -th coefficient of the Fourier expansion of  $v(t)$  for a single tone excitation of frequency  $\omega_p \in \mathcal{S}$ .

By substituting the Fourier transform of the circuit parameters and the steady-state representation of the circuit variables into the integral equation (7), we have:

$$\sum_{n,k,p} [G_k + j((n+k)\omega_c + \omega_p)C_k] V_{(p,n)} \delta(\omega - (n+k)\omega_c - \omega_p) + I_{(p,n)} \delta(\omega - n\omega_c - \omega_p) = J_p \delta(\omega - \omega_p) \quad (11)$$

where  $s = j\omega$ . The previous equation is obtained by computing the integral in (7) as follows.

$$\int_{-\infty}^{+\infty} \sum_{n,k} G_k \delta(\omega - \omega' - k\omega_c) V_{(p,n)} \delta(\omega' - n\omega_c - \omega_p) d\omega' = \sum_{n,k} G_k V_{(p,n)} \delta(\omega - (n+k)\omega_c + \omega_p). \quad (12)$$

It is important to remark that (11) is exact since it is obtained from the discrete kernel  $K(s-s')$ . However, its numerical solution requires to approximate the spectra in (9) and (10) with finite sums collecting the first  $(2N+1)$  harmonics of the Fourier series for each excitation  $\omega_p$ , leading to the following matrix approximation of (11),

$$(\mathbf{\Sigma} + \mathbf{\Omega}\mathbf{\Gamma})\mathbf{X} + \mathbf{Y} = \mathbf{\Theta} \quad (13)$$

where  $\mathbf{\Sigma}$ ,  $\mathbf{\Omega}$ , and  $\mathbf{\Gamma}$  are complex circular matrices of dimension  $(2N+1) \times (2N+1)$  and,  $\mathbf{\Omega} = \text{diag}([-jN\omega_c, \dots, jN\omega_c])$ . The unknown voltage and current variables are collected into the vetctors  $\mathbf{X} = [V_{(-N,p)}, \dots, V_{(N,p)}]^T$  and  $\mathbf{Y} = [I_{(-N,p)}, \dots, I_{(N,p)}]^T$ . The excitation vector is  $\mathbf{\Theta} = [0, \dots, J(\omega_p), \dots, 0]^T$ .

The above equation can be considered as an augmented formulation of the MNA [10]-[11], for which the time-varying elements are represented in the  $f$ -domain as multiport elements defined by suitable matrices accounting for the harmonic coupling introduced by their time-varying activity.

It is important to notice that a similar procedure must be applied to the second row of the MNA equation with parameters defined by (5) and (6), leading to a matrix equation similar to (13) but with a larger, i.e., double, dimension.

Once the discrete spectrum of the voltage  $V(\omega)$  is computed numerically, its the corresponding time-domain behavior can be obtained via the following general relation [14]:

$$v(t) \approx \sum_{p=-P}^P \sum_{n=-N}^{+N} V_{n,p} \exp(j\omega_p t) \exp(jn\omega_c t), \quad (14)$$

for any  $\omega_p \in \mathcal{S}$ .

### B. Transient simulation of PLTV circuits

The previous results can be generalized to the case of a generic *aperiodic* excitation  $j(t)$  defined for  $t \in [0, +\infty)$ . In this case, the periodic parameters of the circuit are represented in the Laplace domain. The resulting complex functions  $G(s)$ ,  $C(s)$  and  $L(s)$  are meromorphic functions with an infinite number of poles at  $p_n = jn\omega_c$ , which can be recast in terms of their rational representations e.g., for  $c(t)$  writes,

$$C(s) = \mathcal{L}\{c(t)\}(s) = \sum_{n=-\infty}^{+\infty} \frac{C_n}{s - jn\omega_c}. \quad (15)$$

By substituting (15) in (7) and after some manipulations, we derive:

$$\sum_n [G_n + s_0 C_n] V(s_0 - jn\omega_c) + I(s_0) = J(s_0). \quad (16)$$

where  $s_0 \in \mathbb{C}$  is a generic point in the Laplace domain.

It is important to notice that also in this case, the spectral relationship in (16) is exact, since the kernel of any PLTV system turns out to be a meromorphic function yielding:

$$\int_{\sigma-j\infty}^{\sigma+j\infty} \sum_n \frac{G_n}{s - s' - jn\omega_c} V(s') ds' = \sum_n G_n V(s - jn\omega_c). \quad (17)$$

In order to solve (16) numerically, the sum over  $n$  is truncated to consider the first  $(2N + 1)$  harmonics of the expansion. Also,  $s_0$  is replaced by a set of complex frequencies  $\mathcal{S} = \{\sigma + jp2\pi\Delta_f\}$  for  $-P \leq p \leq P$  in which the terms  $\sigma = 2\log(P)/T_w$  and  $\Delta_f = f_{max}/P$  are defined according to the numerical inverse Laplace transform algorithm [14].

For any  $s_0 \in \mathcal{S}$  the above equation can be interpreted as the linear system in (13) with the same  $\Sigma$  and  $\Gamma$  matrices, whereas the diagonal matrix  $\Omega = \text{diag}([s_0 - jN\omega_c, \dots, s_0 + jN\omega_c])$ . The solution of the mentioned linear system provides the current and voltage sampled spectra yielding the time-domain responses of the circuit, e.g.,

$$v(t) \approx \Delta_f \sum_{p=-P}^P \sum_{n=-N}^{+N} V_{n,p} \exp(s_p t) \exp(jn\omega_c t). \quad (18)$$

The proposed methodology is applied to the steady-state and transient analysis of the current  $i(t)$  and voltage  $v(t)$  responses to the sinusoidal excitation  $j(t) = \sin(2\pi 250t)u(t)$  A (see Fig. 2). The curves in the figure show an excellent agreement between the solution provided by the proposed spectral approach and the result of the time-domain simulation of the circuit in Matlab Simulink.

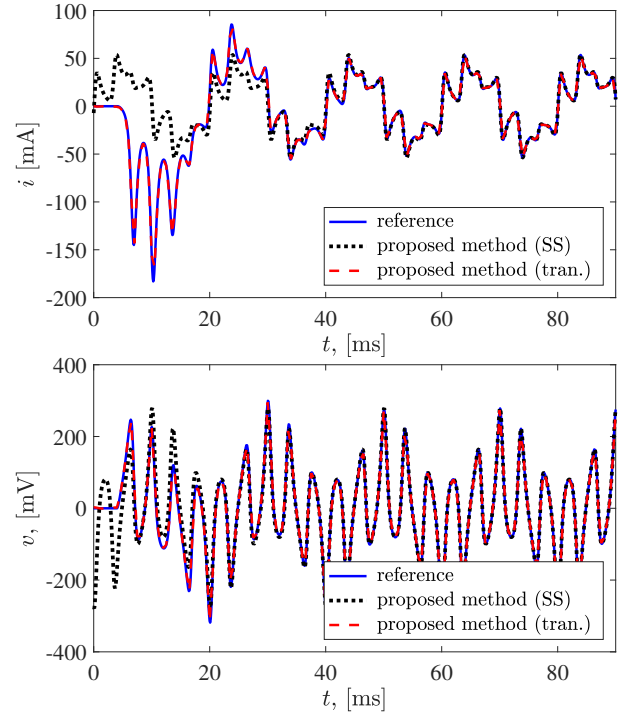


Fig. 2. Current  $i(t)$  (top panel) and voltage  $v_c(t)$  (bottom panel) responses of the PLTV circuit of Fig. 1. The results of the proposed spectral approach for the steady-state (dashed black, SS label) and transient (dashed red, tran. label) analyses are compared with the results provided by the reference time-domain simulation of the circuit (blue). The parameters  $P = 1$  and  $N = 60$  for the steady-state analysis and  $P = 180$  and  $N = 120$  for the transient analysis, respectively.

## IV. EXAMPLE 2: TIME-VARYING NETWORK

This section proposes a second application of the proposed spectral approach to the solution of the LTV network of Fig. 3 based on the alternative state space representation. This simple circuit consists of the interconnection of standard elements and a time-varying conductance with the aperiodic behavior  $g_2(t) = g_0[u(t - a) - u(t - a - b)]$ .

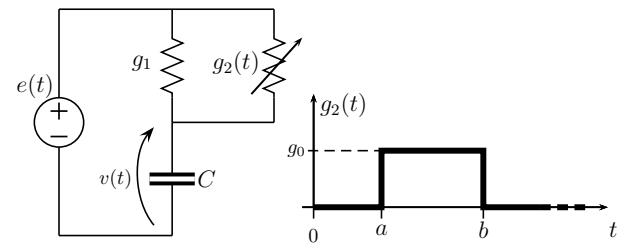


Fig. 3. Second example consisting of a simple LTV circuit with a time-varying conductance  $g_2(t)$ . The circuit elements take the following values:  $e(t) = u(t)$  V,  $g_1 = g_0 = 30 \Omega$ ,  $a = 1$  s,  $b = 3$  s and  $C = 0.1$  F.

The voltage  $v(t)$  across the capacitor can be described by means of (3) where  $A(t) = -(g_1 + g_2(t))/C$  and  $B(t) = (g_1 + g_2(t))/C$  are scalar coefficients. The above definition leads to the following corresponding integral representation:

$$(sC + g_1)V + \underbrace{\int_{\sigma-j\infty}^{\sigma+j\infty} G(s-s')V(s')ds'}_{I_1} = \tilde{E}(s), \quad (19)$$

where,

$$\tilde{E}(s) = GE(s) + \int_{\sigma-j\infty}^{\sigma+j\infty} G(s-s')E(s')ds'. \quad (20)$$

Equation (19) is a Fredholm integral equation of the second kind where the kernel  $G(s, s')$  is defined as:

$$G(s-s') = \mathcal{L}\{g(t)\}(s-s') = \begin{cases} g_0 \cdot (b-a) & \text{for } s = s', \\ g_0 \cdot \frac{\exp(-a(s-s')) - \exp(-b(s-s'))}{(s-s')} & \text{otherwise.} \end{cases} \quad (21)$$

Different from the previous example (which is a PLTV circuit), the kernel  $K(s, s') = G(s-s')$  in (21) is an entire function. The point  $s = s'$  is a removable discontinuity and there not exist a trivial closed-form solution of the integral  $I_1$  in (19). An approximation is however achieved by replacing the integral with a finite sum defined over a set of complex frequencies  $\mathcal{S} = \{\sigma + jn2\pi\Delta_f\}$ ,  $-N \leq n \leq N$ , yielding:

$$\int_{\sigma-j\infty}^{\sigma+j\infty} G(s-s')V(s')ds' \approx \Delta_f \sum_{n=-N}^{+N} G(s-s_n)V(s_n). \quad (22)$$

for any  $s_n \in \mathcal{S}$ .

By substituting (22) in (19) we have, for a given complex frequency  $s_0 \in \mathcal{S}$ ,

$$\mathcal{M}(s_0)V(s_0) + \sum_{k=-N}^{+N} \mathcal{K}(s_0, s_k)V(s_k) = \tilde{E}(s_0), \quad (23)$$

where  $\mathcal{M}(s) = (sC + g_1)$  and  $\mathcal{K}(s, s_n) = \Delta_f G(s-s_n)$ .

Similar to the case of a PLTV circuit, the above equation can be easily recast in terms of the linear problem:

$$(\Sigma + \Gamma)\mathbf{X} = \Theta, \quad (24)$$

where  $\mathbf{X} = [V(s_{-N}), \dots, V(s_N)]^T$  and  $\Theta = [\tilde{E}(s_{-N}), \dots, \tilde{E}(s_N)]^T$  are column vectors collecting the sampled spectra of the voltage  $V(s)$  and of the excitation  $\tilde{E}(s)$ , respectively. Also,  $\Sigma = \text{diag}([\mathcal{M}(s_{-N}), \dots, \mathcal{M}(s_N)])$  is a diagonal matrix and  $\Gamma_{i,j} = \mathcal{K}(s_{i-N-1}, s_{j-N-1})$  is a full-coupled circular matrix accounting for the harmonic coupling introduced by the time-varying parameter.

Once the discrete samples of the voltage  $V(s)$  are computed from the inversion of the linear system, the corresponding transient response  $v(t)$  can be readily obtained via the standard numerical inverse Laplace transform:

$$v(t) \approx \sum_{n=-N}^N V_p \exp(s_n t), \quad (25)$$

for any  $s_n \in \mathcal{S}$ .

Figure 4 collects the transient response of the voltage  $v(t)$  to the excitation  $e(t) = u(t)$  V and its corresponding spectrum. The curves obtained via the proposed spectral approach are compared with the closed-form time-domain solution of the first order example circuit. The results in the figure highlight the capability of the proposed method to accurately predict both the spectrum and the transient behavior of this class of circuits.

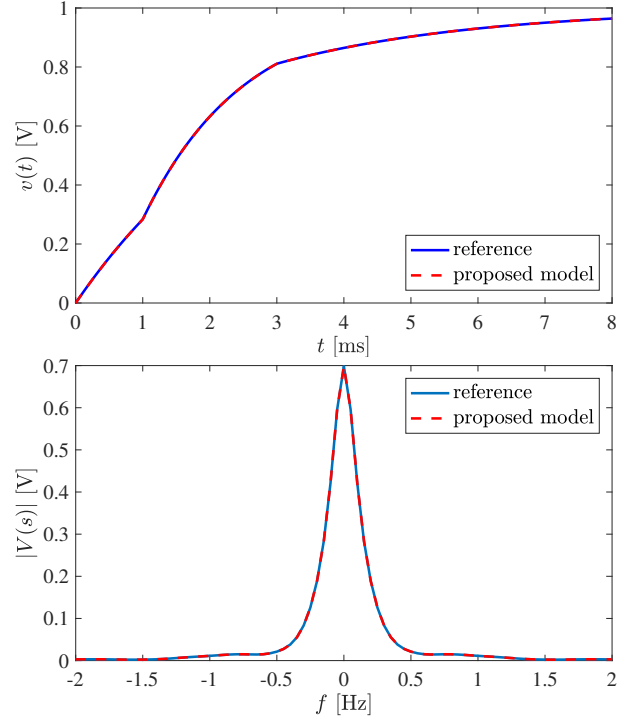


Fig. 4. Transient response of the voltage  $v(t)$  (top panel) of the LTV circuit of Fig. 3 and its corresponding spectrum (bottom panel). The responses obtained via the proposed spectral approach with  $N = 100$  (dashed red curves) are compared with the analytical solution of the circuit (blue curves).

## V. CONCLUSIONS

This paper presented a preliminary study of the application of a spectral technique for the steady-state and transient analyses of linear time-varying circuits. The proposed approach is based on an integral formulation derived from the governing circuit equations in frequency-domain. The strength and the accuracy of the methodology have been demonstrated on two examples consisting of a parallel resonator with time-varying periodical parameters and an aperiodic first order circuit.

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