Fusion of partial orderings for decision problems in Quality Management

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## Papers index

# Fusion of partial orderings for decision problems in Quality Management

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## ABSTRACT

**Purpose –** In a rather common problem for the Quality Management field, (i) a set of *judges* express their individual (*subjective*) judgments about a specific *attribute*, which is related to some *objects* of interest, and (ii) these judgments have to be fused into a *collective* one. This paper develops a new technique where individual judgments – which are expressed in the form of *partial* preference orderings, including the more/less preferred objects only – are fused into a collective judgment, which is expressed in the form of a *ratio* scaling of the objects of interest. An application example concerning the design of a civilian aircraft seat is presented.

**Design/methodology/approach –** The proposed technique borrows the architecture and the underlying postulates from the Thurstone's *Law of Comparative Judgment* (LCJ), adopting a more user-friendly response mode, which is based on (partial) preference orderings instead of paired-comparison relationships. By aggregating and processing these orderings, an overdefined system of equations can be constructed and solved through the *Generalized Least Squares* method. Apart from a ratio scaling of the objects of interest, this approach makes it possible to estimate the relevant uncertainty, by propagating the uncertainty of input data.

**Findings –** Preliminary results show the effectiveness of the proposed technique, even when preference orderings are rather "incomplete", i.e., they include a relatively limited number of objects, with respect to those available.

**Research limitations/implications –** Thanks to the relatively simple and practical response mode, the proposed technique is applicable to a variety of practical contexts, such as telephone and street interviews. Although preliminary results are promising, the technique will be tested in a more organic way, considering several factors (e.g., number of judges, number of objects, degree of completeness of preference orderings, degree of agreement of judges, etc.).

**Originality/value –** Even though the scientific literature includes many techniques that are inspired by the LCJ, the proposed one is characterized by two important novelties: (i) it is based on a more user-friendly response mode and (ii) it allows to obtain a ratio scaling of objects with a relevant uncertainty estimation.

**Paper type**: Research paper

**Keywords**: Group decision making, Law of comparative judgment, Partial preference ordering, Generalized least squares

## INTRODUCTION

A problem that is rather common to several scientific field is articulated as follows (Keeney and Raiffa, 1993; Franceschini et al., 2007; Coaley, 2014):

- a set of *objects* ($o_1$, $o_2$, …) should be compared on the basis of the degree of a specific *attribute*;
- a set of *judges* ($j_1$, $j_2$, …) individually express their *subjective* judgments on these objects;
- these judgments should be fused into a single *collective judgment*, which is usually expressed in the form of a *scaling*, i.e., assignment of numbers to the objects, according to a conventional rule/method (De Vellis, 2016).

With reference to the Quality Management field, possible examples of this problem are: (i) fusing judgments related to the customer satisfaction of set of competing products, or (ii) fusing judgments by reliability/maintenance engineers on the severity of potential process failures, etc..

Figure 1 shows a pedagogical representation of the problem of interest, in which four final consumers (i.e., judges) have to express their judgments on the taste (i.e., attribute) of three types of candies (i.e., objects). It general, judges may refrain from judging part of the objects, when lacking adequate knowledge of them (e.g., see judgments with "???").

The scientific literature encompasses a plurality of fusion techniques, which differ from each other for at least three features: (i) the *response mode* for collecting subjective judgments; (ii) the underlying *rationale* of the fusion technique, and (iii) the *form* of the resulting collective judgment. For an exhaustive discussion of the existing techniques in various fields (e.g., Quality Management, Multi-Criteria Decision Making, etc.), we refer the reader to the vast literature and reviews (Coaley, 2014; De Vellis, 2016).

Regardless of the peculiarities of the individual fusion techniques, a key element for their success is the simplicity of response mode (Franceschini et al., 2007; Harzing et al., 2009). For example, various studies show that comparative judgments of objects (e.g., "$o_i$ is more/less preferred than $o_j$") are simpler and more reliable than judgments in absolute terms (e.g., "the degree of the attribute of $o_i$ is low/intermediate/high") (Harzing et al., 2009; Edwards, 1957).

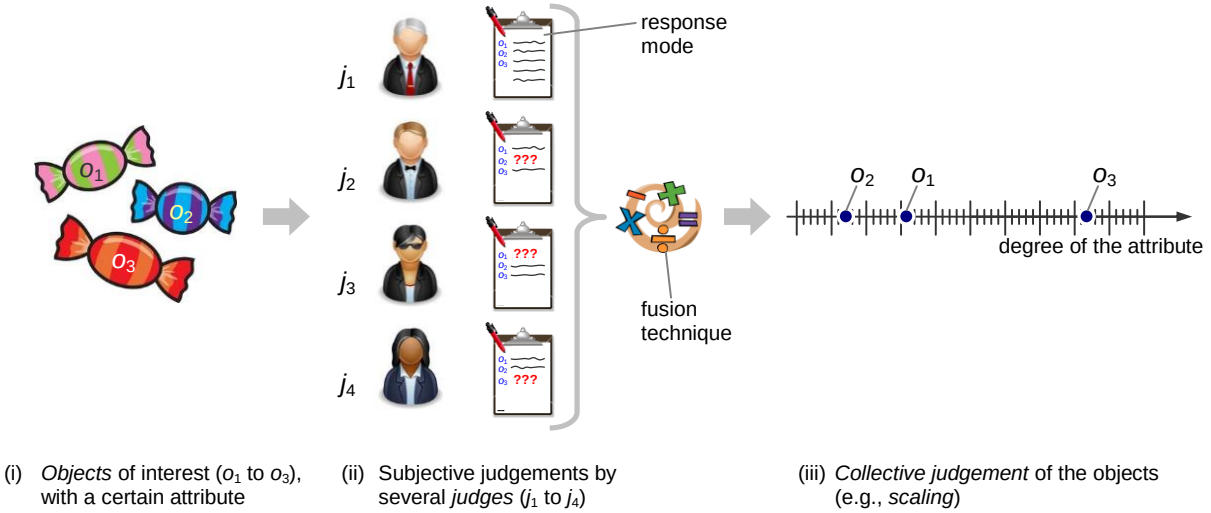



Figure 1 – Pedagogical representation of the problem of interest.

As to the typology of collective judgments, we note that they are often treated as if they were defined on a *ratio* scale – i.e., a scale with non-arbitrary zero and meaningful distance – even when they actually are not; e.g., rankings or ordinal-scale values of the objects are often improperly "promoted" to ratio-scale values, in the moment in which they are combined with other indicators through weighted sums, geometric averages, or – more in general – statistics that are permissible to ratio-scale values only (Roberts, 1979).

In a recent paper, the authors have developed a technique, denominated "*ZM*-technique", that combines the Thurstone's *Law of Comparative Judgment* (LCJ) (Thurstone, 1927; Edwards, 1957) with a response mode based on preference orderings (Franceschini and Maisano, 2018). The resulting collective judgment is expressed in the form of a *ratio* scaling, which can be constructed without any conceptually prohibited "promotion". An important requirement of the *ZM*-technique is that, apart from "regular" objects (i.e., $o_1$, $o_2$, …, $o_n$), preference orderings also include two "dummy" or "anchor" objects: i.e., $o_Z$, which corresponds to the *absence* of the attribute of interest, and $o_M$, which corresponds to the *maximum-imaginable* degree of the attribute (Franceschini and Maisano, 2018).

The *ZM*-technique requires judges to formulate *linear* preference orderings, i.e., orderings including all (regular and dummy) objects, according to a hierarchical sequence with relationships of *strict preference* (“>”) and/or *indifference* (“~”) (Nederpelt and Kamareddine, 2004). This is certainly a limitation, as it makes the response mode unsuitable for some practical contexts where ranking a number of objects can be problematic. It has also been observed that, when formulating preference orderings, judges tend to focus on the more/less preferred objects, providing more reliable judgments about them, to the detriment of the remaining objects (Lagerspetz, 2016; Harzing et al., 2009). Another limitation of the *ZM*-technique – and the traditional LCJ too (Montag, 2006) – is the impossibility to estimate the uncertainty related to the resulting scaling of objects.

The above considerations raise the following research question: “How could the *ZM*-technique be modified so as to (1) make the response more user-friendly and reliable and (2) determine a (statistically sound) estimate of the uncertainty related to the solution?”.

The aim of this paper is to address the previous research question, proposing a new technique that overcomes the limitations of the *ZM*-technique while preserving the basic principles. The new technique replaces linear preference orderings with “incomplete” orderings, which are focussed exclusively on the more/less preferred objects. Borrowing the language from Mathematics’ Order Theory, these other orderings can be classified as *partial*, i.e., apart from strict preference and indifference relationships, they may also contain *incomparability* relationships among (some of) the objects (Nederpelt and Kamareddine, 2004).

The rest of the paper is organized into five sections. Section “Background information” briefly recalls the LCJ and *ZM*-technique. Section “Methodology” illustrates the new technique, which includes the construction of an overdetermined system of equations and its solution through the *Generalized Least Squares* (GLS) method (Kariya and Kurata, 2004). This technique also allows to estimate the uncertainty related to the solution, “propagating” the uncertainty of input data. Section “Application example” applies the new technique to a real-life example, concerning the design of a civilian aircraft seat. Section “Conclusions” summarizes the original contributions of this paper and its practical implications, limitations and suggestions for future research. Further information on the GLS method is contained in the appendix.

## BACKGROUND INFORMATION

This section "prepares the field" to better understand the proposed technique and is organized in two subsections, which respectively recall the LCJ and the *ZM*-technique.

### *Thurstone's LCJ*

Thurstone (1927) postulated the existence of a *psychological continuum*, i.e., an abstract and unknown unidimensional scale, in which objects are positioned depending on the degree of a certain *attribute* – i.e., a specific feature of the objects, which evokes a subjective response in each judge. The position of a generic *i*-th object ($o_i$) is postulated to be distributed normally, in order to reflect the intrinsic judge-to-judge variability: $o_i \sim N(x_i, \sigma_i^2)$, where $x_i$ and $\sigma_i^2$ are the unknown mean value and variance related to the degree of the attribute of that object. Considering two generic objects, $o_i$ and $o_j$, it can be asserted that:

$$o_i - o_j \sim N(x_i - x_j,\ \sigma_i^2 + \sigma_j^2 - 2 \cdot \rho_{ij} \cdot \sigma_i \cdot \sigma_j), \tag{1}$$

where $\rho_{ij}$ is the Pearson coefficient, denoting the correlation between the positioning of objects $o_i$ and $o_j$. The probability that the position of $o_i$ in the psychological continuum is higher than that of $o_j$ can be expressed as:

$$p_{ij} = P(o_i - o_j > 0) = 1 - \Phi\left[\frac{0 - (x_i - x_j)}{\sqrt{\sigma_i^2 + \sigma_j^2 - 2 \cdot \rho_{ij} \cdot \sigma_i \cdot \sigma_j}}\right], \tag{2}$$

$\Phi$ being the cumulative distribution function of the standard normal distribution $z \sim N(0, 1)$.

The LCJ (*case V*) includes the following additional simplifying assumptions (Thurstone, 1927; Edwards, 1957): $\sigma_i^2 = \sigma^2\ \forall i$, $\rho_{ij} = \rho, \forall i, j$, and $2 \cdot \sigma^2 \cdot (1 - \rho) = 1$. Eq. 2 can therefore be expressed as:

$$p_{ij} = P(o_i - o_j > 0) = 1 - \Phi[-(x_i - x_j)]. \tag{3}$$

Although $p_{ij}$ is unknown, it can be estimated using the information contained in a set of (subjective) judgments by a number ($m$) of judges (Thurstone, 1927). Precisely, each judge expresses his/her judgment for each paired comparison (i.e., $\forall i, j$), through relationships of *strict preference* (e.g., "$o_i > o_j$" or "$o_i < o_j$") or *indifference* (e.g., "$o_1 \sim o_2$"). Then, for each judge who prefers $o_i$ to $o_j$, a frequency indicator $f_{ij}$ is incremented by one unit. In the case the two objects are considered indifferent, $f_{ij}$ is conventionally incremented by 0.5, so that:

$$f_{ij} = m_{ij} - f_{ji}, \tag{4}$$

$m_{ij}$ being the total number of judges who express their judgment for the *i*,*j*-th paired comparison. In general, $m_{ij} \leq m$ since judges may sometimes refrain from expressing their judgments on some of the possible paired comparisons. We remark that the condition in Eq. 4 is a *sine qua non* for the application of the LCJ (Thurstone, 1927).

The observed proportion of judges that prefer $o_i$ to $o_j$ can be used to estimate the unknown probability $p_{ij}$:

$$\hat{p}_{ij} = \frac{f_{ij}}{m_{ij}}. \tag{5}$$

Of course, the relationship of complementarity $\hat{p}_{ij} = 1 - \hat{p}_{ji}$ holds.

Returning to Eq. 3, it can be expressed as:

$$\hat{p}_{ij} = 1 - \Phi[-(x_i - x_j)], \tag{6}$$

from which:

$$x_i - x_j = -\Phi^{-1}(1 - \hat{p}_{ij}). \tag{7}$$

In general, objects are judged differently by judges; however, if all judges express the same judgment, the model is no more viable ( $\hat{p}_{ij}$ values of 1.00 and 0.00 would correspond to $-\Phi^{-1}(1-\hat{p}_{ij})$ values of $\pm\infty$). A simplified approach for tackling this problem is associating values of $\hat{p}_{ij} \geq 0.977$ with $-\Phi^{-1}(1 - 0.977) = 1.995$ and values of $\hat{p}_{ij} \leq 0.023$ with $-\Phi^{-1}(1 - 0.023) = -1.995$. More sophisticated solutions to deal with this issue have been proposed (Edwards, 1957).

Extending the reasoning to all possible paired comparisons for which $m_{ij} \geq 1$ (i.e., at least one judge expresses his/her own judgment), the relevant $\hat{p}_{ij}$ values can be determined and the following system of equations can be constructed:

$$\begin{cases} \vdots \\ x_i - x_j + \Phi^{-1}\left(1 - \hat{p}_{ij}\right) = 0 \qquad \forall i, j : m_{ij} \geq 1. \\ \vdots \end{cases} \tag{8}$$

Since, the rank of the system is lower than the number ($n$) of unknowns of the problem (i.e., $x_1$, $x_2$, …, $x_n$) – and the system itself would be indeterminate (Thurstone, 1927) – the following conventional condition is introduced:

$$\sum_{i=1}^{n} x_i = 0 \,. \tag{9}$$

Eqs. 8 and 9 are then aggregated into a new system, which is *over-determined* (i.e., it has rank *n* while the total number of equations (*q*) is higher than *n*) and *linear* with respect to the unknowns:

$$\begin{cases} \begin{bmatrix} \vdots \\ x_i - x_j + \Phi^{-1}\left(1-\hat{p}_{ij}\right) = 0 \qquad \forall i, j : m_{ij} \geq 1 \\ \vdots \end{bmatrix} \\ \sum_{i=1}^{n} x_i = 0 \end{cases} . \tag{10}$$

This system can be expressed in matrix form as:

$$\begin{cases} \vdots \\ \sum_{k=1}^{n}\left(a_{hk} \cdot x_k\right) - b_h = 0 \quad \forall h \in [0, q] \\ \vdots \end{cases} \quad \Rightarrow \quad \boldsymbol{A} \cdot \boldsymbol{X} - \boldsymbol{B} = \boldsymbol{0} \,, \tag{11}$$

$\boldsymbol{X} = [x_1, x_2, \ldots, x_n]^T \in R^{n\times 1}$ being the column vector containing the unknowns of the problem, $a_{hk}$ being a generic element of matrix $\boldsymbol{A} \in R^{q\times n}$, and $b_h$ being a generic element of vector $\boldsymbol{B} \in R^{n\times 1}$. For details on the construction of $\boldsymbol{A}$ and $\boldsymbol{B}$, see (Gulliksen, 1956).

In the case each judge expresses his/her judgment on the totality of the $C_2^n = n \cdot (n-2)/2$ paired comparisons, the system in Eq. 10 is "complete" – i.e., with $q = C_2^n + 1$ equations – and can be solved in a closed form as (Thurstone, 1927):

$$\hat{x}_j = \sum_{i=1}^{m} \Phi^{-1}\left(1-\hat{p}_{ij}\right) \qquad \forall j \,. \tag{12}$$

The LCJ unfortunately has some limitations, including the following ones:

1. The response mode is relatively tedious for judges;
2. The LCJ results into an *interval* scaling, i.e., objects are defined on a scale with meaningful distance but arbitrary zero point (Thurstone, 1927; Roberts, 1979);
3. The solution can be determined only when the system of equations is "complete";
4. No uncertainty estimation is provided.

***ZM-method***

This technique has been proposed to overcome some of the limitations of the LCJ (Franceschini and Maisano, 2018). A significant drawback of the LCJ response mode is that paired comparisons can be tedious and complex to manage, due to the fact that much repetitious information is required from judges. This problem can be overcome asking each judge to formulate a *preference ordering*, i.e., a sequence of objects in order of preference (more preferred ones in the top positions and less preferred ones in the bottom ones).

Apart from regular objects ($o_1$, $o_2$, …, $o_n$), judges should include two *dummy* objects in their orderings: one ($o_Z$) corresponding to the *absence* of the attribute of interest and one ($o_M$) corresponding to the *maximum-imaginable* degree of the attribute, consistently with the current technological and socio-economic context (Franceschini and Maisano, 2018). When dealing with these special objects, two important requirements should be considered by judges:

1. $o_Z$ should be positioned at the bottom of a preference ordering, i.e., there should not be any other object with preference lower than $o_Z$. In the case the attribute of another object is judged to be absent, that object will be considered indifferent to $o_Z$ and positioned at the same hierarchical level.
2. $o_M$ should be positioned at the top of a preference orderings, i.e., there should not be any other object with preference higher than $o_M$. In the case the attribute of another object is judged to be the maximum-imaginable, that object will be considered indifferent to $o_M$ and positioned at the same hierarchical level.

Next, the preference orderings of judges can be turned into paired-comparison data (e.g., the four-object ordering $(o_3 \sim o_1) > o_2 > o_4$ is turned into the $C_2^4 = 6$ paired-comparison relationships: "$o_1 > o_2$", "$o_1 \sim o_3$", "$o_1 > o_4$", "$o_2 < o_3$", "$o_2 > o_4$", and "$o_3 > o_4$"; it can be noticed that this response mode forces judges to be *transitive* (e.g., if "$o_1 > o_2$" and "$o_2 > o_4$", then "$o_1 > o_4$").

Next, the traditional LCJ can be applied to the resulting paired-comparison data and a scaling ($x$) of the objects can be determined (Eq. 12). Through the following transformation, the resulting scaling ($x$) is transformed into a new one ($y$), which is defined in the conventional range [0, 100]:

$$\hat{y}_i = \hat{y}_i\left(\hat{\boldsymbol{X}}\right) = 100 \cdot \frac{\hat{x}_i - \hat{x}_Z}{\hat{x}_M - \hat{x}_Z} \quad \forall i\,, \tag{13}$$

where: $\hat{x}_Z$ and $\hat{x}_M$ are the scale values of $o_Z$ and $o_M$, resulting from the LCJ; $\hat{x}_i$ is the scale value of a generic *i*-th object, resulting from the LCJ; $\hat{y}_i$ is the scale value of a generic *i*-th object in the new scale *y*. This transformation can also be expressed in vector form as:

$$\hat{\boldsymbol{Y}} = \hat{\boldsymbol{Y}}\left(\hat{\boldsymbol{X}}\right) = \left[\hat{y}_1\left(\hat{\boldsymbol{X}}\right), \hat{y}_2\left(\hat{\boldsymbol{X}}\right), \ldots\right]^T, \tag{14}$$

being $\hat{\boldsymbol{Y}}$ a column vector whose components result from a system of *n* decoupled equations. Since scale *y* "inherits" the *interval* property from scale *x* and has a conventional zero point that corresponds to the absence of the attribute (i.e., $\hat{y}_Z$), it can be reasonably considered as a *ratio* scale, without any conceptually prohibited "promotion". We note that the two dummy objects, $o_Z$ and $o_M$, are used to "anchor" the *x* scale to the *y* scale (Paruolo et al., 2013).

Although the *ZM*-technique simplifies the response mode and allows to obtain a ratio scaling, it still does not solve other relevant limitations of the traditional LCJ:

- The procedure is not applicable to the system in Eq. 10 when it is not "complete" (i.e., there is at least one (*i*, *j*) paired comparison for which $m_{ij} = 0$).
- It does not contemplate neither the variability of $\hat{p}_{ij}$ values, which are actually treated as deterministic parameters (not probabilistic ones), nor the "propagation" of this variability on the $\hat{\boldsymbol{X}}$ solution (and therefore on the "transformed" solution, $\hat{\boldsymbol{Y}}$).

In fact, since $f_{ij}$ is determined considering a sample of $m_{ij}$ paired comparisons (as illustrated in section "Thurstone's LCJ"), it will be distributed binomially; $\hat{p}_{ij}$ is the best estimator of $p_{ij}$, according to the information available. In formal terms:

$$f_{ij} \sim B[\mu_{f_{ij}} \approx m_{ij} \cdot \hat{p}_{ij}, \sigma^2_{f_{ij}} \approx m_{ij} \cdot \hat{p}_{ij} \cdot (1 - \hat{p}_{ij})], \tag{15}$$

In the hypothesis that $m_{ij} \cdot \hat{p}_{ij} \geq 5$, when $0 \leq \hat{p}_{ij} \leq 0.5$, or $m_{ij} \cdot \left(1 - \hat{p}_{ij}\right) \geq 5$, when $0.5 < \hat{p}_{ij} \leq 1$, the following approximations can be reasonably introduced (Ross, 2014):

$$\begin{aligned} f_{ij} &\sim N[\mu_{f_{ij}} \approx m_{ij} \cdot \hat{p}_{ij}, \sigma^2_{f_{ij}} \approx m_{ij} \cdot \hat{p}_{ij} \cdot (1 - \hat{p}_{ij})] \\ p_{ij} &\sim N\left[\mu_{p_{ij}} \approx \hat{p}_{ij}, \sigma^2_{p_{ij}} \approx \frac{\hat{p}_{ij} \cdot (1 - \hat{p}_{ij})}{m_{ij}}\right] \end{aligned} \quad . \tag{16}$$

It is worth remarking that, even when all judges express their judgments for all the possible paired comparisons (i.e., $m_{ij} = m \ \forall i, j$), the variance of $p_{ij}$ may change from one paired comparison to one other, as it also depends on the relevant $\hat{p}_{ij}$ value.

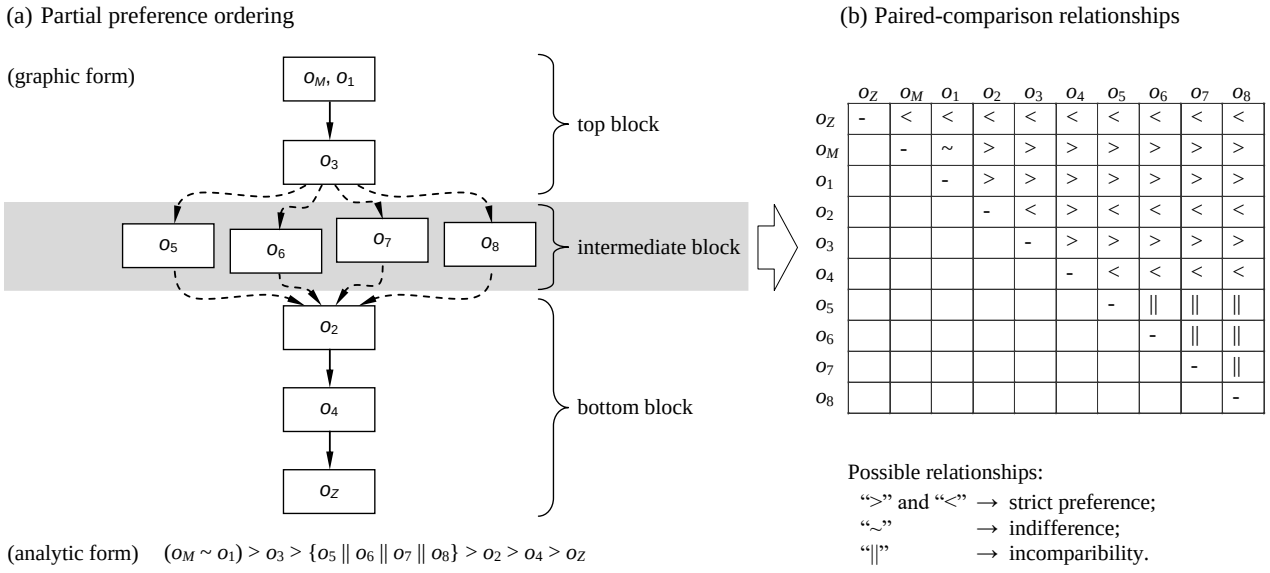
## METHODOLOGY

### *Response-mode simplification*

Although the formulation of preference orderings is less tedious and complex to manage than the direct formulation of paired-comparison relationships, it still may be problematic for some practical situations, e.g., asking judges to rank more than a handful of objects during a telephone or street interview may put a very high demand on their cognitive abilities (Harzing et al., 2009; Lenartowicz and Roth, 2001).

To further simplify the response mode, judges could formulate "incomplete" orderings of the more and/or less preferred objects only, neglecting the remaining ones. These orderings can be decomposed into three blocks: (i) a block including the *top* objects (i.e., the more preferred ones, plus $o_M$), (ii) a block including the *bottom* objects (i.e., the less preferred ones, plus $o_Z$), and (iii) a block including the *intermediate* objects. Surely the objects in the intermediate block will not be comparable to each other (i.e., it cannot be asserted that one object is more/less/equally preferred to one other), but their hierarchical level will be (1) lower than that of the objects in the top block and (2) higher than that of the objects in the bottom block. The resulting preference orderings can be classified as *partial* since – apart from the relationships of *strict preference* and *indifference* – they may also contain *incomparability* relationships among pairs of objects (Nederpelt and Kamareddine, 2004). Figure 2(a) contains a fictitious partial ordering of $n = 10$ (regular and dummy) objects, which is divided into the three afore-described blocks.

Any generic partial ordering can be translated into paired-comparison relationships. Among the $C_2^{n=10} = 45$ possible paired-comparison relationships in the example in Figure 2(b), thirty-nine are of *strict preference* (">" or "<") or *indifference* ("~"), while the remaining six are of incomparability ("||") and concern exclusively the elements in the intermediate block ($o_5$, $o_6$, $o_7$, $o_8$). We note that the objects in the top and bottom blocks are mutually ordered and translated into paired-comparison relationships of strict preference and indifference. An even more simplified response mode could be that one in which each judge merely identifies the more or less preferred objects, without ordering them; this other form of judgment can be translated into a partial ordering too, which also includes mutual relationships of incomparability among the objects in the top and bottom blocks.




| | $o_Z$ | $o_M$ | $o_1$ | $o_2$ | $o_3$ | $o_4$ | $o_5$ | $o_6$ | $o_7$ | $o_8$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $o_Z$ | - | < | < | < | < | < | < | < | < | < |
| $o_M$ | | - | ~ | > | > | > | > | > | > | > |
| $o_1$ | | | - | > | > | > | > | > | > | > |
| $o_2$ | | | | - | < | > | < | < | < | < |
| $o_3$ | | | | | - | > | > | > | > | > |
| $o_4$ | | | | | | - | < | < | < | < |
| $o_5$ | | | | | | | - | ‖ | ‖ | ‖ |
| $o_6$ | | | | | | | | - | ‖ | ‖ |
| $o_7$ | | | | | | | | | - | ‖ |
| $o_8$ | | | | | | | | | | - |

Figure 2 – (a) Example of partial preference ordering with corresponding *top*, *intermediate* and *bottom* blocks; (b) the partial ordering is turned into paired-comparison relationships.

Returning to the general problem, all the (partial) preference orderings that have been formulated by $m$ judges can be translated into a number of paired-comparison relationships. For each paired comparison, there will therefore be $m_{ij}$ "usable" relationships for determining $\hat{p}_{ij}$. In fact, the ($m_{ij}$) relationships that contribute to the estimate of $p_{ij}$ values are those of strict preference and indifference, while the remaining ($m$ – $m_{ij}$) relationships of incomparability do not contribute to this estimate; regarding the example in Figure 2, $m_{ij} = 45 - 6 = 39$. In the case $m_{ij} = 0$, $p_{ij}$ cannot be estimated.

### *GLS solution*

In general, the system in Eq. 10 will not necessarily be "complete", as the number of equations ($q$) could be lower than $C_2^n + 1$ (i.e., for any paired comparison with $m_{ij} = 0$, no equation can be formulated) and therefore cannot be solved through the LCJ.

The literature dealt with the problem of solving such "incomplete" systems through the *Ordinary Least Squares* (OLS) method. For example, Gulliksen (1956) discusses some approximate numerical methods for the OLS solution formula to Eq. 11 (Kariya and Kurata, 2004; Ross, 2014):

$$\hat{\boldsymbol{X}} = \left(\boldsymbol{A}^T \cdot \boldsymbol{A}\right)^{-1} \cdot \boldsymbol{A}^T \cdot \boldsymbol{B}. \tag{17}$$

Also, it can be demonstrated that, in the case in which the equation system is "complete", the LCJ solution coincides with the OLS one – i.e., that one minimizing the sum of the squared residuals related to the equations in Eq. 11 (Gulliksen, 1956):

$$\sum_{h=1}^{q}\left[\sum_{k=1}^{n}\left(a_{hk}\cdot x_k\right)-b_h\right]^2, \tag{18}$$

*n* being the number of elements in $\hat{\boldsymbol{X}}$, and *q* being the total number of equations available. In general, the OLS solution is possible even for "incomplete" systems, as long as $q \geq n$; this condition is easily met in practice (Gulliksen, 1956).

Even though the OLS method provides an effective solution to the problem of interest, it does not provide any practical estimate of the uncertainty associated with the elements of $\hat{\boldsymbol{X}}$. In fact, although it is possible to calculate the covariance matrix of $\hat{\boldsymbol{X}}$ as:

$$\Sigma_{\boldsymbol{X}} = \left(\boldsymbol{A}^T\cdot\boldsymbol{A}\right)^{-1}, \tag{19}$$

it is of no practical use for this specific problem, as the uncertainties of the $\hat{\boldsymbol{X}}$ elements are identical and not affected by the real uncertainty of input data (i.e., $\hat{p}_{ij}$ values, see section "*ZM*-method") (Gulliksen, 1956). This limitation can be overcome using the *Generalized Least Squares* (GLS) method, which is more articulated than the OLS method as it includes several additional steps (see the qualitative representation in Figure 3).

The idea of applying the GLS to the problem of interest in the "incomplete" case had already been advanced several decades ago by Arbuckle and Nugent (1973), who contemplated this and other goodness-of-fit criteria, such as *maximum likelihood*. These techniques, however, have not been applied extensively, probably due to some computational constraints that are nowadays overcome. Additionally, the GLS solution proposed by Arbuckle and Nugent (1973) was combined with a "classic" response mode, based on the direct formulation of paired-comparison relationships.

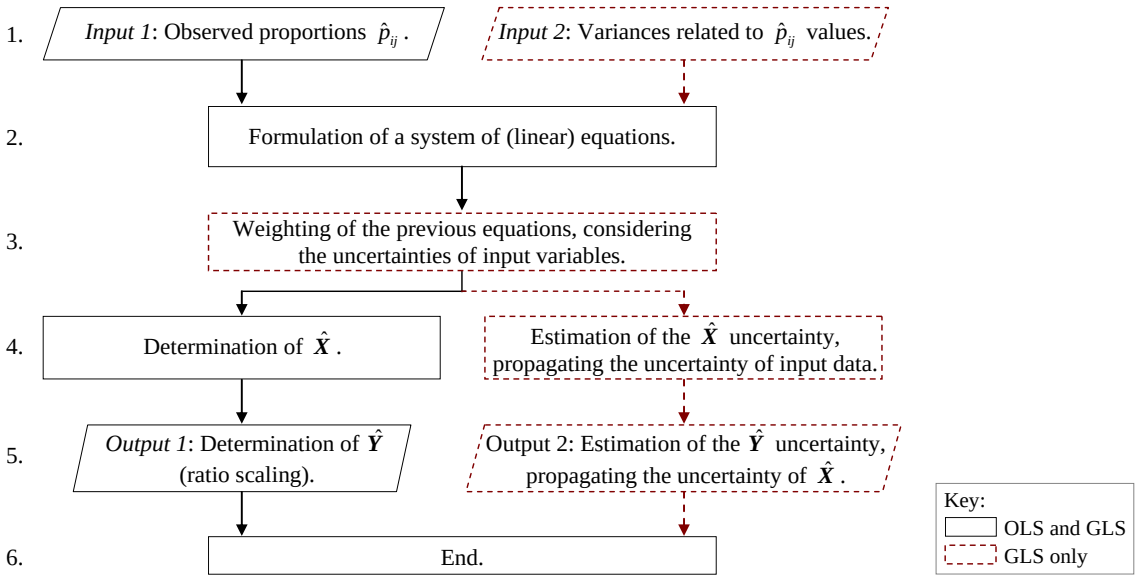



Figure 3 – Flow chart representing the main steps of the OLS and GLS solution to the problem of interest; it can be noticed that the GLS solution includes several additional steps (see dashed blocks) with respect to the OLS one.

From a technical point of view, the GLS method allows obtaining a solution that minimizes the weighted sum of the squared residuals related to the equations in Eq. 11, i.e.:

$$\sum_{h=1}^{q} w_h \cdot \left[ \sum_{k=1}^{n} \left( a_{hk} \cdot x_k \right) - b_h \right]^2 , \tag{20}$$

in which weights ($w_h$) take into account the uncertainty in the $\hat{p}_{ij}$ values. It can be demonstrated that, for a generic equation related to a generic *ij*-th paired comparison: $w_h = \left[ \frac{\partial \Phi^{-1}\left(1-\hat{p}_{ij}\right)}{\partial \hat{p}_{ij}} \right]^2 \Big/ \sigma^2_{p_{ij}}$ (Arbuckle and Nugent, 1973).

Next, weights are aggregated into a (squared) matrix $\boldsymbol{W}$, whose construction is illustrated in the "Appendix" section, and $\boldsymbol{X}$ can be estimated as:

$$\hat{\boldsymbol{X}} = \left( \boldsymbol{A}^T \cdot \boldsymbol{W} \cdot \boldsymbol{A} \right)^{-1} \cdot \boldsymbol{A}^T \cdot \boldsymbol{W} \cdot \boldsymbol{B} . \tag{21}$$

Combining Eqs. 21 and 14, the final (ratio) scaling $\hat{\boldsymbol{Y}}$ can be obtained as:

$$\hat{\boldsymbol{Y}} = \hat{\boldsymbol{Y}}[\hat{\boldsymbol{X}}] = \hat{\boldsymbol{Y}}[(\boldsymbol{A}^T \cdot \boldsymbol{W} \cdot \boldsymbol{A})^{-1} \cdot \boldsymbol{A}^T \cdot \boldsymbol{W} \cdot \boldsymbol{B}] . \tag{22}$$

Next, the uncertainty related to the elements in $\hat{\boldsymbol{Y}} = \left[\hat{y}_1, \hat{y}_2, \cdots\right]^T \in R^{n \times 1}$ can be determined by applying the relationship:

$$\Sigma_{\boldsymbol{Y}} = \boldsymbol{J}_{\hat{\boldsymbol{X}}} \cdot [(\boldsymbol{A}^T \cdot \boldsymbol{W} \cdot \boldsymbol{A})^{-1}] \cdot \boldsymbol{J}_{\hat{\boldsymbol{X}}}^T . \tag{23}$$

where $\boldsymbol{J}_{\hat{\boldsymbol{X}}} \in R^{(q-1)\times(q-1)}$ is a Jacobian matrix containing the partial derivatives related to the equations of the system in Eq. 14, with respect to the elements of $\hat{\boldsymbol{X}}$. Assuming that the $p_{i,j}$ and $\hat{y}_i$ values are approximately normally distributed, a 95% confidence interval related to each $\hat{y}_i$ value can be computed as:

$$\hat{y}_i \pm U_{\hat{y}_i} = \hat{y}_i \pm 2 \cdot \sigma_{\hat{y}_i} \quad \forall i\,, \tag{24}$$

$U_{\hat{y}_i}$ being the so-called *expanded uncertainty* of $\hat{y}_i$ with a coverage factor $k$ = 2 and $\sigma_{\hat{y}_i} = \sqrt{\Sigma_{\boldsymbol{Y},(i,i)}}$ (JCGM 100:2008, 2008).

## APPLICATION EXAMPLE

The proposed technique is applied to the design of a civilian aircraft seat. The goal is to prioritize the customer requirements (CRs) in Table 1 (i.e., objects), according to their importance (i.e., attribute) for $m$ = 20 regular air passengers (i.e., judges).

Table 2 contains $m$ "complete" linear orderings by judges, assuming that they have no difficulty in managing both the regular and dummy objects. These orderings are then translated into a number of paired-comparison relationships (i.e., $C_2^{12} = 66$ for each preference ordering, resulting in total 66·20 =1320 paired-comparison relationships) and the LCJ is applied, producing the scaling in Table 5(a) (see also the graphical representation in Figure 5). These results are already referred to the conventional scale ($y$), which is included in the range [0, 100].

Table 1 – List of the major CRs related to an aircraft seat, from the perspective of passengers.

| Abbr. | Description |
|---|---|
| $o_1$ | Comfortable (does not give you back ache) |
| $o_2$ | Enough leg room |
| $o_3$ | Comfortable when you recline |
| $o_4$ | Does not hit person behind when you recline |
| $o_5$ | Comfortable seat belt |
| $o_6$ | Seat belt feels safe |
| $o_7$ | Arm rests not too narrow |
| $o_8$ | Arm rest folds right away |
| $o_9$ | Does not make you sweat |
| $o_{10}$ | Does not soak up a spilt drink |
| $o_{11}$ | Hole in tray for coffee cup |
| $o_{12}$ | Magazines can be easily removed from rack |

Table 2 – Linear (or "complete") preference orderings used in the application example.

| Judges | Linear preference orderings |
|---|---|
| $j_1$ | $o_M > (o_1 \sim o_2) > (o_5 \sim o_6 \sim o_7) > o_3 > o_4 > (o_{10} \sim o_8) > (o_{11} \sim o_9) > (o_Z \sim o_{12})$ |
| $j_2$ | $(o_1 \sim o_5 \sim o_7 \sim o_M) > (o_9 \sim o_6) > (o_2 \sim o_{11} \sim o_8) > (o_4 \sim o_{12} \sim o_{10}) > o_3 > o_Z$ |
| $j_3$ | $(o_1 \sim o_M) > (o_3 \sim o_2) > (o_6 \sim o_5) > o_7 > (o_8 \sim o_9 \sim o_4) > (o_{11} \sim o_{12} \sim o_Z \sim o_{10})$ |
| $j_4$ | $(o_1 \sim o_2 \sim o_7 \sim o_M) > (o_3 \sim o_5) > o_6 > (o_8 \sim o_9 \sim o_{10}) > (o_{11} \sim o_{12}) > o_4 > o_Z$ |
| $j_5$ | $(o_1 \sim o_2 \sim o_5 \sim o_M) > (o_3 \sim o_6 \sim o_9 \sim o_4) > (o_7 \sim o_8) > (o_{11} \sim o_{12} \sim o_Z \sim o_{10})$ |
| $j_6$ | $o_M > (o_1 \sim o_5 \sim o_6) > o_7 > (o_2 \sim o_9 \sim o_3) > (o_{11} \sim o_8) > (o_4 \sim o_{10}) > (o_Z \sim o_{12})$ |
| $j_7$ | $(o_2 \sim o_7 \sim o_M) > o_1 > (o_5 \sim o_8 \sim o_6) > (o_9 \sim o_{10} \sim o_3) > o_4 > (o_{12} \sim o_Z \sim o_{11})$ |
| $j_8$ | $(o_1 \sim o_5 \sim o_M) > (o_6 \sim o_7 \sim o_9 \sim o_{12} \sim o_2) > (o_8 \sim o_{11}) > (o_3 \sim o_4) > (o_Z \sim o_{10})$ |
| $j_9$ | $(o_1 \sim o_4 \sim o_M) > (o_5 \sim o_7 \sim o_9) > o_2 > o_6 > (o_{10} \sim o_8) > (o_{11} \sim o_{12} \sim o_3) > o_Z$ |
| $j_{10}$ | $(o_2 \sim o_5 \sim o_6 \sim o_M) > o_7 > o_1 > o_3 > o_{11} > o_9 > (o_8 \sim o_{10}) > (o_{12} \sim o_Z \sim o_4)$ |
| $j_{11}$ | $(o_1 \sim o_M) > (o_6 \sim o_2) > (o_7 \sim o_5) > o_3 > o_4 > o_9 > o_{11} > o_{12} > (o_{10} \sim o_Z \sim o_8)$ |
| $j_{12}$ | $(o_2 \sim o_7 \sim o_M) > o_1 > (o_9 \sim o_3) > (o_8 \sim o_6 \sim o_4) > o_5 > (o_{11} \sim o_{12} \sim o_Z \sim o_{10})$ |
| $j_{13}$ | $(o_1 \sim o_2 \sim o_5 \sim o_8 \sim o_M) > (o_7 \sim o_3 \sim o_6) > (o_9 \sim o_{11} \sim o_4) > (o_{12} \sim o_Z \sim o_{10})$ |
| $j_{14}$ | $(o_2 \sim o_5 \sim o_6 \sim o_M) > o_1 > o_3 > o_9 > o_7 > o_4 > (o_{10} \sim o_{11}) > (o_{12} \sim o_Z \sim o_8)$ |
| $j_{15}$ | $(o_2 \sim o_6 \sim o_7 \sim o_M) > o_1 > o_9 > o_8 > (o_{11} \sim o_{12} \sim o_5) > o_3 > o_4 > (o_Z \sim o_{10})$ |
| $j_{16}$ | $(o_1 \sim o_M) > (o_2 \sim o_6) > o_7 > o_9 > o_5 > (o_4 \sim o_8) > o_{11} > o_3 > (o_{12} \sim o_Z \sim o_{10})$ |
| $j_{17}$ | $(o_1 \sim o_2 \sim o_M) > (o_7 \sim o_5) > (o_8 \sim o_9 \sim o_{10} \sim o_4) > (o_6 \sim o_{11} \sim o_{12}) > (o_Z \sim o_3)$ |
| $j_{18}$ | $(o_1 \sim o_2 \sim o_3 \sim o_M) > o_9 > (o_7 \sim o_8 \sim o_4 \sim o_5) > o_{10} > (o_{11} \sim o_6) > (o_Z \sim o_{12})$ |
| $j_{19}$ | $(o_1 \sim o_2 \sim o_6 \sim o_9 \sim o_M) > (o_3 \sim o_7 \sim o_4 \sim o_5) > o_8 > o_{11} > (o_{12} \sim o_{10}) > o_Z$ |
| $j_{20}$ | $o_M > (o_1 \sim o_6) > o_7 > (o_5 \sim o_9 \sim o_2) > (o_3 \sim o_{11} \sim o_8) > (o_4 \sim o_{12}) > o_{10} > o_Z$ |

Total no. of usable paired-comparison relationships: 1320 of 1320;
$o_1$ to $o_{12}$ are the regular objects, while $o_Z$ to $o_M$ are the dummy objects;
">" and "~" respectively depict the *strict preference* and *indifference* relationships;
$(o_i \sim o_j \sim \ldots)$ is a generic block containing indifferent objects.

Table 5(a) also shows that, consistently with the considerations in section "Methodology", the results of the LCJ are identical to those obtained by applying the OLS method to the same orderings. On the other hand, the application of the GLS method produces a very close – although non-identical – result. The difference stems from the fact that – unlike LCJ and OLS – the GLS takes into account the uncertainties related to $\hat{p}_{ij}$ values. The GLS solution is therefore superior from both a conceptual and practical point of view.

To study the effectiveness of the GLS in the presence of partial orderings, we have intentionally "degraded" the linear orderings in Table 2, replacing some of the relationships of strict preference (">" and "<") and indifference ("~"), with incomparability relationships ("||"). Precisely, two types of partial orderings have been generated according to the following "degradation criteria" (see also the example in Figure 4):

*Type-t&b* preference orderings, in which the relationships between the *t* more preferred (top) objects – and any other tied object – and those between the *b* less preferred (bottom) objects – and any other tied object – have been preserved ("*t&b*" stands for "top and bottom"). The remaining objects are allocated at an intermediate hierarchical level, which is certainly lower than the top block and higher than the bottom block.

*Type-t* preference orderings, in which the relationships between the *t* more preferred (top) objects – and any other tied object – have been preserved. The remaining objects are

allocated at a lower hierarchical level, with mutual relationships of incomparability. In this case, the bottom block is empty.

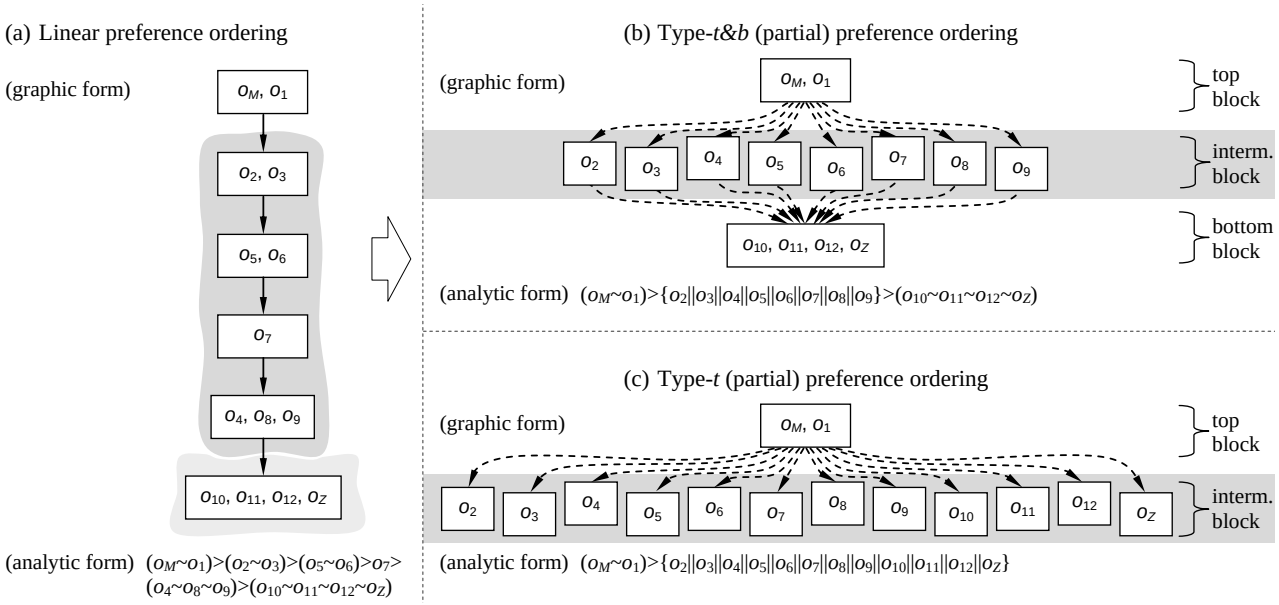



Figure 4 – Example of degradation of the linear preference ordering by $j_5$ (see Table 5) into a type-*t&b* and type-*t* partial preference ordering. In this case, $t = b = 1$.

Table 3 and Table 4 report the resulting type-*t&b* and type-*t* orderings, which are obtained by degrading the linear orderings in Table 2. In this case, the "level of degradation" is significant since $t = b = 1$; as a rough indicator of this level, we can consider the portion of *usable* paired-comparison relationships that can be obtained from the resulting partial preference orderings: i.e., 1150 out of 1320 for type-*t&b* orderings and 726 out of 1320 for type-*t* orderings.

Table 5 and Figure 5 contain the results of the application of the LCJ, OLS and GLS (where applicable) to the paired-comparison relationships that result from the preference orderings in Table 2, Table 3 and Table 4. In all cases, the resulting ($x$) scaling has been turned into a ($y$) scaling, through the transformation in Eq. 13.

The GLS results that are obtained for linear preference orderings (in Table 5(a)) can be used as "gold standard" to evaluate the goodness of the GLS results for degraded (partial) orderings. As for type-*t&b* (partial) orderings, results (in Table 5(b) and Figure 5) are – quite surprisingly – very close to those related to linear orderings, both in terms of accuracy and dispersion. As for type-*t* (partial) orderings, results (in Table 5(c) and Figure 5) worsen considerably, especially for the less preferred objects (see the very wide uncertainty bands). This is probably due by the relatively small amount of usable paired-comparison relationships that concern the less preferred objects.

Table 3 – Type-*t&b* (partial) preference orderings that are obtained by "degrading" the linear orderings in Table 2; both *t* and *b* values have been set to 1.

| Judges | Type-*t&b* (partial) preference orderings |
|---|---|
| $j_1$ | $o_M > \{o_1 \parallel o_2 \parallel o_5 \parallel o_6 \parallel o_7 \parallel o_3 \parallel o_4 \parallel o_{10} \parallel o_8 \parallel o_{11} \parallel o_9\} > (o_Z \sim o_{12})$ |
| $j_2$ | $(o_1 \sim o_5 \sim o_7 \sim o_M) > \{o_9 \parallel o_6 \parallel o_2 \parallel o_{11} \parallel o_8 \parallel o_4 \parallel o_{12} \parallel o_{10} \parallel o_3\} > o_Z$ |
| $j_3$ | $(o_1 \sim o_M) > \{o_3 \parallel o_2 \parallel o_6 \parallel o_5 \parallel o_7 \parallel o_8 \parallel o_9 \parallel o_4\} > (o_{11} \sim o_{12} \sim o_Z \sim o_{10})$ |
| $j_4$ | $(o_1 \sim o_2 \sim o_7 \sim o_M) > \{o_3 \parallel o_5 \parallel o_6 \parallel o_8 \parallel o_9 \parallel o_{10} \parallel o_{11} \parallel o_{12} \parallel o_4\} > o_Z$ |
| $j_5$ | $(o_1 \sim o_2 \sim o_5 \sim o_M) > \{o_3 \parallel o_6 \parallel o_9 \parallel o_4 \parallel o_7 \parallel o_8\} > (o_{11} \sim o_{12} \sim o_Z \sim o_{10})$ |
| $j_6$ | $o_M > \{o_1 \parallel o_5 \parallel o_6 \parallel o_7 \parallel o_2 \parallel o_9 \parallel o_3 \parallel o_{11} \parallel o_8 \parallel o_4 \parallel o_{10}\} > (o_Z \sim o_{12})$ |
| $j_7$ | $(o_2 \sim o_7 \sim o_M) > \{o_1 \parallel o_5 \parallel o_8 \parallel o_6 \parallel o_9 \parallel o_{10} \parallel o_3 \parallel o_4\} > (o_{12} \sim o_Z \sim o_{11})$ |
| $j_8$ | $(o_1 \sim o_5 \sim o_M) > \{o_6 \parallel o_7 \parallel o_9 \parallel o_{12} \parallel o_2 \parallel o_8 \parallel o_{11} \parallel o_3 \parallel o_4\} > (o_Z \sim o_{10})$ |
| $j_9$ | $(o_1 \sim o_4 \sim o_M) > \{o_5 \parallel o_7 \parallel o_9 \parallel o_2 \parallel o_6 \parallel o_{10} \parallel o_8 \parallel o_{11} \parallel o_{12} \parallel o_3\} > o_Z$ |
| $j_{10}$ | $(o_2 \sim o_5 \sim o_6 \sim o_M) > \{o_7 \parallel o_1 \parallel o_3 \parallel o_{11} \parallel o_9 \parallel o_8 \parallel o_{10}\} > (o_{12} \sim o_Z \sim o_4)$ |
| $j_{11}$ | $(o_1 \sim o_M) > \{o_6 \parallel o_2 \parallel o_7 \parallel o_5 \parallel o_3 \parallel o_4 \parallel o_9 \parallel o_{11} \parallel o_{12}\} > (o_{10} \sim o_Z \sim o_8)$ |
| $j_{12}$ | $(o_2 \sim o_7 \sim o_M) > \{o_1 \parallel o_9 \parallel o_3 \parallel o_8 \parallel o_6 \parallel o_4 \parallel o_5\} > (o_{11} \sim o_{12} \sim o_Z \sim o_{10})$ |
| $j_{13}$ | $(o_1 \sim o_2 \sim o_5 \sim o_8 \sim o_M) > \{o_7 \parallel o_3 \parallel o_6 \parallel o_9 \parallel o_{11} \parallel o_4\} > (o_{12} \sim o_Z \sim o_{10})$ |
| $j_{14}$ | $(o_2 \sim o_5 \sim o_6 \sim o_M) > \{o_1 \parallel o_3 \parallel o_9 \parallel o_7 \parallel o_4 \parallel o_{10} \parallel o_{11}\} > (o_{12} \sim o_Z \sim o_8)$ |
| $j_{15}$ | $(o_2 \sim o_6 \sim o_7 \sim o_M) > \{o_1 \parallel o_9 \parallel o_8 \parallel o_{11} \parallel o_{12} \parallel o_5 \parallel o_3 \parallel o_4\} > (o_Z \sim o_{10})$ |
| $j_{16}$ | $(o_1 \sim o_M) > \{o_2 \parallel o_6 \parallel o_7 \parallel o_9 \parallel o_5 \parallel o_4 \parallel o_8 \parallel o_{11} \parallel o_3 \parallel o_{12}\} > (o_Z \sim o_{10})$ |
| $j_{17}$ | $(o_1 \sim o_2 \sim o_M) > \{o_7 \parallel o_5 \parallel o_8 \parallel o_9 \parallel o_{10} \parallel o_4 \parallel o_6 \parallel o_{11} \parallel o_{12}\} > (o_Z \sim o_3)$ |
| $j_{18}$ | $(o_1 \sim o_2 \sim o_3 \sim o_M) > \{o_9 \parallel o_7 \parallel o_8 \parallel o_4 \parallel o_5 \parallel o_{10} \parallel o_{11} \parallel o_6\} > (o_Z \sim o_{12})$ |
| $j_{19}$ | $(o_1 \sim o_2 \sim o_6 \sim o_9 \sim o_M) > \{o_3 \parallel o_7 \parallel o_4 \parallel o_5 \parallel o_8 \parallel o_{11} \parallel o_{12} \parallel o_{10}\} > o_Z$ |
| $j_{20}$ | $o_M > \{o_1 \parallel o_6 \parallel o_7 \parallel o_5 \parallel o_9 \parallel o_2 \parallel o_3 \parallel o_{11} \parallel o_8 \parallel o_4 \parallel o_{12} \parallel o_{10}\} > o_Z$ |

Total no. of usable paired-comparison relationships: 1150 of 1320;
$o_1$ to $o_{12}$ are the regular objects, while $o_Z$ to $o_M$ are the dummy objects;
">", "~" and "||" respectively depict the *strict preference*, *indifference* and *incomparability* relationships;
$\{o_i \parallel o_j \parallel \ldots\}$ is a generic block containing incomparable objects;
$(o_i \sim o_j \sim \ldots)$ is a generic block containing indifferent objects.

Table 4 – Type-*t* (partial) preference orderings that are obtained by "degrading" the linear orderings in Table 2; *t* values have been set to 1.

| Judges | Type-*t* (partial) preference orderings |
|---|---|
| $j_1$ | $o_M > \{o_1 \parallel o_2 \parallel o_5 \parallel o_6 \parallel o_7 \parallel o_3 \parallel o_4 \parallel o_{10} \parallel o_8 \parallel o_{11} \parallel o_9 \parallel o_Z \parallel o_{12}\}$ |
| $j_2$ | $(o_1 \sim o_5 \sim o_7 \sim o_M) > \{o_9 \parallel o_6 \parallel o_2 \parallel o_{11} \parallel o_8 \parallel o_4 \parallel o_{12} \parallel o_{10} \parallel o_3 \parallel o_Z\}$ |
| $j_3$ | $(o_1 \sim o_M) > \{o_3 \parallel o_2 \parallel o_6 \parallel o_5 \parallel o_7 \parallel o_8 \parallel o_9 \parallel o_4 \parallel o_{11} \parallel o_{12} \parallel o_Z \parallel o_{10}\}$ |
| $j_4$ | $(o_1 \sim o_2 \sim o_7 \sim o_M) > \{o_3 \parallel o_5 \parallel o_6 \parallel o_8 \parallel o_9 \parallel o_{10} \parallel o_{11} \parallel o_{12} \parallel o_4 \parallel o_Z\}$ |
| $j_5$ | $(o_1 \sim o_2 \sim o_5 \sim o_M) > \{o_3 \parallel o_6 \parallel o_9 \parallel o_4 \parallel o_7 \parallel o_8 \parallel o_{11} \parallel o_{12} \parallel o_Z \parallel o_{10}\}$ |
| $j_6$ | $o_M > \{o_1 \parallel o_5 \parallel o_6 \parallel o_7 \parallel o_2 \parallel o_9 \parallel o_3 \parallel o_{11} \parallel o_8 \parallel o_4 \parallel o_{10} \parallel o_Z \parallel o_{12}\}$ |
| $j_7$ | $(o_2 \sim o_7 \sim o_M) > \{o_1 \parallel o_5 \parallel o_8 \parallel o_6 \parallel o_9 \parallel o_{10} \parallel o_3 \parallel o_4 \parallel o_{12} \parallel o_Z \parallel o_{11}\}$ |
| $j_8$ | $(o_1 \sim o_5 \sim o_M) > \{o_6 \parallel o_7 \parallel o_9 \parallel o_{12} \parallel o_2 \parallel o_8 \parallel o_{11} \parallel o_3 \parallel o_4 \parallel o_Z \parallel o_{10}\}$ |
| $j_9$ | $(o_1 \sim o_4 \sim o_M) > \{o_5 \parallel o_7 \parallel o_9 \parallel o_2 \parallel o_6 \parallel o_{10} \parallel o_8 \parallel o_{11} \parallel o_{12} \parallel o_3 \parallel o_Z\}$ |
| $j_{10}$ | $(o_2 \sim o_5 \sim o_6 \sim o_M) > \{o_7 \parallel o_1 \parallel o_3 \parallel o_{11} \parallel o_9 \parallel o_8 \parallel o_{10} \parallel o_{12} \parallel o_Z \parallel o_4\}$ |
| $j_{11}$ | $(o_1 \sim o_M) > \{o_6 \parallel o_2 \parallel o_7 \parallel o_5 \parallel o_3 \parallel o_4 \parallel o_9 \parallel o_{11} \parallel o_{12} \parallel o_{10} \parallel o_Z \parallel o_8\}$ |
| $j_{12}$ | $(o_2 \sim o_7 \sim o_M) > \{o_1 \parallel o_9 \parallel o_3 \parallel o_8 \parallel o_6 \parallel o_4 \parallel o_5 \parallel o_{11} \parallel o_{12} \parallel o_Z \parallel o_{10}\}$ |
| $j_{13}$ | $(o_1 \sim o_2 \sim o_5 \sim o_8 \sim o_M) > \{o_7 \parallel o_3 \parallel o_6 \parallel o_9 \parallel o_{11} \parallel o_4 \parallel o_{12} \parallel o_Z \parallel o_{10}\}$ |
| $j_{14}$ | $(o_2 \sim o_5 \sim o_6 \sim o_M) > \{o_1 \parallel o_3 \parallel o_9 \parallel o_7 \parallel o_4 \parallel o_{10} \parallel o_{11} \parallel o_{12} \parallel o_Z \parallel o_8\}$ |
| $j_{15}$ | $(o_2 \sim o_6 \sim o_7 \sim o_M) > \{o_1 \parallel o_9 \parallel o_8 \parallel o_{11} \parallel o_{12} \parallel o_5 \parallel o_3 \parallel o_4 \parallel o_Z \parallel o_{10}\}$ |
| $j_{16}$ | $(o_1 \sim o_M) > \{o_2 \parallel o_6 \parallel o_7 \parallel o_9 \parallel o_5 \parallel o_4 \parallel o_8 \parallel o_{11} \parallel o_3 \parallel o_{12} \parallel o_Z \parallel o_{10}\}$ |
| $j_{17}$ | $(o_1 \sim o_2 \sim o_M) > \{o_7 \parallel o_5 \parallel o_8 \parallel o_9 \parallel o_{10} \parallel o_4 \parallel o_6 \parallel o_{11} \parallel o_{12} \parallel o_Z \parallel o_3\}$ |
| $j_{18}$ | $(o_1 \sim o_2 \sim o_3 \sim o_M) > \{o_9 \parallel o_7 \parallel o_8 \parallel o_4 \parallel o_5 \parallel o_{10} \parallel o_{11} \parallel o_6 \parallel o_Z \parallel o_{12}\}$ |
| $j_{19}$ | $(o_1 \sim o_2 \sim o_6 \sim o_9 \sim o_M) > \{o_3 \parallel o_7 \parallel o_4 \parallel o_5 \parallel o_8 \parallel o_{11} \parallel o_{12} \parallel o_{10} \parallel o_Z\}$ |
| $j_{20}$ | $o_M > \{o_1 \parallel o_6 \parallel o_7 \parallel o_5 \parallel o_9 \parallel o_2 \parallel o_3 \parallel o_{11} \parallel o_8 \parallel o_4 \parallel o_{12} \parallel o_{10} \parallel o_Z\}$ |

Total no. of usable paired-comparison relationships: 726 of 1320;
$o_1$ to $o_{12}$ are the regular objects, while $o_Z$ to $o_M$ are the dummy objects;
">", "~" and "||" respectively depict the *strict preference*, *indifference* and *incomparability* relationships;
$\{o_i \parallel o_j \parallel \ldots\}$ is a generic block containing incomparable objects;
$(o_i \sim o_j \sim \ldots)$ is a generic block containing indifferent objects.

Table 5 – Results of the application of several techniques (i.e., LCJ, OLS and GLS) to the orderings in Table 2; GLS solutions are associated with relevant extended uncertainties (according to Eq. 24).

| **Objects** | **(a) Linear/complete orderings** | | **(b) type-*t&b* (partial)** | **(b) type-*t* (partial)** |
|---|---|---|---|---|
| | LCJ=OLS | GLS | GLS | GLS |
| $o_Z$ | 0.0 | 0.0 ±7.2 | 0.0 ±7.2 | 0.0 ±24.1 |
| $o_M$ | 100.0 | 100.0 ±6.7 | 100.0 ±6.7 | 100.0 ±8.8 |
| $o_1$ | 93.6 | 90.2 ±8.7 | 87.8 ±8.9 | 84.2 ±12.3 |
| $o_2$ | 85.1 | 83.4 ±8.1 | 86.5 ±9.0 | 82.6 ±12.6 |
| $o_3$ | 46.1 | 47.2 ±7.6 | 46.0 ±10.6 | 21.8 ±26.3 |
| $o_4$ | 37.7 | 37.7 ±7.9 | 47.4 ±10.6 | 23.4 ±25.7 |
| $o_5$ | 73.7 | 71.1 ±7.7 | 73.9 ±9.2 | 64.5 ±15.0 |
| $o_6$ | 68.5 | 67.7 ±7.5 | 67.0 ±9.7 | 54.3 ±17.3 |
| $o_7$ | 73.6 | 70.5 ±7.7 | 70.9 ±9.3 | 59.6 ±15.9 |
| $o_8$ | 40.3 | 41.6 ±7.8 | 40.9 ±10.1 | 20.6 ±26.8 |
| $o_9$ | 55.3 | 55.1 ±7.8 | 50.5 ±11.9 | 20.9 ±26.7 |
| $o_{10}$ | 19.3 | 20.9 ±8.3 | 17.7 ±9.9 | 0.0 ±36.2 |
| $o_{11}$ | 23.7 | 26.9 ±8.3 | 29.0 ±10.6 | 0.0 ±36.2 |
| $o_{12}$ | 13.4 | 14.8 ±8.7 | 11.9 ±9.8 | 0.0 ±36.2 |

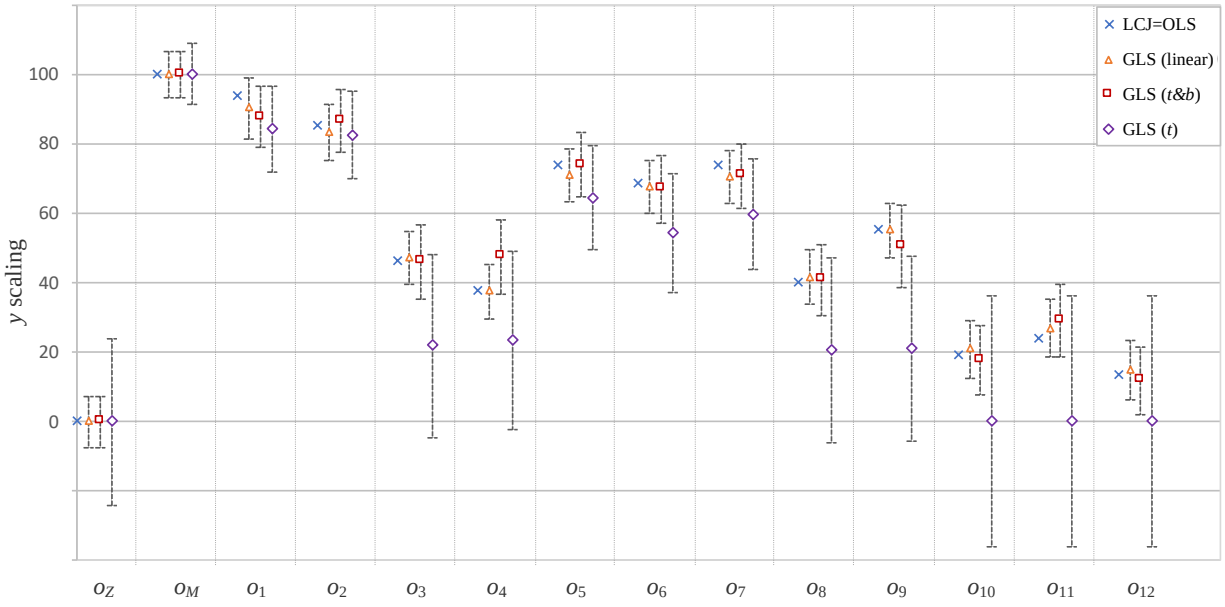



Figure 5 – Graphical representation of the results in Table 5.

These preliminary results have a significant practical implication: even when adopting a simplified response mode like type-*t&b* partial orderings, relatively accurate results can be obtained. On the contrary, the adoption of type-*t* partial orderings produces acceptable results only for the more preferred objects.

## CONCLUSIONS

The proposed technique allows to fuse multiple *partial* preference orderings into a ratio scaling with a relevant uncertainty estimation. Apart from regular objects, these orderings will also include two dummy objects, to univocally represent the zero and the maximum-possible degree of the attribute on a conventional ratio scale. This technique represents an important improvement over the technique proposed in (Franceschini and Maisano, 2018), whose application is limited to *linear* orderings exclusively.

From a technical point of view, the proposed technique is based on the formulation of a system of equations – borrowing the underlying postulates/assumptions of the LCJ – and its solution through the GLS method. From a practical point of view, the new response mode makes the technique more versatile and adaptable to a variety of contexts in which the concentration effort of judges cannot realistically be too high (e.g., telephone or street interviews).

Based on the above considerations, the proposed technique reasonably represents an appropriate response to the previously formulated research question: "How could the *ZM*-technique be modified, so as to (1) make the response more user-friendly and reliable and (2) determine a (statistically sound) estimate of the uncertainty related to the solution?".

Preliminary results show that the technique is largely automatable, computationally efficient and provides relatively accurate results, even when preference orderings are significantly "incomplete". Additionally, it seems that much better results can be obtained when partial orderings contain both the more and the less preferred elements (i.e., type-*t&b* orderings).

Regarding the future, we will test the new technique in a more organic way. Precisely, we plan to investigate the accuracy of the solution depending on various factors, such as (i) "level of degradation" of the (partial) preference orderings, (ii) number of judges, (iii) number of objects, (iv) degree of agreement between judges, etc.

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## APPENDIX

This section illustrates in more detail the application of the GLS method to the problem of interest. From an operational point of view, the GLS requires the definition of a (squared) weight matrix ($\boldsymbol{W}$), which encapsulates the uncertainty related to the equations of the system. A practical way to define $\boldsymbol{W}$ is to apply the *Multivariate Law of Propagation of Uncertainty* (MLPU) to the system in Eq. 10, referring to the input variables affected by uncertainty (Kariya and Kurata, 2004); these variables can be collected in the column vector $\boldsymbol{\xi}$. Precisely, $\boldsymbol{W}$ can be determined propagating the uncertainty of the elements in $\boldsymbol{\xi}$ to the equations of the system:

$$\boldsymbol{W} = \left[\boldsymbol{J}_{\xi} \cdot \sum_{\xi} \cdot \boldsymbol{J}_{\xi}^{T}\right]^{-1}, \tag{A1}$$

where $\boldsymbol{J}_{\boldsymbol{\xi}}$ is the Jacobian matrix containing the partial derivatives of the first members of Eq. 10, with respect to the elements in $\boldsymbol{\xi}$, and $\boldsymbol{\Sigma}_{\boldsymbol{\xi}}$ is the covariance matrix of $\boldsymbol{\xi}$.

By applying the GLS method to the system in Eq. 11, a final estimate of $\boldsymbol{X}$ can be obtained as (Kariya and Kurata, 2004):

$$\hat{\boldsymbol{X}} = \left(\boldsymbol{A}^{T} \cdot \boldsymbol{W} \cdot \boldsymbol{A}\right)^{-1} \cdot \boldsymbol{A}^{T} \cdot \boldsymbol{W} \cdot \boldsymbol{B}. \tag{A2}$$

The uncertainty of the solution can be estimated through a covariance matrix $\Sigma_{\boldsymbol{X}}$, which can be obtained by applying the following relationship:

$$\Sigma_{\boldsymbol{X}} = \left(\boldsymbol{A}^{T} \cdot \boldsymbol{W} \cdot \boldsymbol{A}\right)^{-1}. \tag{A3}$$

$\Sigma_{\boldsymbol{X}}$ – unlike the homologous matrix resulting from the OLS method (in Eq. 19) – is of considerable practical use, since it is obtained by propagating the real uncertainty of input data.

Focussing on the problem of interest, the vector containing the input variables affected by uncertainty is $\boldsymbol{\xi} = [\ldots, p_{ij}, \ldots]^{T} \in R^{(q-1)\times 1}$. On the other hand, the partial derivatives in the

Jacobian matrix $\boldsymbol{J}_{\boldsymbol{\xi}} \in R^{(q-1)\times(q-1)}$ can be determined in a closed form, by approximating terms $\Phi^{-1}(1 - \hat{p}_{ij})$ (see Eq. 10) through the following formula ( Aludaat and Alodat, 2008):

$$\Phi^{-1}(1-\hat{p}_{i,j}) \approx k\sqrt{\frac{-\ln[1-(1-2\cdot\hat{p}_{i,j})^2]}{\sqrt{\pi/8}}} \qquad \begin{cases} 0 \le \hat{p}_{i,j} \le 0.5 \rightarrow k = 1 \\ 0.5 < \hat{p}_{i,j} \le 1 \rightarrow k = -1 \end{cases}, \tag{A4}$$

from which:

$$\begin{aligned} &\frac{\partial\left[\Phi^{-1}(1-\hat{p}_{i,j})\right]}{\partial\hat{p}_{i,j}} \approx \left|\frac{\sqrt{2}\cdot(2\cdot\hat{p}_{i,j}-1)}{\sqrt{-2\cdot\sqrt{2\cdot\pi}\cdot\ln(-4\cdot\hat{p}_{i,j}^2+4\cdot\hat{p}_{i,j})}\cdot\hat{p}_{i,j}\cdot\left(1-\hat{p}_{i,j}\right)}\right| && \text{for } \hat{p}_{i,j} \neq 0.5 \\ &\frac{\partial\left[\Phi^{-1}(1-\hat{p}_{i,j})\right]}{\partial\hat{p}_{i,j}} \approx 2.506628 && \text{for } \hat{p}_{i,j} = 0.5 \end{aligned} \tag{A5}$$

The matrix $\boldsymbol{\Sigma}_{\boldsymbol{\xi}} \in R^{(q-1)\times(q-1)}$ diagonally contains the variances related to the input variables, i.e., $\hat{p}_{ij}$ terms:

$$\sigma_{p_{ij}}^2 = \frac{\hat{p}_{ij}\cdot(1-\hat{p}_{ij})}{m_{ij}}. \tag{A6}$$

The relevant covariances can be neglected, upon the reasonable assumption that the estimates of different $p_{ij}$ values are (statistically) independent from each other.

Next, it is possible to determine the matrix $\boldsymbol{W}$ (Eq. A1) and, subsequently, $\hat{\boldsymbol{X}}$ (Eq. A2) with the relevant uncertainty (Eq. A3); this solution is defined on an interval scale ($x$), as illustrated in the "Background information" section.

Through the transformation in Eq. 14, the $x$ scaling can be transformed into a new one ($y$), which is included in the conventional range [0, 100]. The uncertainty related to the elements in $\hat{\boldsymbol{Y}} = \left[\hat{y}_1, \hat{y}_2, \cdots\right]^T \in R^{n\times1}$ can be determined by applying the relationship:

$$\Sigma_{\boldsymbol{Y}} = \boldsymbol{J}_{\hat{\boldsymbol{X}}} \cdot \Sigma_{\boldsymbol{X}} \cdot \boldsymbol{J}_{\hat{\boldsymbol{X}}}^T, \tag{A7}$$

where $\boldsymbol{J}_{\hat{\boldsymbol{X}}} \in R^{(q-1)\times(q-1)}$ is the Jacobian matrix containing the partial derivatives related to the equations of the system in Eq. 14, with respect to the elements of $\hat{\boldsymbol{X}}$ (demonstration omitted).