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# On the generalized sum of the symmetric q-integers

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**Abstract** Here we will show that the symmetric q-integers of the q-calculus have a generalized sum which is also the generalized sum that we can find in the  $\kappa$ -calculus proposed by G. Kaniadakis.

**Keywords** q-calculus, q-integers, Kaniadakis  $\kappa$ -entropy.

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**Introduction** In a previous work [1], we have discussed the group of the q-integers as defined by q-calculus. In the notation given in the book by Kac and Cheung [2], the q-integers are:

$$(1) \quad [n] = \frac{q^n - 1}{q - 1} = 1 + q + q^2 + \dots + q^{n-1}.$$

In [1], we defined the generalized sum of the group as:

$$(2) \quad [m] \oplus [n] = [m] + [n] + (q - 1)[m][n]$$

As a consequence, we have that the q-integers (1) with operation (2) form a multiplicative group. The generalized sum (2) is similar to the generalized sum that we find for the Tsallis entropies of independent systems [3].

In the q-calculus [2], it is also defined the symmetric q-integer in the following form (here we use a notation different from that given in the Ref.2):

$$(3) \quad [n]_s = \frac{q^n - q^{-n}}{q - q^{-1}}$$

Repeating the approach used in [1], we can determine the group of the symmetric q-integers.

Let us start from the q-integer  $[m+n]_s$ , which is according to (3):

$$[m+n]_s = \frac{q^{m+n} - q^{-(m+n)}}{q - q^{-1}}$$

and try to find it as a generalized sum of the q-integers  $[m]_s$  and  $[n]_s$ .

By writing  $q = \exp(\log q)$ , the q-integer turns out into a hyperbolic sine:

$$(4) \quad [n]_s = \frac{q^n - q^{-n}}{q - q^{-1}} = \frac{e^{n \log q} - e^{-n \log q}}{q - q^{-1}} = 2 \frac{\sinh(n \log q)}{(q - q^{-1})}$$

Apart from a numerical factor, this is the form of the generalized numbers proposed by G. Kaniadakis in his  $\kappa$ -calculus [4-8].

From (4), we can write also:

$$\frac{1}{2}(q - q^{-1})[n]_s = \sinh(n \log q)$$

Therefore:

$$[m+n]_s = \frac{q^{m+n} - q^{-(m+n)}}{q - q^{-1}} = 2 \frac{\sinh((m+n) \log q)}{(q - q^{-1})}$$

Using the properties:

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y \quad ; \quad \cosh x = \sqrt{1 + \sinh^2 x}$$

we obtain:

$$[m+n]_s = \frac{2}{(q - q^{-1})} [\sinh(m \log q) \cosh(n \log q) + \sinh(n \log q) \cosh(m \log q)]$$

$$[m+n]_s = [m]_s \cosh(n \log q) + [n]_s \cosh(m \log q)$$

$$[m+n]_s = [m]_s \sqrt{1 + \sinh^2(n \log q)} + [n]_s \sqrt{1 + \sinh^2(m \log q)}$$

Let us define:  $k = (q - q^{-1})/2$  and then:  $k[n]_s = \sinh(n \log q)$ .

As a consequence we have the generalized sum of the symmetric q-integers as:

$$(5) \quad [m]_s \oplus [n]_s = [m]_s \sqrt{1 + k^2 [n]_s^2} + [n]_s \sqrt{1 + k^2 [m]_s^2}$$

Let us conclude stressing that (5) is also the generalized sum proposed by G. Kaniadakis in the framework of a calculus [5-8], the details of which are given in [8]. By means of (5), we can repeat the approach given in Ref.1 and study of the group of the symmetric q-integers.

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