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Original
On the generalized sum of the symmetric q-integers / Sparavigna, Amelia Carolina. - ELETTRONICO. - (2018). [10.5281/zenodo.1248959]

Availability:
This version is available at: 11583/2708001 since: 2018-05-22T11:12:23Z

Publisher:
Zenodo

Published
DOI:10.5281/zenodo.1248959

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06 May 2026

On the generalized sum of the symmetric q-integers

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Abstract Here we will show that the symmetric q-integers of the q-calculus have a generalized sum which is also the generalized sum that we can find in the κ-calculus proposed by G. Kaniadakis.

Keywords q-calculus, q-integers, Kaniadakis κ-entropy.

DOI: 10.5281/zenodo.1248959

Written in Turin, on May 17, 2018

Introduction In a previous work [1], we have discussed the group of the q-integers as defined by q-calculus. In the notation given in the book by Kac and Cheung [2], the q-integers are:

$$(1) \quad [n]=\frac{q^n-1}{q-1}=1+q+q^2+\ldots+q^{n-1} \quad .$$

In [1], we defined the generalized sum of the group as:

$$(2) \quad [m]\oplus[n]=[m]+[n]+(q-1)[m][n]$$

As a consequence, we have that the q-integers (1) with operation (2) form a multiplicative group. The generalized sum (2) is similar to the generalized sum that we find for the Tsallis entropies of independent systems [3].

In the q-calculus [2], it is also defined the symmetric q-integer in the following form (here we use a notation different from that given in the Ref.2):

$$(3) \quad [n]_s=\frac{q^n-q^{-n}}{q-q^{-1}}$$

Repeating the approach used in [1], we can determine the group of the symmetric q-integers. Let us start from the q-integer $[m+n]_s$, which is according to (3):

$$[m+n]_s=\frac{q^{m+n}-q^{-(m+n)}}{q-q^{-1}}$$

and try to find it as a generalized sum of the q-integers $[m]_s$ and $[n]_s$.

By writing $q=\exp(\log q)$, the q-integer turns out into a hyperbolic sine:

$$(4) \quad [n]_s=\frac{q^n-q^{-n}}{q-q^{-1}}=\frac{e^{n\log q}-e^{-n\log q}}{q-q^{-1}}=2\frac{\sinh(n\log q)}{(q-q^{-1})}$$

Apart from a numerical factor, this is the form of the generalized numbers proposed by G. Kaniadakis in his κ-calculus [4-8].

From (4), we can write also:

$$\frac{1}{2}(q-q^{-1})[n]_s=\sinh(n\log q)$$

Therefore:

$$[m+n]_s=\frac{q^{m+n}-q^{-(m+n)}}{q-q^{-1}}=2\frac{\sinh((m+n)\log q)}{(q-q^{-1})}$$

Using the properties:

$$\sinh(x+y)=\sinh x\cosh y+\cosh x\sinh x \quad ; \quad \cosh x=\sqrt{1+\sinh^2 x}$$

we obtain:

$$[m+n]_s=\frac{2}{(q-q^{-1})}[\sinh(m\log q)\cosh(n\log q)+\sinh(n\log q)\cosh(m\log q)]$$

$$[m+n]_s=[m]_s\cosh(n\log q)+[n]_s\cosh(m\log q)$$

$$[m+n]_s=[m]_s\sqrt{1+\sinh^2(n\log q)}+[n]_s\sqrt{1+\sinh^2(m\log q)}$$

Let us define: $k=(q-q^{-1})/2$ and then: $k[n]_s=\sinh(n\log q)$.

As a consequence we have the generalized sum of the symmetric q-integers as:

$$(5) \quad [m]_s\oplus[n]_s=[m]_s\sqrt{1+k^2[n]_s^2}+[n]_s\sqrt{1+k^2[m]_s^2}$$

Let us conclude stressing that (5) is also the generalized sum proposed by G. Kaniadakis in the framework of a calculus [5-8], the details of which are given in [8]. By means of (5), we can repeat the approach given in Ref.1 and study of the group of the symmetric q-integers.

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