On the additive group of g-Integers

ON THE ADDITIVE GROUP OF q-INTEGERS

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Abstract Here we will show that the q-integers, that we can find in the q-calculus, are forming an additive group having a generalized sum, which is similar to sum of the Tsallis q-entropy of two independent systems.

Keywords q-calculus, q-integers, Tsallis q-entropy.

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Introduction Many mathematicians have contributed to the calculus that today is known as the q-calculus [1-6]. As a consequence, it is known as "quantum calculus," "time-scale calculus" or "calculus of partitions" too [5]. Moreover, it is expressed by means of different notations or, as told in [5], by different "dialects". Here we will use the approach and the notation given in the book by Kac and Cheung [6].

The aim of this work is that of showing the following. The q-integers are forming a group having a generalized sum, which is similar to sum of the Tsallis q-entropy of two independent systems. Let us start from the definition of the q-integers.

In the q-calculus, the q-difference is simply given by:

$$d_q f = f(qx) - f(x)$$

From this difference, the q-derivative is given as:

$$D_q f = \frac{f(qx) - f(x)}{qx - x}$$

The q-derivative reduces to the Newton's derivative in the limit $q \to 1$.

Let us consider the function $f(x) = x^n$. If we calculate its q-derivative, we obtain:

$$(1) \quad D_q x^n = \frac{(qx)^n - x^n}{qx - x} = \frac{q^n - 1}{q - 1} x^{n-1}$$

Comparing the ordinary calculus, which is giving $(x^n)'=nx^{n-1}$, to Equation (1), we can define the "q-integer" $[n]$ by:

$$(2) \quad [n]=\frac{q^n-1}{q-1}=1+q+q^2+\ldots+q^{n-1}$$

Therefore Equation (1) turns out to be:

$$D_q x^n=[n]\, x^{n-1}$$

As a consequence, the *n*-th q-derivative of $f(x)=x^n$, which is obtained by repeating *n* times the q-derivative, generates the q-factorial:

$$[n]!=[n][n-1]\ldots[3][2][1]$$

Form the q-factorials, we can define q-binomial coefficients:

$$\frac{[n]!}{[m]![n-m]!}$$

This means that we can use the usual Taylor formula, replacing the derivatives by the q-derivatives and the factorials by q-factorials (in a previous work, we have discussed the q-exponential and q-trigonometric functions [7]). Then, in the q-calculus, the q-integer $[n]$ acts as the integer in the ordinary calculus.

We known that the set of integers Z, which consists of the numbers ..., −4, −3, −2, −1, 0, 1, 2, 3, 4, ..., having as operation the addition, is a group. Therefore, let us consider the set of q-integers given by (2) and investigate its group. In particular, we have to determine its operation of addition.

Let us remember that a group is a set *A* having an operation • which is combining the elements of *A*. That is, the operation combines any two elements *a,b* to form another element of the group denoted *a•b*. To qualify (*A*,•) as a group, the set and operation must satisfy the following requirements. *Closure*: For all *a*,*b* in *A*, the result of the operation *a•b* is also in *A*. *Associativity:* For all *a*,*b* and *c* in *A,* it holds (*a•b*)•*c* = *a*•(*b•c*). *Identity element*: An element *e* exists in *A*, such that for all elements *a* in *A*, it is *e•a* = *a•e* = *a*. *Inverse element:* For each *a* in *A*, there exists an element *b* in *A* such that *a•b* = *b•a* = *e*, where *e* is the identity (the notation is inherited from the multiplicative operation).

A further requirement is the c*ommutativity:* For all *a*,*b* in *A*, *a*•*b* = *b*•*a*. In this case, the group is known as an Abelian group.

Therefore, to qualify a group as an Abelian group, the set and operation must satisfy five requirements which are known as the *Abelian group axioms.* A group having a not commutative operation is called a "non-abelian group" or "non-commutative group". For an Abelian group, one may choose to denote the group operation by + and the *identity element* by 0 (*neutral element*) and the inverse element as $-a$ (*opposite element*). In this case, the group is called an additive group.

First, we have to define the operation of addition. It is not the sum that we use for the integers, but it is a generalized sum which obeys the axioms of the group.

Let us start from the q-integer $[m+n]$:

$$[m+n]=\frac{q^{m+n}-1}{q-1}=\frac{1}{q-1}(q^m q^n-1+q^m-q^m)=\frac{1}{q-1}(q^m(q^n-1)+q^m-1)$$

$$[m+n]=\frac{1}{q-1}(q^m(q^n-1)+(q^m-1)+(q^n-1)+(1-q^n))=\frac{1}{q-1}((q^m-1)(q^n-1)+(q^m-1)+(q^n-1))$$

Therefore, we have:

$$(3)\quad [m+n]=[m]+[n]+(q-1)[m][n]$$

Then, we can define the generalized “sum” of the group as:

$$(4)\quad [m]\oplus[n]=[m]+[n]+(q-1)[m][n]$$

(for other examples of generalized sums see [8]):

If we use (4) as the sum, we have the closure of it, because the result of the sum is a q-integer. Moreover, this sum is commutative.

The neutral element is:

$$(5)\quad [0]=\frac{q^0-1}{q-1}=0$$

Let us determine the opposite element $[o]$, so that:

$$[o]\oplus[n]=0$$

$$0=[0]=[o]\oplus[n]=[o]+[n]+(q-1)[o][n]$$

$$-[n]=[o]+(q-1)[o][n]$$

(6) $$[o]=-\frac{[n]}{1+(q-1)[n]}=-\frac{q^n-1}{(q-1)q^n}=\frac{q^{-n}-1}{q-1}=[-n]$$

The opposite element of q-integer $[n]$ is the q-integer of $-n$, that is $[-n]$.
Let us discuss the associativity of the sum.
It is necessary to have:

$$[m]\oplus([n]\oplus[l])=([m]\oplus[n])\oplus[l]$$

Let us calculate:

$$[m]\oplus([n]\oplus[l])=[m]\oplus([n]+[l]+(q-1)[n][l])$$
$$[m]\oplus([n]\oplus[l])=[m]+[n]+[l]+(q-1)[n][l]+(q-1)[m][n]+(q-1)[m][l]+(q-1)^2[m][n][l]$$

And also:

$$([m]\oplus[n])\oplus[l]=([m]+[n]+(q-1)[m][n])\oplus[l]$$
$$([m]\oplus[n])\oplus[l]=[m]+[n]+(q-1)[m][n]+[l]+(q-1)[m][l]+(q-1)[n][l]+(q-1)^2[m][n][l]$$

It is also easy to see that:

$$[m]\oplus[n]\oplus[l]=[m+n+l]$$

As we have shown, the five axioms of an Abelian group are satisfied. In this manner, using the generalized sum given by (4), we have the Abelian group of the q-integers. Let us also note that the generalized sum (4) is similar to the sum that we find in the approach to entropy proposed by Constantino Tsallis.
In 1948 [9], Claude Shannon defined the entropy *S* of a discrete random variable Ξ as the expected value of the information content: $S=\sum_i p_i I_i=-\sum_i p_i \log_b p_i$ [10]. In this expression, *I* is the information content of Ξ, the probability of *i*-event is p_i and *b* is the base of the used logarithm. Common values of the base are 2, the Euler's number e, and 10.
Constantino Tsallis generalized the Shannon entropy in the following manner [11]:

$$S_q = \frac{1}{q-1}\left(1-\sum_i p_i^q\right)$$

Given two independent systems *A* and *B*, for which the joint probability density satisfies:

$$p(A,B) = p(A)\,p(B)$$

the Tsallis entropy gives:

$$(7) \quad S_q(A,B) = S_q(A) + S_q(B) + (1-q)S_q(A)S_q(B)$$

The parameter $(1-q)$, in a certain manner, measures the departure from the ordinary additivity, which is recovered in the limit $q \to 1$.

Actually the group on which is based the Tsallis entropy, and therefore Equation (7), is known as the "multiplicative group" [6,12-13]. As stressed in [14], the use of a group structure allows to determine a class of generalized entropies. Let us note the group of the q-integers, with addition (4), can be considered a "multiplicative group" too.

Let us conclude telling that the main result of the work here proposed is the link to the multiplicative group and the Tsallis entropy. The group of the n-integers had been studied in [15,16] too, but in these articles, a quite different expression for the generalized sum had been proposed. It is given as the "quantum sum" $[x] \oplus [y] = [x] + q^x [y]$, where the link to the Tsallis calculus is less evident.

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