Two conjectures on Ricci-flat Kähler metrics

TWO CONJECTURES ON RICCI-FLAT KÄHLER METRICS

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Abstract. We propose two conjectures about Ricci-flat Kähler metrics:
Conjecture 1: *A Ricci-flat projectively induced metric is flat.*
Conjecture 2: *A Ricci-flat metric on an n-dimensional complex manifold such that the a_{n+1} coefficient of the TYZ expansion vanishes is flat.*

We verify Conjecture 1 (see Theorem 1.1) under the assumptions that the metric is radial and stable-projectively induced and Conjecture 2 (see Theorem 1.2) for complex surfaces whose metric is either radial or complete and ALE. We end the paper by showing, by means of the Simanca metric, that the assumption of Ricci-flatness in Conjecture 1 and in Theorem 1.2 cannot be weakened to scalar-flatness (see Theorem 1.3).

Contents

1. Introduction

An interesting open question in Kähler geometry is concerned with the characterization of Kähler-Einstein projectively induced metrics. Here a Kähler metric g on a complex manifold M is said to be *projectively induced* if there exists a Kähler (isometric and holomorphic) immersion of (M, g) into the complex projective space $(\mathbb{C}P^N, g_{FS})$, $N \leq +\infty$, endowed with the Fubini–Study metric g_{FS}, namely the metric whose associated Kähler form is given in homogeneous coordinates by $\omega_{FS} = \frac{i}{2}\partial\bar{\partial}\log(|Z_0|^2 + \cdots + |Z_N|^2)$.

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When N is finite, the only known examples of complete Kähler-Einstein projectively induced metrics are compact and homogeneous and it is still an open problem to show that these are the only possibilities (see [11], [36], [19], [2], [3], [37]). Indeed one can prove (see, e.g. [12]) that, given a compact simply connected homogeneous Kähler (Einstein) manifold M with integral Kähler form ω, then the Kodaira map $k : M \to \mathbb{C}P^N$ suitably normalized is a Kähler immersion. One can see [12] that the assumptions of simply-connectedness and compactness of M and the integrality of ω are necessary (this excludes for example the compact flat torus to be projectively induced). Moreover, if the Kähler form ω of a compact and simply-connected homogeneous Kähler manifold (M, g) is integral then there exists a positive integer k such that kg is projectively induced (see [37] for a proof based on semisimple Lie groups and Dynkin diagrams). This last assertion is valid also for noncompact simply-connected homogeneous Kähler (Einstein) manifolds by considering instead of the Kodaira map the coherent states map (coming from the theory of geometric quantization) and by allowing the ambient space to be infinite dimensional, namely by considering Kähler immersion into $\mathbb{C}\mathrm{P}^\infty$ (see [25]). Nevertheless there exist complete and nonhomogeneous projectively induced Kähler-Einstein metrics on Cartan-Hartogs domains with *negative* (constant) scalar curvature (see [27]).

Notice that in the noncompact case, due for example to the fact that $\lambda\omega$ is always integral provided M is contractible, the structure of the set of the positive real numbers $\lambda \in \mathbb{R}^+$ for which λg is projectively induced is in general less trivial than in the compact case (where it is always discrete). For example, in the noncompact symmetric case one has the following (see also [26] for the more general case of bounded homogeneous domains):

Theorem A (Theorem 2 in [27]) *Let Ω be an irreducible bounded symmetric domain endowed with its Bergman metric g_B. Then there exist a positive real number a and an integer k (both depending on Ω) such that $(\Omega, \lambda g_B)$ admits an equivariant Kähler immersion into $\mathbb{C}P^\infty$ if and only if λ belongs to the set*

$$\{a, 2a, \dots, ka\} \cup (ka, +\infty).$$

From this theorem it follows that the only irreducible bounded symmetric domain where λg_B is projectively induced for all $\lambda > 0$ is the complex hyperbolic space. More generally, for a homogeneous bounded domain (Ω, g) we have that λg is projectively induced for all $\lambda > 0$ if and only if $(\Omega, g) = \mathbb{C}H^{n_1}_{\lambda_1} \times \cdots \times \mathbb{C}H^{n_r}_{\lambda_r}$, where $\mathbb{C}H^{n_j}_{\lambda_j} = (\mathbb{C}H^{n_j}, \lambda_j g_{hyp})$ ([12], Theorem 4).

Inspired by these results, we give the following definition:

A Kähler metric g is said to be stable-projectively induced *if there exists $\epsilon > 0$ such that λg is projectively induced for all $\lambda \in (1-\epsilon, 1+\epsilon)$. A Kähler metric is said to be* unstable *if it is not stable-projectively induced.*

Obviously a Kähler metric on a compact complex manifold is always unstable and Theorem A shows that there exists metrics g which are projectively induced and unstable and which become stable-projectively induced by multiplying them for a suitable constant. Notice also that the flat metric g_0 on the complex Euclidean space $\mathbb{C}^n$ is stable-projectively induced by the map $\Psi : \mathbb{C}^n \to \mathbb{C}P^\infty$, $\Psi(z) = \left(\dots, \sqrt{\frac{1}{m_j!}} z^{m^j}, \dots\right)$ (see [7]). Consequently, many examples of stable-projectively induced metrics can be constructed on those complex manifolds M which admit a holomorphic immersion into $\mathbb{C}^n$ (e.g. Stein manifolds) by simply taking the restriction of the flat metric g_0 to M.

For the case of Ricci-flat metrics, namely Kähler-Einstein metrics with Einstein constant zero, D. Hulin [20] proves that a compact Kähler-Einstein manifold Kähler immersed into $\mathbb{C}P^N$ has positive scalar curvature. This result implies for example that a Calabi-Yau manifold does not admit a Kähler immersion into $\mathbb{C}P^N$. On the other hand there are many interesting examples of Ricci-flat metrics on noncompact complex manifolds, for example the celebrated *Taub-NUT metric*, defined as the family (constructed by C. Le Brun) of complete Kähler forms on $\mathbb{C}^2$ given by $\omega_m = \frac{i}{2}\partial\bar{\partial}\Phi_m$, for $m \geq 0$, where $\Phi_m(u,v) = u^2 + v^2 + m(u^4 + v^4)$ and u and v are implicitly defined by $|z_1| = e^{m(u^2+v^2)}u$, $|z_2| = e^{m(v^2-u^2)}v$ (notice that for $m = 0$ one gets the flat metric on $\mathbb{C}^2$). Then one can prove [30] that for $m > \frac{1}{2}$ there does not exist a Kähler immersion of $(\mathbb{C}^2, \omega_m)$ into $\mathbb{C}P^\infty$.

Thus, we believe the validity of the following conjecture:

Conjecture 1. *A Ricci-flat projectively induced metric is flat.*

In this paper we verify Conjecture 1 under the assumptions that the metric involved is stable-projectively induced and restricting ourselves to *radial* Kähler metrics, i.e. those admitting a Kähler potential Φ which depends only on the sum $|z|^2 = |z_1|^2 + \cdots + |z_n|^2$ of the local coordinates' moduli. Our first result is then the following:

Theorem 1.1. *The only Ricci-flat, stable-projectively induced and radial Kähler metric is the flat one.*

Notice that without assuming the Ricci-flatness the thesis of the previous theorem does not hold. For example the radial non-flat Kähler metric $g = \frac{i}{2}\partial\bar{\partial}(|z|^2 + |z|^4)$ on $\mathbb{C}$ is stable-projectively induced being the pull-back of the flat metric on $\mathbb{C}^2$ via the embedding $z \mapsto (z, z^2)$.

The requirement that a Kähler metric is projectively induced is a somehow strong assumption. Thus it is natural to try to approximate a Kähler metric g on a complex manifold M with projectively induced ones. In the last two decades a lot of work has been done in this direction both in the noncompact and compact case. Roughly speaking, if the Kähler form ω associated to g is integral, then for every positive integer m one can construct a holomorphic map $\varphi_m : M \to \mathbb{C}\mathrm{P}^{N_m}$ into an N_m-dimensional ($N_m \leq \infty$) complex projective space such that

$$\lim_{m\to\infty} \frac{1}{m}\varphi_m^* g_{FS} = g.$$

More precisely, under suitable assumptions (automatically satisfied in the compact case) (see, e.g. [1]) there exists a smooth function ϵ_{mg} on M, depending on m and on the metric g, such that

$$\varphi_m^*\omega_{FS} = m\omega + \frac{i}{2}\partial\bar\partial \log \epsilon_{mg}$$

and admitting the so called *Tian–Yau–Zelditch expansion* (TYZ in the sequel)

$$\epsilon_{mg}(x) \sim \sum_{j=0}^{\infty} a_j(x) m^{n-j}, \tag{1}$$

where $a_0(x) = 1$ and $a_j(x)$, $j = 1, \dots$ are smooth functions on M depending on the curvature and its covariant derivatives at x of the metric g (see [42] for details). In particular, Z. Lu [31] computed the first three coeffcients:

$$\begin{cases} a_1(x) = \frac{1}{2}\rho \\ a_2(x) = \frac{1}{3}\Delta\rho + \frac{1}{24}(|R|^2 - 4|\mathrm{Ric}|^2 + 3\rho^2) \\ a_3(x) = \frac{1}{8}\Delta\Delta\rho + \frac{1}{24}\mathrm{divdiv}(R, \mathrm{Ric}) - \frac{1}{6}\mathrm{divdiv}(\rho\mathrm{Ric}) + \\ \qquad + \frac{1}{48}\Delta(|R|^2 - 4|\mathrm{Ric}|^2 + 8\rho^2) + \frac{1}{48}\rho(\rho^2 - 4|\mathrm{Ric}|^2 + |R|^2) + \\ \qquad + \frac{1}{24}(\sigma_3(\mathrm{Ric}) - \mathrm{Ric}(R, R) - R(\mathrm{Ric}, \mathrm{Ric})), \end{cases} \tag{2}$$

where ρ, R, Ric denote respectively the scalar curvature, the curvature tensor and the Ricci tensor of (M, g), and we are using the following notations (in local coordinates $z_1, \dots, z_n$):

$$\begin{aligned} &|D'\rho|^2 = g^{j\bar i}\frac{\partial\rho}{\partial z_i}\frac{\partial\rho}{\partial \bar z_j}, \\ &|D'\mathrm{Ric}|^2 = g^{\alpha\bar i}g^{j\bar\beta}g^{\gamma\bar k}\mathrm{Ric}_{i\bar j,k}\overline{\mathrm{Ric}_{\alpha\bar\beta,\gamma}}, \\ &|D'R|^2 = g^{\alpha\bar i}g^{j\bar\beta}g^{\gamma\bar k}g^{l\bar\delta}g^{\epsilon\bar p}R_{i\bar j k\bar l,p}\overline{R_{\alpha\bar\beta\gamma\bar\delta,\epsilon}}, \\ &\mathrm{divdiv}(\rho\mathrm{Ric}) = 2|D'\rho|^2 + g^{\beta\bar i}g^{j\bar\alpha}\mathrm{Ric}_{i\bar j}\frac{\partial^2\rho}{\partial z_\alpha\partial\bar z_\beta} + \rho\Delta\rho, \\ &\mathrm{divdiv}(R, \mathrm{Ric}) = -g^{\beta\bar i}g^{j\bar\alpha}\mathrm{Ric}_{i\bar j}\frac{\partial^2\rho}{\partial z_\alpha\partial\bar z_\beta} - 2|D'\mathrm{Ric}|^2 + \\ &\qquad\qquad + g^{\alpha\bar i}g^{j\bar\beta}g^{\gamma\bar k}g^{l\bar\delta}R_{i\bar j,k\bar l}R_{\beta\bar\alpha\delta\bar\gamma} - R(\mathrm{Ric}, \mathrm{Ric}) - \sigma_3(\mathrm{Ric}), \\ &R(\mathrm{Ric}, \mathrm{Ric}) = g^{\alpha\bar i}g^{j\bar\beta}g^{\gamma\bar k}g^{l\bar\delta}R_{i\bar j k\bar l}\mathrm{Ric}_{\beta\bar\alpha}\mathrm{Ric}_{\delta\bar\gamma}, \\ &\mathrm{Ric}(R, R) = g^{\alpha\bar i}g^{j\bar\beta}g^{\gamma\bar k}g^{\delta\bar p}g^{q\bar\epsilon}\mathrm{Ric}_{i\bar j}R_{\beta\bar\gamma p\bar q}R_{k\bar\alpha\epsilon\bar\delta}, \\ &\sigma_3(\mathrm{Ric}) = g^{\delta\bar i}g^{j\bar\alpha}g^{\beta\bar\gamma}\mathrm{Ric}_{i\bar j}\mathrm{Ric}_{\alpha\bar\beta}\mathrm{Ric}_{\gamma\bar\delta}, \end{aligned} \tag{3}$$

where the $g^{j\bar{i}}$'s denote the entries of the inverse matrix of the metric (i.e. $g_{k\bar{i}}g^{j\bar{i}} = \delta_{kj}$), " ,p" represents the covariant derivative in the direction $\frac{\partial}{\partial z_p}$ and we are using the summation convention for repeated indices.

The reader is also referred to [23] and [24] for a recursive formula for the coefficients a_j's and an alternative computation of a_j for $j \leq 3$ using Calabi's diastasis function (see also [40] for a graph-theoretic interpretation of this recursive formula).

Due to Donaldson's work (cfr. [13, 14, 1]) in the compact case and respectively to the theory of quantization in the noncompact case (see, e.g. [6, 9, 10]), it is natural to study metrics with the coefficients of the TYZ expansion being prescribed. In this regard Z. Lu and G. Tian [32] (see also [16] and [4] for the symmetric and homogenous case respectively) prove that the PDEs $a_j = f$ ($j \geq 2$ and f a smooth function on M) are elliptic and that if the logterm of the Bergman and Szegö kernel of the unit disk bundle over M vanishes then $a_k = 0$, for $k > n$ (n being the complex dimension of M). The study of these PDEs makes sense regardless of the existence of a TYZ expansion and so given any Kähler manifold (M, g) it makes sense to call the a_j's the *coefficients associated to metric* g. In the noncompact case in [30] one can find a characterization of the flat metric as a Taub-Nut metric with $a_3 = 0$ while Feng and Tu [17] solve a conjecture formulated in [41] by showing that the complex hyperbolic space is the only Cartan-Hartogs domain where the coefficient a_2 is constant. In a recent paper [28] the first author together with M. Zedda prove that a locally hermitian symmetric space with vanishing a_1 and a_2 is flat.

In this paper we address the following:

Conjecture 2. *A Ricci-flat metric on an n-dimensional complex manifold such that $a_{n+1} = 0$ is flat.*

In the following theorem, which represents our second result, we verify Conjecture 2 for (compact or noncompact) complex surfaces under the assumption that the metric is either ALE (Asymptotically Locally Euclidean) or radial.

Roughly speaking, an n-dimensional complete Riemannian manifold (M, g) is said to be ALE if there exists a compact subset $K \subset M$ such that $M \setminus K$ is diffeomorphic to the quotient of $\mathbb{R}^n \setminus B_R(0)$ (the ball of radius $R > 0$) by a finite group $G \subset O(n)$, and such that the metric g on this open subset tends to the flat euclidean metric at infinity. For the exact definition and construction of ALE Kähler metrics, which are interesting both from the mathematical and the physical point of view, the reader is referred to the foundational paper [22] (see also [5], [18], [34], [33]): in this paper we will need just the fact that the norm of the curvature tensor of such metrics vanishes at infinity.

Theorem 1.2. *Let (M, g) be a Ricci-flat Kähler surface such that the third coefficient a_3 of the TYZ expansion vanishes. Assume that one of the following two conditions holds true:*

1. *g is complete and ALE (asymptotically locally Euclidean);*
2. *g is radial.*

Then g is flat.

We end the paper by showing that the assumption of Ricci-flatness in Conjecture 1 and in Theorem 1.2 cannot be weakened to scalar-flatness. Indeed we prove the following:

Theorem 1.3. *The Simanca metric g_S on the blown-up $\mathbb{C}\mathrm{P}^2 \sharp \overline{\mathbb{C}\mathrm{P}^2}$ of $\mathbb{C}\mathrm{P}^2$ at one point is an ALE complete radial projectively induced scalar flat (and not Ricci-flat) metric with vanishing a_3.*

The paper is organized as follows. In Section 2 we recall the definition and properties of Calabi's diastasis function, which is the main tool for the proof of our results, and we apply Calabi's theory to radial metrics defined on open domains of $\mathbb{C}^n \setminus \{0\}$ obtaining Lemma 2.2, a fundamental tool in this paper. Finally, Section 3 and 4 are dedicated to the proofs of Theorem 1.1 and Theorems 1.2 and 1.3 respectively.

2. Radial projectively induced metrics

In order to prove our theorems we need to recall the definition of Calabi's diastasis function and some of its properties. Let (M, g) be a Kähler manifold with a local Kähler potential Φ, i.e. such that $\omega = \frac{i}{2}\partial\bar{\partial}\Phi$, where ω is the Kähler form associated to g. A Kähler potential is not unique, but it is defined up to an addition of the real part of a holomorphic function. If g (and hence Φ) is assumed to be real analytic, by duplicating the variables z and $\bar{z}$, Φ can be complex analytically continued to a function $\hat{\Phi}$ defined in a neighbourhood U of the diagonal containing $(p, \bar{p}) \in M \times \bar{M}$ (here $\bar{M}$ denotes the manifold conjugated to M).

Then the *diastasis function* $\mathrm{D}_p(z)$ for g is defined to be the unique Kähler potential around p given by

$$D_p(z) = \hat{\Phi}(z, \bar{z}) + \hat{\Phi}(p, \bar{p}) - \hat{\Phi}(z, \bar{p}) - \hat{\Phi}(p, \bar{z}).$$

By shrinking U if necessary we can assume that D_p is defined on U.

As shown by the statement of the following lemma, the diastasis turns out to be an important tool to study projectively induced metrics.

Lemma 2.1 (Calabi [7])**.** *Let (M, g) be a Kähler manifold. There exists a neighborhood of a point $p \in M$ that admits a Kähler immersion into $(\mathbb{C}\mathrm{P}^N, g_{FS})$, with*

$N \leq \infty$, if and only if the metric g is 1-resolvable at p of rank at most N. If M is connected the 1-resolvability does not depend on the point chosen. Moreover, if M is simply-connected and g is 1-resolvable at a point then there exists a global Kähler immersion from (M, g) into $(\mathbb{C}\mathrm{P}^N, g_{FS})$.

A Kähler metric with diastasis $D_p(z)$ is 1-*resolvable at p of rank N* if the matrix $B_{i,j}$, defined by considering the expansion around the point p of the function $e^{D_p(z)} - 1 = \sum_{m_i, m_j \in \mathbb{N}^n} B_{i,j}(z-p)^{m_i}(\overline{z} - \bar{p})^{m_j}$, is positive semidefinite and its rank is N. Here, z^{m_j} denotes the monomial in n variables $\prod_{\alpha=1}^n z_\alpha^{m_{\alpha,j}}$ and we arrange every n-tuple of nonnegative integers as a sequence $m_j = (m_{1,j}, \dots, m_{n,j})$ such that $m_0 = (0, \dots, 0)$, $|m_j| \leq |m_{j+1}|$ for all positive integer j and all the m_j's with the same $|m_j|$ using lexicographic order.

In particular, we are going to study metrics which admit a Kähler potential $\Phi : U \to \mathbb{R}$ that depends only on the sum of the local coordinates' moduli defined on a domain that does not contain the origin. Namely, there exists $f : \tilde{U} \to \mathbb{R}$, $\tilde{U} \subset \mathbb{R}^+$, such that

$$f(x) = \Phi(z), \ z = (z_1, \dots, z_n), \tag{4}$$

where

$$\tilde{U} = \{x = |z|^2 = |z_1|^2 + \cdots + |z_n|^2 \mid z \in U\}.$$

Unlike the case in which the origin is contained in the domain of definition of the diastasis, the matrix $B_{i,j}$ is not diagonal, so it is more difficult to apply Lemma 2.1 (see, e.g. [29] for the case on which the origin is contained). The following lemma is the key ingredient for the proof of our results.

Lemma 2.2. *Let $n \geq 2$ and $p = (s, 0, \dots, 0)$, with $s \in \mathbb{R}$, $s \neq 0$, be a point of the complex domain $U \subset \mathbb{C}^n \setminus \{0\}$ on which is defined a radial metric g with radial Kähler potential $\Phi : U \to \mathbb{R}$ and corresponding diastasis $D_p : U \to \mathbb{R}$. Let $f : \tilde{U} \to \mathbb{R}$ defined by (4) and, for $h \in \mathbb{N}$, let $g_h : \tilde{U} \to \mathbb{R}$ given by:*

$$g_h(x) = \frac{d^h e^{f(x)}}{dx^h} e^{-f(x)}. \tag{5}$$

Assume that the entries of the following infinite matrix

$$\left(det \left(\frac{1}{i!j!} \frac{\partial^{i+j}(e^{D_p(z)} g_h(|z|^2))}{\partial z_1^i \partial \overline{z}_1^j} \right)_{0 \leq i,j \leq l} \right)_{l,h \in \mathbb{N}} \tag{6}$$

are positive when evaluated in p. Then the metric g is 1-resolvable at p of infinite rank.

Proof. Let $z = (z_1, z_2, \dots, z_n) = (z_1, z^*)$, let $m_i = (m_{1,i}, m_i^*) \in \mathbb{N}^n$ and let $D_p(z)$ be the diastasis function. We observe that if $m_i^* \neq m_j^* \in \mathbb{N}^{n-1}$ then

$$\frac{\partial^{|m_i|+|m_j|}}{\partial z^{m_i} \partial \overline{z}^{m_j}} (e^{D_p(z)} - 1)\Big|_p = 0. \tag{7}$$

In fact, by definition of diastasis, $D_p(z)$ is the the sum of the Kähler potential $f(|z|^2)$, the constant $f(s^2)$ and the real part of a holomorphic function which depends only on z_1 and which is equal to $-2f(s^2)$ if evaluated in s. Therefore

$$\frac{\partial^{|m_i|+|m_j|}(e^{D_p(z)} - 1)}{\partial z_1^{m_{1,i}} \partial \overline{z}_1^{m_{1,j}} \partial z^{*m_i^*} \partial \overline{z}^{*m_j^*}}\Big|_p = \frac{\partial^{|m_i|+m_{1,j}}}{\partial z_1^{m_{1,i}} \partial \overline{z}_1^{m_{1,j}} \partial z^{*m_i^*}} (z^{*m_j^*} e^{D_p(z) - f} \frac{d^{|m_j|}}{dx^{|m_j|}} e^f)\Big|_p, \tag{8}$$

where $x = |z|^2$. From which we can deduce obviously (7) and also

$$\frac{\partial^{|m_j^*|+|m_j^*|}}{\partial z^{*m_j^*} \partial \overline{z}^{*m_j^*}} (e^{D_p(z)} - 1)\Big|_p = m_j^*! g_{|m_j^*|}(s^2). \tag{9}$$

Now, notice that in order to check if a metric is 1-resolvable, we are free to change the above arrangement of the multiindices m_i's, because this has just the effect to apply the same permutation to both rows and columns of the matrix $B_{i,j} = \frac{1}{m_i! m_j!} \frac{\partial^{|m_i|+|m_j|}(e^{D_p(z)} - 1)}{\partial z^{m_i} \partial \overline{z}^{m_j}}\Big|_p$ defined by the expansion around the point p of the function $(e^{D_p(z)} - 1)$, and then yields a similar matrix, which is positive definite if and only if the original one is. In particular, let us change the ordering of the m_i's as follows: $m_0 = (0, \dots, 0)$, $|m_j| \leq |m_{j+1}|$ for all positive integer j, if $|m_i| = |m_j|$ and $m_{1,i} > m_{1,j}$ then $i < j$.

With this order, the square submatrix E_h of $B_{i,j}$ relative to multi-indices m_i such that $|m_i| \leq h$ assumes the following form

$$\left(\begin{array}{c|c} A_h & 0 \\ \hline 0 & D_h \end{array}\right) \tag{10}$$

where A_h is the square matrix relative to multi-indices m_i such that $|m_i| < h$ or $|m_i| = h$ and $m_{1,i} \neq 0$, while D_h is the matrix relative to multi-indices m_i such that $|m_i| = h$ and $m_{1,i} = 0$. Indeed, if $|m_i| < h$ and $|m_j| = h, m_{1,j} = 0$, then $m_i^* \neq m_j^*$ because, if not, we would have $|m_i| \geq |m_j|$; similarly we clearly have $m_i^* \neq m_j^*$ provided $|m_i| = |m_j| = h$ and $m_{1,i} \neq 0, m_{1,j} = 0$. This, by (7), explains the null blocks in (10). Moreover, it follows again by (7), combined with the fact that $m_i \neq m_j$, $|m_i| = |m_j| = h$ and $m_{1,i} = m_{1,j} = 0$ imply $m_i^* \neq m_j^*$, that D_h is diagonal (and the entries on the diagonal are described by (9)) Now, if every matrix E_h is positive definite, namely if for every positive integer h the matrix A_h is positive definite and the entries of D_h are positive, the metric examined is 1-resolvable at p of infinite rank.

Since we obtain the entries of D_h by multiplying $g_h|_p$ for a positive constant, these are positive for every integer h if and only if the entries of the first row ($l = 0$) of the matrix (6), given by $e^{D_p(z)}g_h,\ h = 0, 1, \dots$, are positive.

Now we consider the matrix A_h and we change again the order of the m_i's as follows: $|m^*_j| < |m^*_{j+1}|$ for all positive integer j, if $|m^*_i| = |m^*_j|$ and m^*_i precedes m^*_j with respect to the lexicographical order or if $m^*_i = m^*_j$ and $m_{1,i} < m_{1,j}$ then $i < j$. Then, after the corresponding rows and columns exchanges on A_h and by using (7) we obtain a block matrix of the following form:

$$\begin{pmatrix} M^h_0 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & M^h_{|m^*_j|} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & M^h_{h-1} \end{pmatrix}$$

where M^h_k are square matrices whose main diagonal belongs to the main diagonal of the whole matrix and, for the same reason, are themselves block matrices of the same type. By (8), each block of M^h_k is equal to

$$\Big(\frac{1}{i!j!}\frac{\partial^{i+j}(e^{D_p(z)}g_k)}{\partial z_1^i \partial \overline{z}_1^j}\Big)_{0\le i,j\le h-k}$$

multiplied by a positive constant. Therefore, by using Sylvester's criterion, if the entries from the second row onwards of the matrix (6) are positive, A_h is positive definite for every integer h. $\square$

Corollary 2.3. *Under the same assumptions of Lemma 2.2, if there exists $x \in \tilde{U}$ and $h \in \mathbb{N}$ such that the function given by (5) is negative, namely $g_h(x) < 0$, then the metric g is not projectively induced.*

Proof. If follows by combining Lemma 2.1, Lemma 2.2 and the observation that the entries of the first row of the matrix (6) are given by $e^{D_p(z)}g_h(|z|^2)$, $h \in \mathbb{N}$. $\square$

3. Proof of Theorem 1.1

In order to prove Theorem 1.1 we need the following (well-known) classification of the potentials of radial Ricci-flat metrics (cfr. [8]).

Lemma 3.1. *Let U be a complex domain of $\mathbb{C}^n$ equipped with a radial Kähler Ricci-flat metric g. Then there esist $\lambda \in \mathbb{R}^+$ and $\epsilon = -1, 0, 1$ such that the function $f : \tilde{U} \to \mathbb{R}$ defined by (4) has the following expression*

$$f(x) = \lambda \int (\epsilon x^{-n} + 1)^{\frac{1}{n}} dx. \qquad (11)$$

Proof. The Kähler form ω associated to g reads as:

$$\omega = \frac{i}{2}\sum_{\alpha,\bar\beta} g_{\alpha\bar\beta} dz_\alpha \wedge d\bar z_\beta = \frac{i}{2}\partial\bar\partial\Phi,$$

Sinces g Ricci-flat its Ricci form vanishes, namely

$$\rho = -i\partial\bar\partial \log\det(g) = 0 \tag{12}$$

where

$$g = (g_{\alpha\bar\beta}) = \begin{pmatrix} f' + f''|z_1|^2 & f''\bar z_1 z_2 & \dots & f''\bar z_1 z_n \\ f''\bar z_2 z_1 & f' + f''|z_2|^2 & \dots & f''\bar z_2 z_n \\ & & \dots & \\ f''\bar z_n z_1 & f''\bar z_n z_2 & \dots & f' + f''|z_n|^2 \end{pmatrix}.$$

Thus, one easily sees that

$$\det(g) = (f')^{n-1}(f' + f''x),\ x = |z|^2.$$

If we denote $\Psi(x) = \log\det(g)$, equation (12) is equivalent to the following equations

$$\frac{\partial^2\Psi}{\partial z_\alpha \partial\bar z_\beta} = \Psi''\bar z_\alpha z_\beta = 0\ (\alpha \neq \beta),\quad \frac{\partial^2\Psi}{\partial z_\alpha\partial\bar z_\alpha} = \Psi' + \Psi''|z_\alpha|^2 = 0,\ \alpha, \bar\beta = 1, \dots, n.$$

This yields $\Psi' = 0$, i.e.

$$\log\det(g) = \log[(f')^{n-1}(f' + f''x)] = c,$$

for some constant c.

Setting $f' = y$ and $\tilde c = e^c > 0$, we get

$$y^{n-1}(y + xy') = y^n + xy'y^{n-1} = y^n + \frac{x}{n}(y^n)' = \tilde c$$

which rewrites as the following linear O.D.E. in $\xi = y^n$

$$\xi' = -\frac{n}{x}\xi + \tilde c\frac{n}{x}.$$

Therefore, one finds

$$y^n = \xi = Cx^{-n} + \tilde c$$

that is

$$f' = (Cx^{-n} + \tilde c)^{\frac{1}{n}}$$

and then the general solution is

$$f(x) = \int (Cx^{-n} + \tilde c)^{\frac{1}{n}} dt,\ C \in \mathbb{R}, \tilde c > 0, \tag{13}$$

which is equivalent to (11) after a change of variables. $\square$

Remark 3.2. It is known that the metrics corresponding to the Kähler potentials (11) are non-complete and non-flat except in the case of the Euclidean metric ($\epsilon = 0$).

Proof of Theorem 1.1. Let us denote by ω_ϵ the Kähler form corresponding to the potential (11) with $\lambda = 1$, namely

$$\omega_\epsilon = \frac{i}{2}\partial\bar{\partial} f_\epsilon, \tag{14}$$

where

$$f_\epsilon(x) = \int (\epsilon x^{-n} + 1)^{\frac{1}{n}} dx, \ \epsilon = -1, 0, 1. \tag{15}$$

Notice that ω_ϵ is flat either for $n = 1$ or $\epsilon = 0$. We will show that for $n \geq 2$ we have the following:

(a) $\lambda\omega_{-1}$ is not projectively induced for any $\lambda \in \mathbb{R}^+$;
(b) $\lambda\omega_1$ is not projectively induced for any $\lambda \in \mathbb{R}^+ \setminus \mathbb{Z}$.

Then the proof of Theorem 1.1 will follow by the very definition of stable-projectively induced metric.

A simple computation shows that the function $g_3(x)$ (namely (5) for $h = 3$) for the potential $f = \lambda f_{-1}$ is given by:

$$g_3(x) = \lambda \frac{(x^n - 1)^{\frac{1-2n}{n}}}{x^3} \big(\lambda^2 (x^n - 1)^{\frac{2+2n}{n}} + 3\lambda (x^n - 1)^{\frac{1+n}{n}} - (x^n(n+1) - 2)\big).$$

Hence, one has $\lim_{x \to 1^+} g_3(x) = -\infty$ and the proof of (a) follows by Corollary 2.3.

In order to prove (b) we first show by induction that the function $g_h(x)$ for the potential $f = \lambda f_1$ is given by:

$$g_h(x) = \frac{\lambda}{x^h} \Big(\Psi(x) \prod_{j=1}^{h-1} \big(\lambda\Psi(x) - j\big) + \varphi_h(x) x\Big), \tag{16}$$

where $\Psi(x) = (x^n + 1)^{1/n}$ and $\varphi_h \in C^\infty([0, +\infty))$. This statement is trivially true for g_1, because it is equal to $\frac{\lambda}{x}\Psi$. The functions g_h can be defined recursively as

$$g_{h+1} = g_h' + g_1 g_h,$$

where $g_1 = f'$. Hence

$$g_{h+1} = \frac{\lambda}{x^{h+1}} \big(\Psi \prod_{j=1}^{h} (\lambda\Psi - j) + \varphi_{h+1} x\big),$$

with

$$\varphi_{h+1} = \frac{d}{dx}\big(\Psi \prod_{j=1}^{h-1} (\lambda\Psi - j)\big) + (1 - h + \lambda\Psi)\varphi_h + \varphi_h' x \ \in C^\infty([0, +\infty))$$

and (16) is proved. Therefore, if $\lambda \in \mathbb{R}^+ \setminus \mathbb{Z}$

$$\lim_{x\to 0^+} g_{[\lambda]+2}(x) = -\infty,$$

where $[\lambda]$ denotes the integral part of λ. Thus, Corollary 2.3 implies (b) and this concludes the proof of the theorem. □

Notice that we are able to extend the proof of (b) also for some fixed integer values of λ with a case by case analysis. For example when $\lambda = 1$ one obtains the following table which expresses $g_h(x)$, for suitable values of x, depending on the dimension n of the domain, for $n = 2, 3, 4, 5$:

x	h	n	$g_h(x)$
$3/4$	7	2	$-\frac{12294367331}{2373046875}$
$3/4$	5	3	≈ -2.81
$3/4$	5	4	≈ -10.3
$6/5$	4	5	≈ -0.14

Moreover, for any n, we have:

$$g_4(1) = 2^{\frac{1-3n}{n}}(8(2^{\frac{1}{n}})^3 - 24(2^{\frac{1}{n}})^2 + 30(2^{\frac{1}{n}}) - 15 + 8n2^{\frac{1}{n}} - 9n)$$

which is seen to be negative for $n \geq 6$. We believe (in accordance with Conjecture 1) that $\lambda\omega_1$ is not projectively induced for all integer values of λ even if we are not able to provide a general proof.

Notice that for $n = 2$ and $\epsilon = 1$ one can explicitly express a Kähler potential for the Kähler metric ω_1 on $\mathbb{C}^2 \setminus \{0\}$, namely

$$f_1(x) = \sqrt{x^2+1} + \log x - \log(1 + \sqrt{x^2+1}),\ x = |z_1|^2 + |z_2|^2 \tag{17}$$

If M denotes the blow-up of $\mathbb{C}^2$ at the origin and E denotes the exceptional divisor one can prove (see [15]) that there exists a complete Ricci-flat and ALE Kähler metric g_{EH} on M whose restriction to $\mathbb{C}^2 \setminus \{0\}$ has Kähler potential given by (17). This metric is known in the literature as the Eguchi–Hanson metric and denoted here by g_{EH}.

Therefore as a byproduct of our analysis one gets the following:

Corollary 3.3. *The Eguchi–Hanson metric g_{EH} is not projectively induced.*

Remark 3.4. Notice that if one will be able to prove that λg_{EH} is not projectively induced for all $\lambda > 0$ (in accordance with our conjecture), this will provide an example of Ricci-flat and complete Kähler metric which does not admit a Kähler immersion into any finite or infinite dimensional complex space form (the reader is referred to [29] for details related to this issue).

4. Proof of Theorem 1.2 and Theorem 1.3

Proof of Theorem 1.2. By (2) the assumption $a_3 = 0$ implies $\Delta|R|^2 = 0$. By a celebrated result of Yau [39] (being M complete) (M, g) does not admit a nonconstant positive harmonic function. Hence $|R|^2$ is constant. Being g an ALE metric $|R|^2 = 0$ and so the metric g is forced to be flat. This proves 1.

In order to prove 2. it is enough to show that the vanishing of the term a_3 for the Kähler metric g associated to the Kähler form ω_ϵ given by (14) implies g is flat, i.e. either $n = 1$ or $\epsilon = 0$. Since

$$R_{i\bar{j}k\bar{l}} = \frac{\partial^2 g_{i\bar{l}}}{\partial z_k \partial \bar{z}_j} - \sum_{pq} g^{p\bar{q}} \frac{\partial g_{i\bar{p}}}{\partial z_k} \frac{\partial g_{q\bar{l}}}{\partial \bar{z}_j}. \tag{18}$$

one easily sees that the non-vanishing components of the curvature tensor of the metric g at $(z_1, 0, \ldots, 0)$ are:

$$R_{1\bar{1}1\bar{1}} = 2f''_\epsilon + 4f'''_\epsilon|z_1|^2 + f''''_\epsilon|z_1|^4 - \tfrac{1}{(f'_\epsilon + f''_\epsilon|z_1|^2)}(2f''_\epsilon + f'''_\epsilon|z_1|^2)^2|z_1|^2,$$
$$R_{1\bar{1}i\bar{i}} = f''_\epsilon + f'''_\epsilon|z_1|^2 - \tfrac{1}{f'_\epsilon}(f''_\epsilon)^2|z_1|^2,$$
$$R_{i\bar{i}i\bar{i}} = 2R_{i\bar{i}j\bar{j}} = 2f''_\epsilon,$$

where $i, j \neq 1$ and $i \neq j$.

Therefore, after a straightforward but long computation, taking into account the curvature tensors symmetries and the invariance of $|R|^2$ under unitary transformations, we get

$$|R|^2 = n(n-1)(n+1)(n+2)\epsilon^2(|z|^{2n} + \epsilon)^{-2(n+1)/n}.$$

Since

$$\Delta|R|^2 = g^{1\bar{1}}(\frac{d|R|^2}{dx} + \frac{d^2|R|^2}{dx^2}x) + (n-1)g^{i\bar{i}}\frac{d}{dx}|R|^2, \ x = |z|^2$$

this yields (by Ricci-flatness)

$$a_3 = \Delta|R|^2 = 2n(n-1)(n+2)(n+1)^2\epsilon^2((|z|^{2n} + \epsilon)^{-3(n+1)/n}(|z|^{2n}(n+3) - n\epsilon),$$

which vanishes either for $\epsilon = 0$ or $n = 1$. □

In order to prove Theorem 1.3 we recall the definition of Simanca's metric.

Let $M = \mathbb{C}\mathrm{P}^2 \sharp \overline{\mathbb{C}\mathrm{P}^2}$ be the blow-up of $\mathbb{C}^2$ at the origin and denote by E the exceptional divisor. Let (z_1, z_2) be the standard coordinates of $\mathbb{C}^2$. In [35] Simanca constructs a scalar flat Kähler complete (not Ricci-flat) metric g on M, whose Kähler potential on $M \setminus E = \mathbb{C}^2 \setminus \{0\}$ can be written as

$$\Phi_S(|z|^2) = |z|^2 + \log|z|^2. \tag{19}$$

Proof of Theorem 1.3. The holomorphic map

$$\varphi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}\mathrm{P}^\infty$$

given by

$$(z_1, z_2) \mapsto (z_1, z_2, \ldots, \sqrt{\frac{j+k}{j!k!}} z_1^j z_2^k, \ldots),\ j+k \neq 0,$$

is a Kähler immersion from $(\mathbb{C}^2 \setminus \{0\}, g_S)$ into $(\mathbb{C}\mathrm{P}^\infty, g_{FS})$, where g_S denotes the restriction of the Simanca metric g_S to $\mathbb{C}^2 \setminus \{0\}$. Indeed

$$\varphi^*\omega_{FS} = \frac{i}{2}\partial\bar\partial \log \sum_{j,k\in\mathbb{N},\ j+k\neq 0} \frac{j+k}{j!k!}|z_1|^{2j}|z_2|^{2k} = \frac{i}{2}\partial\bar\partial \log(|z|^2 e^{|z|^2}) = \frac{i}{2}\partial\bar\partial \Phi_S = \omega_S$$

Since $M = \mathbb{C}\mathrm{P}^2 \sharp \overline{\mathbb{C}\mathrm{P}^2}$ is simply-connected it follows by Lemma 2.1 that φ extends to a Kähler immersion from (M, g_S) into $(\mathbb{C}\mathrm{P}^\infty, g_{FS})$. It remains to show that $a_3 = 0$.

By (2) and (3) and taking into account that $a_2 = 0$ and hence $|R|^2 = 4|Ric|^2$ (see [28, Example 1]) one gets:

$$\begin{aligned} a_3 = \frac{1}{24}\big(&-2g^{\alpha\bar i}g^{j\bar\beta}g^{\gamma\bar k}Ric_{i\bar j,k}\overline{Ric_{\alpha\bar\beta,\gamma}} + g^{\alpha\bar i}g^{j\bar\beta}g^{\gamma\bar k}g^{l\bar\delta}Ric_{i\bar j,k\bar l}R_{\beta\bar\alpha\delta\bar\gamma} - \\ &-g^{\alpha\bar i}g^{j\bar\beta}g^{\gamma\bar k}g^{\delta\bar p}g^{q\bar\epsilon}R_{\beta\bar\gamma p\bar q}R_{k\bar\alpha\epsilon\bar\delta}Ric_{i\bar j} - 2g^{\alpha\bar i}g^{j\bar\beta}g^{\gamma\bar k}g^{l\bar\delta}R_{i\bar jk\bar l}Ric_{\beta\bar\alpha}Ric_{\delta\bar\gamma}\big). \end{aligned} \tag{20}$$

Since a_3 is invariant under unitary transformations, we only need to compute a_3 in $(z_1, 0)$. By (19) we have

$$g = \begin{pmatrix} 1 + \frac{|z_2|^2}{(|z_1|^2+|z_2|^2)^2} & -\frac{z_2\bar z_1}{(|z_1|^2+|z_2|^2)^2} \\ -\frac{z_1\bar z_2}{(|z_1|^2+|z_2|^2)^2} & 1 + \frac{|z_1|^2}{(|z_1|^2+|z_2|^2)^2} \end{pmatrix}$$

so that, for $z_2 = 0$,

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \frac{|z_1|^2+1}{|z_1|^2} \end{pmatrix}, \quad g^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{|z_1|^2}{|z_1|^2+1} \end{pmatrix}.$$

Combining this with (18) we deduce that the unique components different from zero when evaluated at $(z_1, 0)$ are:

$$\begin{aligned} R_{1\bar11\bar1} &= 2\Phi_S'' + 4\Phi_S'''|z_1|^2 + \Phi_S''''|z_1|^4 - \tfrac{1}{(\Phi_S'+\Phi_S''|z_1|^2)}(2\Phi_S'' + \Phi_S'''|z_1|^2)^2|z_1|^2 = 0 \\ R_{1\bar12\bar2} &= \Phi_S'' + \Phi_S'''|z_1|^2 - \tfrac{1}{\Phi_S'}(\Phi_S'')^2|z_1|^2 = \tfrac{1}{|z_1|^2(|z_1|^2+1)} \\ R_{2\bar22\bar2} &= 2\Phi_S'' = -\tfrac{2}{|z_1|^4} \end{aligned}$$

By recalling that $Ric_{i\bar j} = -\frac{\partial^2 \log\det g}{\partial z_i \partial \bar z_j}$ one gets:

$$Ric = \begin{pmatrix} -\frac{1}{(|z_1|^2+1)^2} & 0 \\ 0 & \frac{1}{(|z_1|^2+1)|z_1|^2} \end{pmatrix}$$

By definition $Ric_{i\bar j,k} = \partial_k Ric_{i\bar j} - Ric_{p\bar j}\Gamma^p_{ki}$, where Γ^p_{ki} are Christoffel's symbols, given by $\Gamma^p_{ki} = g^{p\bar q}\frac{\partial g_{i\bar q}}{\partial z_k}$.

A straightforward computation gives that the unique first covariant derivatives different from zero are

$Ric_{1\bar{1},1} = \frac{2}{(|z_1|^2+1)^3}\bar{z}_1$
$Ric_{2\bar{2},1} = Ric_{1\bar{2},2} = -\frac{2}{|z_1|^2(|z_1|^2+1)^2}\bar{z}_1$

Finally, we compute only the following second covariant derivatives (by definition $Ric_{i\bar{j},k\bar{l}} = \partial_{\bar{l}}\partial_k Ric_{i\bar{j}} + \Gamma^q_{ki}\Gamma^{\bar{p}}_{\bar{l}\bar{j}}Ric_{q\bar{p}} - \Gamma^p_{ki}\partial_{\bar{l}}Ric_{p\bar{j}} - \partial_{\bar{l}}\Gamma^p_{ki}Ric_{p\bar{j}} - \Gamma^{\bar{p}}_{\bar{l}\bar{j}}\partial_k Ric_{i\bar{p}}$).

$Ric_{1\bar{1},2\bar{2}} = Ric_{2\bar{1},1\bar{2}} = \frac{4(4|z_1|^2-1)}{|z_1|^2(|z_1|^2+1)^6} - \frac{1}{|z_1|^2(|z_1|^2+1)^3}$
$Ric_{2\bar{2},1\bar{1}} = Ric_{1\bar{2},2\bar{1}} = \frac{4(4|z_1|^2-1)}{|z_1|^2(|z_1|^2+1)^6} + \frac{1}{|z_1|^2(|z_1|^2+1)^3}$
$Ric_{2\bar{2},2\bar{2}} = -\frac{4}{|z_1|^2(|z_1|^2+1)^2}$

Substituting in (4), after a long but straightforward computation one gets $a_3 = 0$, and we are done. □

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