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De Giorgi's approach to hyperbolic Cauchy problems: the case of nonhomogeneous equations

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Abstract

In this paper we discuss an extension of some results obtained by E. Serra and P. Tilli, in [15, 16], concerning an original conjecture by E. De Giorgi ([4, 5]) on a purely minimization approach to the Cauchy problem for the defocusing nonlinear wave equation. Precisely, we show how to extend the techniques developed by Serra and Tilli for homogeneous hyperbolic nonlinear PDEs to the nonhomogeneous case, thus proving that the idea of De Giorgi yields in fact an effective approach to investigate general hyperbolic equations.

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1 Introduction

In this paper we present an extension, to the case of nonhomogeneous equations, of some recent results obtained in [15, 16] on a *minimization approach* to hyperbolic Cauchy problems. This approach was originally suggested by E. De Giorgi through a conjecture ([4, 5]), essentially proved in [15] and then extended to an abstract setting in [16] (see also [14, 20, 24] and references therein).

More precisely, we introduce a suitable variant of this method in order to investigate hyperbolic PDEs having the formal structure of

$$w''(t,x) = -\nabla\mathcal{W}\big(w(t,\cdot)\big)(x) + f(t,x), \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \tag{1}$$

with two prescribed initial conditions

$$w(0,x) = w_0(x), \qquad w'(0,x) = w_1(x), \qquad x \in \mathbb{R}^n. \tag{2}$$

Here, as in [16], $\nabla\mathcal{W}$ is the Gâteaux derivative of a functional $\mathcal{W} : \mathrm{W} \to [0,\infty)$ (W is some Banach space of functions in $\mathbb{R}^n$, typically a Sobolev space), the main novely being that we allow for a function $f(t,x)$ in (1), that acts as a forcing term (a source) in the resulting PDE.

The idea behind De Giorgi's approach is to obtain solutions of hyperbolic Cauchy problems as limits (when $\varepsilon \downarrow 0$) of the minimizers w_ε of a sequence of suitable functionals F_ε of the Calculus of Variations, defined as integrals in space-time of a suitable Lagrangian with an exponential weight. De Giorgi's conjecture, in its original formulation [4], concerns the defocusing NLW equation

$$w'' = \Delta w - |w|^{p-2}\, w \qquad (p \geq 2), \tag{3}$$

which falls within the general scheme (1) if we let

$$\mathcal{W}(v) = \int_{\mathbb{R}^n} \left(\frac{1}{2}|\nabla v|^2 + \frac{1}{p}|v|^p \right) dx \qquad \text{and} \qquad f \equiv 0.$$

If w_ε denotes the minimizer of the convex functional in space-time

$$F^h_\varepsilon(w) := \int_0^\infty e^{-t/\varepsilon} \left(\frac{\varepsilon^2}{2} \int_{\mathbb{R}^n} |w''(t,x)|^2\, dx + \mathcal{W}\big(w(t,\cdot)\big) \right) dt \tag{4}$$

subject to the *boundary* conditions (2), De Giorgi conjectured that $w_\varepsilon \to w$, where w solves (3) and satisfies (2), now meant as *initial* conditions of the Cauchy problem (for more details see [4, 10, 15]). This conjecture was essentially proved in [15] (see also [20]), and then generalized in [16] with an abstract version of the result, which shows that the NLW equation (3) can be replaced with the abstract equation (1) for quite general functionals $\mathcal{W}$, but still in the homogeneous case where $f \equiv 0$.

Of course, when a nontrivial source $f(t,x)$ is present in (1), the functional F^h_ε defined in (4) (being independent of f) is no longer appropriate: instead of F^h_ε, a natural choice is to minimize, subject to the boundary conditions (2), the functional

$$F_\varepsilon(w) := F^h_\varepsilon(w) - F^s_\varepsilon(w) \tag{5}$$

where F^s_ε is the linear functional

$$F^s_\varepsilon(w) := \int_0^\infty \int_{\mathbb{R}^n} e^{-t/\varepsilon}\, f_\varepsilon(t,x) w(t,x)\, dx\, dt \tag{6}$$

and $f_\varepsilon(t,x)$ is a suitable approximation of $f(t,x)$. Intuitively, this can be justified by the following heuristic argument: if w_ε is a minimizer of F_ε subject to (2), by elementary computations one can check that the Euler-Lagrange equations for F_ε reduce to

$$\varepsilon^2\, w_\varepsilon''''(t,x) - 2\varepsilon w_\varepsilon'''(t,x) + w_\varepsilon''(t,x) = -\nabla\mathcal{W}\big(w_\varepsilon(t,\cdot)\big)(x) + f_\varepsilon(t,x). \tag{7}$$

Now the connection with (1) is clear: when $\varepsilon \downarrow 0$, assuming that $f_\varepsilon \to f$ and $w_\varepsilon \to w$, one formally obtains (1) (coupled with (2)) in the limit (of course choosing $f_\varepsilon = f$ would seem most natural, but unfortunately this is not possible, as we shall explain later).

In this paper we show that this procedure can be carried out successfully, under the sole assumption that $f \in L^2_{\mathrm{loc}}([0,\infty); L^2)$, and under very general assumptions on the functional $\mathcal{W}$ (namely Assumption 2.1 and (17), as in [16]). Our results are summarized in Theorem 2.3, which is a natural development of the research program initiated in [15, 16] (in fact, letting $f \equiv 0$ in Theorem 2.3, one obtains all the results of [16] as a particular case). In order to illustrate the wide variety of nonhomogeneous equations covered by Theorem 2.3 we refer to Section 8 which, being independent of the technical parts of the paper, can serve as a supplement to this introduction.

We wish to emphasize that this is not just a technical extension of the results in [16]. Indeed, in [15, 16] the main ingredient to obtain estimates on the minimizers w_ε is a control (uniform in ε) of the quantity

$$\mathcal{E}_\varepsilon(t) := \frac{1}{2}\int_{\mathbb{R}^n} |w'_\varepsilon(t,x)|^2\,dx + \varepsilon^{-2}\int_0^\infty s\,e^{-s/\varepsilon}\,\mathcal{W}\big(w_\varepsilon(t+s)\big)\,ds, \tag{8}$$

the so called *approximate energy*, to be compared (in view of $w_\varepsilon \to w$) to

$$\mathcal{E}(t) := \frac{1}{2}\int_{\mathbb{R}^n} |w'(t,x)|^2\,dx + \mathcal{W}\big(w(t)\big),$$

the natural energy for a solution of (1), which is formally preserved when $f \equiv 0$. Now, contrary to $\mathcal{E}(t)$ which depends only on $w'(t)$ and $\mathcal{W}\big(w(t)\big)$, we see that the potential term in (8) (the integral involving $\mathcal{W}$) depends on the values of $\mathcal{W}\big(w_\varepsilon(\tau)\big)$ for all $\tau \geq t$: following [23], we say that this term is "acausal".

This acausality is deep-seated: since (7) is of the fourth order in t, prescribing *two* initial conditions as in (2) is not enough to uniquely determine the evolution of $w_\varepsilon(t)$. On the other hand, w_ε is obtained as a minimizer of F_ε subject to (2), and the minimization procedure certainly selects, among the infinitely many solutions of (7)&(2), one with special features (such as the finiteness of $F_\varepsilon(w_\varepsilon)$, which is trivial for a minimizer, but does not follow from (7)&(2)). Thus, the fact that the global-in-time behaviour of $w_\varepsilon(s)$ is relevant for the approximate energy $\mathcal{E}_\varepsilon(t)$ is not surprising. Note, however, that the function $\varepsilon^{-2}se^{-s}$ in (8) is a *probability measure* on $s > 0$, which concentrates at $s = 0$ when $\varepsilon \downarrow 0$: therefore, the second integral in (8) is just an (acausal) average of $\mathcal{W}\big(w_\varepsilon(\tau)\big)$ for $\tau \geq t$, which concentrates around $\tau = t$ for small ε (so that, heuristically, acausality becomes negligible when $\varepsilon \downarrow 0$, as long as smoothness is assumed).

Now, in the nonhomogeneous case where $f \not\equiv 0$, the approximate energy $\mathcal{E}_\varepsilon(t)$ (as defined in (8)) is again the natural object to estimate. But we see from (5) and (6) that the presence of f_ε may strongly influence the behaviour of $w_\varepsilon(t)$, possibly in a *global* (hence also acausal) way: and this is in contrast with the limit problem (1)&(2), where the solution $w(T)$ depends only on $f(t)$ restricted to $t \in [0,T]$, in a strictly causal way.

This calls for some new ideas, in addition to those introduced in [15, 16], in order to obtain strong enough estimates on w_ε, pass to the limit in (7), and obtain sharp energy estimates as in (16). Therefore, in our proofs, we shall mainly focus on these new aspects, referring to [16] for those lemmas or computations which do not require significant changes.

In [16], where $f \equiv 0$, it is proved that $\mathcal{E}'_\varepsilon(t) \leq 0$, so that $\mathcal{E}_\varepsilon(t) \leq \mathcal{E}_\varepsilon(0) \leq \mathcal{E}(0) + o(1)$, and this is the key to all the subsequent estimates for $w_\varepsilon(t)$, uniform in ε and t. Here, instead, the presence of the forcing term f prevents any apriori monotonicity, and every bound for $\mathcal{E}_\varepsilon(t)$ will depend on f itself. In fact, $\mathcal{E}'_\varepsilon(t)$ depends on f (more precisely on f_ε) in a nonlocal, acausal way (see Section 5), and this requires new strategies and a careful analysis based (among the other things) on the tools introduced in Section 3. Indeed, we can obtain estimates only over bounded time intervals and up to some residual terms, which however can be proved to vanish in the limit when $\varepsilon \downarrow 0$.

In the light of these considerations, it appears that letting $f_\varepsilon = f$ in (6) (though formally correct) is not appropriate, and some nontrivial approximation $f_\varepsilon \to f$ is therefore mandatory. Moreover, we wish to work with $f \in L^2_{\mathrm{loc}}([0,\infty);L^2)$ (which is the natural assumption if one seeks solutions of (1) with *finite energy* – see e.g. [3, 8]), while the integral in (5), in order to be defined, requires some restriction on the growth of $\|f_\varepsilon(t)\|_{L^2}$. The crucial properties of f_ε (necessary for the control of the residual terms in our estimates), as listed in Lemma 6.1, may look awkward: fortunately, however, a neat and simple procedure is available to construct f_ε from f, as described in the proof of the lemma.

The full strength of Theorem 2.3 is obtained under the structural assumption (17), which forces the evolution equation (1) to be semilinear (albeit of arbitrary order in space, including wave equations with the fractional Laplacian – see Section 8).

It should be pointed out, however, that assumption (17) is required only in item (e) of Theorem 2.3 (the passage to the limit in (7) to obtain (1)): all the other claims of the theorem (items (a)–(d), including estimates and convergence to a function that satisfies the energy inequality) are valid in the much wider setting of Assumption 2.1, which is typically satisfied by any reasonable functional of the Calculus of Variations (not necessarily convex, and possibly nonlocal). As shown in Section 8, this broad framework includes wave equations with the p-Laplacian such as (62) and nonlocal evolutions like the Kirchhoff equation (63), for which the existence of global weak solutions is an open problem: the validity of items (a)–(d) of Theorem 2.3 for these equations suggests a possible new strategy in this direction, since no counterexample is known to the claim of item (e), for which (17) is just a sufficient condition.

Of course, in several concrete examples where (17) is satisfied (e.g. the NLW equation (61)) the existence of global weak solutions provided by Theorem 2.3 is not new (for an overview of other techniques we refer the reader to [3, 7, 13, 17, 18, 19, 21, 22, 23] and references therein). However, we point out that the variety of different examples of equations that can be treated by this unifying approach is remarkable, and we believe that this variational technique would deserve further investigations.

In the applications, it would also be interesting to consider the same kind of problems on a bounded time interval $[0,T]$, as done in [20] for the NLW equation with no forcing term. In fact, the techniques of our paper can be adapted to handle the case where $t \in [0,T]$: this, however, would call for several minor modifications, and therefore we will not pursue this issue here.

Finally, we recall that suitable variants of this variational approach to evolution problems have recently been developed to study other kind of equations: we refer the reader to [1, 2, 9] for applications to parabolic equations, and to [6] (and references therein) for the application to ODE systems.

Remark on Notation. If $g = g(t,x)$, we write $g(t)$ or equivalently $g(t,\cdot)$ to denote the function of x that is obtained fixing t. We also write g', g'' etc. to denote partial derivatives with respect to t, while differential operators like ∇, Δ etc. are referred to the space variables only. Concerning function spaces, we agree that $L^p = L^p(\mathbb{R}^n)$, $H^m = H^m(\mathbb{R}^n)$ etc., the domain $\mathbb{R}^n$ being understood. Finally, $\langle\cdot,\cdot\rangle$ denotes a duality pairing (usually clear from the context), while $(\cdot,\cdot)_H$ denotes the inner product in a Hilbert space H.

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2 Functional setting and main results

The abstract equation (1) and the functional (4) are defined in terms of the abstract functional $\mathcal{W}$. In order to develop our approach, the properties that $\mathcal{W}$ must safisfy are the same as in [16], and can be summarized as follows.

Assumption 2.1. The functional $\mathcal{W} : L^2 \to [0, \infty]$ is lower semicontinuous in the weak topology of L^2. Moreover, we assume that its domain, i.e. the set of functions

$$\mathrm{W} := \{v \in L^2 : \mathcal{W}(v) < \infty\}, \tag{9}$$

is a Banach space such that

$$C_0^\infty \hookrightarrow \mathrm{W} \hookrightarrow L^2 \qquad \text{(dense embeddings)}. \tag{10}$$

Finally, $\mathcal{W}$ is Gâteaux differentiable on W and its derivative $\nabla\mathcal{W} : \mathrm{W} \to \mathrm{W}'$ satisfies

$$\|\nabla\mathcal{W}(v)\|_{\mathrm{W}'} \leq C\big(1 + \mathcal{W}(v)^\theta\big), \qquad \forall v \in \mathrm{W}, \tag{11}$$

for suitable constants $C \geq 0$ and $\theta \in (0, 1)$. □

Remark 2.2. This assumption (in particular, inequality (11)) is typical of Dirichlet-type functionals like $\mathcal{W}(v) = \|\nabla^k v\|_{L^p}^p$ with $p > 1$ (in this case W is a suitable Sobolev space). We refer to Section 8 for some examples. Here we just point out that Assumption 2.1 is *additively stable*, i.e. if two functionals satisfy Assumption 2.1, then so does their sum (for further remarks on this assumption, see [16]).

Theorem 2.3. *Let $\mathcal{W}$ be a functional satisfying Assumption 2.1 and w_0, $w_1 \in \mathrm{W}$. Let also $f \in L^2_{loc}([0, \infty), L^2)$. Then, there exists a sequence (f_ε), converging to f in $L^2_{loc}([0, \infty), L^2)$, such that:*

(a) ***Minimizers****. For every $\varepsilon \in (0, 1)$, the functional F_ε defined by (5) has a minimizers w_ε, among all functions in $H^2_{loc}([0, \infty); L^2)$ that satisfy (2).*

(b) ***Estimates****. For every $T > 0, \tau \geq 0$, there exist constants C_T, $C_{\tau,T}$ independent of ε such that*

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^n} \big(|w'_\varepsilon(t, x)|^2 + |w_\varepsilon(t, x)|^2 \big)\, dx \leq C_T, \tag{12}$$

$$\int_\tau^{\tau+T} \mathcal{W}\big(w_\varepsilon(t)\big)\, dt \leq C_{\tau,T}, \qquad \forall T > \varepsilon, \tag{13}$$

$$\int_0^T \|w''_\varepsilon(s)\|^2_{\mathrm{W}'}\, ds \leq C_T. \tag{14}$$

(c) ***Convergence****. Every sequence w_{ε_i} (with $\varepsilon_i \downarrow 0$) admits a subsequence which is convergent in the weak topology of $H^1_{loc}([0, +\infty); L^2)$ to a function w that satisfies (2) (where the latter condition is meant as an equality in W'). In addition,*

$$w' \in L^\infty_{loc}([0, \infty); L^2) \qquad \text{and} \qquad w'' \in L^2_{loc}([0, \infty); \mathrm{W}'). \tag{15}$$

(d) ***Energy inequality***. *Letting*

$$\mathcal{E}(t) := \frac{1}{2}\int_{\mathbb{R}^n} |w'(t,x)|^2\,dx + \mathcal{W}\big(w(t)\big),$$

there holds

$$\mathcal{E}(t) \le \left(\sqrt{\mathcal{E}(0)} + \sqrt{\frac{t}{2}\int_0^t\int_{\mathbb{R}^n} |f(s,x)|^2\,dx\,ds}\right)^2, \qquad \text{for a.e.} \quad t \ge 0. \tag{16}$$

(e) ***Weak solution of*** (1). *Assuming furthermore that, for some real numbers $m > 0$, $\lambda_k \ge 0$, $p_k > 1$, $\mathcal{W}$ takes the form of*

$$\mathcal{W}(v) = \frac{1}{2}\|v\|^2_{\dot{H}^m} + \sum_{0\le k<m} \frac{\lambda_k}{p_k}\int_{\mathbb{R}^n} |\nabla^k v(x)|^{p_k}\,dx, \tag{17}$$

then the limit function w satisfies

$$\int_0^\infty\int_{\mathbb{R}^n} w'(t,x)\varphi'(t,x)\,dx\,dt = \int_0^\infty \big\langle \nabla\mathcal{W}\big(w(t)\big),\varphi(t)\big\rangle\,dt - \int_0^\infty\int_{\mathbb{R}^n} f(t,x)\varphi(t,x)\,dx\,dt \tag{18}$$

for every $\varphi \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$, namely, solves (1) *in the sense of distributions.*

Remark 2.4. Note that the functional defined in (17) satisfies Assumption 2.1 with $\mathrm{W} = \{v \in H^m : \nabla^k v \in L^{p_k},\ 0 \le k < m\}$ (for details see [16]). Recall also that $\|v\|_{\dot{H}^m}$ is the L^2 norm of $|\xi|^m\,\widehat{v}(\xi)$, where $\widehat{v}$ is the Fourier transform of v. The typical case is $m \in \mathbb{N}$ when $\|v\|_{\dot{H}^m}$ reduces to $\|\nabla^m v\|_{L^2}$.

Remark 2.5. Throughout, solutions of (1)-(2) obtained via Theorem 2.3, are called *variational solutions.*

Remark 2.6. We mention that several variants of Theorem 2.3 can be proved. For instance, one can introduce nonconstant coefficients in (17), possibly exploiting some Gårding type inequalities to keep $\mathcal{W}$ coercive. Also, one can consider more general lower-order terms (with proper convexity and growth assumptions) like powers of single partial derivatives. In any case, the main point is that $\mathcal{W}$ be quadratic (and coercive) in the highest order terms, namely that equation (1) be semilinear. Furthermore, as pointed out in [15, 16], the minimization approach can be adapted, without significative changes, to the case of a generic (sufficiently smooth) open set $\Omega \subset \mathbb{R}^n$ with Dirichlet or Neumann boundary conditions.

We point out that $\mathcal{E}$ is formally preserved by variational solutions of (1)-(2), in the sense that

$$\mathcal{E}(t) = \mathcal{E}(0) + \int_0^t (f(s), w'(s))_{L^2}\,ds, \qquad \forall t \ge 0 \tag{19}$$

follows from formal differentiation. However, we are not able to prove enough regularity for such solutions in order to solve the long-standing problem of the energy conservation for weak solutions of (1). Anyway, a formal Grönwall argument based on (19) reveals that the energy estimate (16) is close to being optimal.

3 A preliminary tool: the average operator

The study of integrals with an exponential weight plays a central role in our investigation. Therefore, it is worth recalling the definition of *average* operator, introduced in [16].

Definition 3.1. The *average operator* is the linear operator that associates any measurable function $h : [0, \infty] \to [0, \infty]$ with the function $\mathcal{A}h$, given by

$$\mathcal{A}h\,(t) := \int_t^\infty e^{-(s-t)}\,h(s)\,ds, \qquad t \geq 0.$$

We also recall that, if $\mathcal{A}h\,(0) < \infty$, then $\mathcal{A}h$ is absolutely continuous on intervals $[0, T]$, for all $T > 0$, and that

$$(\mathcal{A}h)' = \mathcal{A}h - h. \tag{20}$$

In addition, one can iterate the action of $\mathcal{A}$, thus obtaining

$$\mathcal{A}^2 h\,(t) := \mathcal{A}(\mathcal{A}h)\,(t) = \int_t^\infty e^{-(s-t)}\,(s-t)\,h(s)\,ds \tag{21}$$

(for details see [16]). Finally, we note that $\mathcal{A}h$ is well defined (and all the previous properties are valid) even when h is a changing sign function, provided that it satisfies $\mathcal{A}|h|\,(0) < \infty$.

Now, we show some relevant results that will be widely used in the sequel.

Lemma 3.2. *Let $h : [0, \infty) \to [0, \infty)$ be a function such that $\mathcal{A}h\,(0) < \infty$. Then, for every $\tau \geq 0$ and every $\delta > 0$*

$$\int_\tau^{\tau+\delta} \mathcal{A}h\,(s)\,ds = \int_\tau^{\tau+\delta} h(s)\,ds + \mathcal{A}h\,(\tau+\delta) - \mathcal{A}h\,(\tau). \tag{22}$$

If, in addition, $\mathcal{A}^2 h\,(0) < \infty$, then

$$\int_\tau^{\tau+\delta} \mathcal{A}^2 h\,(s)\,ds = \int_\tau^{\tau+\delta} h(s)\,ds + \mathcal{A}h\,(\tau+\delta) - \mathcal{A}h\,(\tau) + \mathcal{A}^2 h\,(\tau+\delta) - \mathcal{A}^2 h\,(\tau). \tag{23}$$

Proof. Integrating (20) on $[\tau, \tau+\delta]$ one obtains (22) and then, simply iterating the same argument for $\mathcal{A}^2$, there results (23). $\square$

Lemma 3.3. *For every $\alpha > 1$ there exists a constant $C_\alpha > 0$ such that for all $h \in H^1_{loc}([0, \infty); L^2)$*

$$\mathcal{A}\|h(\cdot)\|^2_{L^2}\,(t) \leq \alpha \|h(t)\|^2_{L^2} + C_\alpha \mathcal{A}\|h'(\cdot)\|^2_{L^2}\,(t), \qquad \forall t \geq 0. \tag{24}$$

Proof. Let $t \geq 0$ and $a > t$. By assumption, for a.e. $x \in \mathbb{R}^n$ the function $h(\,\cdot\,, x)$ belongs to $H^1((t, a))$. Then, integrating by parts and using Cauchy-Schwarz,

$$\begin{aligned}\int_t^a e^{-s}\,|h(s,x)|^2\,ds &\leq e^{-t}\,|h(t,x)|^2 + 2\int_t^a e^{-s}\,h(s,x)h'(s,x)\,ds \\ &\leq e^{-t}\,|h(t,x)|^2 + 2\left(\int_t^a e^{-s}\,|h(s,x)|^2\,ds\right)^{1/2}\left(\int_t^a e^{-s}\,|h'(s,x)|^2\,ds\right)^{1/2}.\end{aligned}$$

Since $2\sqrt{bc} \leq \nu b + \frac{1}{\nu}c$ for every $\nu > 0$, we can split the last product and, for any choice of $\nu < 1$, we find that

$$\int_t^a e^{-s}\,|h(s,x)|^2\,ds \leq \frac{1}{1-\nu}e^{-t}\,|h(t,x)|^2 + \frac{1}{\nu(1-\nu)}\int_t^a e^{-s}\,|h'(s,x)|^2\,ds.$$

Now, integrating over $\mathbb{R}^n$ and letting $a \to \infty$, we obtain (24), where $\alpha = \frac{1}{1-\nu}$ and $C_\alpha = \frac{1}{\nu(1-\nu)} = \frac{\alpha^2}{\alpha-1}$. □

Lemma 3.4. *For every $\beta > 1$ there exists a constant $C_\beta > 0$ such that for all $h \in H^1_{loc}([0,\infty); L^2)$*

(25) $$\mathcal{A}^2\|h(\cdot)\|^2_{L^2}(t) \leq \beta\|h(t)\|^2_{L^2} + C_\beta\left(\mathcal{A}\|h'(\cdot)\|^2_{L^2}(t) + \mathcal{A}^2\|h'(\cdot)\|^2_{L^2}(t)\right), \qquad \forall t \geq 0.$$

Proof. Let again $t \geq 0$ and $a > t$. Setting $\tau = s - t$ yields

$$\int_t^a e^{-(s-t)}(s-t)\,|h(s,x)|^2\,ds = \int_0^{a-t} e^{-\tau}\tau\,|g(\tau,x)|^2\,d\tau,$$

with $g(\tau,x) = h(\tau+t,x)$. Then, arguing as in the proof of the previous lemma, we see that

$$\int_0^{a-t} e^{-\tau}\tau\,|g(\tau,x)|^2\,d\tau \leq \int_0^{a-t} e^{-\tau}\,|g(\tau,x)|^2\,d\tau +$$

$$+ 2\left(\int_0^{a-t} e^{-\tau}\tau\,|g(\tau,x)|^2\,d\tau\right)^{1/2}\left(\int_0^{a-t} e^{-\tau}\tau\,|g'(\tau,x)|^2\,d\tau\right)^{1/2}.$$

Now, by Young inequality, for every $\nu \in (0,1)$

$$\int_0^{a-t} e^{-\tau}\tau\,|g(\tau,x)|^2\,d\tau \leq \frac{1}{1-\nu}\int_0^{a-t} e^{-\tau}\,|g(\tau,x)|^2\,d\tau + \frac{1}{\nu(1-\nu)}\int_0^{a-t} e^{-\tau}\tau\,|g'(\tau,x)|^2\,d\tau.$$

Hence, integrating over $\mathbb{R}^n$, changing the variables back and letting $a \to \infty$, we have

$$\mathcal{A}^2\|h(\cdot)\|^2_{L^2}(t) \leq \alpha\mathcal{A}\|h(\cdot)\|^2_{L^2}(t) + C_\alpha\mathcal{A}^2\|h'(\cdot)\|^2_{L^2}(t)$$

(where $\alpha = \frac{1}{1-\nu}$ and $C_\alpha = \frac{1}{\nu(1-\nu)}$). Finally, combining with (24) and setting $\beta = \alpha^2$ and $C_\beta = \alpha C_\alpha$, we obtain (25). □

Remark 3.5. Setting $t = 0$ in Lemma 3.3 we recover [15, Lemma 2.3]. In addition, note that we do not claim that any integral appearing in (24) and (25) is necessarily finite.

4 Minimizers and first properties

The search of the minimizers mentioned in the previous sections is actually performed on an auxiliary functional. For a given a function $\phi : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$, define

(26) $$J_\varepsilon(u) := H_\varepsilon(u) - S(u),$$

where

$$H_\varepsilon(u) := \int_0^\infty e^{-t}\left(\frac{1}{2\varepsilon^2}\int_{\mathbb{R}^n}|u''(t,x)|^2\,dx + \mathcal{W}\big(u(t)\big)\right)dt,$$

and

$$S(u) := \int_0^\infty \int_{\mathbb{R}^n} e^{-t}\, \phi(t,x) u(t,x)\, dx\, dt. \tag{27}$$

One can see that J_ε is equivalent to F_ε in the sense that, setting $\phi(t,x) = f_\varepsilon(\varepsilon t, x)$, there results $F_\varepsilon(w) = \varepsilon J_\varepsilon(u)$, whenever u and w are related by the change of variable $u(t,x) = w(\varepsilon t, x)$. Hence, properly scaling the boundary conditions (namely, as in (32)), the existence of minimizers w_ε for F_ε is equivalent to the existence of minimizers u_ε for J_ε and, in particular,

$$w_\varepsilon(t,x) = u_\varepsilon(t/\varepsilon, x), \qquad t \geq 0, \quad x \in \mathbb{R}^n.$$

On the other hand, in contrast to F_ε, the weight in J_ε does not depend on ε, thus simplifying the investigation.

For functions $v = v(t,x)$, it is convenient to define the weighted L^2-norm

$$\|v\|_{\mathcal{L}}^2 := \int_0^\infty \int_{\mathbb{R}^n} e^{-t}\, |v(t,x)|^2\, dx\, dt,$$

with the proviso that we regard it as a functional with values in $[0, +\infty]$.

Throughout, for fixed ε, we make the following assumptions on $\phi(t,x)$:

$$\exists\, T^* \in \left(0, \varepsilon^{-3/2}\right] \qquad \text{such that} \qquad \phi(t,x) = 0, \quad \forall t > T^*, \tag{28}$$

$$\|\phi\|_{\mathcal{L}} \leq \varepsilon, \tag{29}$$

$$\varepsilon \int_0^t \mathcal{A}^2 \|\phi(\cdot)\|_{L^2}^2 (s)\, ds \leq \gamma(\varepsilon t + t_\varepsilon) + \varepsilon^2 \quad \forall t \geq 0, \tag{30}$$

where $t_\varepsilon > 0$ satisfies $\lim_{\varepsilon \downarrow 0} t_\varepsilon = 0$ while

$$\gamma(t) := \int_0^t \|f(s)\|_{L^2}^2\, ds, \quad t \geq 0, \tag{31}$$

quantifies the growth in time of the forcing term $f \in L^2_{loc}([0,\infty); L^2)$ of (1).

Proposition 4.1. *Let $w_0, w_1 \in \mathrm{W}$ (with W defined by (9)) and $\varepsilon \in (0,1)$. Then, under Assumption 2.1, J_ε admits a minimizer u_ε in the class of functions $u \in H^2_{loc}([0,\infty); L^2)$ satisfying the boundary conditions*

$$u(0) = w_0, \qquad u'(0) = \varepsilon w_1. \tag{32}$$

Moreover,

$$H_\varepsilon(u_\varepsilon) \leq \mathcal{W}(w_0) + \varepsilon C. \tag{33}$$

In order to prove Proposition 4.1, we recall the following facts (for more see [15, Lemma 2.3]).

Lemma 4.2. *If $u \in H^2_{loc}([0,\infty);L^2)$, then*

$$\|u'\|^2_{\mathcal{L}} \le 2\,\|u'(0)\|^2_{L^2} + 4\,\|u''\|^2_{\mathcal{L}} \tag{34}$$

and

$$\|u\|^2_{\mathcal{L}} \le 2\,\|u(0)\|^2_{L^2} + 8\,\|u'(0)\|^2_{L^2} + 16\,\|u''\|^2_{\mathcal{L}}. \tag{35}$$

Proof of Proposition 4.1. Let M be the set of functions in $H^2_{\text{loc}}([0,\infty);L^2)$ satisfying (32). If $u \in M$, then $S(u)$ is finite by (28), so that $J_\varepsilon(u)$ is well defined (possibly equal to $+\infty$). If $J_\varepsilon(u)$ is finite, then, since $\mathcal{W} \ge 0$, the finiteness of $H_\varepsilon(u)$ implies that the last integral in (35) is finite, and using Cauchy-Schwarz, (29) and (35) we have

$$|S(u)| \le \|\phi\|_{\mathcal{L}}\|u\|_{\mathcal{L}} \le \varepsilon C\left(1 + \|u''\|_{\mathcal{L}}\right),$$

where C takes into account (via (32)) also the L^2 norms of $u(0)$ and $u'(0)$. Moreover, from the definition of J_ε and last inequality we have

$$J_\varepsilon(u) \ge \frac{1}{2\varepsilon^2}\|u''\|^2_{\mathcal{L}} + \int_0^\infty e^{-t}\mathcal{W}\big(u(t)\big)\,dt - \varepsilon C\left(1 + \|u''\|_{\mathcal{L}}\right), \tag{36}$$

so that $\|u''\|_{\mathcal{L}}$ can be controlled in terms of $J_\varepsilon(u)$: using again (34) and (35), we see that J_ε is coercive in M with respect to the topology of $H^2_{\text{loc}}([0,\infty);L^2)$, so that every minimizing sequence has a subsequence weakly convergent in $H^2_{\text{loc}}([0,\infty);L^2)$, which also preserves (32). The weak semicontinuity of $H_\varepsilon(u)$ (building on Assumption 2.1) was proved in [16, proof of Lemma 3.1]: since $S(u)$ is a weakly continuous functional, the existence of a minimizer u_ε is established.

Now set $\psi(t,x) := w_0(x) + \varepsilon t w_1(x)$, and observe that $\psi \in M$ and $\psi'' \equiv 0$. Moreover in [16, proof of Lemma 3.1] it is proved that

$$H_\varepsilon(\psi) = \int_0^\infty e^{-t}\,\mathcal{W}\big(\psi(t)\big)\,dt \le \mathcal{W}(w_0) + C\varepsilon,$$

while by a direct computation, using Cauchy-Schwarz and (29), one has

$$-S(\psi) \le \Big(\|w_0\|_{L^2} + \sqrt{2}\,\varepsilon\|w_1\|_{L^2}\Big)\,\|\phi\|_{\mathcal{L}} \le C\varepsilon.$$

Thus $J_\varepsilon(\psi) \le \mathcal{W}(w_0) + C\varepsilon$, and then also $J_\varepsilon(u_\varepsilon) \le \mathcal{W}(w_0) + C\varepsilon$ since u_ε is a minimizer. So, in particular, $J_\varepsilon(u_\varepsilon) \le C$: combining with (36) (written with $u = u_\varepsilon$), by Young's inequality one can easily obtain $\|u_\varepsilon''\|_{\mathcal{L}} \le \varepsilon C$ as a byproduct. This, in turn, can be plugged into (36) (with $u = u_\varepsilon$) to estimate the last term, thus finding

$$J_\varepsilon(u_\varepsilon) \ge \frac{1}{2\varepsilon^2}\|u_\varepsilon''\|^2_{\mathcal{L}} + \int_0^\infty e^{-t}\mathcal{W}\big(u_\varepsilon(t)\big)\,dt - \varepsilon C.$$

Finally, (33) follows from the last inequality, recalling that $J_\varepsilon(u_\varepsilon) \le \mathcal{W}(w_0) + C\varepsilon$. □

Remark 4.3. In the sequel we will always assume that $\varepsilon \in (0,1)$, as in Proposition 4.1.

Now, we introduce some notation. Given a minimizer u_ε of J_ε, we define

$$\mathcal{W}_\varepsilon(t) := \mathcal{W}\big(u_\varepsilon(t)\big), \quad \forall t \geq 0, \qquad D_\varepsilon(t) := \frac{1}{2\varepsilon^2}\|u_\varepsilon''(t)\|_{L^2}^2, \quad \text{for a.e. } t > 0, \tag{37}$$

and

$$L_\varepsilon(t) := D_\varepsilon(t) + \mathcal{W}_\varepsilon(t).$$

We also set

$$\Phi_\varepsilon(t) := \big(\phi(t), u_\varepsilon'(t)\big)_{L^2}, \qquad \forall t \geq 0,$$

and define the *kinetic energy* function as

$$K_\varepsilon(t) := \frac{1}{2\varepsilon^2}\|u_\varepsilon'(t)\|_{L^2}^2, \qquad \forall t \geq 0. \tag{38}$$

Note that K_ε is absolutely continuous on intervals $[0, T]$, with $T > 0$, and that

$$K_\varepsilon'(t) = \frac{1}{\varepsilon^2}\big(u_\varepsilon'(t), u_\varepsilon''(t)\big)_{L^2}, \qquad \text{for a.e. } t > 0.$$

Proposition 4.4. *Let w_0, w_1 and $\mathcal{W}$ satisfy the assumptions of Proposition 4.1 and let u_ε be a minimizer of J_ε. Then, for every $g \in C^2([0,\infty))$ constant for large t and with $g(0) = 0$,*

$$\begin{aligned}\int_0^\infty e^{-t}\,\big(g'(t) - g(t)\big)L_\varepsilon(t)\,dt + \int_0^\infty e^{-t}\,g(t)\Phi_\varepsilon(t)\,dt +& \\ - \int_0^\infty e^{-t}\,\big(4g'(t)D_\varepsilon(t) + g''(t)K_\varepsilon'(t)\big)\,dt &= g'(0)R(u_\varepsilon),\end{aligned} \tag{39}$$

where the remainder

$$R(u_\varepsilon) := \varepsilon\int_0^\infty e^{-t}\,t\,\big(-\big\langle\nabla\mathcal{W}\big(u_\varepsilon(t)\big), w_1\big\rangle + (\phi(t), w_1)_{L^2}\big)\,dt \tag{40}$$

satisfies the estimate

$$|R(u_\varepsilon)| \leq C\varepsilon. \tag{41}$$

Proof. We proceed exactly as in [16, proof of Proposition 4.4]. For small δ, we use the diffeomorphism $\varphi_\delta(t) := t - \delta g(t)$ to define $U_\delta(t) := u_\varepsilon\big(\varphi_\delta(t)\big) + t\varepsilon\delta g'(0)w_1$, which is an admissible competitor of u_ε in the minimization of J_ε, since it satisfies the initial conditions (32). Then, since u_ε is a minimizer and $U_\delta = u_\varepsilon$ when $\delta = 0$, one has

$$\frac{\partial}{\partial\delta}J_\varepsilon(U_\delta)\bigg|_{\delta=0} = \frac{\partial}{\partial\delta}H_\varepsilon(U_\delta)\bigg|_{\delta=0} - \frac{\partial}{\partial\delta}S(U_\delta)\bigg|_{\delta=0} = 0, \tag{42}$$

which (computing the derivatives) yields (39). Indeed, the derivative of $H_\varepsilon(U_\delta)$ has been computed in [16, proof of Proposition 4.4], and it produces all the terms in (39) except, of course, the integral of Φ_ε and the integrand involving $\phi(t)$ in (40). On the other hand, recalling (27), using (28), (29) and dominated convergence one can check that

$$\begin{aligned}\frac{\partial}{\partial\delta}S(U_\delta)\bigg|_{\delta=0} &= \int_0^\infty e^{-t}\int_{\mathbb{R}^n}\phi(t,x)\big(-g(t)u_\varepsilon'(t,x) + t\varepsilon g'(0)w_1(t,x)\big)\,dx\,dt \\ &= -\int_0^\infty e^{-t}g(t)\Phi_\varepsilon(t)\,dt + \varepsilon g'(0)\int_0^\infty e^{-t}t\,\big(\phi(t), w_1\big)_{L^2}\,dt,\end{aligned}$$

whence (42) reduces to (39).

Finally, combining (11) and (33) as in [16], one has

$$\left|\int_0^\infty e^{-t}\, t\, \langle \nabla\mathcal{W}\big(u_\varepsilon(t)\big), w_1\rangle \, dt\right| \le C\left(1+\int_0^\infty e^{-t}\, t\mathcal{W}^\theta(u_\varepsilon(t))\, dt\right)$$
$$\le C\big(1+H_\varepsilon(u_\varepsilon)\big) \le C(1+\varepsilon),$$

while from Cauchy-Schwarz and (29)

$$\left|\int_0^\infty e^{-t}\, t\, \big(\phi(t), w_1\big)_{L^2}\, dt\right| \le \|w_1\|_{L^2}\int_0^\infty e^{-t}\, t\, \|\phi(t)\|_{L^2}\, dt \le \|w_1\|_{L^2}\left(\int_0^\infty e^{-t}\, t^2\, dt\right)^{\frac{1}{2}} \|\phi\|_{\mathcal{L}} \le C\varepsilon$$

and hence inequality (41) is satisfied. □

This result has an immediate consequence.

Corollary 4.5. *Using the notation of Section 3 for the operator $\mathcal{A}$, one has*

$$\mathcal{A}^2 L_\varepsilon(0) + 4\mathcal{A}D_\varepsilon(0) - \mathcal{A}L_\varepsilon(0) = \mathcal{A}^2\Phi_\varepsilon(0) - R(u_\varepsilon) \tag{43}$$

and

$$\mathcal{A}^2 L_\varepsilon(t) + 4\mathcal{A}D_\varepsilon(t) - \mathcal{A}L_\varepsilon(t) = \mathcal{A}^2\Phi_\varepsilon(t) - K'_\varepsilon(t), \qquad \textit{for a.e. } t>0. \tag{44}$$

Proof. Recalling (21), (43) is formally obtained choosing $g(t)=t$ in (39), but this goes beyond the assumptions of Proposition 4.4. However, as shown in [16, proof of Corollary 4.5], it suffices to approximate $g(t)=t$ from below, by suitable functions g_k satisfying the assumptions of Proposition 4.4, and pass to the limit in (39). Since one can arrange for $g'_k(0)=1$, only the integrals on the left hand side of (39) are actually involved, and the one with Φ_ε (the only novelty with respect to [16, Corollary 4.5]) passes to the limit by dominated convergence, using (28) and (29).

Finally, also (44) is proved exactly as in [16, proof of Corollary 4.7] (the only novelty being the term with Φ_ε that can be treated as described above), and we omit the details. We just mention that (44) (if written with T in place of t) is formally obtained choosing $g(t)=(t-T)^+$ in (39): then $g''(t)$ is a Dirac delta at $t=T$, which produces the last term in (44). □

5 The approximate energy

Now we study the *approximate energy*, a quantity that has been first introduced in [16] and whose investigation is crucial for the proof of our main results.

Definition 5.1. Let u_ε be a minimizer of J_ε obtained via Proposition 4.1. The *approximate energy* associated with u_ε is the function $E_\varepsilon:[0,\infty)\to[0,\infty)$ defined by

$$E_\varepsilon(t) := \frac{1}{2\varepsilon^2}\int_{\mathbb{R}^n} |u'_\varepsilon(t,x)|^2\, dx + \int_t^\infty e^{-(s-t)}(s-t)\,\mathcal{W}\big(u_\varepsilon(s)\big)\, ds. \tag{45}$$

Remark 5.2. Recalling (21), (37) and (38), (45) reads

$$E_\varepsilon(t) = K_\varepsilon(t) + \mathcal{A}^2\mathcal{W}_\varepsilon(t), \qquad t\ge 0.$$

In addition we note that, in view of (8), $\mathcal{E}_\varepsilon(t) = E_\varepsilon(t/\varepsilon)$.

The value of E_ε at $t=0$ can be estimated simply using (43).

Lemma 5.3 (Estimate for $E_\varepsilon(0)$)**.** *We have*

$$E_\varepsilon(0) \le \frac{1}{2}\|w_1\|_{L^2}^2 + \mathcal{W}(w_0) + C\sqrt{\varepsilon}. \tag{46}$$

Proof. From (32), $E_\varepsilon(0) = \frac{1}{2}\|w_1\|_{L^2}^2 + \mathcal{A}^2\mathcal{W}_\varepsilon(0)$. Since $\mathcal{A}^2\mathcal{W}_\varepsilon(0) \le \mathcal{A}^2 L_\varepsilon(0)$, from (43) we obtain

$$\mathcal{A}^2\mathcal{W}_\varepsilon(0) \le \mathcal{A}^2\Phi_\varepsilon(0) + \mathcal{A}L_\varepsilon(0) - R(u_\varepsilon).$$

Now, as $\mathcal{A}L_\varepsilon(0) = H_\varepsilon(u_\varepsilon)$, combining the previous inequality with (33) and (41) yields

$$\mathcal{A}^2\mathcal{W}_\varepsilon(0) \le \mathcal{A}^2\Phi_\varepsilon(0) + \mathcal{W}(w_0) + C\varepsilon. \tag{47}$$

Moreover, using first (28) and then (29) we have

$$|\mathcal{A}^2\Phi_\varepsilon(0)| \le T^*\int_0^\infty e^{-t}\left(|u_\varepsilon'(t)|, |\phi(t)|\right)_{L^2} dt \le T^*\|\phi\|_{\mathcal{L}}\|u_\varepsilon'\|_{\mathcal{L}} \le C\frac{\|u_\varepsilon'\|_{\mathcal{L}}}{\sqrt{\varepsilon}}.$$

Since from (34), (32) and (33) we have

$$\|u_\varepsilon'\|_{\mathcal{L}}^2 \le C\varepsilon^2\|w_1\|_{L^2}^2 + C\int_0^\infty e^{-t} D_\varepsilon(t)\,dt \le C\varepsilon^2\left(1 + H_\varepsilon(u_\varepsilon)\right) \le C\varepsilon^2,$$

we find that $|\mathcal{A}^2\Phi_\varepsilon(0)| \le C\sqrt{\varepsilon}$. Hence, plugging back into (47), (46) is proved. □

Furthermore, we establish an upper bound for the time evolution of the approximate energy.

Proposition 5.4 (Approximate energy estimate)**.** *For every $\beta > 1$, there exists a constant $C_\beta > 0$ such that for every $T \ge 0$*

$$\sqrt{E_\varepsilon(T/\varepsilon)} \le \sqrt{E_\varepsilon(0)} + \left(\sqrt{\varepsilon C_\beta} + \sqrt{T\beta/2}\right)\sqrt{\gamma(T+t_\varepsilon) + \varepsilon^2} \qquad \forall\varepsilon \in (0,1). \tag{48}$$

In particular, for every $T \ge 0$ there exists C_T such that

$$E_\varepsilon(t/\varepsilon) \le C_T \qquad \forall\varepsilon\in(0,1), \quad \forall t \in [0,T]. \tag{49}$$

To prove Proposition 5.4, we must previously compute the derivative of E_ε.

Lemma 5.5. *The approximate energy E_ε is absolutely continuous on every interval* $[0,T]$*, and*

$$E_\varepsilon'(t) = -3\mathcal{A}D_\varepsilon(t) - \mathcal{A}^2 D_\varepsilon(t) + \mathcal{A}^2\Phi_\varepsilon(t), \qquad \textit{for a.e. } t \ge 0. \tag{50}$$

Proof. Arguing as in [16, proof of Theorem 4.8] one can see that

$$E_\varepsilon'(t) = K_\varepsilon'(t) - \mathcal{A}L_\varepsilon(t) + \mathcal{A}D_\varepsilon(t) + \mathcal{A}^2 L_\varepsilon(t) - \mathcal{A}^2 D_\varepsilon(t).$$

Then (44) can be used to eliminate $K_\varepsilon'(t)$, and (50) follows. □

Now, it also is convenient to recall, without proof, a well-known variant of the Grönwall's lemma (see e.g. [3, Proposition 2.3.1]).

Lemma 5.6. *Let $c : [a,b] \to \mathbb{R}$ be a positive, differentiable and nondecreasing function. Let also u and v be two nonnegative functions such that $u \in C^0([a,b])$ and $v \in L^1([a,b])$. If we assume that c, u and v satisfy*

$$u(t) \le c^2(t) + 2\int_a^t v(s)\sqrt{u(s)}\,ds, \qquad \forall t \in [a,b],$$

then there results

$$\sqrt{u(t)} \le c(t) + \int_a^t v(s)\,ds, \qquad \forall t \in [a,b].$$

Proof of Proposition 5.4. First, recall that by definition

$$\mathcal{A}^2\Phi_\varepsilon\,(t) = \int_t^\infty e^{-(s-t)}(s-t)\,\big(\phi(s), u_\varepsilon'(s)\big)_{L^2}\,ds.$$

Now, observing that $e^{-(s-t)}(s-t)$ is a probability kernel on $[t,\infty)$, (50) implies

$$E_\varepsilon'(t) \le -3\mathcal{A}D_\varepsilon\,(t) - \mathcal{A}^2 D_\varepsilon\,(t) + N_\phi(t)\left(\mathcal{A}^2\|u_\varepsilon'(\cdot)\|_{L^2}^2\,(t)\right)^{1/2} \tag{51}$$

where $N_\phi(t) = \left(\mathcal{A}^2\|\phi(\cdot)\|_{L^2}^2\,(t)\right)^{1/2}$. By Lemma 3.4, applied with $h = u_\varepsilon'$, for every $\beta > 1$ there exists a constant $C_\beta > 0$ such that

$$\left(\mathcal{A}^2\|u_\varepsilon'(\cdot)\|_{L^2}^2\,(t)\right)^{1/2} \le \sqrt{\beta}\,\|u_\varepsilon'(t)\|_{L^2} + \sqrt{C_\beta}\,\left(\mathcal{A}\|u_\varepsilon''(\cdot)\|_{L^2}^2\,(t) + \mathcal{A}^2\|u_\varepsilon''(\cdot)\|_{L^2}^2\,(t)\right)^{1/2}.$$

Since $\|u_\varepsilon''(\cdot)\|_{L^2}^2\,(t) = 2\varepsilon^2 D_\varepsilon(t)$, multiplying by $N_\phi(t)$ and using Young's inequality we find

$$N_\phi(t)\left(\mathcal{A}^2\|u_\varepsilon'(\cdot)\|_{L^2}^2\,(t)\right)^{1/2} \le \sqrt{\beta}\,N_\phi(t)\|u_\varepsilon'(t)\|_{L^2} + C_\beta\varepsilon^2 N_\phi(t)^2 + \big(3\mathcal{A}D_\varepsilon(t) + \mathcal{A}^2 D_\varepsilon(t)\big)$$

(where C_β has been possibly redefined). Plugging into (51), we obtain

$$E_\varepsilon'(t) \le \sqrt{\beta}\,N_\phi(t)\|u_\varepsilon'(t)\|_{L^2} + C_\beta\varepsilon^2 N_\phi(t)^2 \le \sqrt{2\beta}\,\varepsilon N_\phi(t)\sqrt{E_\varepsilon(t)} + C_\beta\varepsilon^2 N_\phi(t)^2$$

and then, integrating,

$$E_\varepsilon(t) \le E_\varepsilon(0) + C_\beta\varepsilon^2\int_0^t N_\phi^2(s)\,ds + \sqrt{2\beta}\,\varepsilon\int_0^t N_\phi(s)\sqrt{E_\varepsilon(s)}\,ds.$$

Now, setting

$$u(t) = E_\varepsilon(t), \qquad v(t) = \varepsilon\sqrt{\beta/2}N_\phi(t), \qquad c(t)^2 = E_\varepsilon(0) + C_\beta\,\varepsilon^2\int_0^t N_\phi^2(s)\,ds,$$

assumptions of Lemma 5.6 are satisfied and thus for every $t \ge 0$

$$\sqrt{E_\varepsilon(t)} \le \left(E_\varepsilon(0) + C_\beta\varepsilon^2\int_0^t N_\phi^2(s)\,ds\right)^{1/2} + \varepsilon\sqrt{\beta/2}\int_0^t N_\phi(s)\,ds.$$

Therefore,

$$\sqrt{E_\varepsilon(t)} \le \sqrt{E_\varepsilon(0)} + \sqrt{C_\beta}\,\varepsilon\sqrt{\int_0^t N_\phi^2(s)\,ds} + \varepsilon\sqrt{\beta/2}\int_0^t N_\phi(s)\,ds$$

and, applying Cauchy-Schwarz in the last integral, we find

$$\sqrt{E_\varepsilon(t)} \le \sqrt{E_\varepsilon(0)} + \sqrt{\varepsilon}\left(\sqrt{C_\beta} + \sqrt{t\beta/2}\right)\sqrt{\varepsilon \int_0^t N_\phi^2(s)\,ds}.$$

On the other hand, (30) gives

$$\varepsilon \int_0^t N_\phi^2(s)\,ds = \varepsilon \int_0^t \mathcal{A}^2 \|\phi(\cdot)\|_{L^2}^2\,(s)\,ds \le \gamma(\varepsilon t + t_\varepsilon) + \varepsilon^2$$

so that, setting $t = T/\varepsilon$, we obtain (48). Then (49) is immediate, since the right hand side of (48) is increasing with respect to T; moreover, $t_\varepsilon \downarrow 0$ (decreasingly), β can be fixed (e.g. $\beta = 2$) and $E_\varepsilon(0) \le C$ by (46). □

6 Proof of Theorem 2.3: parts (a) and (b)

Now, we can use the tools developed in the previous sections in order to to prove the first parts of Theorem 2.3.

Preliminarily, we need a result on the approximation of functions in $L^2_{\text{loc}}([0,\infty); L^2)$.

Lemma 6.1. *For every function $f \in L^2_{loc}([0,\infty); L^2)$ there exists a sequence $(f_\varepsilon) \subset L^2_{loc}([0,\infty); L^2)$ satisfying the following properties:*

- *(i) as $\varepsilon \downarrow 0$, $f_\varepsilon \to f$ in $L^2([0,T]; L^2)$ and $\|f_\varepsilon\|_{L^2([0,T];L^2)} \uparrow \|f\|_{L^2([0,T];L^2)}$, for every $T > 0$;*
- *(ii) $\operatorname{supp}\{f_\varepsilon\} \subset [t_\varepsilon, T_\varepsilon] \times \mathbb{R}^n$, with $t_\varepsilon > 0$ and $T_\varepsilon < \infty$;*
- *(iii) as $\varepsilon \downarrow 0$, $t_\varepsilon \downarrow 0$ and $T_\varepsilon \uparrow \infty$, and moreover $\varepsilon T_\varepsilon \le \sqrt{\varepsilon}$, $e^{-t_\varepsilon/\varepsilon}\left(1 + \frac{T_\varepsilon}{\varepsilon}\right) \le \varepsilon^3$;*
- *(iv) for every $\varepsilon \in (0,1)$, $\displaystyle\int_{t_\varepsilon}^{T_\varepsilon} \|f_\varepsilon(t)\|_{L^2}^2\,dt \le 1/\varepsilon$;*
- *(v) for every $\varepsilon \in (0,1)$, $\displaystyle\int_0^\infty e^{-t}\,\|f_\varepsilon(\varepsilon t)\|_{L^2}^2\,dt \le \varepsilon^3$.*

Proof of Lemma 6.1. Defining

$$f_\varepsilon(t,x) = \chi_{(t_\varepsilon,T_\varepsilon)}(t)\,f(t,x),$$

it is clear that (i) and (ii) are satisfied, as soon as $t_\varepsilon \downarrow 0$ and $T_\varepsilon \to +\infty$. We first construct T_ε. The function

$$\Gamma : [0,\infty) \to [0,\infty), \qquad \Gamma(t) := \int_0^t \left(1 + \|f(s)\|_{L^2}^2\right)\,ds$$

is continuous, increasing and surjective, and therefore the same is true of its inverse Γ^{-1}. Letting, for instance, $T_\varepsilon = \min\left\{\Gamma^{-1}(1/\varepsilon), 1/\sqrt{\varepsilon}\right\}$, we have that $T_\varepsilon \to +\infty$ and that the first part of (iii) is satisfied, as well as (iv), since

$$\int_{t_\varepsilon}^{T_\varepsilon} \|f_\varepsilon(s)\|_{L^2}^2\,ds = \int_0^{T_\varepsilon} \|f(s)\|_{L^2}^2\,ds < \Gamma(T_\varepsilon) \le 1/\varepsilon.$$

Finally, we see that

$$\int_0^\infty e^{-t}\,\|f_\varepsilon(\varepsilon t)\|_{L^2}^2\,dt = \frac{1}{\varepsilon}\int_0^\infty e^{-t/\varepsilon}\,\|f_\varepsilon(t)\|_{L^2}^2\,dt = \frac{1}{\varepsilon}\int_{t_\varepsilon}^{T_\varepsilon} e^{-t/\varepsilon}\,\|f_\varepsilon(t)\|_{L^2}^2\,dt \le \frac{e^{-t_\varepsilon/\varepsilon}}{\varepsilon^2}$$

having used (iv). Hence, to fulfill (v), it suffices to have $e^{-t_\varepsilon/\varepsilon} \le \varepsilon^5$ for every $\varepsilon \in (0,1)$, which is achieved choosing for instance $t_\varepsilon = k\sqrt{\varepsilon}$ with k large enough. Finally, since $T_\varepsilon \le 1/\sqrt{\varepsilon}$, the same choice can also guarantee the second inequality in (iii). □

The previous lemma has an important corollary.

Corollary 6.2. *Let $f \in L^2_{loc}([0,\infty); L^2)$ and (f_ε) be a sequence obtained via Lemma 6.1. If we fix $\varepsilon \in (0,1)$, then the function*

$$\phi(t,x) := f_\varepsilon(\varepsilon t, x), \qquad t \ge 0, \quad x \in \mathbb{R}^n, \tag{52}$$

satisfies (28)–(30).

Proof. First, one can easily see that (28) and (29) are direct consequences of properties (ii), (iii) and (v) of Lemma 6.1.

On the other hand, if one applies Lemma 3.2 with $\tau = 0$, $\delta = t$ and $h(t) = \phi(t) = f_\varepsilon(\varepsilon t)$, then

$$\int_0^t \mathcal{A}^2\|\phi(\cdot)\|_{L^2}^2\,(s)\,ds \le \int_0^t \|\phi(s)\|_{L^2}^2\,ds + \mathcal{A}\|\phi\|_{L^2}^2\,(t) + \mathcal{A}^2\|\phi\|_{L^2}^2\,(t).$$

Now, from (ii) of Lemma 6.1 (with some changes of variable) we find that

$$\begin{aligned}\mathcal{A}\|\phi\|_{L^2}^2\,(t) + \mathcal{A}^2\|\phi\|_{L^2}^2\,(t) &= \int_0^{T^*} e^{-s}\,(1+s)\,\|\phi(s+t)\|_{L^2}^2\,ds \\ &= \varepsilon^{-1}\int_0^{T_\varepsilon} e^{-s/\varepsilon}\left(1+\tfrac{s}{\varepsilon}\right)\|f_\varepsilon(s+\varepsilon t)\|_{L^2}^2\,ds.\end{aligned}$$

If we split the integral in two parts, then, from (iii) and (iv) in Lemma 6.1,

$$\varepsilon^{-1}\int_{t_\varepsilon}^{T_\varepsilon} e^{-s/\varepsilon}\,\left(1+\tfrac{s}{\varepsilon}\right)\,\|f_\varepsilon(s+\varepsilon t)\|_{L^2}^2\,ds \le \varepsilon^{-1}e^{-t_\varepsilon/\varepsilon}\,\left(1+\tfrac{T_\varepsilon}{\varepsilon}\right)\int_{t_\varepsilon}^{T_\varepsilon}\|f_\varepsilon(s+\varepsilon t)\|_{L^2}^2\,ds \le \varepsilon,$$

while, recalling that $e^{-x}\,(1+x) \le 1$ for every $x \ge 0$,

$$\varepsilon^{-1}\int_0^{t_\varepsilon} e^{-s/\varepsilon}\,\left(1+\tfrac{s}{\varepsilon}\right)\,\|f_\varepsilon(s+\varepsilon t)\|_{L^2}^2\,ds \le \varepsilon^{-1}\int_0^{t_\varepsilon}\|f_\varepsilon(s+\varepsilon t)\|_{L^2}^2\,ds = \varepsilon^{-1}\int_{\varepsilon t}^{\varepsilon t+t_\varepsilon}\|f_\varepsilon(s)\|_{L^2}^2\,ds.$$

Therefore, recalling the definition of γ given by (31),

$$\int_0^t \mathcal{A}^2\|\phi(\cdot)\|_{L^2}^2\,(s)\,ds \le \int_0^t \|\phi(s)\|_{L^2}^2\,ds + \varepsilon^{-1}\int_{\varepsilon t}^{\varepsilon t+t_\varepsilon}\|f_\varepsilon(s)\|_{L^2}^2\,ds + \varepsilon \le \varepsilon^{-1}\gamma(\varepsilon t+t_\varepsilon) + \varepsilon$$

and (30) follows. □

We can now prove the first part of Theorem 2.3.

Proof of Theorem 2.3: part (a). Let (f_ε) be a sequence obtained via Lemma 6.1. If we set (52) in (27), (28)–(30) and all the hypothesis of Proposition 4.1 are satisfied and hence we obtain a minimizer u_ε, in the class of functions $u \in H^2_{\text{loc}}([0,\infty); L^2)$ subject to (32), that fulfills (33). Now, as

$$F_\varepsilon(w) = \varepsilon J_\varepsilon(u) \qquad \text{whenever} \qquad u(t,x) = w(\varepsilon t, x),$$

if w_ε is defined by

$$w_\varepsilon(t,x) = u_\varepsilon(t/\varepsilon, x) \qquad t \geq 0, \quad x \in \mathbb{R}^n, \tag{53}$$

then it is the required minimizer. □

Remark 6.3. We notice that, under the assumptions of Theorem 2.3 and (52), (53) provides a direct connection between the minimizers of J_ε obtained via Proposition 4.1 and the minimizers of F_ε. Throughout, we will repeatedly use this relation and, in particular, the fact that, setting (52) with (f_ε) obtained via Lemma 6.1, all the results proved in Sections 4&5 are valid. We will also tacitly assume that the hypothesis of Theorem 2.3 are satisfied.

The proof of item (b) of Theorem 2.3 requires two further auxiliary results.

Lemma 6.4 (Euler-Lagrange equation of u_ε)**.** *If $\eta(t,x) = \varphi(t)h(x)$, where $h \in \mathrm{W}$ and $\varphi \in C^{1,1}([0,\infty))$ satisfies $\varphi(0) = \varphi'(0) = 0$, then*

$$\frac{1}{\varepsilon^2}\int_0^\infty e^{-t}\left(u_\varepsilon''(t), \eta''(t)\right)_{L^2} dt = \int_0^\infty e^{-t}\left(-\left\langle \nabla\mathcal{W}\big(u_\varepsilon(t)\big), \eta(t)\right\rangle + \big(f_\varepsilon(\varepsilon t), \eta(t)\big)_{L^2}\right) dt. \tag{54}$$

Moreover, the same conclusion holds if $\eta \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$.

Proof. When $\eta(t,x) = \varphi(t)h(x)$, (54) is obtained letting $g(\delta) = J_\varepsilon(u_\varepsilon + \delta\eta)$ and observing that $g'(0) = 0$, since u_ε is a minimizer of J_ε and $u_\varepsilon + \delta\eta$ is an admissible competitor. The case where $\eta \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$ follows by a density argument (see [16, proof of Lemma 5.1] for more details). The novelty here, with respect to [16], is just the term with f_ε in (54), which originates from the additional term $S(u)$ in (26). □

Lemma 6.5 (Representation formula for u_ε'')**.** *For all $h \in \mathrm{W}$*

$$\frac{1}{\varepsilon^2}\big(u_\varepsilon''(\tau), h\big)_{L^2} = -\mathcal{A}^2\omega_1(\tau) + \mathcal{A}^2\omega_2(\tau), \qquad \textit{for a.e.} \quad \tau > 0, \tag{55}$$

where $\omega_1(\tau) = \left\langle \nabla\mathcal{W}\big(u_\varepsilon(\tau)\big), h\right\rangle$ and $\omega_2(\tau) = \big(f_\varepsilon(\varepsilon\tau), h\big)_{L^2}$.

Proof. For every $h \in W$ and every $\tau > 0$, (55) formally follows from (54) choosing $\eta(t,x) = \varphi(t)h(x)$, with $\varphi(t) = (t-\tau)^+$ so that $\varphi''(t)$ is a Dirac delta at $t = \tau$. Indeed, (55) can be proved rigorously (at every Lebesgue point τ of $\big(u_\varepsilon''(\tau), h\big)_{L^2}$) by approximating $\varphi(t) = (t-\tau)^+$ with $C^{1,1}$ functions, exactly as in [16, proof of (2.11)&(2.16)]. □

Proof of Theorem 2.3: part (b). Note that, by (53) and (38), $K_\varepsilon(t/\varepsilon) = \frac{1}{2}\|w_\varepsilon'(t)\|^2_{L^2}$ and, since $K_\varepsilon(t/\varepsilon) \leq E_\varepsilon(t/\varepsilon)$, (49) entails (12).

On the other hand, arguing as in [16, proof of Theorem 2.4], we see that

$$\int_s^{s+1} \mathcal{W}_\varepsilon(t)\, dt \leq C\big(1 + E_\varepsilon(s-1)\chi_{[1,\infty)}(s)\big), \qquad \forall s \geq 0,$$

so that, setting $s=\tau/\varepsilon$, (with some changes of variable)

$$\int_\tau^{\tau+\varepsilon} \mathcal{W}\big(w_\varepsilon(t)\big)\,dt \le C\varepsilon\big(1+E_\varepsilon(\tau/\varepsilon-1)\chi_{[\varepsilon,\infty)}(\tau)\big) \le C\varepsilon\big(1+C_{\tau-\varepsilon}\chi_{[\varepsilon,\infty)}(\tau)\big), \qquad \forall \tau\ge 0,$$

where $C_{\tau-\varepsilon}$ is the constant provided by (49) (when $T=\tau-\varepsilon$). As this constant is increasing with respect to time, one sees that for every $\tau\ge 0$ and every $T>\varepsilon$

$$\int_\tau^{\tau+T} \mathcal{W}\big(w_\varepsilon(t)\big)\,dt \le C\varepsilon \sum_{i=1}^{[\frac{T}{\varepsilon}]+1} (1+C_{\tau+(i-1)\varepsilon}) \le C(1+C_{T+\tau+1})$$

(where $\left[\frac{T}{\varepsilon}\right]$ denotes the integer part of $\frac{T}{\varepsilon}$), so that (13) is proved.

Finally, we must prove (14). By (11), one can see that $|\omega_1(t)| \le C\|h\|_{\mathrm{W}}\big(1+\mathcal{W}_\varepsilon(t)\big)$ and consequently

$$|\mathcal{A}^2\omega_1\,(t)| \le C\|h\|_{\mathrm{W}}\big(1+E_\varepsilon(t)\big). \tag{56}$$

On the other hand

$$|\mathcal{A}^2\omega_2\,(t)| \le \|h\|_{L^2}\int_t^\infty e^{-(s-t)}\,(s-t)\,\|f_\varepsilon(\varepsilon s)\|_{L^2}\,ds$$

and then, from (10) and Jensen inequality, there results

$$|\mathcal{A}^2\omega_2\,(t)| \le C\|h\|_{\mathrm{W}}\left(\mathcal{A}^2\|f_\varepsilon(\varepsilon\,\cdot)\|^2_{L^2}\,(t)\right)^{1/2}. \tag{57}$$

Combining (56) and (57) with (55), we obtain that

$$\frac{1}{\varepsilon^2}\big|\big(u_\varepsilon''(t),h\big)_{L^2}\big| \le C\|h\|_{\mathrm{W}}\left(1+E_\varepsilon(t)+\big(\mathcal{A}^2\|f_\varepsilon(\varepsilon\,\cdot)\|^2_{L^2}\,(t)\big)^{1/2}\right), \qquad \text{for a.e.} \quad t>0$$

and hence, as (10) entails $L^2\hookrightarrow \mathrm{W}'$, that

$$\frac{1}{\varepsilon^2}\|u_\varepsilon''(t)\|_{\mathrm{W}'} \le C\left(1+E_\varepsilon(t)+\big(\mathcal{A}^2\|f_\varepsilon(\varepsilon\,\cdot)\|^2_{L^2}\,(t)\big)^{1/2}\right), \qquad \text{for a.e.} \quad t>0.$$

Furthermore, in view of (53) and (49), the last inequality reads

$$\|w_\varepsilon''(t)\|_{\mathrm{W}'} \le C\left(1+C_t+\big(\mathcal{A}^2\|f_\varepsilon(\varepsilon\,\cdot)\|^2_{L^2}\,(t/\varepsilon)\big)^{1/2}\right), \qquad \text{for a.e.} \quad t>0. \tag{58}$$

Now, recalling (30) and (31), in view of Corollary 6.2, one finds that for every $T>0$

$$\int_0^T \mathcal{A}^2\|f_\varepsilon(\varepsilon\,\cdot)\|^2_{L^2}\,(t/\varepsilon)\,dt \le \gamma(T+t_\varepsilon)+\varepsilon^2.$$

Therefore, since $t_\varepsilon\downarrow 0$ when $\varepsilon\downarrow 0$, squaring and integrating inequality (58) on $[0,T]$, we get (14). □

Remark 6.6. Due to the presence of the source term in (1), the estimate that we establish on (w_ε'') is much "weaker" than the one obtained in [16] in the homogeneous case. However, as we show below, this does not compromise the proof.

7 Proof of Theorem 2.3: parts (c), (d) and (e)

In the following proofs, we will extract several subsequences from a given sequence of minimizers w_{ε_i}: for ease of notation, however, we will simply denote by w_ε the original sequence, as well as the subsequences we extract. The same holds for all the other quantities depending on ε.

Proof of Theorem 2.3: part (c). Let $T > 0$. By (12) and (14), we see that

$$\|w_\varepsilon\|_{H^1([0,T];L^2)} \leq C_T, \qquad \|w'_\varepsilon\|_{L^\infty([0,T];L^2)} \leq C_T, \qquad \|w'_\varepsilon\|_{H^1([0,T];\mathrm{W}')} \leq C_T.$$

Arguing as in [16, proof of Theorem 2.4], this is sufficient to prove convergence in $H^1([0,T];L^2)$, (2) (with the second condition meant as an equality in W') and (15). □

Proof of Theorem 2.3: part (d). Observe that, letting $l(t) := \mathcal{W}_\varepsilon(t)$ and $m(t) := E_\varepsilon(t) - K_\varepsilon(t)$ in [16, Lemma 6.1], we obtain that for every $T > 0$, $a > 0$, $\delta \in (0,1)$

$$Y(\delta a)\int_{T+\delta a}^{T+a} \mathcal{W}_\varepsilon(t)\,dt + \int_T^{T+a} K_\varepsilon(t)\,dt \leq \int_T^{T+a} E_\varepsilon(t)\,dt,$$

where $Y(z) := \int_0^z e^{-s}\,s\,ds$. Replacing a with a/ε and T with T/ε, with a change of variable, the previous inequality reads

$$Y\left(\frac{\delta a}{\varepsilon}\right)\int_{T+\delta a}^{T+a} \mathcal{W}\big(w_\varepsilon(t)\big)\,dt + \frac{1}{2}\int_T^{T+a} \|w'_\varepsilon(t)\|^2_{L^2}(t)\,dt \leq \int_T^{T+a} E_\varepsilon(t/\varepsilon)\,dt.$$

Hence, from (48), we see that for an arbitrary $\beta > 1$

$$\begin{aligned} Y\left(\frac{\delta a}{\varepsilon}\right)\int_{T+\delta a}^{T+a} \mathcal{W}\big(w_\varepsilon(t)\big)\,dt &+ \frac{1}{2}\int_T^{T+a} \|w'_\varepsilon(t)\|^2_{L^2}(t)\,dt \\ &\leq \int_T^{T+a} \left(\sqrt{E_\varepsilon(0)} + \left(\sqrt{\varepsilon C_\beta} + \sqrt{t\beta/2}\right)\sqrt{\gamma(t+t_\varepsilon)+\varepsilon^2}\right)^2 dt. \end{aligned}$$

Now, when $\varepsilon \downarrow 0$, by definition $Y\left(\frac{\delta a}{\varepsilon}\right) \to 1$, whereas by (46) and (31)

$$\int_T^{T+a} \left(\sqrt{E_\varepsilon(0)} + \left(\sqrt{\varepsilon C_\beta} + \sqrt{t\beta/2}\right)\sqrt{\gamma(t+t_\varepsilon)+\varepsilon^2}\right)^2 dt \to \int_T^{T+a} \left(\sqrt{\mathcal{E}(0)} + \sqrt{t\gamma(t)\beta/2}\right)^2 dt.$$

Consequently, arguing as in [16, proof of Theorem 2.4],

$$\int_{T+\delta a}^{T+a} \mathcal{W}\big(w(t)\big)\,dt + \frac{1}{2}\int_T^{T+a} \|w'(t)\|^2_{L^2}(t)\,dt \leq \int_T^{T+a} \left(\sqrt{\mathcal{E}(0)} + \sqrt{t\gamma(t)\beta/2}\right)^2 dt$$

and, letting $\delta \downarrow 0$ and, subsequently, dividing by a and letting $a \downarrow 0$, we obtain

$$\mathcal{W}\big(w(T)\big) + \frac{1}{2}\|w'(T)\|^2_{L^2} \leq \left(\sqrt{\mathcal{E}(0)} + \sqrt{T\gamma(T)\beta/2}\right)^2, \qquad \text{for a.e.} \quad T \geq 0.$$

Since the inequality is valid for every $\beta > 1$, letting $\beta \downarrow 1$, (16) follows. □

Finally, before proving part (e) of Theorem 2.3, we state the following result, which can be established directly by (54) (see [16, Lemma 6.2]).

Lemma 7.1. *Let w_ε be a minimizer of F_ε. Then, for every function $\varphi \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$, there results*

$$\begin{aligned}\int_0^\infty \big(w_\varepsilon'(t), \varepsilon^2 \varphi'''(t) + 2\varepsilon\varphi''(t) + \varphi'(t)\big)_{L^2}\, dt = \\ = \int_0^\infty \big\langle \nabla\mathcal{W}\big(w_\varepsilon(t)\big), \varphi(t)\big\rangle\, dt - \int_0^\infty \big(f_\varepsilon(t), \varphi(t)\big)_{L^2}\, dt.\end{aligned} \tag{59}$$

Proof of Theorem 2.3: part (e). The goal, now, is to prove that, as $\varepsilon \downarrow 0$, equation (59) reduces to (18). Let $\varphi \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$ and w be the function obtained at point (c). We immediately see that

$$\int_0^\infty \big(w_\varepsilon'(t), \varepsilon^2 \varphi'''(t) + 2\varepsilon\varphi''(t) + \varphi'(t)\big)_{L^2}\, dt \to \int_0^\infty \big(w'(t), \varphi'(t)\big)_{L^2}\, dt.$$

and, by construction, that

$$\int_0^\infty \big(f_\varepsilon(t), \varphi(t)\big)_{L^2}\, dt \to \int_0^\infty \big(f(t), \varphi(t)\big)_{L^2}\, dt.$$

Hence the core of the proof is to show that

$$\int_0^\infty \big\langle \nabla\mathcal{W}\big(w_\varepsilon(t)\big), \varphi(t)\big\rangle\, dt \to \int_0^\infty \big\langle \nabla\mathcal{W}\big(w(t)\big), \varphi(t)\big\rangle\, dt.$$

However, this follows by exploiting (12), (13), (17) and [8, Theorem 5.1] as in [16]. □

8 Examples

For the sake of completeness we show some examples of second order nonhomogeneous hyperbolic equations that satisfy the assumptions of Theorem 2.3.

1. Linear equations. A linear hyperbolic equation (with constant coefficients and without dissipative terms) can be written as

$$w'' = -\sum_{j\in\mathcal{R}} (-1)^{|j|}\, \partial^{2j} w + f, \tag{60}$$

where $\mathcal{R} \subset \mathbb{N}^n$ is a finite set of multi-indices and ∂^{2j} denotes partial differentiation in space with respect to the multi-index $2j$. If we set

$$\mathcal{W}(v) = \frac{1}{2}\sum_{j\in\mathcal{R}} \int_{\mathbb{R}^n} |\partial^j v|^2\, dx,$$

then (1) reads like (60) and, letting $\mathrm{W} = \{v \in L^2 : \partial^j v \in L^2, \forall j \in \mathcal{R}\}$ and $\theta = 1/2$, assumptions of Theorem 2.3 are satisfied (in view of Remark 2.6). In particular, for suitable choices of $\mathcal{R}$, (60) reads:

$$w'' = \Delta w + f, \qquad w'' = \Delta w - w + f, \qquad \text{or} \qquad w'' = -\Delta^2 w + f$$

and than all of these equations (namely, D'Alembert, Klein-Gordon and Plate/Bi-harmonic wave equations, respectively) admit a variational solution.

2. Defocusing NLW equation. The defocusing NLW equation reads

$$w'' = \Delta w - |w|^{p-2}w + f \qquad (p > 1). \tag{61}$$

Here the proper choice of $\mathcal{W}$ is

$$\mathcal{W}(v) = \int_{\mathbb{R}^n} \left(\frac{1}{2}|\nabla v|^2 + \frac{1}{p}|v|^p \right) dx,$$

with $\mathrm{W} = H^1 \cap L^p$ and $\theta = 1 - 1/\max\{2, p\}$, so that the assumptions of Theorem 2.3 are satisfied.

3. Sine-Gordon equation. For the Sine-Gordon equation

$$w'' = \Delta w - \sin w + f$$

the suitable definition of $\mathcal{W}$ is

$$\mathcal{W}(v) = \int_{\mathbb{R}^n} \left(\frac{1}{2}|\nabla v|^2 + 1 - \cos v \right) dx$$

and, letting $\mathrm{W} = H^1$ and $\theta = 1/2$, Theorem 2.3 applies (again in view of Remark 2.6).

4. Quasilinear wave equations. Two examples of quasilinear hyperbolic equations are

$$w'' = \Delta_p w + f \qquad \text{and} \qquad w'' = \Delta_p w - |w|^{q-2}w + f \qquad (p, q > 1,\ p \neq 2). \tag{62}$$

For these equations, good choices of $\mathcal{W}$ are provided by

$$\mathcal{W}(v) = \frac{1}{p}\int_{\mathbb{R}^n} |\nabla v|^p \, dx \qquad \text{and} \qquad \mathcal{W}(v) = \int_{\mathbb{R}^n} \left(\frac{1}{p}|\nabla v|^p + \frac{1}{q}|v|^q \right) dx$$

(respectively). Here, letting $\mathrm{W} = \{v \in L^2 : \nabla v \in L^p\}$ with $\theta = 1 - 1/p$ in the former case and $\mathrm{W} = \{v \in L^2 : \nabla v \in L^p,\ v \in L^q\}$ with $\theta = 1 - 1/\max\{p, q\}$ in the latter case, Theorem 2.3 holds up to item (e). It is an open problem then to establish the existence of a variational solution for both $(62)_1$-(2) and $(62)_2$-(2).

5. Higher order nonlinear equations. An example of higher order hyperbolic equation is the nonlinear vibrating-beam equation

$$w'' = -\Delta^2 w + \Delta_p w - |w|^{q-2}\, w + f \qquad (p,\, q > 1)$$

(see e.g. [11, 12]). The suitable choice of $\mathcal{W}$ here is

$$\mathcal{W}(v) = \int_{\mathbb{R}^n} \left(\frac{1}{2}|\Delta v|^2 + \frac{1}{p}|\nabla v|^p + \frac{1}{q}|v|^q \right) dx$$

and, setting $\mathrm{W} = \{v \in H^2 : \nabla v \in L^p, v \in L^q\}$ with $\theta = 1 - 1/\max\{2, p, q\}$, Theorem 2.3 holds.

6. Kirchhoff equations. These are typical examples of nonlocal problems. For instance, consider the equation

$$w'' = \left(\int_{\mathbb{R}^n} |\nabla w|^2\,dx\right)\Delta w + f. \tag{63}$$

The natural choice of $\mathcal{W}$ is given by

$$\mathcal{W}(v) = \frac{1}{4}\left(\int_{\mathbb{R}^n} |\nabla v|^2\,dx\right)^2,$$

and, setting $\mathrm{W} = H^1$ and $\theta = 3/4$, Theorem 2.3 applies except for part (e), which consequently remains an open problem.

7. Wave equations with fractional Laplacian. Further examples of nonlocal problems are provided by hyperbolic equations involving the fractional Laplacian, as for instance

$$w'' = -(-\Delta)^s - \lambda|w|^{p-2}\,w + f \qquad (0 < s < 1,\, \lambda \geq 0,\, p > 1).$$

Here, if one takes the functional

$$\mathcal{W}(v) = c_{n,s}\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \frac{\lambda}{p}\int_{\mathbb{R}^n} |v|^p\,dx,$$

which is the natural energy associated to the fractional Laplacian, then setting $\mathrm{W} = H^s \cap L^p$ and $\theta = 1 - 1/\max\{2,p\}$ (when $\lambda > 0$, or $\mathrm{W} = H^s$ and $\theta = 1/2$ when $\lambda = 0$) one sees that the assumptions of Theorem 2.3 are satisfied.

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