Bernoulli Numbers: from Ada Lovelace to the Debye Functions

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Abstract: Jacob Bernoulli owes his fame for the numerous contributions to calculus and for his discoveries in the field of probability. Here we will discuss one of his contributions to the theory of numbers, the Bernoulli numbers. They were proposed as a case study by Ada Lovelace in her analysis of Menabrea's report on Babbage Analytical Engine. It is probable that it was this Lovelace's work, that inspired Hans Thirring in using the Bernoulli numbers in the calculus of the Debye functions.

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Introduction

Jacob Bernoulli (1655-1705) was one of the many prominent mathematicians in the Bernoulli family, a family of merchants and scholars, originally from Antwerp that resettled in Basel, Switzerland. Another member and prominent scholar of this family was Daniel Bernoulli that worked in fluid dynamics, publishing his results in the book Hydrodynamica in 1738.

Jacob Bernoulli owes his fame for the numerous contributions to calculus; along with his brother Johann, was one of the founders of the calculus of variations. Jacob's most important contribution is generally considered in the field of probability. In fact, he derived the first version of the law of large numbers in his work Ars Conjectandi [1].

Here we will discuss one of his contributions to the theory of numbers, the Bernoulli numbers. They were proposed as a case study by Ada Lovelace in her analysis of Menabrea's report on Babbage Analytical Engine. Among their several uses, we have the calculus of the Debye functions, in theory of the lattice specific heat. Probably, their use had been inspired by the Lovelace's work.

Bernoulli and the integers

One of the numbers that Jacob Bernoulli investigated was e. In 1683, Bernoulli discovered the constant e by studying a question about compound interests, which required him to find the value of the following limit, having as a value e:

(1) $$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right) = e$$

Number e is an important mathematical constant that is the base of the natural logarithm. It is approximately equal to 2.71828. Sometimes, this number is called Euler's number after Leonhard Euler, but it is not to be confused with γ, the Euler–Mascheroni constant, sometimes called simply Euler's constant. In fact, number e is also known as the Napier's constant (see Appendix for e). However, here our aim is discussing the Bernoulli's numbers.

The manner in which Bernoulli discovered then is discussed in detail in [2] by K. Ward. Bernoulli was working on the sums of the natural number power series. He noted certain facts about them and discovered a close formula by means of which the sum of a given power of natural number is calculated. This formula requires some special numbers, that is, the Bernoulli numbers. In his original formulae, Bernoulli used for then the letters A,B,C, etc.. Ward gives us the following formula:

$$\sum_{k=1}^{n} k^m = B_0 \frac{1}{m+1} n^{m+1} - B_1 n^m + \frac{B_2}{2}\binom{m}{1} n^{m-1} + \frac{B_4}{4}\binom{m}{3} n^{m-3} + \dots \qquad (2)$$

In (2), we have (with p and q integers): $\binom{p}{q} = \frac{p!}{q!(p-q)!}$.

Ada Lovelace and the Bernoulli numbers

Augusta Ada King-Noel, Countess of Lovelace (née Byron; 1815–1852) was an English mathematician known for her work on Charles Babbage's mechanical computer, the Analytical Engine. Babbage (1791-1871) is considered a computer pioneer, because of the design of this first automatic computing engine.

In 1840, Giovanni Plana of University of Turin invited Babbage for a talk about his Analytical Engine, in occasion of the Second Congress of Italian Scientists invited by the King Carlo Alberto of Savoy, congress that was held at the Academy of Science. "La presentazione appassionò gli scienziati italiani e proseguì in seminari ristretti. Particolarmente interessati a questi seminari, nei quali per la prima volta si discusse di concatenamento delle operazioni, potremmo dire di programmazione, furono il fisico Mossotti e Luigi Menabrea" [3].

Luigi Menabrea, an engineer who became a Prime Minister of Italy, wrote an article upon the Babbage's lecture in French, and this transcript was published in the Bibliothèque Universelle de Genève in October 1842. Babbage's friend Charles Wheatstone commissioned Ada Lovelace to translate Menabrea's paper into English. She added some notes to the translation, with a discussion on the calculation of Bernoulli numbers [4,5]. So we have that the Bernoulli numbers had been recognized as important for calculations that we can obtain by concatenated operations.

In reading the Ref.[5], we can see that Ada Lovelace proposed one of this operations, that, as we will see, is used for the Debye functions (see Figure 1).

We will terminate these Notes by following up in detail the steps through which the engine could compute the Numbers of Bernoulli, this being (in the form in which we shall deduce it) a rather complicated example of its powers. The simplest manner of computing these numbers would be from the direct expansion of

$$\frac{x}{\epsilon^x - 1} = \frac{1}{1 + \frac{x}{2} + \frac{x^2}{2\cdot 3} + \frac{x^3}{2\cdot 3\cdot 4} + \&c.} \tag{1.}$$

which is in fact a particular case of the development of

$$\frac{a + bx + cx^2 + \&c.}{a' + b'x + c'x^2 + \&c.}$$

mentioned in Note E. Or again, we might compute them from the well-known form

$$\mathrm{B}_{2n-1} = 2 \cdot \frac{1\cdot 2\cdot 3 \ldots 2n}{(2\pi)^{2n}} \cdot \left\{1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \cdots\right\} \tag{2.}$$

Figure 1: Snapshot from Ref.5.

Note that from the formula (2) in the Figure 1, we have the following values. For $n = 1$, we have $B_1 = 1/6$ and for $n = 2$, $B_3 = 1/30$. Then, for $n = 3$, $B_5 = 1/42$, and so on. The notation that Ada Lovelace used is different from the modern one. Let us call these numbers B^A_{2n-1}.

A modern view: According to [6,7] there are two definitions for the Bernoulli numbers. In the modern use, they are written as B_n. But in the older literature they were written as B^*_n. These numbers are a special case of the Bernoulli Polynomial $B_n(x)$ or $B^*_n(x)$, where $B_n = B_n(0)$ and $B^*_n = B^*_n(0)$. The older definition of the Bernoulli numbers is given using equation:

(3) $$\frac{x}{e^x - 1} + \frac{x}{2} - 1 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} B^*_n x^{2n}}{(2n)!} = \frac{B^*_1}{2!}x^2 - \frac{B^*_2}{4!}x^4 + \frac{B^*_3}{6!}x^6 + \ldots$$

or

(4) $$1 - \frac{x}{2}\cot\left(\frac{x}{2}\right) = \sum_{n=1}^{\infty} \frac{B^*_n x^{2n}}{(2n)!} = \frac{B^*_1}{2!}x^2 + \frac{B^*_2}{4!}x^4 + \frac{B^*_3}{6!}x^6 + \ldots$$

In the modern use, the Bernoulli number s are given by the general of the generating function:

(5) $$\frac{xe^{nx}}{e^x - 1} = \sum_{m=1}^{\infty} \frac{B_m(n)\, x^m}{m!}$$

The choice n=0 and n=1 lead to:

(6) $$n=0 \rightarrow \frac{x}{e^x-1} = \sum_{m=0}^{\infty} \frac{B_m x^m}{m!}$$

(7) $$n=1 \rightarrow \frac{x}{1-e^{-x}} = \sum_{m=0}^{\infty} \frac{B_m (-x)^m}{m!}$$

The formula for the case n=0 is the case used by Ada Lovelace (see Figure 1).

As told in [6], the modern Bernoulli numbers appears in the calculation of the Debye functions, which are:

(8) $$\int_0^x \frac{t^n dt}{e^t-1} = x^n \left[\frac{1}{n} - \frac{x}{2(n+1)} + \sum_{k=1}^{\infty} \frac{B_{2k} x^{2k}}{(2k+n)(2k!)} \right]$$

where $|x|<2\pi$ and B_n are the Bernoulli numbers.

Let us see when and where this modern use is mentioned for the first time.

Debye theory: The first use of the Bernoulli numbers in the Debye theory of the specific heat was given by Hans Thirring in 1913, as told in [8,9]. In solid state physics, the Debye model is a method developed by Peter Debye in 1912.

From the Debye model we can estimate the phonon contribution to the specific heat in a solid [10]. The model correctly predicts the low temperature dependence of the heat capacity, and, like the Einstein model, it is giving the Dulong–Petit law at high temperatures. As told in [9], for intermediate temperatures, Thirring proposed a specific heat given by:

(9) $$c = k \left\{ 1 + \sum_{n=1}^{\infty} (-1)^n B_n \frac{(2n-1)}{(2n)!} \left(\frac{hv}{kT} \right)^{2n} \right\}$$

where $hv/kT<2\pi$, h is the Planck constant and k the Boltzmann constant. B_n are the Bernoulli numbers. In (9), v is the frequency of a linear oscillator.

In the Ref.9, B_1 is equal to 1/6. In fact, it is giving the following for the heat capacity:

(10) $$C = k \left\{ 3N - \frac{B_1}{2} \frac{h^2}{k^2 T^2} \sum_{v} v^2 + \dots \right\} = 3Nk \left\{ 1 - \frac{1}{12} \frac{h^2}{k^2 T^2} \frac{1}{3N} \sum_{v} v^2 + \dots \right\}$$

It means that in [9], it was an old notation used for the Bernoulli numbers.
Let us write Eq.10 with the notation of Ada Lovelace of these numbers:

$$c = k\left\{1 + \sum_{n=1}^{\infty} (-1)^n B_{2n-1}^A \frac{(2n-1)}{2n!} \left(\frac{h\nu}{kT}\right)^{2n}\right\} \tag{11}$$

Probably, their use had been inspired to Thirring by the Lovelace's work.
In general, the expression of the heat capacity is:

$$c = k\left\{1 + \sum_{n=1}^{\infty} (-1)^n B_{2n-1}^A \frac{(2n-1)}{2n!} \left(\frac{h}{kT}\right)^{2n} \mu_{2n}\right\} \tag{12}$$

In (12), we have $\mu_n = \sum_{\nu} \nu^n$ [11].

Appendix: Number *e* has a very interesting history [12]. In [12], it is told that the number *e* first comes into mathematics in a very minor way. This was in 1618 when, in an appendix to Napier's work on logarithms, a table appeared giving the natural logarithms of various numbers. A few years later, in 1624, Briggs gave a numerical approximation to the base 10 logarithm of *e* but did not mention this number in his work. The next possible occurrence of *e* is dubious, when, in 1647, Saint-Vincent computed the area under a rectangular hyperbola. Certainly by 1661, Huygens understood the relation between the rectangular hyperbola and the logarithm. Huygens made another advance in 1661. He defined a curve that he calls "logarithmic", that in today terminology is an exponential curve, having the form $y = k\ a^x$. In 1668, Nicolaus Mercator published Logarithmotechnia which contains the series expansion of log(1+x). In this work Mercator uses the term "natural logarithm" for the first time for logarithms to base *e*. As told in [12], it is curious that, since the works on logarithms had come so close to recognizing the number *e*, that *e* was first "discovered" not through the notion of logarithm but rather through a study of compound interest, as did in 1683 Jacob Bernoulli looked at the problem of compound interest [12].

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