Countering the False Myth of Democracy: Boosting Compressed Sensing Performance with Maximum-energy Approach

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***Abstract*—Compressed Sensing (CS) is an effective way to sample a signal at a sub-Nyquist rate, i.e., by using a number of measurements smaller than the number of samples required when using the standard Nyquist approach. Measurements are obtained as linear projections of input signals along random sensing vectors. CS has been often regarded as a *democratic* method, in the sense that each measurement contributes to signal reconstruction with a similar amount of information.**

In this paper, by combining empirical observations with results from recent papers, we propose a different point of view, and show that CS is an *oligarchic* approach where performance is basically set by the measurements with the highest energy. This allows us to propose a new CS-based approach that bases the reconstruction on the maximum-energy measurements only and improves the compression performance with respect to classical approaches.

I. Introduction

The recently introduced signal acquisition paradigm known as Compressed Sensing (CS) [1] is capable to add signal compression capabilities directly at the acquisition phase. This paves the way to the design of analog-to-information conversion stages that match the resource they need with the actual information content of the signal [2], [3], [4], with a possible reduction in power consumption. In addition a limited form of lossy compression is obtained almost for free [5]. From this last point of view, the most peculiar aspect of CS, is the radical shift in the complexity balancing. In fact, compression becomes a computationally simple task, being based on the linear projection of the input signal over a set of typically random sensing vectors. On the contrary, signal recovery from compressed measurements requires the solution of a computationally hard convex optimization problem [6]. This makes CS particularly appealing in scenarios like Body Area Sensor Networks (BASNs) [7], [8] where many simple, miniaturized, battery-powered sensor nodes used for the acquisition of biological signals, can take advantage from the compression capabilities available at reduced cost offered by CS techniques, while the more power-hungry decoding stage is executed on the gateway where energy issues are less relevant.

The aim of this paper is to focus on the compression performance offered by CS. In particular we will consider the role played by measurements. CS has been many times considered a *democratic* system in the sense that each measurement carries the same amount of information to the signal reconstruction algorithm. This is certainly true in the theoretical setting in which the concept is developed, i.e., when definitions and derivations concern with worst-case analyses to deliver mathematical guarantees, and no system optimization is sought. Yet, it is well known that such an approach gives only extremely loose bounds on real performance and that actual design of effective CS systems should follow different guidelines.

As an example, as already observed in recent papers, in practical cases the higher the magnitude of a measurement, the higher the amount of information it brings to the final reconstruction [9]. Hence, CS is actually an *oligarchic* system, where the measurements with the highest energy decide reconstruction quality.

This paper is to proposes a new compression strategy named *maximum-energy CS* based on this property. The encoder computes a large number of candidate measurements, but transmits to the decoder only those with the largest energy. This ensures a boost in compression performance. To apply this method one has to cope with the fact that both the measurements and the indication of which measurements among all candidates are used has to be encoded and transmitted, with a clear overhead. Even considering this, in practical cases, performance in terms of bit required for a given reconstruction quality improves with respect to standard CS.

The paper is organized as follows. In Sec. II the CS theory will be briefly reviewed, along with the concept of CS democracy and CS oligarchy. Then, in Sec. III the new coding approach is described, while some experimental results are proposed in Sec. IV. Finally, we draw the conclusion.

II. Compressed Sensing Framework

Let us assume that $x \in \mathbb{R}^n$ is an instance of the input signal composed by the n samples $x_0, x_1, \ldots, x_{n-1}$ collected at Nyquist frequency f_N. Roughly speaking, x is a discrete-time representation of the input signal, but could also stands for its digital representation if assuming that the x_j are actually defined in the subspace of $\mathbb{R}$ created by a quantization function. The entire CS framework is based on the *sparsity* assumption, i.e., each instance x can be represented by a linear combination of few vectors of a proper basis $\Psi \in \mathbb{R}^{n\times n}$ such that $x = \Psi\alpha$, where $\alpha \in \mathbb{R}^n$ is a coefficients vector with no more then $\kappa \ll n$ non-zero elements. In this case we say that the class of input signals is κ-sparse. The CS encodes information by projecting x on a set of m usually randomly generated sensing sequences $a_j \in \mathbb{R}^n$, $j = 0, 1, \ldots, m-1$ (with $m < n$ in order to enable signal *compression*), and arranged as the rows of a sensing matrix $A \in \mathbb{R}^{m\times n}$.

Mathematically, m measurements $y_0, y_1, \ldots, y_{m-1}$ are generated and collected in a measurement vector $y \in \mathbb{R}^m$ as

$$y = Ax = A\Psi\alpha. \qquad (1)$$

Recovers x from y is an ill-posed inverse problem, i.e., it has an infinite number of solutions. The decoder stage, given y (sent by the encoder stage) and both A and Ψ (where A is *a priori* shared knowledge), reconstruct the input signal as $\hat{x} = \Psi\hat{\alpha}$ exploiting the sparsity assumption: over the infinite set of vectors $\hat{\alpha}$ mapped by $A\Psi$ on y, we select the sparsest one. To this aim, the most common approach is the solution of the convex optimization problem given by the following basic pursuit denoising (BPDN) formulation [1], [6].

$$\begin{aligned} &\min_{\hat{\alpha}} \quad \|\hat{\alpha}\|_1 \\ &\text{s.t.} \quad \|A\Psi\hat{\alpha} - y\|_2^2 \le \varepsilon^2 \end{aligned} \qquad (2)$$

where $\|\cdot\|_p$ stands for standard p-norm, and ε is used to take into account possible source of noise in the process such as non-idealities in the sensing circuit, or even the quantization noise when assuming the x_j belong to a quantized set.

Guarantees on the correct reconstruction of x are based on some properties of A. In particular the CS theory states that the reconstruction error $\hat{x} - x$ is vanishing with probability one if A satisfies the so called Restricted Isometry Property (RIP) [10], [11] and when the number of measurements $m > O(\kappa \log(n/\kappa))$ [6]. The easiest way to ensure RIP is to generate the rows of A as instances of independent and identically distributed (i.i.d.) Gaussian (or Sub-Gaussian) random variables.

Based on this framework, many papers advocate an alleged *democratic* behavior of the CS based on the (correct) mathematical observation that, by replacing the generic random a_j sensing vector with another random one, the RIP property is still verified. The (wrong) conclusion is that any sensing vector can be replaced with another, similar one without any change in system performance.

This myth of democracy, however, has been countered by many empirical observation. In [9] authors observe that, indicating with y_j the generic j-th element of y, performance in terms of reconstructed signal quality increases when replacing measurements presenting a low energy (i.e., that for which $\|y_j\|_2$ is small) with new ones with increased $\|y_j\|_2$. In [12] a statistical matching between x and y_j is proposed with the aim of increasing $\|y_j\|_2$, on the average. This approach, known as *rakeness-based CS*, relies on the knowledge of the second-order statistic of the signal to acquire and has been proven to be very effective in increasing reconstruction quality. This reveals that CS is an *oligarchic* system: measurements with high energy have an important role in signal reconstruction.

The approach proposed here does not need the second-order prior, it is not adapted offline to the average features of the signal, but it automatically adjusts measurement choice to the characteristic of each individual signal instance. From this point of view it is much more flexible than the rakeness-based approach though this comes at the cost of an overhead that may limit compression performance.

III. Boosting CS performance: the maximum-energy approach

The proposed maximum-energy approach is described as follows, starting from a shared knowledge of M different (random) sensing vectors $a_j \in \mathcal{A}$, $j = 0, 1, \ldots, M-1$ between encoder and decoder stage.

1) The encoder computes M candidate measurements by using all the M sensing vector in $\mathcal{A}$.
2) Among the M candidate measurements, the m ones (with $m < M$) with the highest energy are identified and selected. Let $\mathcal{J}$ be the set of indexes j corresponding to the sensing vectors generating them.
3) Both $\mathcal{J}$ and the m selected measurements are sent to the decoder.
4) The decoder stage solves (2) by composing y as the vector of selected measurement, and A as the collection (ordered accordingly to y) of the sensing vector a_j identified by $\mathcal{J}$.

The proposed system is that depicted in Fig. 1, where $\mathcal{A}$ is an *a priori* shared information (e.g., it is generated by two replicas of the same Pseudo-Random Number Generator initialized by the same seed), while actual information transferred from the encoder to the decoder is given by the values of y as in the standard CS, but also by the encoding of side-information $\mathcal{J}$.

Note that the presence of this overhead has consequences when testing the performance of the proposed algorithm, since the amount of information in $\mathcal{J}$ and in y should be measured by using the same unit and added the each other.

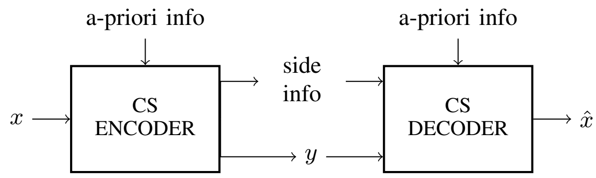


Fig. 1. The encoder-channel-decoder model with additional signal path.

Let us indicate with b_x the number of bit required to code each sample of x. In particular, if x is already a quantized signal, b_x is the number of bit used; if x is an analog signal affected by noise (due, for example, to the analog stage processing it) and with an estimated signal-to-noise ratio SNR, we can consider b_x by using the classic equivalence $\mathrm{SNR} = 6.02 b_x + 1.76\mathrm{dB}$. We define this SNR as the target SNR we would like to achieve in reconstruction.

In conclusion, the value of b_x is enough to fully characterize a Nyquist system, since from it, it is possible to get both signal reconstruction quality in terms of SNR, and the total number of bit used as $n\, b_x$.

In a CS system, signal is reconstructed by using (2) given y and A. Signal retrieval is characterized by a reconstruction SNR (RSNR) as

$$\mathrm{RSNR} = \left(\frac{\|x\|_2}{\|x - \hat{x}\|_2} \right)_{\mathrm{dB}}.$$

To assess the amount of information transmitted note that coding the side information means identifying one out of $\binom{M}{m}$ possible subsets of m elements out of a larger set of M elements. By means on combinatorial encoding one may do so using $\lceil \log_2 \binom{M}{m} \rceil$ bit [13, p. 27–30].

Accuracy required for the quantization of y is still an open problem in CS theory [14]. In standard CS, it is known that measurements, due to the central limit theorem, have a zero-mean Gaussian distribution, whose standard deviation is $\sqrt{n}$ times that of x. This means that, in standard CS system, a conservative choice to get a measurement quantization noise aligned with the input signal one could be the value $b_y = b_x + \log_2 \sqrt{n}$.

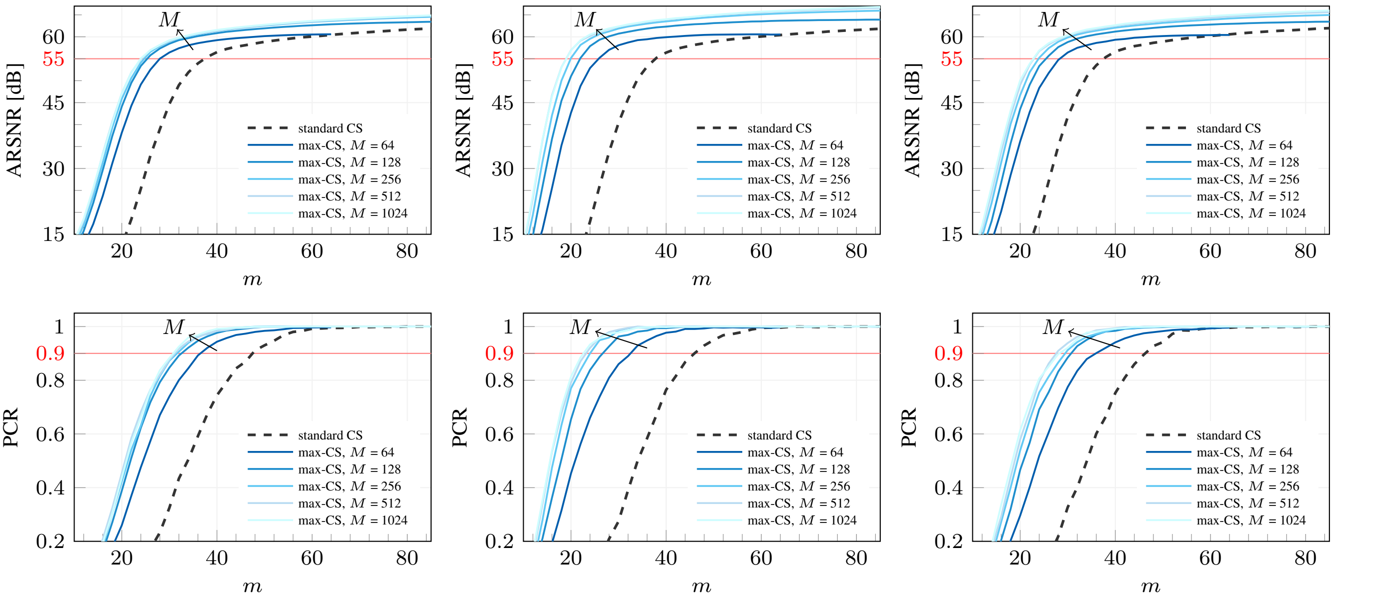
Though the proposed approach modifies the measurements distribution by choosing maximum magnitude values, in the following we consider the conservative assumption to use the same b_y that allow in a standard CS system to align measurement with input signal SNR.

This said Nyquist acquisition, standard CS, and the maximum-energy CS can be compared by using the following figures of merit:

- average RSNR for a proper class of signal (ARSNR);
- probability of correct reconstruction (PCR), defined as the probability the a generic instance x is reconstructed with a quality at least equal to the target SNR;
- compression ratio $\mathrm{CR}^{\mathrm{bit}}$ in terms of number of bits. This is defined as the ratio between the number of bits used in the encoder (that is $m\, b_y$ for a standard CS, $m\, b_y + \lceil \log_2 \binom{M}{m} \rceil$ for the proposed maximum-energy CS) to achieve the target SNR, and the number of bit used by a pure Nyquist system ($n\, b_x$).

IV. Experimental results

With the aim of testing performance of the identified algorithm, we propose in this section results from Montecarlo simulation of a synthetic CS system. Each input signal instance x is randomly drawn with a sparsity level $\kappa = 6$ using a sparsity basis Ψ given by the discrete-cosine transform (DCT) with $n = 128$. Each x has been perturbed by a white noise whose power is such that signal-to-noise ratio (SNR) is $60\,\mathrm{dB}$ to emulate non-idealities of the sensing stage or a $\approx 10\,\mathrm{bit}$ quantization noise. Due to this, we set a target quality



(a) (b) (c)

Fig. 2. Montecarlo comparison for 6-sparse signals with $n = 128$ between performance of standard CS (dashed) and maximum-energy CS (solid) in terms of both ARSNR (top) and PCR (bottom) for zero localization signals (a), medium localization LP signals (b) and for high localization HP signals (c).

equal to 55 db. This corresponds to $b_x = 9$ bit. The total number of bit used in a Nyquist system is given by $n\, b_x = 1152$ bit.

Tree different input signal classes have been considered. In the first one (referred to as no-localization) each x is generated by randomly drawn κ column of Ψ with associated amplitudes as instances of zero-mean unity-variance Gaussian process.

Additionally, two localized classes, a high-pass (HP) one and low-pass (LP) one, have been considered. In details, a synthetic κ-sparse localized vector is generated starting from an n-dimensional random Gaussian vector x' with zero mean and correlation matrix C^x. Then, being $\alpha' = \Psi^{-1}x'$, a κ-sparse coefficient vector α is generating by considering the κ largest modulus entries of α' and setting all others to zero. Finally, we set $x = \Psi\alpha$.

This is based on the observation that, if the eigenvalues of C^x are not identical, then x' is localized and this property is approximately propagated through sparsification since the $n-\kappa$ smallest components of α' are discarded. We consider here an exponential correlation profile with $C^x_{i,j} = r^{|i-j|}$ with two different value of r. One is $r = 0.81$ corresponding to a medium localization LP and one, $r = -0.96$, for high localization HP behavior.

Each random instance x generated in one of the aforementioned ways is encoded by (1) and decoded with the BPDN problem in (2) by using the SPGL1 tool[1]. Both the standard CS and the maximum-energy CS are considered. In the first case, A is generated by collecting m sensing vectors randomly drawn as instances of an i.i.d. Gaussian random variables; in the second one, $\mathcal{A}$ is composed by M random vectors, of which only m are used for computing y in according to the largest modulus observed. In both cases we assume that $b_y = 12$.

Figure 2 shows system performance in terms of both ARSNR (top plots) and PCR (bottom plots) for all considered classes of signals as a function of the measurement vector cardinality m where for the PCR figure of merit the target SNR is fixed to 55 dB. Two target values, equal to ARSNR = 55 dB and PCR = 0.9 are highlighted to help performance comparison. It is clear from the plots how the maximum-energy CS outperforms standard CS in terms of signal reconstruction quality given m, or equivalently, in terms of m required to get a target signal reconstruction quality, for all considered cases. This is more evident when considering the two localized examples, and confirms the non-democratic behavior of CS systems.

However, since the maximum-energy CS take advantage from an information overhead given by $\mathcal{J}$, the only fair comparison can be made in terms of amount of information required for a target quality. To keep this into account, Table I proposes, limited to the medium localized LP signal, a comparison for different M values of the proposed method performance with that of the standard CS in terms of $\mathrm{CR}^{\mathrm{bit}}$, accordingly to the definition of the previous section. The considered value of m is the one, accordingly to Figure 2, that ensures the target figure of merit ARSNR = 55 dB or PCR = 0.9.

TABLE I
BITWISE COMPRESSION RATIOS OF MAXIMUM-ENERGY CS AND STANDARD CS WHERE STRAIGHTFORWARD ENCODING OF $n = 128$ SAMPLES WOULD REQUIRE 1152 bit.

ARSNR = 55 dB

maximum-energy CS				standard CS		
m	M	$m\, b_y + \left\lceil \log_2 \binom{M}{m} \right\rceil$	$\mathrm{CR}^{\mathrm{bit}}$	m	$m\, b_y$	$\mathrm{CR}^{\mathrm{bit}}$
26	64	372	3.10	38	456	2.53
22	128	346	3.33	38	456	2.53
20	256	338	3.41	38	456	2.53
19	512	342	3.37	38	456	2.53

PCR = 0.9

maximum-energy CS				standard CS		
m	M	$m\, b_y + \left\lceil \log_2 \binom{M}{m} \right\rceil$	$\mathrm{CR}^{\mathrm{bit}}$	m	$m\, b_y$	$\mathrm{CR}^{\mathrm{bit}}$
34	64	469	2.46	46	552	2.01
28	128	430	2.68	46	552	2.01
25	256	415	2.78	46	552	2.01
24	512	425	2.71	46	552	2.01

[1] online available at https://www.math.ucdavis.edu/ mpf/spgl1/

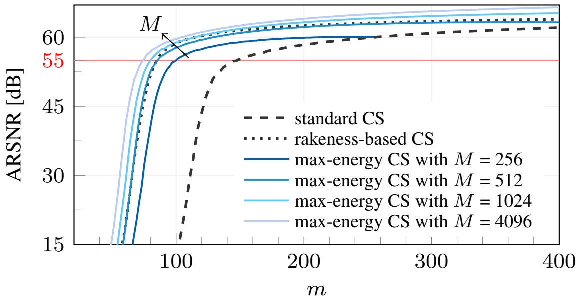


Fig. 3. Montecarlo comparison for 25-sparse signals with $n = 512$ between performance of standard CS (dashed), rakeness based CS (dotted) and maximum-energy CS (solid) in terms of both ARSNR for medium localization LP signals.

Accordingly to this point of view, optimal performance is achieved for an intermediate value of M. This is actually expected. If M is too low, it is clear that the number of candidate measurements is not enough to take full advantage from the maxim-energy approach. As M increases, also performance in terms of signal quality reconstruction increases due to the higher probability to include high-energy measurements in y. However, when M is too large, the overhead to encode $\mathcal{J}$ is too large with respect to the gain in signal quality reconstruction, and performance in terms of $\mathrm{CR}^{\mathrm{bit}}$ drops.

As a final step, we propose the comparison between the maximum-energy approach and the rakeness approach proposed in [12][2]. Both methods aim at maximizing performance (i.e., maximizing reconstruction quality given m, or minimizing m given a target reconstruction quality) based on the maximization of the energy measurements. However, the first one is based on an *a-posteriori* maximization: independently of the input signal, many measurements are taken, and only the most useful ones are considered. On the contrary, the second approach is based on an *a-priori* maximization: the sensing vectors a_j are randomly generated by a stochastic process maximizing the expected energy of (1). The advantage of the first approach is to be signal-agnostic: it can be applied without any a-priori knowledge on x, but has the drawback of the encoding of $\mathcal{J}$. The disadvantage of second approach is that it can be applied only to input signal that are localized, and also to require to known in advance its statistical characterization, but no additional information needs to be encoded.

In Figure 3 the two approaches, along with the standard CS, are compared in terms of ARSNR for a LP localized signal where $\kappa = 25$ and $n = 512$. Both rakeness CS and maximum-energy CS clearly outperform standard CS due to the localization of the signal. Furthermore, assuming M large enough, maximum-energy CS has better performance, being an a-posteriori maximization much easier to do and effective than an a-priori one. Indeed, the advantage of the maximum-energy CS of not requiring any information on the input signal in advance is paid in terms of $\mathrm{CR}^{\mathrm{bit}}$. System performance in terms of $\mathrm{CR}^{\mathrm{bit}}$ to reach the target figure of merit $\mathrm{ARSNR} = 55\,\mathrm{dB}$ is compared in Table II. Despite the fact that the number of measurement m required to reach the target quality is (almost) always lower with the maximum-energy approach, the required information overhead makes, for this particular case, the rakeness approach more convenient.

[2]Matlab code online available at http://cs.signalprocessing.it

TABLE II
BITWISE COMPRESSION RATIOS OF MAXIMUM-ENERGY CS, RAKENESS-BASED CS AND STANDARD CS FOR ARSNR $\geq 55\,\mathrm{dB}$.

maximum-energy CS			standard CS		rakeness CS	
m	M	$\mathrm{CR}^{\mathrm{bit}}$	m	$\mathrm{CR}^{\mathrm{bit}}$	m	$\mathrm{CR}^{\mathrm{bit}}$
100	256	3.19	149	2.58	87	4.41
87	512	3.35	149	2.58	87	4.41
81	1024	3.35	149	2.58	87	4.41
75	4096	3.21	149	2.58	87	4.41

V. Conclusion

In this paper we have introduced a new approach for optimizing CS performance based on the computation of a large number of candidate measurements, and on the transmission/storage only of that with the higher energy. This maximum-energy CS approach is proved to be extremely effective in inreasing reconstruction performance with respect to standard CS approach. Furthermore, even considering the overhead given by the transmission to the decoder stage of the information on which candidate measurements have been actually used, this system could greatly improve performance in terms of signal compression.

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